Borrowing constraints, housing tenure choice and buy-to-let investors: An assignment model*

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Abstract

We study the effect of borrowing constraints in an assignment model of the housing market. When constraints apply symmetrically to all households, these lead to lower prices but unchanged housing consumption. When households can invest their own wealth, borrowing constraints will in general result in lower house prices and higher housing consumption for unconstrained households, while housing consumption of constrained households may fall. Binding borrowing constraints result in profitable arbitrage possibilities for buy-to-let investors. They can buy houses that are preferred by constrained households unable to finance them, and make them available as rental housing. In an equilibrium with free entry of such investors, house prices and the allocation of houses to households is the same as without borrowing constraints.

Keywords: borrowing constraints, housing tenure, arbitrage, buy-to-let investment, assignment models.

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1 Introduction

To the annoyance of many would-be homeowners with medium incomes and modest wealth, binding borrowing constraints are present in many booming housing markets all over the world. Although the presence of such constraints is motivated by the desire of banks (and authorities) to mitigate mortgage default risks, which may be in the interest of households as foreclosure can be a traumatic experience, many of the constrained households would like to bid more for their dream-house than banks allow them to do and feel constrained rather than protected.

Being unable to buy a house means that one has to rent, often at a cost that is comparable to that of the mortgage payments that would otherwise have been due. Moreover, the combination of a growing supply of rental housing and rapidly increasing prices of owner-occupied housing of comparable quality could suggest that buy-to-let investors drive up house prices, thereby worsening the situation for would-be owner-occupiers. Moreover, the rents to be paid may be comparable to or even higher than the mortgage payments that would have been due in case the household would have been allowed to buy it, which may call into question the motivation of the borrowing constraints.

Concerns like these appear to be present in many booming urban housing markets and it is not easy to evaluate them. It is clear that mitigating the risk of default and foreclosure is socially desirable, but so is the presence of affordable (owner-occupied) housing. There is the impression that these two concerns are in conflict with each other when rents are comparable or even higher than user costs for similar housing. On the one hand, one may argue that buy-to-let investors come to the rescue of households facing tight borrowing constraints, by allowing them to occupy their desired houses. On the other hand there is the concern that these investors are removing the opportunities of these households to become owner-occupiers by overbidding them on the housing market and thereby driving up the prices[1].

This paper sheds light on these issues by developing an assignment model for an urban housing market in which households that differ in income compete for a given number of owner-occupied houses differing in quality[2] and price. A given stock of housing is a realistic feature of housing markets in many core urban areas and that has implications for the way the housing market functions. The model we use was originally formulated in Braid (1981) where a rental market is analyzed in which the number of houses of any given quality level is fixed. Here the focus is on owner-occupied housing, implying that the user cost takes the place of the rent.

We contrast the assignment model with an alternative model proposed by Muth (1969) that considers housing as a standard commodity that is available at a constant price per unit in arbitrary quantities for all households. By choosing a particular num-

[1]Note that owner-occupiers have the opportunity to realize wealth gains associated with increasing house prices and often gradually attain full ownership by using self-amortizing mortgage loans.

[2]A scalar measure of quality is assumed.

ber of units, households determine the quality of their house individually and instead of a continuum of interrelated markets – one for each quality level – there is essentially only one market for housing services underlying the submarkets for qualities. Muth's model is therefore easier to handle, but this is mainly due to the unrealistic assumption that housing is malleable. That is, the model assumes implicitly that the housing stock is rebuilt from scratch in every period.

Associated with this are differences in the predictions of the two models in a number of relevant cases. Perhaps the most important one is the reaction to an income shock. With malleable housing supply, all households increase their housing consumption in response to a higher income. With fixed housing supply this is impossible, but prices of existing houses increase nevertheless, implying that households have to pay more for the same housing after the shock. Clearly, the latter possibility has relevance in many urban housing markets, where the built environment is determined by history. Changes are possible, but often take much time and effort, not just because renovation or demolition followed by replacement are in themselves complicated processes, but also because of hold-up problems, nimby-behavior and preservation measures. Although the assumption of a fixed housing stock may be regarded as the opposite extreme of that of perfectly malleable housing, it is much closer to reality in urban housing markets, at least in the short or medium run.

We then introduce borrowing constraints into the model. The starting point of the analysis is a simple case in which there is only one group of households, all households face the same constraint and possess no wealth. We show that in this case the borrowing constraint has no impact on the allocation of households to housing – that is, each household lives in the same house as in the market without the borrowing constraint – and that user costs will be lower if the constraint is binding for some households with a lower income. In particular, if the households with the lowest incomes that are participating in the market experience a binding constraint, all households in the market will realize lower user cost of housing. This implies that the borrowing constraint will benefit some or even all households and harms none of them. This conclusion contrasts with the alternative (Muth) model with malleable housing in which borrowing constraints may harm some households and benefit no one. However, a paradoxical feature of the model is that constrained households are no longer in equilibrium, but have a marginal willingness to pay for housing that exceeds its market price. They are thus willing to pay for relaxation of the constraint, even though this would imply that in the resulting equilibrium their welfare is lower.

These conclusions change if borrowing constraints differ among households, for instance because some have wealth that may be invested in owner-occupied housing to circumvent or soften the borrowing constraint. The reason is that in this more general situation the assignment rule may change due to the presence of borrowing constraints. That is, two households that are otherwise similar will be assigned to houses of different qualities if they face different borrowing constraints. As a consequence constrained households may now be assigned to houses of lower quality compared to the equilibrium without constraints, while unconstrained households may be assigned to housing of higher quality. The lower housing consumption of constrained households has a negative impact on their welfare. Still, it can be shown that in the more general situation with heterogeneous constraints or tastes, the borrowing constraints decrease the average housing price as long as they are binding for some households and do not increase the price for any level of housing quality. However, the welfare effects for constrained households are now ambiguous.

