On the Combination of Biased Members *

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August 6, 2023

Abstract

We consider the information transmission problem within organization, and especially we focus on how to combine biased subordinates to elicit truthful information from them. We assume that the directions of subordinates' biases are common knowledge, but not their sizes. This leads to the results that homogeneous combination of subordinates are better than heterogeneous one. This is because in the case of homogeneous subordinates, the effect of one's false report might be accelerated by another false report and this anxiety reduces an incentive for Subordinates to send a false report.

*I gratefully acknowledge Junichiro Ishida, Akifumi Ishihara, Kohei Kawamura, Shintaro Miura, Hideo Owan, and Hitoshi Sadakane for helpful comments. This research is financially supported from the Japan Society for the Promotion of Science (Nos. 20K01547, 19H01471, and 21H00697). Of course, any remaining errors are my own.

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1 Introduction

In their seminal book, Cyert and March [7] consider the organization as a communication system and emphasize the fact that there is some bias in the information system because some members of the organization often attempt to manipulate information for manipulating the decision of the organization. The purpose of this paper is to consider the information transmission problem within organization by using a cheap talk model à la Crawford and Sobel [6]. Particularly, we focus on how to combine biased members to elicit truthful information from them. The key element is the uncertainty about the size of members' biases.

Krishna and Morgan [16] is a seminal paper dealing with strategic information transmission between a receiver and multiple senders. Krishna and Morgan consider a sequential cheap talk model with two senders and compare between the case of heterogeneous senders (i.e., their biases are in opposite directions) and the case of homogeneous senders (i.e., their biases are in the same direction). [1] Their main results are that, under some mild conditions, heterogeneous senders are superior to homogeneous senders in the receiver's viewpoint.

One of the crucial assumptions in Krishna and Morgan is that senders' biases are common knowledge among players. It is often the case, however, that we know which direction our colleagues want to induce the organization's decision, but we do not know the exact strength of their willingness. This paper adopts this type of assumption which is often more natural in real organization; we assume that *the directions* of senders' biases are common knowledge, while *their exact sizes* are not. We show that this change of setting completely reverses the conclusion.

To be more precise, we consider the game played by one boss and multiple subordinates. Subordinates are the senders of cheap talk messages, while Boss is the receiver. We consider a model with binary state. We also consider two types of Subordinates: upward biased Subordinates and downward biased Subordinates. The former is those whose biases are distributed over a positive region, while the latter those whose biases are distributed over a negative region. Our main results are that Boss prefers completely homogeneous combination of Subordinates to any other combination of Subordinates.

[1] Krishna and Morgan consider a one-dimensional state space. For multi-dimensional state space, see Battaglini [2].

A rough intuition of the results are as follows: In the case of any combination other than completely homogeneous one, even if one Subordinate would send a false report, he expects the other Subordinate's report would be conflicting, so his report would be discounted by Boss. This makes it easier for him to send a false report. On the other hand, in the case of completely homogeneous subordinates, the effect of one's false report might be accelerated by another false report. In result, this anxiety reduces Subordinate's incentive to send a false report. This is a new mechanism for information transmission within organization.

1.1 Related Literature

Cheap talk models with one sender whose bias is uncertain have been analyzed by Morgan and Stocken [23], Dimitrakas and Sarafidis [11], Li and Madarász [20] for static situations, and Sobel [32], Benabou and Laroque [3], Morris [24] for dynamic situations. Cheap talk models with two senders of which biases are uncertain also have been analyzed by Li [18, 19], Karakoç [15]. However, they have not addressed the combination of directions of senders' biases.

The combination of directions of senders' biases has been addressed by Rantakari [28, 29] and Shimizu [30].[2] Rantakari [28, 29] consider a model in which sender's private information pertains to the probability of his own project or idea which is independent from that of another sender. On the other hand, in our model senders' private information is one common state of the world and thus they are completely correlated. Shimizu [31] is the most closely related paper to the present one. I investigate the combination of directions of senders' biases when they are imperfectly informed of the realized state by adopting Austen-Smith [1] as baseline model. The present model is simpler in the sense that a state space is binary. Instead, it allows much more general environments, especially concerning bias distributions and the number of Subordinates.

The hommogenuity/heteroenuity in principal-agent relationships has been discussed by many papers.[3] Most of them, however, consider the hom-

[2] Mechtenberg and Münster [22] discuss the combination of directions of sender's and strategic mediator's biases.

[3] There are mainly two lines of research: those which deal with separation of decision and implementation (Blanes i Vidal and Möller [34], Bester and Krähmer [4], Landier et al. [17], Marino et al. [21], Van den Steen [10], Ishihara and Miura [13], and Itoh and

mogenuity/heteroenuity between principal and agent. The exceptions are Prasad and Tomaino [27], Prasad and Tanase [26], and Rantakari [28, 29] referred above. The first two papers examine the divergence between two agents. Their models are, however, much different from ours in the sense that there is no asymmetric information in Prasad and Tomaino and there is no strategic information transmission in Prasad and Tanase.

1.2 Organization of the Paper

In Section 2, we present our model and refers to strategy and equilibrium concept in our setting. In Section 3, we present a basic result in the case of 2 Subordinates and uniformly distributed biases. In Section 4, we show that the previous results are extended to more general environments, more precisely, those with any arbitrary number of Subordinates and more generally distributed biases. In Section 5. We show the robustness of our results by discussing the possibility of their extensions to more general environments. In Section 6, we conclude the paper. In Appendix, we present a few proofs omitted in the main text.

2 Model

2.1 Environment

There are $n+1$ players where $n \geq 2$. Player 0 is called Boss (female), who is the receiver of cheap talk messages. Players $1, 2, \ldots, n$ called Subordinates (males), who are the senders of cheap talk. The state of the world is denoted by t, which takes binary values 0 or 1 with equal probabilities. $a \in \mathbb{R}$ is an action chosen by Boss. For ease of exposition, we denote the state space and the action space by $T := \{0, 1\}$ and $A := \mathbb{R}$, respectively.

We assume that player i's payoff function is $U_i(t, b_i, a) := -(t + b_i - a)^2$ where b_i is called player i's bias. We assume that $b_0 = 0$ for normalization. This implies that Boss's best response is to choose the action equal to Boss's belief on $t = 1$. Subordinate i's bias is distributed by a distribution function $F_i(b_i)$, specified in detail later. We assume that $t, b_1, b_2, \ldots, b_n$ are mutually independent.

Morita [14]) and those which deal with information acquisition (Szalay [33], Hori [12], Che and Kartik [5], Van den Steen [9], Omiya et al. [25], and de Bettigniesand Zábojnìk [8]).

We assume b_0 is common knowledge, while b_i for $i \neq 0$ is Subordinate i's private information. A distribution function F_i for $i \neq 0$ is, however, assumed to be common knowledge.

According to Krishna and Morgan [16], we consider a sequential information transmission protocol. The timing of the game is as follows:

Stage 0: A state t is realized, which is only observed by Subordinates. Also, for each $i \neq 0$, b_i is realized, which is only observed by Subordinate i.

Stage i **(**$i = 1, \ldots, n$**):** Subordinate i publicly announces a cheap talk message $m_i \in M_i$, which is heard by Boss and other Subordinates.

Stage $(n+1)$**:** Based on the received message profile, Boss chooses an action $a \in A$.

We assume that each Subordinate's message space M_i contains 2 or more messages.

Let F be a baseline distribution function. We focus on F satisfying the following assumptions.

Assumption 1

- F is continuous.
- $F\left(\frac{1}{4}\right) > 0$ and $F\left(\frac{1}{2}\right) < 1$.
- Supp $F \subseteq [0, 1]$.

The first assumption is made for the existence of pure strategy equilibrium. The second assumption is made for excluding degenerate cases. Roughly speaking, a random variable is widely distributed, not clustered at a particular point. The third assumption is the most critical. The most important thing is that it never takes a negative value.

By using a baseline distribution function, we define two types of Subordinates as follows. The first type Subordinates are called *upward biased*, where their bias distributions F_i is equal to F. The second type Subordinates are called *downward biased* where their bias distributions satisfy $F_i(b) = 1 - F(-b)$ for any b. Combined with the assumption that bias distributions are common knowledge, these assumptions implies that the other players know the direction of a Subordinate while they do not know the realized value. To put it differently, it is common knowledge that a bias of upward or downward biased Subordinate is non-negative or non-positive, respectively. Also, it is assumed that Subordinates are symmetric except their bias directions. We assume $F_1(b) = F(b)$ without loss of generality.

2.2 Equilibrium

In order to avoid technical difficulties, we focus on Subordinates' pure strategies. Generally, Subordinate i pure strategy is a function μ_i such that $\mu_i : T \times B_i \times M_1 \times \cdots \times M_{i-1} \to M_i$. We, however, focus on the class of Subordinates' pure strategies that are characterized by some thresholds. b_+ or b_- is a threshold for upward or downward biased Subordinates, respectively. For ease of exposition, we denote them such that $b_+ \geq 0$ and $b_- \geq 0$. In this class, there are two messages each of which corresponds to each state. We denote a message corresponding to $t = 0$ or 1 by $m_i = 0$ or 1, respectively.