Finally, we show that binding borrowing constraints imply the possibility for arbitrage by buy-to-let investors. The reason is that the constrained households' willingness to pay for housing exceeds the marginal price it faces. Hence they are willing to pay more for higher quality housing than they are currently allowed to do. Buy-to-let investors can solve their problem by giving them access to the higher quality housing, albeit only as renters. These agents can buy the higher quality houses at a price that exceeds the market level and let them to (until then) constrained households at their marginal willingness to pay. With free entry of such investors and negligible intermediation costs, the market will move towards an equilibrium in which housing consumption and expenditure are the same as in the absence of binding borrowing constraints, but where all households experiencing binding constraints on the owner-occupied market have become tenants. The restrictions on housing consumption have effectively been removed, but so have the lower housing prices implied by the borrowing constraints.

Related Literature. Assignment models have been applied for a long time to allocation of workers over jobs, see Sattinger (1993) for a review. The first application of such a model to the housing market, as far as we know, is Braid (1981), which is the starting point of the model of the present paper. Braid studies a rental market in which houses differ in one-dimensional quality and households all have the same tastes, but may differ in income. The assignment model of Landvoigt et al. (2015) differs mainly from this setup in that they consider households who divide their wealth, rather then their income, between the purchase price of owner-occupied housing and consumption of a composite good in the current period. They expand their model to a more quantitative version in which heterogeneous households maximize intertemporal utility subject to an intertemporal budget constraint as well as a down-payment borrowing constraint for housing, which they then take to the data. However, they do not formally discuss how the heterogeneity of households and the presence of borrowing constraints affects the allocation of households over housing in equilibrium.

The setup of the assignment model in Määttänen and Terviö (2014) and Määttänen and Terviö (2021), is similar to that in Landvoigt et al. (2015) in that they consider households facing a single-period constraint in which they have to distribute their wealth over owner-occupied housing and a composite consumption good. They start from an initial equilibrium allocation of the household population over the housing stock and then study the impact of change in the income distribution. In general such a change implies that some households move from one house to another and use the revenues from selling the initial house to help finance the next one. The analysis focuses on the impact of mean- and order preserving spreads of the distribution which have the special feature of leaving each household's rank in the income distribution unchanged. In the assignment model such changes leave the allocation of households over houses unchanged - hence there is no trade in response to it - and only results in a change in house prices. That is, after the change in the income distribution for each household the preferred house is the same as before, but its price is now different. The authors show that if all incomes increase, all house prices will also increase. They note that Braid (1981)'s model is essentially a version of their model in which the price of the initial house is absent from the wealth constraint.

Few households have enough wealth to purchase a house without having to borrow. The availability of mortgage loans is therefore an important determinant of the homeownership rate. Banks usually impose restrictions on the size of mortgage loans in order to limit the probability of default. They typically look at the loan-to-value ratio (LTV) and the ratio between the mortgage payments and household income at the time of purchase. An upper bound on the LTV implies a lower bound on the downpayment, amount of equity that has to be invested by the household. This constraint implies a hurdle for would-be homeowners that affects tenure choice (see Brueckner (1986)). With additional security, provided, for instance, by mortgage insurance, banks are willing to relax the downpayment constraint. The Netherlands is a relevant case, because there such insurance is offered under attractive conditions by a semi-government agency (NHG). With NHG insurance, households are able to finance the purchase of a house completely by a mortgage loan at a somewhat lower interest rate. However, to mitigate default risk, the mortgage payment-to-income ratio (PTI) is used as a yardstick that restricts the size of the loan that is available. Hence in the Netherlands the LTV constraint is of negligible importance for households qualifying for NHG, while the PTI constraint is binding for many. In the US and the UK, where mortgage insurance is usually expensive, the LTV or downpayment constraint is often the main hurdle. An important difference between the two is that the downpayment constraint forces many households to postpone ownership until enough has been saved to pay the downpayment, whereas the PTI constraint allows immediate access to homeownership albeit only for modest housing if income is relatively low.

Linneman and Wachter (1989) is one of the first empirical studies of the impact of borrowing constraints on tenure choice. The authors assume that the LTV and PTI constraints are simultaneously present. Simple elaboration of these constraints leads to maximum feasible values of the purchase prices of houses which depend on income and wealth. They estimate unrestricted housing demand by concentrating on households for which these constraints are not binding and use the estimation results to compute the gap between the price of desired and actual housing for the other households. This allows them to quantify the impact of both constraints. Interestingly, they go on to consider the impact of these constraints on tenure choice by estimating a discrete choice model and find that the frequency of renting is much higher among households that are unable to realize their desired housing consumption under owner-occupation. This suggests that such households have better possibilities of realizing their desired housing consumption on the rental market. One possible reason for this is that rents are lower than the user cost of comparable owner-occupied housing. From the point of view of economic theory this is somewhat implausible as one would expect arbitrage between both types of tenure.[3] A second explanation, in line with our analysis of borrowing constraints, holds that the rental market allows these households to realize, or at least come closer to, their desired housing consumption because there are no constraints on rent-to-income ratios. Duca et al. (2011) indeed show that borrowing constraints help to explain price-to-rent ratios.

Later literature on the impact of borrowing constraints does more carefully consider endogeneity and selection issues, which turn out to be non-trivial. For instance, Blickle and Brown (2019) study the impact of intra-family wealth transfers in Switzerland, a country with a large and diversified rental housing market. They find that such transfers lead to a sizable increase in the propensity to own but do not induce receivers to move to larger houses or more attractive locations. This suggest that the main effect of such transfers is the relaxation of borrowing constraints. Kinghan et al. (2022) exploit a policy reform in Ireland to provide direct empirical evidence that macroprudential LTV regulations reduce LTV ratios of affected borrowers. Both papers obtain causal identification by focusing on the effects for affected households, but as a result cannot address the effect of borrowing constraints on the homeownership rate or buy-to-let investment.

The most important mechanism that prevents a simple aggregation of individual effects is that borrowing constraints reduce house prices, as cross-country studies such as Akinci and Olmstead-Rumsey (2018) show. Lower house prices potentially allow other, unconstrained households to access the housing market. Vigdor (2006) is one of the few papers that considers such spillover effects of binding borrowing constraints. He develops a theoretical model that predicts that such constraints can depress house prices and points out that this implies that the relaxation of such constraints may have detrimental welfare effects for some households. Using vetaran status as an instrument for better access to credit, he provides empirical evidence that supports this hypothesis. Using the same reform as Kinghan et al. (2022), Acharya et al. (2022) show that borrowing constraints result in spillovers in mortgage credit from low-income to high-income borrowers and from urban to rental areas. As a result,

[3]We should note that arbitrage arguments are less forceful on the housing market, which is in many respects imperfect, than on many asset markets, see Glaeser and Gyourko (2007).

house price growth fell more for smaller properties in hot housing markets. None of these papers considers buy-to-let investment.

In an attempt to avoid econometric complications, Fuster and Zafar (2021) conduct a stated choice experiment in which respondents were asked their willingness to pay for housing, in case they would have to move to a different city with similar house prices on short notice. The financial conditions under which the purchase of the new house should be realized differ in the various parts of the experiment. They find that current homeowners tend to choose similar houses, while current renters, who also have to imagine that they purchase a house, often opt for smaller houses. Lowering the required downpayment has a much larger impact on the willingness to pay of renters. If households are allowed to choose the downpayment, renters usually take lower values than owners. And if the respondents receive a large cash windfall, renters would spend a larger fraction of it on the downpayment. All these results point to the importance of borrowing constraints in tenure choice, but cannot straightforwardly be aggregated to equilibrium responses.

2 The return of the private rental sector in the Netherlands

To illustrate the empirical relevance of the analysis, this section contains a brief discussion of some recent developments in the Dutch housing market which triggered the analysis of the present paper. However, note that buy-to-let investment occurs in many other housing markets – such as the United Kingdom – and is possibly also related there with the impact of borrowing constraints.

In the second half of the 20^{th} century the Dutch housing market was characterized by a large social housing sector and a growing share of owner-occupied housing. Since social housing was subsidized and full mortgage deductibility lowered the user costs of owner-occupiers, there was little room left for the private rental sector. As a consequence, this part of the housing market, which was the most important one until the 1950s, was continuously shrinking over time.[4] A reasonable prediction in the 1990s would have been that the provision of short term housing for expats would be its only market niche in the long run. However, things have changed since then.

In the Netherlands house prices have been almost continuously increasing since the mid-1980s and, after a dip associated with the Global Financial Crisis and the euro crisis, reached unprecedented values. Since 2000, mortgage lenders have – under pressure of the government and consumers authorities, introduced a Code of Conduct that was tightened in 2011[5]. Although the principle of the Code is 'comply

[4]The slow disappearance of the sector despite its lack of profitability was due in large part to extensive protection of the rights of existing tenants. They could not be forced to depart, while rent control ensured the prolonged attractiveness of continuation of the contract.

[5]Rouwendal and Petrat (2022) show that the tightening of the Code of Conduct had a negative

or explain' its rules only allow for exceptions in specific cases since the tightening took place in 2011 and in more recent years some of its rules have been transformed into official government regulations that are strictly binding for these lenders. The most important aspect of the code of conduct is that home buyers should not be allowed to get a loan implying a mortgage payment to income ratio that threatens their possibility of realizing other necessary consumer expenses[6]. With high demand pressure and strongly increasing prices, the implied mortgage qualification constraint becomes binding for a growing share of households. As a result there emerged a concern for the position of (what are referred to as) medium income households, that is households that are not eligible for social housing, but whose borrowing capacity is not sufficient to allow them access to (a reasonable part of) the owner-occupied markets in large cities, especially Amsterdam. For such households, rental housing in the private sector is an obvious alternative.

It is perhaps not a coincidence that since the early 2000s the room for the private rental sector has increased. It was determined that the determination of rents that exceeded a particular threshold could be left to market forces. That is, rent control no longer referred to such housing. Moreover, tenant protection was reduced in this part of the market. Temporary contracts became common. Activities of private investors in rental housing, also from those of foreign origin, were welcomed by the national government. However, not only professional investors entered the market, many private individuals possessing some wealth decided to invest in rental housing, expecting a stable flow of revenues and on top of that the possibility to realize a handsome indirect return via continued price increases[7].