Using this notation, we define a strategy for an upward biased Subordinate i as follow. When $t = 1$ is realized, he always sends a truthful message $m_i = 1$ in any equilibrium since he has no incentive to lie. On the other hand, when $t = 0$ is realized, the size of his bias matters. When his bias is sufficiently small such that $b_i \leq b_+$, he sends a truthful message $m_i = 0$. When his bias is sufficiently large such that $b_i > b_+$, he sends a false message $m_i = 1$. Note that this strategy is history-independent and order-independent.

Similarly, we define a strategy for a downward biased Subordinate i as follows. When $t = 0$ is realized, he always sends a truthful message $m_i = 0$ in any equilibrium. On the other hand, when $t = 1$ and $|b_i| \leq b_-$, he sends a truthful message $m_i = 1$. When $t = 1$ and $|b_i| > b_-$, he sends a false message $m_i = 0$. This strategy is also history-independent and order-independent.

These thresholds b_+ and b_- can be said to measure the degree of organizational governance concerning the Subordinates' incentives to manipulate information relevant to the organization decision. In other words, the larger the equilibrium thresholds are, the more efficient information Boss can elicit from Subordinates.

Generally, Boss's pure strategy is a function α such that $\alpha : M_1 \times \cdots \times M_n \to A$. It is, however, much more simply written when Subordinates follow history-independent and order-independent threshold strategies defined above. Boss considers $m_i = 1$ (resp. $m_i = 0$) dubious when it is sent by upward (resp. downward) biased Subordinate i. It follows that Boss's best response is depending upon the numbers of dubious messages. To be more precise, Boss's best response is defined as $a\left(\tilde{k},\tilde{\ell}\right)$ where $\tilde{k}$ is the number of dubious messages sent by upward biased Subordinates and $\tilde{\ell}$ is the number of dubious messages sent by downward biased Subordinates.

We adopt Perfect Bayesian equilibrium as solution concept. A PBE associated with the threshold strategy, which we simply call it a *threshold strategy equilibrium* hereafter, is defined a triple, $(b_+, b_-, a(\cdot,\cdot))$.

I would like to make a few remarks. First, we can prove that there exists no fully revealing PBE in this environment. This is done in Appendix A. Second, our focus on the threshold strategy equilibrium is not restrictive. To be exact, in Appendix B, we show that any PBE is essentially outcome-equivalent to some threshold strategy equilibrium.

2.3 Organization mode

In this paper, we call a combination of Subordinates as *organization mode*. To be more precise, organization mode (n,k) means a combination of Subordinates in which there are n Subordinates in total, and among them, k Subordinates are upward biased and $(n-k)$ Subordinates are downward biased. Particularly, in the case of $n=2$, organization mode $(2,2)$ is called as *homogeneous (Subordinates) mode*, while organization mode $(2,1)$ is called as *heterogeneous (Subordinates) mode*. Due to the symmetricity referred above, it suffices to only consider the case of $k \geq \frac{n}{2}$ without loss of generality.

Hereafter, by n in organization mode (n,k), we mean that there are n Subordinates active in information transmission. For example, when there are 2 Subordinates, but only Subordinate 1 plays an influential strategy and the other Subordinate always babbles, this organization mode is $(1,1)$.

In this paper, we compare threshold strategy equilibria among various organization modes. We denote an equilibrium variable x under organization mode (n,k) by $x^{(n,k)}$.

3 2 Subordinates with Uniformly Distributed Biases

In this section, we consider the case in which there are 2 Subordinates with uniformly distributed biases, i.e., $n=1,2$ and $F(b)=b$ for $b\in[0,1]$.

3.1 Single Subordinate Mode

First, we consider organization mode $(1, 1)$. As referred previously, this is the situation in which only one Subordinate plays an influential threshold strategy and the other Subordinate always babbles. This mode can be also interpreted as the situation in which Boss intentionally ignores a particular Subordinate's message.

We obtain

$$\begin{aligned} a^{(1,1)}(0,0) &= 0, \\ a^{(1,1)}(1,0) &= \frac{1}{1+\left(1-b_{+}^{(1,1)}\right)}. \end{aligned}$$

The Subordinate's incentive condition requires

$$-\left(0+b_{+}^{(1,1)}-a^{(1,1)}(0,0)\right)^{2}=-\left(0+b_{+}^{(1,1)}-a^{(1,1)}(1,0)\right)^{2},$$

which boils down to

$$b_{+}^{(1,1)}=\frac{a^{(1,1)}(1,0)}{2}.$$

Therefore, $b_{+}^{(1,1)}$ is the solution of the following equation:

$$G^{(1,1)}(b):=b\,(1-b)-\left(\frac{1}{2}-b\right)=0.$$

Solving this, we obtain

$$b_{+}^{(1,1)}=1-\frac{1}{\sqrt{2}}\fallingdotseq 0.29289.$$

The receiver's equilibrium expected payoff is

$$\begin{aligned} EU_{0}^{(1,1)} &= -\frac{1}{2}\left(1-b_{+}^{(1,1)}\right)\left(a^{(1,1)}(1,0)\right)^{2}-\frac{1}{2}\left(1-a^{(1,1)}(1,0)\right)^{2} \\ &\fallingdotseq -0.20711. \end{aligned}$$

3.2 Homogeneous Subordinates Mode

Next, we consider organization mode $(2,2)$. We obtain

$$\begin{aligned} a^{(2,2)}(0,0) &= a^{(2,2)}(1,0) = 0, \\ a^{(2,2)}(2,0) &= \frac{1}{1+\left(1-b_{+}^{(2,2)}\right)^{2}}. \end{aligned}$$

Subordinate's message is influential only if he is pivotal, in other words, the other Subordinate sends a false message. Therefore, the Subordinates' incentive condition requires that

$$-\left(0+b_{+}^{(2,2)}-a^{(2,2)}(1,0)\right)^{2} = -\left(0+b_{+}^{(2,2)}-a^{(2,2)}(2,0)\right)^{2},$$

which boils down to

$$b_{+}^{(2,2)} = \frac{a^{(2,2)}(2,0)}{2}.$$

Therefore, $b_{+}^{(2,2)}$ is the solution of the following equation:

$$G^{(2,2)} := b\,(1-b)^{2} - \left(\frac{1}{2}-b\right) = 0.$$

Numerically, it is verified that $b_{+}^{(2,2)} \fallingdotseq 0.3522$. The receiver's equilibrium expected payoff is

$$\begin{aligned} EU_{0}^{(2,2)} &= -\frac{1}{2}\left(1-b_{+}^{(2,2)}\right)^{2}\left(a^{(2,2)}(2,0)\right)^{2} - \frac{1}{2}\left(1-a^{(2,2)}(2,0)\right)^{2} \\ &\fallingdotseq -0.14780. \end{aligned}$$

3.3 Heterogeneous Subordinates Mode

Lastly, we consider organization mode $(2,1)$. We obtain

$$\begin{aligned} a^{(2,1)}(0,1) &= 0, \\ a^{(2,1)}(1,1) &= \frac{\left(1-b_{-}^{(2,1)}\right)}{\left(1-b_{-}^{(2,1)}\right)+\left(1-b_{+}^{(2,1)}\right)} \\ a^{(2,1)}(1,0) &= 1. \end{aligned}$$

Then, by the Subordinates' incentive conditions, we obtain

$$b_+^{(2,1)} = \frac{a^{(2,1)}(1,1)}{2},$$
$$b_-^{(2,1)} = \frac{1 - a^{(2,1)}(1,1)}{2}.$$

Therefore, $b_+^{(2,1)}$ is the solution of the following equation:

$$G^{(2,1)}(b) := b(1-b) - \left(\frac{1}{2} - b\right)\left\{1 - \left(\frac{1}{2} - b\right)\right\} = 0.$$

Solving this, we obtain

$$b_+^{(2,1)} = b_-^{(2,1)} = \frac{1}{4}.$$

The receiver's equilibrium expected payoff is

$$\begin{aligned} EU_0^{(2,1)} &= -\frac{1}{2}\left(1 - b_+^{(2,1)}\right)\left(a^{(2,1)}(1,1)\right)^2 - \frac{1}{2}\left(1 - b_-^{(2,1)}\right)\left(1 - a^{(2,1)}(1,1)\right)^2 \\ &= -0.1875. \end{aligned}$$

3.4 Comparison Results

Comparing the above three organization modes, we obtain the following comparison results:

- $b_+^{(2,1)} < b_+^{(1,1)} < b_+^{(2,2)}$
- $EU_0^{(1,1)} < EU_0^{(2,1)} < EU_0^{(2,2)}$

As for the comparison between homogeneous mode and heterogeneous mode, the advantage in the equilibrium threshold leads to the advantage in the Boss's payoff. This is basically similar as in the comparison between homogeneous mode and single mode. However, in the comparison between heterogeneous mode and single mode, the order is revered. This is because, in heterogeneous mode, the effect of having more information sources dominates the effect of smaller threshold.

4 Multiple Subordinates with Generally Distributed Biases

In this section, we consider the case of $n \geq 2$ with generally distributed biases.