Indeed the private rental sector in the Netherlands has shown a remarkable revival in recent years. Van der Harst and de Vries (2019) show that the turning point occurred in 2012, one year after the tightening of the Code of Conduct. In the municipality of Amsterdam, where social housing dominated the market until the 1990s and owner-occupied housing reached a share of 30% only in the 2010, the growth of the private rental sector started to reduce the share of owner-occupied housing again in the period 2017-2019.[8] To illustrate, in 2019 47% of the recent movers in Amsterdam entered a house in the private rental sector.[9] The growth continued in more recent years. In 2021 more than half of the housing moves in Amsterdam ended in a private rental house.[10]

impact on Dutch house prices at the time of the euro crisis.

[6]In the Netherlands guidelines are provided by NIBUD, an institute specialized in the study of consumer expenses. These guidelines are part of the Code of Conduct for mortgage underwriting used by the banks.

[7]Lankhuizen and Rouwendal (2020) report gross direct returns of 6% in the Amsterdam metropolitan area and somewhat lower values in the municipality of Amsterdam.

[8]From 32.4% in 2017 to 30.8%. See Berkers and Dignum (2020).

[9]Berkers and Dignum (2020)

[10]54.6% while private rental housing now covers 30.5% of the housing stock, a larger share than the owner-occupied sector.

The rapid increase in private rental housing caused concern among especially medium-income would-be homebuyers that they were outbid from market segments that until then were accessible to them. Houses that switched from owner-occupied to private rental are often apartments. Moreover, these apartments are often located in the Center and South of Amsterdam, the most popular and expensive parts of the city.

In response to this concern the municipality looked for possibilities to limit the activity of buy-to-let investors. Initially, it was hard to find appropriate instruments. An obligation for first owners of newly constructed houses to live in these houses themselves was a modest start.[11] In 2021, the transfer tax on houses was changed. Since the beginning of that year, first-time buyers were exempted, while other buyers who would live in the house have to pay 2% and those who do not have to pay 8%. Moreover, the Dutch parliament is considering measures to better protect the position of tenants in the private sector.

The change in attitude towards investors in private rental housing is largely driven by the concern that they overbid medium-income would-be homeowners and drive them out of the market for owner-occupied housing. The public discussion about the measures that are introduced or considered is dominated by sentiments about the bad circumstances of these medium-income households. Initially, the main sentiment was that rental housing should be made available for them, but when the private rental sector exploded, a feeling emerged that the position of would-be buyers became increasingly problematic. However, a systematic analysis of the pros and cons of the developments in the market and the policy proposals is lacking.

3 Two models of housing market allocation

In this section we develop an analysis of tenure choice on a housing market with borrowing constraints. In subsection 3.1 the general setup of the housing market will be introduced. Subsection 3.2 discusses Muth (1960)'s model of the housing market which considers housing essentially as a conventional (in microeconomic textbooks) homogeneous commodity. This model is standard in the economic literature. In subsection 3.3 we take into account that housing is durable and that its stock is given, at least in the short run. The model discussed there is essentially that of Braid (1981), although that study considers a rental market, while here the model is applied to owner-occupied housing.

3.1 The setup

We consider a market with a population of households who all have identical preferences over housing q and other consumption c that can be described by the utility

[11] See https://www.amsterdam.nl/wonen-leefomgeving/vastgoedprofessionals/woningbouw-transformatie/verhuurverbod-zelfbewoningsplicht/

function

$$u = u(q, c). \tag{1}$$

The utility function u is increasing in both arguments, quasi-concave and twice differentiable. Housing and other consumption are both normal goods.[12]

Households maximize utility subject to a budget constraint. The budget will be referred to as income, but it should really be interpreted as the amount of money the household is willing to spend on consumption (of housing and other goods) in the period we consider.[13] Households differ in incomes. The distribution of income is given by the strictly increasing and continuously differentiable function $F(y)$, which has positive support on the interval $[y^{min}, y^{max}]$. We denote the density function by $f(y)$. The total number of households equals B, where $B = F(y^{max})$.

Households maximize utility subject to the budget constraint

$$c + p(q) = y, \tag{2}$$

where $p(q)$ denotes the user cost of housing and y is the available budget. Think of the user cost $p(q)$ as a function of the sales price of the house. More specifically it is the product of the market value and the opportunity cost of the capital invested in the house, plus costs of maintenance and taxes minus the expected increase in value of the house. A common specification in the literature is: $p(q) = \gamma P(q) - E(\Delta P/(1 + r))$, where $P(q)$ denotes the sales (transaction) price, γ reflects the various cost items (maintenance, insurance, taxes), ΔP is the (expected) change in the price of the house and r the rate of discount. In what follows, we focus on the user costs that equilibrate the market in the current period, without paying attention to its composition.[14] Note that it is not assumed that the user cost $p(q)$ (or the transaction price $P(q)$) is linear in the amount of housing. That is, the marginal price of housing $\pi(q) = \partial p/\partial q$ may depend on the quantity of housing services consumed.[15]

Households have an outside option, for instance renting social housing or living in temporary housing. We include this in the model by assuming there exists a combination of housing consumption q^* and user cost p^* that is available for every

[12] A population with identical tastes is clearly restrictive. In Appendix B we discuss an extension of the model to situations with households differing in tastes.

[13] This budget can be thought of as being determined in an intertemporal utility maximizing framework, jointly with savings. See, for instance, Deaton and Muellbauer (1980). In this framework uncertainty about future house prices and the expected wealth effects of housing transactions can also be taken into account.

[14] Note that the first part of the user cost, $\gamma P(q)$, are mainly to out-of-pocket expenses like taxes, mortgage interest payments and maintenance, whereas expected price changes are not. The implication is that monetary outlays on housing can exceed the user cost when house prices are expected to increase. Borrowing restrictions, such as the mortgage qualification constraint discussed later in this paper refer to monetary expenses. Note also that there is in general not a one-to-one correspondence between users cost $p(q)$ and transaction price $P(q)$.