4.1 Equilibrium Conditions and Payoff

The receiver's strategy is given by

$$a^{(n,k)}(\tilde{k}, \tilde{\ell}) = \begin{cases} 0 & \text{if } \tilde{k} < k, \tilde{\ell} = n-k, \\ \frac{\left(1-F\left(b_-^{(n,k)}\right)\right)^{n-k}}{\left(1-F\left(b_-^{(n,k)}\right)\right)^{n-k}+\left(1-F\left(b_+^{(n,k)}\right)\right)^{k}} & \text{if } \tilde{k} = k, \tilde{\ell} = n-k, \\ 1 & \text{if } \tilde{k} = k, \tilde{\ell} < n-k. \end{cases}$$

Then, the senders' thresholds are

$$\begin{aligned} b_+^{(n,k)} &= \frac{a^{(n,k)}(k, n-k)}{2}, \\ b_-^{(n,k)} &= \frac{1 - a^{(n,k)}(k, n-k)}{2}. \end{aligned}$$

Therefore, $b_+^{(n,k)}$ is the solution of the following equation:

$$G^{(n,k)}(b) := b\left(1 - F\left(b\right)\right)^k - \left(\frac{1}{2} - b\right)\left(1 - F\left(\frac{1}{2} - b\right)\right)^{n-k} = 0.$$

Remark 1 In the case of general distributions, the uniqueness of the equilibrium threshold is no longer guaranteed. For example, consider the F such that

$$F(b) = \begin{cases} 0.04b & \text{if } b < \frac{1}{4}, \\ 0.01 & \text{if } \frac{1}{4} \leq b < 0.365, \\ 49b - 17.875 & \text{if } 0.365 < b < 0.385, \\ 0.99 & \text{if } 0.385 < b \leq \frac{1}{2}, \\ 0.02b + 0.98 & \text{if } b > \frac{1}{2}. \end{cases}$$

In this case, $G^{(2,1)}(b) = 0$ has three solutions: $b \fallingdotseq 0.25126, 0.37866, 0.49505$.

The receiver's equilibrium expected payoff is

$$
\begin{aligned}
EU_0^{(n,k)}\left(b_+^{(n,k)}\right) = &-\frac{1}{2}\left(1-F\left(b_+^{(n,k)}\right)\right)^k\left(a^{(n,k)}(k,n-k)\right)^2\\
&-\frac{1}{2}\left(1-F\left(b_-^{(n,k)}\right)\right)^{n-k}\left(1-a^{(n,k)}(k,n-k)\right)^2.
\end{aligned}
$$

By using the equilibrium condition, this is also written as

$$EU_0^{(n,k)}\left(b_+^{(n,k)}\right) = -b_+^{(n,k)}\left(1-F\left(b_+^{(n,k)}\right)\right)^k, \tag{1}$$

$$EU_0^{(n,k)}\left(b_+^{(n,k)}\right) = -\left(\frac{1}{2}-b_+^{(n,k)}\right)\left(1-F\left(\frac{1}{2}-b_+^{(n,k)}\right)\right)^{n-k}. \tag{2}$$

Particularly, the latter expression leads to the following simple expression:

$$EU_0^{(n,n)}\left(b_+^{(n,n)}\right) = b_+^{(n,n)} - \frac{1}{2}. \tag{3}$$

This makes later analysis much easier.

4.2 Superiority of Completely Homogeneous Subordinate Mode

First, we confirm the existence of equilibrium and the region where equilibrium thresholds are lying.

Proposition 1 For any $n \geq 2$ and any k such that $n \geq k \geq \frac{n}{2}$, there exists $b_+^{(n,k)}$, and moreover,

- $0 < b_+^{(n,k)} < \frac{1}{2}$, especially
- $\frac{1}{4} < b_+^{(n,n)} < \frac{1}{2}$.

Proof:
For any $n \geq 2$ and any k such that $n \geq k \geq \frac{n}{2}$,

- $G^{(n,k)}(0) < 0$ and
- $G^{(n,k)}(b) > 0$ for $b \geq \frac{1}{2}$

hold, which implies that $b_+^{(n,k)}$ exists and $b_+^{(n,k)} \in \left(0, \frac{1}{2}\right)$. Especially,

- $G^{(n,n)}(b) < 0$ for $b \leq \frac{1}{4}$

holds, which implies $b_+^{(n,n)} \in \left(\frac{1}{4}, \frac{1}{2}\right)$. ∎

Next, consider the comparison among completely homogeneous modes. The next result shows that completely homogeneous mode with full participation of Subordinates has an equilibrium in which Subordinates' thresholds and Boss's payoff are lager than any one in homogeneous mode with partial participation of Subordinates.

Proposition 2 For any $n \geq 2$ and any $b_+^{(n-1,n-1)}$, there exists $b_+^{(n,n)}$ such that $b_+^{(n,n)} > b_+^{(n-1,n-1)}$ and $EU_0^{(n,n)}\left(b_+^{(n,n)}\right) > EU_0^{(n-1,n-1)}\left(b_+^{(n-1,n-1)}\right)$.

Proof:

$$G^{(n,n)}\left(b_+^{(n-1,n-1)}\right) = -\left(\frac{1}{2} - b_+^{(n-1,n-1)}\right) F\left(b_+^{(n-1,n-1)}\right) < 0,$$

holds. Combined with the fact that $G^{(n,n)}\left(\frac{1}{2}\right) > 0$ and Proposition 1, this implies the existence of $b_+^{(n,n)}$ such that $b_+^{(n,n)} > b_+^{(n-1,n-1)}$. Combined with (3), this directly implies $EU_0^{(n,n)}\left(b_+^{(n,n)}\right) > EU_0^{(n-1,n-1)}\left(b_+^{(n-1,n-1)}\right)$. ∎

As a corollary to this proposition, we obtain the following limit results:

Corollary 1 There exists a strictly increasing sequence $\left\{b_+^{(n,n)}\right\}_{n\geq 2}$ such that

- $\lim_{n\to\infty} b_+^{(n,n)} = \frac{1}{2}$ and
- $\lim_{n\to\infty} EU_0^{(n,n)}\left(b_+^{(n,n)}\right) = 0$.

Proof:
This corollary follows from the fact that, for any small $\varepsilon > 0$, there exists $\bar{n}$ such that $G^{(n,n)}\left(\frac{1}{2} - \varepsilon\right) < 0$ for any $n > \bar{n}$. ∎

Lastly, consider the comparison among completely homogeneous mode and any other mode, given n. In doing so, the next lemma is very helpful.

Lemma 1 For any $n \geq 2$, any k such that $n-1 \geq k \geq 1$, and any $b_+^{(n,k)} \in \left[\frac{1}{4}, \frac{1}{2}\right)$, there exists $b_+^{(n,n)}$ such that $EU_0^{(n,n)}\left(b_+^{(n,n)}\right) > EU_0^{(n,k)}\left(b_+^{(n,k)}\right)$.

Proof:
Throughout the proof, we denote

- $x := b_+^{(n,k)}$,
- $y := 1 - F(x)$, and
- $z := 1 - F\left(\frac{1}{2} - x\right)$.

Then, the equilibrium condition $G^{(n,k)}\left(b_+^{(n,k)}\right) = 0$ is written as $xy^k = \left(\frac{1}{2} - x\right) z^{n-k}$. Then, $z < 1$, which follows from Assumption 1, implies that $\frac{1}{2} - xy^k > x$, and $y \leq z$, which follows from the assumption $b_+^{(n,k)} \in \left[\frac{1}{4}, \frac{1}{2}\right)$, implies that $x \geq \frac{y^{n-k}}{2\left(y^{n-k}+y^k\right)}$. The Boss's equilibrium payoff is also written as $EU_0^{(n,k)}\left(b_+^{(n,k)}\right) = -xy^k$. By using these facts, we obtain

$$\begin{aligned}
G^{(n,n)}\left(EU_0^{(n,k)}\left(b_+^{(n,k)}\right) + \frac{1}{2}\right) &= G^{(n,n)}\left(\frac{1}{2} - xy^k\right) \\
&= \left(\frac{1}{2} - xy^k\right)\left(1 - F\left(\frac{1}{2} - xy^k\right)\right)^n - xy^k \\
&\leq \left(\frac{1}{2} - xy^k\right) y^n - xy^k \\
&= \frac{1}{2}y^n - xy^k\left(1 + y^n\right) \\
&\leq \frac{1}{2}y^n - y^k\left(1 + y^n\right)\frac{y^{n-k}}{2\left(y^{n-k} + y^k\right)} \\
&= -\frac{y^n\left(1 - y^{n-k}\right)\left(1 - y^k\right)}{2\left(y^{n-k} + y^k\right)} \\
&< 0.
\end{aligned}$$

Combined with the fact that $G^{(n,n)}\left(\frac{1}{2}\right) > 0$, this implies the existence of $b_+^{(n,n)}$ such that $b_+^{(n,n)} > EU_0^{(n,k)}\left(b_+^{(n,k)}\right) + \frac{1}{2}$. Combined with (3), this implies

$EU_0^{(n,n)}\left(b_+^{(n,n)}\right) > EU_0^{(n,k)}\left(b_+^{(n,k)}\right)$. ∎

Based on this lemma, we can prove the next proposition, which means that homogeneous mode has an equilibrium in which Subordinates thresholds and Boss's expected payoffs are larger than any equilibrium in any other organizational mode.

Proposition 3 For any $n \geq 2$, any k such that $n-1 \geq k \geq \frac{n}{2}$, and any $b_+^{(n,k)}$, there exists $b_+^{(n,n)}$ such that $b_+^{(n,n)} > b_+^{(n,k)}$ and $EU_0^{(n,n)}\left(b_+^{(n,n)}\right) > EU_0^{(n,k)}\left(b_+^{(n,k)}\right)$.