[15] In later subsections we will encounter situations in which the function $p(q)$ is not differentiable at some points.

household. Hence the reservation utility is $u^*(y) = u(q^*, y - p^*)$ and households will only participate in the (primary) housing market studied here if this offers them a higher utility than the outside option.

3.2 Malleable housing; Muth's model

In this subsection we disregard the durability of housing and assume that houses are supplied on a market with perfect competition according to a cost function $C(q)$. We focus on the case in which this function is linear in the housing quality:[16]

$$C(q) = bq. \tag{3}$$

The price of a house with quality q is then equal to

$$p(q) = bq, \tag{4}$$

which implies that the budget constraint is linear in housing consumption q. Maximization of utility subject to this constraint leads to a housing demand function

$$q = h(q, y). \tag{5}$$

We assume throughout that housing is a normal good: $\partial h/\partial y > 0$. In equilibrium all households are able to realize their demand for housing, which means that the distribution of housing reflects the distribution of income. More precisely, for any pair of households the one with the highest income will always consume more housing services.

Households will participate in the market if indirect utility $v(b, y) = u(h(b, y), y - bh(b, y))$ exceeds reservation utility u^*. The critical income y^c that is needed to participate in the housing market is defined implicitly by the equality: $v(b, y^c) = u^*(y^c)$. If this income level is smaller than y^{min} we take y^{min} as the critical income level.

For later reference, we note some other aspects of the equilibrium. All households with an income at least equal to y^c occupy a house. The household with the critical income inhabits a house of quality $q^{min} = h(b, y^c)$. Denoting the distribution of housing as $G(q)$, we must then have

$$G(h(b, y)) = F(y) - F(y^c). \tag{6}$$

Housing consumption and income are thus aligned in the sense that the ordering of households on the basis of their consumption of regular housing is identical to their ordering in the basis of their incomes. If both distributions are differentiable and we denote their derivatives as $f(y) = \partial F/\partial y$ and $g(q) = \partial G/\partial q$, we must in equilibrium

[16] This implies no loss of generality. If the cost is increasing in housing quality produced, we can define quality so that the amount of housing services is proportional to the production cost.

have $g(q)dq = f(y)dy$, and since all households are on their demand curves, this implies that the slope of the Engel curve for housing is

$$\partial q/\partial y = f(y)/g(q). \tag{7}$$

An important advantage of Muth's model is that it can easily deal with heterogeneity of tastes, borrowing constraints and buy-to-let investors as will now be discussed. The reason is that the housing stock adjusts easily to the implied changes in demand. If households differ in preferences, they can simply order the amount of housing that is suitable for them at the prevailing price. Those with more intense preferences for housing simply order a larger house. To make things concrete, if there are G groups of households $g = 1...G$ and households in each group have identical tastes, but may differ in income, we can describe the market for each group in the same way as was done above. We get G market segments that have the same properties as the single one analyzed above.

Borrowing constraints impose a ceiling on housing demand. To analyze their impact we start by considering a simple situation in which housing expenditure is limited to a particular share μ of household income and there is no rental housing available. In this case, households experiencing a binding constraint must reduce their housing demand to the level that is compatible with the constraint. That is, borrowing constraints reduce housing consumption but have no impact on the price of housing. This remains true if we consider a more general model in which households may have some wealth available to invest in housing. They can use this wealth to relax the borrowing constraint and if their wealth is large enough, avoid it completely. Again, the constraints do not have any impact on housing prices, but the average amount of housing consumption is reduced as long as the constraint is binding for some households. No household will ever increase its housing consumption because other households are experiencing binding restrictions.

However, the presence of binding borrowing constraints provides arbitrage opportunities for investors in rental housing. By offering rental housing of the qualities that are preferred by restricted households, but out of reach for them due to the borrowing constraints, they can make a profit. The reason is that the constrained households have a marginal willingness to pay for housing that exceeds the marginal production cost b. If enough investment in rental housing is realized, and the additional cost of making the housing available as rental housing is negligible, the market will return to the equilibrium without borrowing constraints as far as housing consumption and expenditure are concerned, but all households experiencing binding borrowing constraints in the owner-occupied market have shifted to the private rental sector.

3.3 Durable housing

In the previous subsection we have assumed that all houses are created in the period considered. In reality most of the housing dates back from the past and only a small percentage is constructed per period. To capture this, we will now consider the situation in which housing supply is completely fixed. That is, the housing stock – the distribution of houses, each with a given number of housing services – is given. This setup is similar to Braid (1981) who considered a rental market. Here we assume the market refers to the market for owner-occupied housing in which the user cost occupies the role of rent.

As in the previous section, houses are available in a continuum of varieties and the distribution function of the quality of housing is still denoted $G(q)$. However, in contrast with the previous section, G is now given. It is assumed to have support on an interval $[q^{min}, q^{max}]$ and to be strictly increasing and continuously differentiable. The number of houses is $S, S = G(q^{max})$. We assume that the number of households is at least equal to the number of houses: $B \geq S$. The critical value y^c is now thus determined by the condition that only S households can own a house:

$$B - F(y^c) = S. \tag{8}$$

In equilibrium, user costs will be such that households with incomes below the critical value y^c will choose the outside option. Households with higher incomes compete with each other for the available housing instead of ordering housing construction. The following lemma characterizes the resulting assignment of houses to households.

Lemma 1 (Assignment rule)**.** *In equilibrium, the assignment follows the continuous function*

$$y(q) = y^c + F^{-1}(G(q)). \tag{9}$$

Proof. First, it is easy to see that $p(q)$ must be increasing and continuous in q.[17] That is, the user cost of housing is in the present framework a possibly nonlinear, but always increasing, function of the housing services offered by a house. Second, since housing is a normal good, a household with a higher income will in equilibrium consume more housing than a household with a lower income.[18] These two observations imply the assignment rule. This function is continuous because (9) implies that the change in income dy/dq is

$$\frac{dy}{dq} = \frac{g(q)}{f(y)}, \tag{10}$$

where $g(q)$ and $f(y)$ are continuous and positive. □

[17]Suppose a house with a better quality is less expensive than that of a lower quality. Then there will be no household choosing the lower quality house. Suppose there is a discontinuity in the house price function. Then the marginal price of housing is infinitely high at the point of the discontinuity, which means that there will be no demand for housing with quality just above the point of discontinuity.

[18]Consider a pair of households with different incomes and assume that the one with the highest income consumes less housing than the one with the lowest income. Then the high-income household is able to compensate the low-income household for moving to the lower quality house and still reach a higher utility. Hence they can engage in a transaction that is beneficial to both, which shows that that initial situation is incompatible with equilibrium. See Appendix A.1 for details.

In equilibrium the ranking of households on the basis of housing consumption thus corresponds to the ranking of households on the basis of income. However, in this setting there is no reason to suppose that housing expenditure in equilibrium will be proportional to the quality of housing services consumed, q. The main purpose of this section is to find the equilibrium user cost function $p(q)$ on this market.

The user cost function can be derived as follows. The household with the critical income must be indifferent between housing of the lowest quality and the outside option:

$$u(q^{min}, y^c - p(q^{min}) = u^*(y^c). \tag{11}$$

This equation pins down the value $p(q^{min})$, the user cost of housing of the lowest quality.

The following lemma characterizes the equilibrium user cost function.

Lemma 2 (Equilibrium prices)**.** *In equilibrium, the user cost function is given by the first-order differential equation*

$$\pi(q) = \frac{\partial p}{\partial q} = M(q, y(q) - p(q)), \tag{12}$$

with initial condition $p(q^{min})$ *from* (11)*, in which* $M(q, c) = (\partial u/\partial q)/(\partial u/\partial c)$*, and in which* $y(q)$ *follows from the assignment rule in* (9)*.*

Proof. The outside option pins down the user cost at q^{min} and y^c. The user cost at higher qualities is determined by the requirement that the slope of the price function, the marginal price of housing $\pi(q)$, must be equal to the marginal rate of substitution $M(q, c)$. $p(q)$ is differentiable because $M(q, c)$, $p(q)$, and $y(q)$ are continuous in their arguments. □

We can thus trace out the housing price function by making use of the equilibrium condition and the assignment rule. That is, starting from the critical income y^c, the lowest housing quality q^{min}, and its price $p(q^{min})$, (9) determines the income associated with each housing quality and then (12) determines the equilibrium price for each housing quality.

Let us now consider the special case in which the equilibrium price function is linear, as in Muth's model. This happens if the marginal price of housing is constant: $\partial\pi(q)/\partial q = 0$. It requires the marginal willingness to pay for housing to be constant as well. It can be shown (see Appendix A.2) that this implies that we must have

$$\frac{\partial M/\partial q}{\partial M/\partial(y - p)} = \pi - \frac{g(q)}{f(y)}, \tag{13}$$

for all possible combinations of q and $y - p(q)$. This condition requires that the distributions of income and housing are aligned to the marginal willingness to pay in a very specific way. In the model with malleable housing $g(q)$ adjusts so that (13) is always satisfied. However, with $g(q)$ fixed, we lose this flexibility.[19]

One way of interpreting this is that with a given stock of housing there needs to be equality of supply and demand for every housing quality. That is, instead of a single market for housing services that can be used to construct any desired quality of housing, we now have a continuum of markets for housing with given quality.[20] Equilibrium then requires a separate price for each submarket. This is the reason that we cannot expect the price function $p(q)$ to be linear when the housing stock is given.[21]

3.4 Effects of an income increase compared

To appreciate the difference between Muth's model and the assignment model, it is useful to briefly discuss some comparative statics. What happens if all incomes increase by the same percentage so that the ranking of the households on the basis of their income remains unchanged? In Muth's housing market all households increase their housing consumption, while the price per unit of housing remains unchanged. Moreover, some households formerly choosing the outside option will now order an owner-occupied house, thereby increasing the size of the housing stock.

Now consider the assignment model. Here the size of the housing stock remains unchanged. Since the critical income increases, the price of housing of the lowest quality increases.[22] All households occupying a house in the given stock in the initial situation want to increase their housing consumption and bid up the price of housing. However, since the stock of housing is given and their position in the income distribution did not change, they end up in the same house, while paying a higher user cost. The reason is that housing of minimum quality has become more expensive, while on top of that the marginal willingness to pay for housing has increased for every value of housing consumption. Because of the latter effect, the price (user cost) increase will itself be increasing in q. This contrasts sharply in the market with malleable housing, but it seems to describe rather well what happens in many urban housing markets when household incomes increase.

[19]For instance, with Cobb-Douglas utility $u(q, c) = q^{\alpha}c^{\beta}$ the left-hand side of (13) is equal to $(y - p(q))/q$ and one may choose $p(q)$ for every q so that (13) is satisfied, but the first derivative of this price function will only by coincidence be equal to the constant π, implying a contradiction.