Proof:

$$\begin{aligned} & G^{(n,n)}\left(b_+^{(n,k)}\right) \\ & = -\left(\frac{1}{2}-b_+^{(n,k)}\right)\left[1-\left(1-F\left(b_+^{(n,k)}\right)\right)^{n-k}\left(1-F\left(\frac{1}{2}-b_+^{(n,k)}\right)\right)^{n-k}\right] \\ & < 0, \end{aligned}$$

holds. Combined with the fact that $G^{(n,n)}\left(\frac{1}{2}\right) > 0$ and Proposition 1, this implies the existence of $\bar{b}_+^{(n,n)}$ such that $\bar{b}_+^{(n,n)} > b_+^{(n,k)}$.

On the other hand, Proposition 1 means that $b_+^{(n,k)} \in \left(0, \frac{1}{2}\right)$. If $b_+^{(n,k)} \in \left[\frac{1}{4}, \frac{1}{2}\right)$, Lemma 1 directly implies the existence of $\hat{b}_+^{(n,n)}$ such that $EU_0^{(n,n)}\left(\hat{b}_+^{(n,n)}\right) > EU_0^{(n,k)}\left(b_+^{(n,k)}\right)$. If $b_+^{(n,k)} \in \left(0, \frac{1}{4}\right)$, then there exists $b_+^{(n,n-k)} = \frac{1}{2} - b_+^{(n,k)} \in \left(\frac{1}{4}, \frac{1}{2}\right)$. Moreover, by (1) and (2), we obtain $EU_0^{(n,n-k)}\left(b_+^{(n,n-k)}\right) = EU_0^{(n,k)}\left(b_+^{(n,k)}\right)$. Therefore, again, Lemma 1 implies the existence of $\hat{b}_+^{(n,n)}$ such that $EU_0^{(n,n)}\left(\hat{b}_+^{(n,n)}\right) > EU_0^{(n,k)}\left(b_+^{(n,k)}\right)$.

By these facts and (3), it is clear that $\max\left\{\bar{b}_+^{(n,n)}, \hat{b}_+^{(n,n)}\right\}$ satisfies the statement of Proposition. ∎

4.3 Other Comparison Results

The comparison between equilibrium thresholds in other organization modes belonging to the same n is clear.

Proposition 4 For any $n \geq 2$, any k such that $n-1 \geq k \geq \frac{n}{2}$, and any $b_+^{(n,k)}$, there exists $b_+^{(n,k+1)}$ such that $b_+^{(n,k+1)} > b_+^{(n,k)}$.

Proof:

$$\begin{aligned}
&G^{(n,k+1)}\left(b_+^{(n,k)}\right) \\
&= -\left(\frac{1}{2}-b_+^{(n,k)}\right)\left(1-F\left(\frac{1}{2}-b_+^{(n,k)}\right)\right)^{n-k-1}\left[1-\left(1-F\left(b_+^{(n,k)}\right)\right)\left(1-F\left(\frac{1}{2}-b_+^{(n,k)}\right)\right)\right] \\
&< 0,
\end{aligned}$$

holds. Combined with the fact that $G^{(n,k+1)}\left(\frac{1}{2}\right) > 0$ and Proposition 1, this implies the existence of $b_+^{(n,k+1)}$ such that $b_+^{(n,k+1)} > b_+^{(n,k)}$. ∎

An intuition of the comparison results about equilibrium thresholds is as follows. The key is the sensitivity of Boss's response to false messages. For example, consider an upward biased Subordinate. If his bias is small, he wants to induce the Boss's action a little bit upward, but he does not want to induce the Boss's action too far. Then, the more sensitive to false messages Boss' response becomes, the less incentive to manipulate her decision Subordinates has. This is the tips for eliciting truthful information via cheap talk communication.

However, the comparison of Boss's equilibrium payoff is less clear-cut. This is because $b_+^{(n,k+1)} > b_+^{(n,k)}$ implies $b_-^{(n,k+1)} < b_-^{(n,k)}$. In other words, an increase in the number of upward biased Subordinate makes all upward biased Subordinates more disciplined, while a decrease in the number of downward biased Subordinate makes all downward biased Subordinates less disciplined. This is a trade-off that Boss has to deal with.

For the later analysis in this subsection, we restrict our attention to the equilibrium thresholds larger than or equal to $\frac{1}{4}$. The existence of such a threshold is guaranteed by Proposition 4 and $b = \frac{1}{4}$ necessarily satisfies $G^{\left(n,\frac{n}{2}\right)}(b) = 0$.

Since $-b\left(1-F(b)\right)^{k+1} > -b\left(1-F(b)\right)^{k}$ holds for any $b \in \left(0, \frac{1}{2}\right)$, (1) and Proposition 4 imply that

$$\frac{d}{db}\left[-b\left(1-F(b)\right)^{k}\right] \geq 0 \quad \forall b \in \left[\frac{1}{4}, \frac{1}{2}\right)$$

k / n	1	2	3	4	5	6
1	0.292893211	0.25				
2		0.352201038	0.319448451	0.25		
3			0.417477852	0.40612275	0.386369991	0.25
4				0.461113738	0.459000081	0.456444304
5					0.482027692	0.481637276
6						0.491503242

Table 1: Equilibrium Thresholds in the Case of Uniform Distribution

is a sufficient condition for the existence of $b_+^{(n,k+1)}$ such that $EU_0^{(n,k+1)}\left(b_+^{(n,k+1)}\right) > EU_0^{(n,k)}\left(b_+^{(n,k)}\right)$. If F has a density function f, then this is equivalent to

$$kbf(b) - (1 - F(b)) \geq 0 \quad \forall b \in \left[\frac{1}{4}, \frac{1}{2}\right).$$

This implies that, in the case of the uniform distribution, there exists such of $b_+^{(n,k+1)}$ for any $n \geq 6$ and for any k such that $n - 1 \geq k \geq \frac{n}{2}$.

Remark 2 The above sufficient condition is satisfied only is $f(b)$ is sufficiently large. As a counter-example, consider the following baseline distribution function:

$$F(b) = \begin{cases} \frac{b}{100000000} & \text{if } 0 \leq b \leq \frac{1}{2} \\ b + \frac{99999999}{100000000}(b-1) & \text{if } \frac{1}{2} < b \leq 1. \end{cases}$$

Given this distribution function, we can numerically verify that $EU_0^{(40,20)}\left(b_+^{(40,20)}\right) > EU_0^{(40,21)}\left(b_+^{(40,21)}\right)$, while $b_+^{(40,20)} < b_+^{(40,21)}$.

On the other hand, for the case of uniform distribution and lower n, we can numerically derive the equilibrium thresholds and Boss's equilibrium payoffs (Tables 1 and 2). It then follows that the existence of such $b_+^{(n,k+1)}$ is guaranteed for any $n \geq 2$ in the case of the uniform distribution.

Proposition 5 Suppose the baseline distribution is the uniform distribution over $[0, 1]$. Then, for any $n \geq 2$, any k such that $n - 1 \geq k \geq \frac{n}{2}$, and any $b_+^{(n,k)} \geq \frac{1}{4}$, there exists $b_+^{(n,k+1)}$ such that $b_+^{(n,k+1)} > b_+^{(n,k)}$ and $EU_0^{(n,k+1)}\left(b_+^{(n,k+1)}\right) > EU_0^{(n,k)}\left(b_+^{(n,k)}\right)$.

k / n	1	2	3	4	5	6
1	-0.207106778	-0.187500049				
2		-0.147798875	-0.147952681	-0.140625018		
3			-0.082522194	-0.08506431	-0.089273623	-0.105459195
4				-0.03888629	-0.039318918	-0.039844098
5					-0.017972343	-0.018025566
6						-0.008496802

Table 2: Boss's Equilibrium Payoffs in the Case of Uniform Distribution

5 Extensions

We can extend our previous results to other environments.

5.1 Heterogeneous Bias Distributions

We can drop the assumption that baseline bias distributions are common between Subordinates in different indexes, as long as we keep the assumption of symmetricity between upward and downward biased Subordinates in the same index. We relegate the formal results to Appendix C. The next example illustrates how crucial to our results the latter assumption.

Example 1 Consider the case of $n = 2$. We denote upward and downward biased Subordinates 2 by $2+$ and $2-$, respectively. The bias distribution of each Subordinate is

- $b_i \sim U\left[0, w_i\right]$ for $i = 1, 2+$, and
- $b_{2-} \sim U\left[-w_{2-}, 0\right]$.

We assume $w_i \in \left(\frac{1}{2}, 1\right)$ for $i = 1, 2+, 2-$.

It is verified that, if $w_{2-} = w_{2+}$, then Boss prefers the homogeneous mode to the heterogeneous mode. Moreover, we can show that there exists $\hat{w}_{2-} \in \left(\frac{1}{2}, w_{2+}\right)$ such that

$$\begin{aligned} EU_0^{(+,+)} > EU_0^{(+,-)} & \quad \text{if } w_{2-} > \hat{w}_{2-}, \\ EU_0^{(+,+)} < EU_0^{(+,-)} & \quad \text{if } w_{2-} < \hat{w}_{2-}. \end{aligned}$$

As w_2 deceases, downward biased Subordinate 2 has more tendency to send a truthful message, while upward biased Subordinate 1 becomes less disciplined. This is the trade-off for Boss.