[20]Compare the plural in the title of Landvoigt et al. (2015).

[21]Our assumptions imply that the second order condition for utility maximization is satisfied with a linear budget constraint. With a nonlinear housing price function the budget constraint is also nonlinear. Appendix A.3 shows that the second-order condition remains satisfied.

[22]The reason is that the normality of housing implies that the willingness to pay for an increase in housing consumption from q^* to q^{min} increases.

4 Borrowing constraints

The next step is the introduction of borrowing constraints. We start in subsection 4.1 by considering a uniform restriction on the share of user cost in household income and generalize this in subsection 4.2 to an arbitrary distribution of maximum purchase prices or user costs that may depend on household income. Throughout this section we assume all households have the same preferences.

4.1 A mortgage qualification constraint

The analysis thus far has assumed that households are not restricted in their choice behavior, except by the budget constraint. Many households need a mortgage loan to finance the purchase of their house and mortgage payments are an important element of their user cost. Lenders usually impose restrictions on the size of these loans. As discussed in the introduction, in the Netherlands the ratio of mortgage payment to income is the most important indicator used by the lenders, and we will now consider the implications of such a constraint. In this subsection we do so in a simple way: we impose that user cost can at most be equal to a fraction μ of income for all households:[23]

$$p(q) \leq \mu y. \tag{14}$$

We refer to this restriction as the mortgage qualification constraint.

To set the stage and get a first impression of the impact of constraint (14) on house prices, we consider the housing price as a function of income. The assignment rule (9) is a continuous and increasing relationship between housing quality and income and we write its inverse as $q(y)$. Using this, we derive the user cost of housing as a function of income, $p(q(y))$. Since the equilibrium housing price is increasing in quality and, according to the assignment rule, quality is increasing in income, the user cost $p(q)$ must also be increasing in income, with slope

$$\frac{dp}{dy} = \pi(q)\frac{dq}{dy} = \pi(q)\frac{f(y)}{g(q(y))}. \tag{15}$$

Figure 1 illustrates the relationship between user cost and income and the mortgage qualification constraint (14). The equilibrium price function in the initial situation (without borrowing constraints) is $p(q(y))$. In the situation pictured, the constraint is binding for incomes between y^* and y': for such incomes the user cost implied by $p(q(y))$ is higher than μy. This picture may suggest that after the introduction of the constraint the equilibrium price function will coincide with the constraint on the interval $[y^*, y']$ while for incomes higher than y' the initial price function is still valid. As will be shown now, this conjecture is not valid. Instead, the constrained price function will start departing from the borrowing constraint from some $y'' < y'$ onwards, as illustrated by the grey bended line in Figure 1.

[23] We will later consider cases in which households can face different constraints.

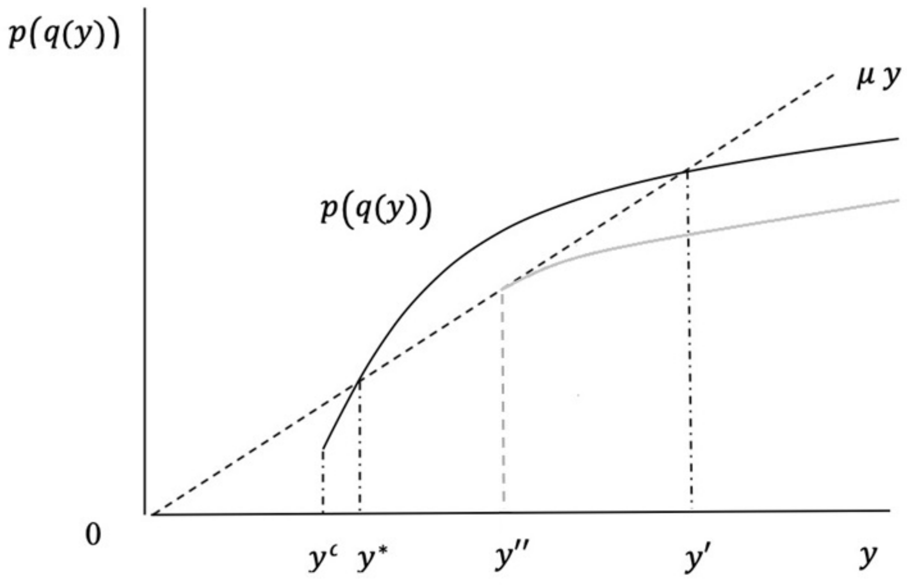

Figure 1: The mortgage qualification constraint and the price of housing

We assume that initially the market is in equilibrium and there is no borrowing constraint, and consider what changes if such a constraint is imposed. We will refer from now on to the price function derived in the previous section for the situation without borrowing constraints as $p^m(q)$ and to the one that is relevant with the mortgage qualification constraint present as $p^{bc}(q)$. We will first consider the case in which the borrowing constraint starts to bind at $y^* \geq y^c$, similar to the situation in Figure 1, and present the case at which the borrowing constraint already binds at y^c afterwards.