5.2 Slightly Overlapping Biases

We assumed that the bias support for upward biased Subordinate does not overlap with one for downward biased Subordinate. The previous results would not change, however, even if a small overlapping would be introduced.

Example 2 Consider the case of $(2,1)$. We assume that the baseline distribution function is the uniform distribution over $[-z, 1-z]$ where $z \in \left[0, \frac{1}{2}\right)$. If $z < \frac{1}{4}$, the strategy profile described in Subsection 3.3 also constitutes an equilibrium.

5.3 Simultaneous Information Transmission Protocol

The previous results would not change even if we assume the cheap talk stage would proceed according to a simultaneous protocol. This is because each Subordinate follows history-independent and order-independent strategy.

5.4 Common Knowledge among Subordinates

The equilibria we focus on would also remain even if their biases were common knowledge among Subordinates as long as we keep the assumption that Boss does not know. Of course, the latter assumption is crucial.

6 Conclusion

We consider the information transmission problem within organization, and especially we focus on how to combine biased subordinates to elicit truthful information from them. We assume that the directions of Subordinates' biases are common knowledge, but not their sizes. This leads to the results that completely homogeneous combination of Subordinates are better than any other combination. This is because in the case of completely homogeneous Subordinates, the effect of one's false report might be accelerated by another false report and this anxiety reduces an incentive for Subordinates to send a false report. This is a new mechanism for information transmission within organization.

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A Non-Existence of Fully Revealing PBE

Given μ, we define $m_i^*(t, b_1, \ldots, b_i)$ recursively as follows:

$$m_1^*(t, b_1) := \mu_1(t, b_1)$$
$$m_i^*(t, b_1, \ldots, b_i) := \mu_i(t, b_i, m_1^*(t, b_1), \ldots, m_{i-1}^*(t, b_1, \ldots, b_{i-1})) \quad \text{for } i > 1.$$

Using this notation, We define a fully revealing strategy profile.

Definition 1 (μ^*, α^*) is *fully revealing* if $\alpha^*(m_1^*(t, b_1), \ldots, m_n^*(t, b_1, \ldots, b_n)) = t$ for any $t \in T$ and any $b_1 \in B_1, \ldots, b_n \in B_n$.

Similarly, given μ, we define $\gamma_i(t, m_1, \ldots, m_i, b_{i+1}, \ldots, b_n)$ as follows:

$$\gamma_i(t, m_1, \ldots, m_i, b_{i+1}, \ldots, b_n) := (\hat{m}_{i+1}, \ldots, \hat{m}_n)$$

where

$$\begin{aligned} \hat{m}_{i+1} &= \mu_{i+1}(t, b_{i+1}, m_1, \ldots, m_i) \\ \hat{m}_{i+2} &= \mu_{i+2}(t, b_{i+2}, m_1, \ldots, m_i, \hat{m}_{i+1}) \\ &\vdots \\ \hat{m}_n &= \mu_n(t, b_n, m_1, \ldots, m_i, \hat{m}_{i+1}, \ldots, \hat{m}_{n-1}). \end{aligned}$$

Proposition 6 For any $n \geq 2$, there is no fully revealing PBE.

Proof:
Suppose to the contrary that there exists a fully revealing PBE (μ^*, α^*). For any i, we pick up $\hat{b}_i \in B_i$ such that $|\hat{b}_i| > \frac{1}{2}$. With a slight abuse of notation, we simply denote $m_i^*(t) := m_i^*(t, \hat{b}_1, \ldots, \hat{b}_i)$ and $\gamma_i(t, m_1, \ldots, m_i) := \gamma_i(t, m_1, \ldots, m_i, \hat{b}_{i+1}, \ldots, \hat{b}_n)$.

Lemma 2 For any $i < n$ and any $m_1 \in M_1 \ldots, m_{i-1} \in M_{i-1}$, there exists no $\hat{m}_i \in M_i$ such that

$$\alpha^*(m_1, \ldots, m_{i-1}, \hat{m}_i, \gamma_i(1, m_1, \ldots, m_{i-1}, \hat{m}_i)) = 1 \text{ and}$$
$$\alpha^*(m_1, \ldots, m_{i-1}, \hat{m}_i, \gamma_i(0, m_1, \ldots, m_{i-1}, \hat{m}_i)) = 0.$$

Proof:
Suppose to the contrary that there exists $(\hat{m}_1, \ldots, \hat{m}_i)$ such that

$$\alpha^*(\hat{m}_1, \ldots, \hat{m}_i, \gamma_i(1, \hat{m}_1, \ldots, \hat{m}_i)) = 1 \text{ and}$$
$$\alpha^*(\hat{m}_1, \ldots, \hat{m}_i, \gamma_i(0, \hat{m}_1, \ldots \hat{m}_i)) = 0.$$

We prove that this makes contradiction for any $i < n$ by backward induction.

Step 1: When $i = n - 1$ and Subordinate n is upward biased, since $U_n(0, \hat{b}_n, a) > U_n(0, \hat{b}_n, 0)$ for any $a \in (0, 1]$, he has no incentive to follow $\gamma_{n-1}(0, \hat{m}_1, \ldots \hat{m}_{n-1}) = \mu_n^*(0, \hat{b}_n, \hat{m}_1, \ldots \hat{m}_{n-1})$ at $t = 0$.

Step 2: When $i = n - 1$ and Subordinate n is downward biased, since $U_n(0, \hat{b}_n, a) > U_n(0, \hat{b}_n, 1)$ for any $a \in [0, 1)$, he has no incentive to follow $\gamma_{n-1}(1, \hat{m}_1, \ldots \hat{m}_{n-1}) = \mu_n^*(1, \hat{b}_n, \hat{m}_1, \ldots \hat{m}_{n-1})$ at $t = 1$.

Step 3: When $i < n-1$, the claim of Lemma holds for $i+1$, and Subordinate $i + 1$ is upward biased, since $U_{i+1}(0, \hat{b}_{i+1}, a) > U_{i+1}(0, \hat{b}_{i+1}, 0)$ for any $a \in (0, 1]$, it must follow

$$\alpha^*(\hat{m}_1, \ldots, \hat{m}_i, m_{i+1}, \gamma_{i+1}(0, \hat{m}_1, \ldots, \hat{m}_i, m_{i+1})) = 0 \quad \forall m_{i+1} \in M_{i+1}.$$

On the other hand, there must exists $\hat{m}_{i+1} \in M_{i+1}$ such that

$$\alpha^*(\hat{m}_1, \ldots, \hat{m}_i, \hat{m}_{i+1}, \gamma_{i+1}(1, \hat{m}_1, \ldots, \hat{m}_i, \hat{m}_{i+1})) = 1.$$

This is contradiction to the supposition that the claim of Lemma holds for $i + 1$.

Step 4: When $i < n$, the claim of Lemma holds for $i + 1$, and Subordinate $i + 1$ is downward biased, since $U_i(1, \hat{b}_{i+1}, a) > U_{i+1}(1, \hat{b}_{i+1}, 1)$ for any $a \in [0, 1)$, it must follow

$$\alpha^*(\hat{m}_1, \ldots, \hat{m}_1, m_{i+1}, \gamma_{i+1}(1, \hat{m}_1, \ldots, \hat{m}_1, m_{i+1})) = 1 \quad \forall m_{i+1} \in M_{i+1}.$$

On the other hand, there must exists $\hat{m}_{i+1} \in M_{i+1}$ such that

$$\alpha^*(\hat{m}_1, \ldots, \hat{m}_i, \hat{m}_{i+1}, \gamma_{i+1}(0, \hat{m}_1, \ldots, \hat{m}_i, \hat{m}_{i+1})) = 0.$$

This is contradiction to the supposition that the claim of Lemma holds for $i + 1$.

∎

Lemma 3 For any $i > 1$ and any $t \in T$,

$$\alpha^*(m_1^*(1), \ldots, m_{i-1}^*(1), \gamma_{i-1}(t, m_1^*(1), \ldots, m_{i-1}^*(1))) > 0.$$

Proof:
We prove this by backward induction.

Step 1: When $i = n$ and Subordinate n is upward biased, since $U_n(0, \hat{b}_n, a) > U_n(0, \hat{b}_n, 0)$ for any $a \in (0, 1]$, then, even at $t = 0$, Subordinate n has no incentive to choose $\hat{m}_n$ such that

$$\alpha^*(m_1^*(1), \ldots, m_{n-1}^*(1), \hat{m}_n) = 0.$$

Step 2: When $i = n$ and Subordinate n is downward biased, since $U_n(1, \hat{b}_n, a) > U_n(1, \hat{b}_n, 1)$ for any $a \in [0, 1)$, it must follow

$$\alpha^*(m_1^*(1), \ldots, m_{n-1}^*(1), m_n) = 1 \quad \forall m_n \in M_n.$$

Step 3: When $i < n$, the claim of Lemma holds for $i + 1$, and Subordinate i is upward biased, since $U_i(0, \hat{b}_i, a) > U_i(0, \hat{b}_i, 0)$ for any $a \in (0, 1]$ and

$$\alpha^*(m_1^*(1), \ldots, m_{i-1}^*(1), m_i^*(1), \gamma_i(t, m_1^*(1), \ldots, m_{i-1}^*(1), m_i^*(1)) > 0 \quad \forall t \in T,$$

then, even at $t = 0$, Subordinate has no incentive to choose $\hat{m}_i$ such that

$$\alpha^*(m_1^*(1), \ldots, m_{i-1}^*(1), \hat{m}_i, \gamma_i(0, m_1^*(1), \ldots, m_{i-1}^*(1), \hat{m}_i) = 0.$$

Step 4: When $i < n$, the claim of Lemma holds for $i + 1, \ldots, n$, and Subordinate i is downward biased, since $U_i(1, \hat{b}_i, a) > U_i(1, \hat{b}_i, 1)$ for any $a \in [0, 1)$, it must follow

$$\alpha^*(m_1^*(1), \ldots, m_{i-1}^*(1), m_i, \gamma_i(1, m_1^*(1), \ldots, m_{i-1}^*(1), m_i)) = 1 \quad \forall m_i \in M_i.$$

Then, by Lemma 2, this implies

$$\alpha^*(m_1^*(1), \ldots, m_{i-1}^*(1), m_i, \gamma_i(0, m_1^*(1), \ldots, m_{i-1}^*(1), m_i)) > 0 \quad \forall m_i \in M_i.$$

∎

Proof of Proposition 6: By Lemma 3, Subordinate 1 has no incentive to choose $\mu_1^*(0)$ at $t = 0$.

∎

B Essentially Outcome-Equivalence to Threshold Strategy Equilibrium

In this Appendix, we use the following notations:

- $m^i := (m_1, \dots, m_i)$ where $m^0 = \emptyset$
- $M^i := M_1 \times \cdots \times M_i$ where $M^0 = \emptyset$
- $\mathcal{M}_i(t, m^{i-1}) := \{m_i \in M_i \mid \exists \hat{B}_i \subseteq B_i \text{ s.t. } \lambda(\hat{B}_i) > 0 \text{ and } m_i = \mu_i(t, b_i, m^{i-1})\ \forall b_i \in \hat{B}_i\}$ where λ is Lesbegue measure
- $\mathcal{M}_{i0}(m^{i-1}) := \mathcal{M}_i(0, m^{i-1}) / \mathcal{M}_i(1, m^{i-1})$
- $\mathcal{M}_{i1}(m^{i-1}) := \mathcal{M}_i(1, m^{i-1}) / \mathcal{M}_i(0, m^{i-1})$
- $\mathcal{M}_{im}(m^{i-1}) := \mathcal{M}_i(0, m^{i-1}) \cap \mathcal{M}_i(1, m^{i-1})$
- $\mathcal{H}_i(t) := \{m^{i-1} \in M^{i-1} \mid m_1 \in \mathcal{M}_1(t), m_2 \in \mathcal{M}_2(t, m_1), \dots, m_{i-1} \in \mathcal{M}_{i-1}(t, m^{i-2})$ where $\mathcal{H}_1(t) = \emptyset$
- $\mathcal{C}_i(t, m^i) := \{(m_{i+1}, \dots, m_n) \in M_{i+1} \times \cdots \times M_n \mid m_{i+1} \in \mathcal{M}_{i+1}(t, m^i), m_{i+2} \in \mathcal{M}_{i+2}(t, m^i, m_{i+1}), \dots, m_n \in \mathcal{M}_n(t, m^i, m_{i+1}, \dots, m_{n-1})\}$ where $\mathcal{C}_n(t, m^n) = \emptyset$

To avoid technical difficulties, we assume that each Subordinate's message space is finite.

Assumption 2 $M_1, \dots, M_n$ are finite.

This assumption guarantees that $\mathcal{M}_{i0}(m^{i-1}) \cup \mathcal{M}_{im}(m^{i-1}) \neq \emptyset$ and $\mathcal{M}_{i1}(m^{i-1}) \cup \mathcal{M}_{im}(m^{i-1}) \neq \emptyset$ for any i and any $m^{i-1} \in M^{i-1}$.

In order to express the situation where all Subordinates are active in information transmission, we define the concept of fully active PBE as follows:

Definition 2 A PBE (μ^*, α^*) is *fully active* if, for any i, there exist $t \in T$, $h_i \in \mathcal{H}_i(t)$, $m_i, m_i' \in \mathcal{M}_i(t, h_i)$, $c_i \in \mathcal{C}_i(t, h_i, m_i)$, and $c_i' \in \mathcal{C}_i(t, h_i, m_i')$ such that $\alpha^*(h_i, m_i, c_i) \neq \alpha^*(h_i, m_i', c_i')$.

We prove that any fully active PBE is essentially outcome-equivalent to some threshold strategy equilibrium. This is done by the following series of Lemmas.

Lemma 4 Given any i. Suppose that $\hat{m}^{i-1} \in M^{i-1}$ such that $\hat{m}_1 \in \mathcal{M}_{1m}, \ldots, \hat{m}_{i-1} \in \mathcal{M}_{i-1.m}(\hat{m}^{i-2})$.

(i) If Subordinate i is upward biased, then $\mathcal{M}_{i1}(\hat{m}^{i-1}) = \emptyset$.

(ii) If Subordinate i is downward biased, then $\mathcal{M}_{i0}(\hat{m}^{i-1}) = \emptyset$.

(iii) $\mathcal{M}_{im}(\hat{m}^{i-1}) \neq \emptyset$

Proof:
We prove this by backward induction.

Step 1: Suppose that $\mathcal{M}_{n1}(\hat{m}^{n-1}) \neq \emptyset$. This implies $\alpha^*(\hat{m}^{n-1}, m_n) = \{1\}$ for any $m_n \in \mathcal{M}_{n1}(\hat{m}^{n-1})$. If $\mathcal{M}_{nm}(\hat{m}^{n-1}) \neq \emptyset$, since $\alpha^*(\hat{m}^{n-1}, m_n) \neq 1$ for any $m_n \in \mathcal{M}_{nm}(\hat{m}^{n-1})$, Subordinate n has no incentive to choose any $m_n \in \mathcal{M}_{nm}(\hat{m}^{n-1})$ at $t = 1$. Therefore, $\mathcal{M}_{nm}(\hat{m}^{n-1}) = \emptyset$. This implies $\mathcal{M}_{n0}(\hat{m}^{n-1}) \neq \emptyset$. Nevertheless, since $\alpha^*(\hat{m}^{n-1}, m_n) = 0$ for any $m_n \in \mathcal{M}_{n0}(\hat{m}^{n-1})$, Subordinate n with $b_n > \frac{1}{2}$ has no incentive to choose any $m_n \in \mathcal{M}_{n0}(\hat{m}^{n-1})$ at $t = 0$. This is contradiction, which means we have proven that $\mathcal{M}_{n1}(\hat{m}^{n-1}) = \emptyset$. This implies $\mathcal{M}_{nm}(\hat{m}^{n-1}) \neq \emptyset$.

Step 2: Similarly, we can prove that, if Subordinate n is downward biased, then $\mathcal{M}_{n0}(\hat{m}^{n-1}) = \emptyset$ and $\mathcal{M}_{nm}(\hat{m}^{n-1}) \neq \emptyset$.

Step 3: Suppose that Subordinate $i < n$ is upward biased, the claim of Lemma holds for $i + 1, \ldots n$, and $\mathcal{M}_{i1}(\hat{m}^{i-1}) \neq \emptyset$. This implies $\alpha^*(\hat{m}^{i-1}, m_i, c_i) = 1$ for any $m_i \in \mathcal{M}_{i1}(\hat{m}^{i-1})$ and any $c_i \in \mathcal{C}_i(1, \hat{m}^{i-1}, m_i)$.

Suppose $\mathcal{M}_{im}(\hat{m}^{i-1}) \neq \emptyset$. Then, for any $m_i \in \mathcal{M}_{im}(\hat{m}^{i-1})$, there exists $(\hat{m}_{i+1}, \ldots, \hat{m}_n) \in \mathcal{C}_i(1, \hat{m}^{i-1}, m_i)$ such that $\hat{m}_{i+1} \in \mathcal{M}_{i+1.m}(\hat{m}^i, m_i)$ and $\hat{m}_{i'} \in \mathcal{M}_{i'm}(\hat{m}^{i-1}, m_i, \hat{m}_{i+1}, \ldots, \hat{m}_{i'-1})$ for any $i' > i + 1$. Since $\alpha^*(\hat{m}^{i-1}, m_i, \hat{m}_{i+1}, \ldots, \hat{m}_n) \neq 1$, Subordinate i has no incentive to choose such m_i at $t = 1$. This is contradiction, which means we have proven $\mathcal{M}_{im}(\hat{m}^{i-1}) = \emptyset$

This implies $\mathcal{M}_{i0}(\hat{m}^{i-1}) \neq \emptyset$. Nevertheless, since $\alpha^*(\hat{m}^{i-1}, m_i, c_i) = 0$ for any $m_i \in \mathcal{M}_{i0}(\hat{m}^{i-1})$ and any $c_i \in \mathcal{C}_i(0, \hat{m}^{i-1}, m_i)$, Subordinate i with $b_i > \frac{1}{2}$ has no incentive to choose any $m_i \in \mathcal{M}_{i0}(\hat{m}^{i-1})$ at $t = 0$. This is contradiction, which means we have proven that $\mathcal{M}_{i1}(\hat{m}^{i-1}) = \emptyset$. This implies $\mathcal{M}_{im}(\hat{m}^{i-1}) \neq \emptyset$.

Step 4: Similarly, we can prove that, if Subordinate $i < n$ is downward biased and the claim of Lemma holds for $i+1, \dots n$, then $\mathcal{M}_{i0}(\hat{m}^{i-1}) = \emptyset$ and $\mathcal{M}_{im}(\hat{m}^{i-1}) \neq \emptyset$.

∎

Lemma 5 There exists $a^* \in (0,1)$ such that $a^* = \alpha^*(m)$ for any m such that $m_1 \in \mathcal{M}_{1m}, \dots, m_n \in \mathcal{M}_{nm}(m^{n-1})$.

Proof:
We can find such a^* backward.

Step 1: Given any m^{n-1} such that $m_1 \in \mathcal{M}_{1m}, \dots, m_{n-1} \in \mathcal{M}_{n-1.m}(m^{n-2})$. If Subordinate n is upward biased, he maximizes $\alpha^*(m^{n-1}, m_n)$ at $t = 1$ and we denote the maximum value by $a_n^*(m^{n-1})$. If Subordinate n is downward biased, he minimizes $\alpha^*(m^{n-1}, m_n)$ at $t = 0$ and we denote the minimum value by $a_n^*(m^{n-1})$.

Step 2: Given any $i < n$ and any m^i such that $m_1 \in \mathcal{M}_{1m}, \dots, m_{i-1} \in \mathcal{M}_{i-1.m}(m^{i-1})$. Subordinate i is upward biased, he maximizes $a_{i+1}^*(m^{i-1}, m_i)$ at $t = 1$ and we denote the maximum value by $a_i^*(m^{i-1})$. If Subordinate i is downward biased, he minimizes $a^*(m^{i-1}, m_i)$ at $t = 0$ and we denote the minimum value by $a_i^*(m^{i-1})$.

Step 3: Lastly, we define $a^* := a^1$.

∎

Lemma 6 Given any i. Suppose that $\hat{m}^{i-1} \in M^{i-1}$ such that $\hat{m}_1 \in \mathcal{M}_{1m}, \dots, \hat{m}_{i-1} \in \mathcal{M}_{i-1.m}(\hat{m}^{i-2})$.

(i) If Subordinate i is upward biased, then $\mathcal{M}_{i0}(\hat{m}^{i-1}) \neq \emptyset$.

(ii) If Subordinate i is downward biased, then $\mathcal{M}_{i1}(\hat{m}^{i-1}) \neq \emptyset$.

Proof:
This is because a PBE is fully active. ∎

Lemma 7 Given any i. Suppose that $\hat{m}^{i-1} \in M^{i-1}$ such that $\hat{m}_1 \in \mathcal{M}_{1m}, \ldots, \hat{m}_{i-1} \in \mathcal{M}_{i-1.m}(\hat{m}^{i-2})$.

(i) $\alpha^*(m^{i-1}, m_i, c_i) = 0$ for any $m_i \in \mathcal{M}_{i0}(\hat{m}^{i-1})$ and any $c_i \in \mathcal{C}_i(0, \hat{m}^{i-1}, m_i)$.

(ii) $\alpha^*(m^{i-1}, m_i, c_i) = 1$ for any $m_i \in \mathcal{M}_{i1}(\hat{m}^{i-1})$ and any $c_i \in \mathcal{C}_i(1, \hat{m}^{i-1}, m_i)$.

Proof:
Straightforward. ∎

C Heterogeneous Baseline Distributions

In this appendix, we extend the previous results to the case in which baseline distributions for Subordinates in the different indexes are not common.

Let F_i be a baseline distribution function for Subordinate i. We focus on F_i satisfying the following assumptions.

Assumption 3 For any i, the followings hold:

- F_i is continuous.
- $F_i\left(\frac{1}{4}\right) > 1$ and $F_i\left(\frac{1}{2}\right) < 1$.
- Supp $F_i \subseteq [0, 1]$.

We denote upward and downward biased Subordinates by $+$ and $-$, respectively. For any $n \geq 2$, let $\mathcal{O}^n := \{+, -\}^n$. An organization mode is characterized by (n, o) where $o \in \mathcal{O}^n$. We denote the numbers of upward and downward Subordinates among o by $+(o)$ and $-(o)$, respectively. We also denote the index of ith upward and downward biased Subordinate by $+_i(o)$ and $-_i(o)$, respectively.[4] We define $\bar{\mathcal{O}}^n := \mathcal{O}^n \setminus (\{(+, \ldots, +)\} \cup \{(-, \ldots, -)\})$, which is the set of combinations of Subordinates other than completely homogeneous combinations. Due to the symmetricity, $(n, (-, \ldots, -))$ have symmetric thresholds and gives the same equilibrium payoff to Boss as $(n, (+, \ldots, +))$. Based on this reason, in presenting the results concerning completely homogeneous mode, we only consider $(+ \ldots, +)$, which is denoted by $\hat{o}^n$.

[4] For example, given $n = 3$ and $o = (+, -, +)$, then $+(o) = 2$, $+_1(o) = 1$, $+_2(o) = 3$, $-(o) = 1$, and $-_1(o) = 2$.

C.1 Equilibrium Conditions and Payoff

We denote a message profile by m. We also denote the message profile in which all Subordinates send their dubious messages by $\hat{m}$. Then, the receiver's strategy is as follows: given m,

- if all downward biased Subordinates send 0 and at least one upward biased Subordinate sends message 0, then $a^{(n,o)}(m) = 0$,
- if all Subordinates send their dubious messages, i.e., $m = \hat{m}$, then

  $$a^{(n,o)}(\hat{m}) = \frac{\Pi_{i=1}^{-(o)}\left(1 - F_{-i(o)}\left(b_-^{(n,o)}\right)\right)}{\Pi_{i=1}^{-(o)}\left(1 - F_{-i(o)}\left(b_-^{(n,o)}\right)\right) + \Pi_{i=1}^{+(o)}\left(1 - F_{+i(o)}\left(b_+^{(n,o)}\right)\right)},$$

  and,
- if all upward biased Subordinates send 1 and at least one downward biased Subordinate sends message 1, then $a^{(n,o)}(m) = 1$.

Then, the senders' thresholds are

$$\begin{aligned} b_+^{(n,o)} &= \frac{a^{(n,o)}(\hat{m})}{2}, \\ b_-^{(n,o)} &= \frac{1 - a^{(n,o)}(\hat{m})}{2}. \end{aligned}$$

Therefore, $b_+^{(n,o)}$ is the solution of the following equation:[5]

$$G^{(n,o)}(b) := b\Pi_{i=1}^{+(o)}\left(1 - F_{+i(o)}(b)\right) - \left(\frac{1}{2} - b\right)\Pi_{i=1}^{-(o)}\left(1 - F_{-i(o)}\left(\frac{1}{2} - b\right)\right).$$

The receiver's equilibrium expected payoff is

$$\begin{aligned} EU_0^{(n,o)}\left(b_+^{(n,o)}\right) = &-\frac{1}{2}\Pi_{i=1}^{+(o)}\left(1 - F_{+i(o)}\left(b_+^{(n,o)}\right)\right)\left(a^{(n,o)}(\hat{m})\right)^2 \\ &- \frac{1}{2}\Pi_{i=1}^{-(o)}\left(1 - F_{-i(o)}\left(\frac{1}{2} - b_+^{(n,o)}\right)\right)\left(1 - a^{(n,o)}(\hat{m})\right)^2. \end{aligned}$$

[5] We assume $\Pi_{i=1}^{0} A_i = 1$.

By using the equilibrium condition, this is also written as

$$EU_0^{(n,o)}\left(b_+^{(n,o)}\right) = -b_+^{(n,o)}\Pi_{i=1}^{+(o)}\left(1 - F_{+i(o)}\left(b_+^{(n,o)}\right)\right), \tag{4}$$

$$EU_0^{(n,o)}\left(b_+^{(n,o)}\right) = -\left(\frac{1}{2} - b_+^{(n,o)}\right)\Pi_{i=1}^{-(o)}\left(1 - F_{-i(o)}\left(\frac{1}{2} - b_+^{(n,o)}\right)\right). \tag{5}$$

Particularly, the latter expression leads to the following simple expression:

$$EU_0^{(n,\hat{o}^n)}\left(b_+^{(n,\hat{o}^n)}\right) = b_+^{(n,\hat{o}^n)} - \frac{1}{2}. \tag{6}$$

C.2 Comparison Results

First, we confirm the existence of equilibrium and the region where equilibrium thresholds are lying.

Proposition 7 For any $n \geq 2$ and any $o \in \mathcal{O}^n$, there exists $b_+^{(n,o)}$, and moreover,

- $0 < b_+^{(n,o)} < \frac{1}{2}$, especially
- $\frac{1}{4} < b_+^{(n,\hat{o}_n)} < \frac{1}{2}$.

Proof:
For any $n \geq 2$ and $o \in \mathcal{O}^n$,

- $G^{(n,o)}(0) < 0$, and
- $G^{(n,o)}(b) > 0$ for $b \geq \frac{1}{2}$,

hold, which implies that $b_+^{(n,o)} \in \left(0, \frac{1}{2}\right)$. Especially,

- $G^{(n,\hat{o}_n)}(b) < 0$ for $b \leq \frac{1}{4}$,

holds, which implies $b_+^{(n,\hat{o}_n)} \in \left(\frac{1}{4}, \frac{1}{2}\right)$. ∎

The next result shows that completely homogeneous mode with full participation of Subordinates has an equilibrium in which Subordinates' thresholds and Boss's payoff are lager than any one in homogeneous mode with partial participation of Subordinates.

Proposition 8 Suppose $n \geq 3$. For any $\tilde{i}$ such that $1 \leq \tilde{i} \leq n$, we consider the organizational mode $(n-1, \tilde{o})$ such that

$$\tilde{o}_i = \begin{cases} \hat{o}_i^n & \text{if } i < \tilde{i}, \\ \hat{o}_{i+1}^n & \text{if } i \geq \tilde{i}. \end{cases}$$

Then, for any $b_+^{(n-1,\tilde{o})}$, there exists $b_+^{(n,\hat{o}^n)}$ such that $b_+^{(n,\hat{o}^n)} > b_+^{(n-1,\tilde{o})}$ and $EU_0^{(n,\hat{o}^n)}\left(b_+^{(n,\hat{o}^n)}\right) > EU_0^{(n-1,\tilde{o})}\left(b_+^{(n-1,\tilde{o})}\right)$.

Proof:

$$G^{(n,\hat{o}^n)}\left(b_+^{(n-1,\tilde{o})}\right) = -\left(\frac{1}{2} - b_+^{(n-1,\tilde{o})}\right) F_{\tilde{i}}\left(b_+^{(n-1,\tilde{o})}\right) < 0,$$

holds. Combined with the fact that $G^{(n,\hat{o}^n)}\left(\frac{1}{2}\right) > 0$ and Proposition 7, this implies the existence of $b_+^{(n,\hat{o}^n)}$ such that $b_+^{(n,\hat{o}^n)} > b_+^{(n-1,\tilde{o})}$. Combined with (6), this directly implies $EU_0^{(n,\hat{o}^n)}\left(b_+^{(n,\hat{o}^n)}\right) > EU_0^{(n-1,\tilde{o})}\left(b_+^{(n-1,\tilde{o})}\right)$. ∎

Lastly, consider the comparison between homogeneous mode and any other mode. In doing so, the next lemma is very helpful.

Lemma 8 For any $n \geq 2$, any $o \in \bar{\mathcal{O}}^n$, and any $b_+^{(n,o)} \in \left[\frac{1}{4}, \frac{1}{2}\right)$, there exists $b_+^{(n,\hat{o}^n)}$ such that $EU_0^{(n,\hat{o}^n)}\left(b_+^{(n,\hat{o}^n)}\right) > EU_0^{(n,o)}\left(b_+^{(n,o)}\right)$.

Proof:
Let $k = +(o) < n$. Without loss of generality, we can relabel the order of Subordinates such that

$$o_i = \begin{cases} + & \text{if } i \leq k, \\ - & \text{if } i \geq k+1. \end{cases}$$

Throughout the proof, we denote

- $x := b_+^{(n,o)}$,
- $y_i := 1 - F_i(x)$ for $i = 1, \dots, n$, and
- $z_i := 1 - F_i\left(\frac{1}{2} - x\right)$ for $i = k+1, \dots, n$.

Then, the equilibrium condition $G^{(n,o)}\left(b_+^{(n,o)}\right) = 0$ is written as $xy_1 \cdots y_k = \left(\frac{1}{2} - x\right) z_{k+1} \cdots z_n$. Then, $z_i < 1$ for $i = k+1, \ldots, n$, which follows from Assumption 3, implies that $\frac{1}{2} - xy_1 \cdots y_k > x$, and $y_i \leq z_i$ for $i = k+1, \ldots, n$, which follows from the assumption $b_+^{(n,o)} \in \left[\frac{1}{4}, \frac{1}{2}\right)$, implies that $x \geq \frac{y_{k+1} \cdots y_n}{2(y_1 \cdots y_k + y_{k+1} \cdots y_n)}$. The Boss's equilibrium payoff is also written as $EU_0^{(n,o)}\left(b_+^{(n,o)}\right) = -xy_1 \cdots y_k$. By using these facts, we obtain

$$
\begin{aligned}
G^{(n,\hat{o}^n)} & \left(EU_0^{(n,o)}\left(b_+^{(n,o)}\right) + \frac{1}{2}\right) \\
&= G^{(n,\hat{o}^n)}\left(\frac{1}{2} - xy_1 \cdots y_k\right) \\
&= \left(\frac{1}{2} - xy_1 \cdots y_k\right) \Pi_{i=1}^n \left(1 - F_i\left(\frac{1}{2} - xy_1 \cdots y_k\right)\right) - xy_1 \cdots y_k \\
&\leq \left(\frac{1}{2} - xy_1 \cdots y_k\right) y_1 \cdots y_n - xy_1 \cdots y_k \\
&= y_1 \cdots y_k \left[\frac{y_{k+1} \cdots y_n}{2} - x\left(1 + y_1 \cdots y_n\right)\right] \\
&\leq y_1 \cdots y_k \left[\frac{y_{k+1} \cdots y_n}{2} - \frac{y_{k+1} \cdots y_n \left(1 + y_1 \cdots y_n\right)}{2\left(y_1 \cdots y_k + y_{k+1} \cdots y_n\right)}\right] \\
&= -\frac{y_1 \cdots y_n \left(1 - y_1 \cdots y_k\right)\left(1 - y_{k+1} \cdots y_n\right)}{2\left(y_1 \cdots y_k + y_{k+1} \cdots y_n\right)} \\
&< 0.
\end{aligned}
$$

Combined with the fact that $G^{(n,\hat{o}^n)}\left(\frac{1}{2}\right) > 0$, this implies the existence of $b_+^{(n,\hat{o}^n)}$ such that $b_+^{(n,\hat{o}^n)} > EU_0^{(n,o)}\left(b_+^{(n,o)}\right) + \frac{1}{2}$. Combined with (6), this implies $EU_0^{(n,\hat{o}^n)}\left(b_+^{(n,\hat{o}^n)}\right) > EU_0^{(n,o)}\left(b_+^{(n,o)}\right)$. ∎

Based on this lemma, we can prove the next proposition, which means that homogeneous mode has an equilibrium in which Subordinates thresholds and Boss's expected payoffs are larger than any equilibrium in any other organizational mode.

Proposition 9 For any $n \geq 2$, any $o \in \bar{\mathcal{O}}^n$, and any $b_+^{(n,o)}$, there exists $b_+^{(n,\hat{o}^n)}$ such that $b_+^{(n,\hat{o}^n)} > b_+^{(n,o)}$ and $EU_0^{(n,\hat{o}^n)}\left(b_+^{(n,\hat{o}^n)}\right) > EU_0^{(n,o)}\left(b_+^{(n,o)}\right)$.

Proof:

$$
\begin{aligned}
&G^{(n,\hat{o}^n)}\left(b_+^{(n,o)}\right) \\
&\quad = -\left(\frac{1}{2} - b_+^{(n,o)}\right)\left[1 - \left(\Pi_{i=1}^{-(o)}\left(1 - F_{-i(o)}\left(b_+^{(n,o)}\right)\right)\right)\left(\Pi_{i=1}^{-(o)}\left(1 - F_{-i(o)}\left(\frac{1}{2} - b_+^{(n,o)}\right)\right)\right)\right] \\
&\quad < 0,
\end{aligned}
$$

holds. Combined with the fact that $G^{(n,\hat{o}^n)}\left(\frac{1}{2}\right) > 0$ and Proposition 7, this implies the existence of $\bar{b}_+^{(n,\hat{o}^n)}$ such that $\bar{b}_+^{(n,\hat{o}^n)} > b_+^{(n,o)}$.

On the other hand, Proposition 7 means that $b_+^{(n,o)} \in \left(0, \frac{1}{2}\right)$. If $b_+^{(n,o)} \in \left[\frac{1}{4}, \frac{1}{2}\right)$, Lemma 8 directly implies the existence of $\hat{b}_+^{(n,\hat{o}^n)}$ such that $EU_0^{(n,\hat{o}^n)}\left(\hat{b}_+^{(n,\hat{o}^n)}\right) > EU_0^{(n,o)}\left(b_+^{(n,o)}\right)$. Suppose $b_+^{(n,o)} \in \left(0, \frac{1}{4}\right)$. We define $\tilde{o} \in \bar{\mathcal{O}}^n$ as

$$
\tilde{o}_i = \begin{cases} - & \text{if } o_i = +, \\ + & \text{if } o_i = -. \end{cases}
$$

Then, there exists $b_+^{(n,\tilde{o})}$ such that $b_+^{(n,\tilde{o})} = \frac{1}{2} - b_+^{(n,o)} \in \left(\frac{1}{4}, \frac{1}{2}\right)$. Moreover, by (4) and (5), we obtain $EU_0^{(n,\tilde{o})}\left(b_+^{(n,\tilde{o})}\right) = EU_0^{(n,o)}\left(b_+^{(n,o)}\right)$. Therefore, again, Lemma 8 implies the existence of $\hat{b}_+^{(n,\hat{o}^n)}$ such that $EU_0^{(n,\hat{o}^n)}\left(\hat{b}_+^{(n,\hat{o}^n)}\right) > EU_0^{(n,o)}\left(b_+^{(n,o)}\right)$.

By these facts and (6), it is clear that $\max\left\{\bar{b}_+^{(n,\hat{o}^n)}, \hat{b}_+^{(n,\hat{o}^n)}\right\}$ satisfies the statement of Proposition. ∎