Proposition 1. *Consider the introduction of a borrowing constraint that starts to bind at $y^* \geq y^c$: $p^m(q^{min}) \leq \mu y^c$, and y^* is the smallest $y \geq y^c$ such that in the right-sided neighborhood of y^*, $p^m(q(y)) > \mu y$. Define y'' as the smallest $y > y^*$ for which in the right-sided neighborhood of y'', $M(q(y), (1-\mu)y)f(y)/g(q(y)) < \mu$ if that occurs, and as y^{max} otherwise. Then,*

- *The assignment rule $q(y)$ does not change;*
- *$p^{bc}(q(y)) = p^m(q(y))$ for $y \in [y^c, y^*]$, and $p^{bc}(q(y)) < p^m(q(y))$ for $y \in (y^*, y^{max}]$, so that utility is the same for all households with $y \leq y^*$ and higher for all households with $y > y^*$:*
    1. *$p^{bc}(q(y)) = \mu y$ for $y \in [y^*, y'']$;*
    2. *$p^{bc}(q(y))$ is described by $\pi^{bc}(q(y)) = M(q(y), y - p^{bc}(q(y)))$ with initial condition $p^{bc}(q(y'')) = \mu y''$, for $y \in [y'', y^{**}]$ if y^{**} exists, and for $y \in [y'', y^{max}]$ otherwise, in which y^{**} is the smallest $y > y''$ such that in the right-sided neighborhood of y^{**}, $p^{bc}(q(y)) > \mu y$.*

- *If y^{**} exists, the constraint binds the constrained price function again, and 1. and 2. apply recursively with y^{**} replacing y^* and y'' redefined accordingly.*

Proof. Assume for now that the assignment rule does not change as the result of the mortgage qualification constraint. Then the minimum income of owner-occupiers is the same as in the equilibrium without a borrowing constraint, and because the borrowing constraint becomes binding only later, $p^{bc}(q^{min}) = p^m(q^{min})$. We can now follow the logic of Lemma 2 to show that until income y^*, where the borrowing constraint becomes binding, and the associated housing quality $q(y^*)$, the functions $p^{bc}(q)$ and $p^m(q)$ will coincide.

At income y^*, $p^m(q)$ crosses the borrowing constraint. Using (15), it must thus be that $\pi^m(q(y^*))f(y^*)/g(q(y^*)) > \mu$. A household with an income slightly higher than y^* would thus like to spend a larger income share than μ on housing, but is restricted. The slope of $p^{bc}(q)$ is then thus smaller than $p^m(q)$ and equal to

$$\frac{\partial p^{bc}}{\partial q} = \mu \frac{g(q(y))}{f(y)} < M(q(y), y - p^{bc}(q(y))). \tag{16}$$

Now consider two cases. First, the constraint remains binding. Then (16) continues to describe the slope of $p^{bc}(q)$ and thus $p^{bc}(q(y)) = \mu y$ for all $y \in [y^*, y^{max}]$. Then it follows immediately that $p^{bc}(q(y)) < p^m(q(y))$ for $y > y^*$. Second, the constraint stops to bind: there exists some $y'' > y^*$ for which $p^{bc}(q(y'')) = \mu y''$ and $M(q(y''), y'' - p^{bc}(q(y'')))f(y'')/g(q(y'')) = \mu$, after which the left-hand side becomes smaller. Then $p^{bc}(q(y)) = \mu y$ only for $y \in [y^*, y'']$.

To see that this function does not describe the constrained price function all the way up to y', note that at income y', where the constraint is again just binding, the marginal willingness to pay for housing is smaller than $\mu g(q(y))/f(y)$. Consequently, the constraint cannot be binding at income y'. Hence there must be a lower income, y'', for which this constraint stops being restrictive, unless the constraint is binding up to y^{max}. The marginal willingness to pay for housing is a continuous function of income, so the function $p^{bc}(q(y))$ will not have a kink at y'', which defines y''.

For incomes higher than y'', the slope of $p^{bc}(q)$ is thus equal to the marginal willingness to pay for housing, as long as the constraint does not become binding again, at which point the slope is again given by $\mu g(q(y))/f(y)$. Irrespective of whether that happens, it follows that $p^{bc}(q(y)) < p^m(q(y))$ for $y > y''$.

The households with $y \geq y''$ pay a lower price for the same housing they would have occupied in the unconstrained market equilibrium and they are satisfied with their situation, as the marginal willingness to pay for housing equals the marginal price they face. The households with $y \in (y^*, y'')$ would like to overbid richer households, but the borrowing constraints prevent them from doing so. The households with $y \in [y^c, y^*]$ still face the marginal price that equals their marginal willingness to pay for housing, so are satisfied too, and will not be overbid by households with $y < y^c$. We conclude that the assignment rule does not change.

Because $p^{bc}(q(y))$ and $p^m(q(y))$ already started diverging for $y > y^*$, prices are lower for $y > y^*$. Because the assignment is the same, it follows that utility is the same for all households with $y \leq y^*$ and higher for all households with $y > y^*$.

□

Summarizing, we have shown that the function $p^{bc}(q(y))$ coincides with $p^m(q(y))$ until this function hits the borrowing constraint. Then $p^{bc}(q(y))$ follows the borrowing constraint until, if that happens, the marginal willingness to pay for housing is so low that households prefer to spend less on housing than is allowed by this constraint. This happens at an income y'' that is lower than the income y' at which $p^m(q(y))$ crosses the borrowing constraint. There is a kink in $p^{bc}(q(y))$ at income y^* but not at y''.

Households with income between y^* and y'' want to consume more housing, but are unable to realize this desire. Due to the binding borrowing constraint they pay less for the same house they would have occupied in the unconstrained market equilibrium, so that their utility will be higher. The utility of households with income largher than y'' will also be higher, but their marginal willingness to pay for housing will equal the marginal price, unless the borrowing constraint binds again. The assignment rule of the unconstrained market equilibrium remains valid, because no household can reach a higher utility by deviating from this rule.

Now consider the introduction of a borrowing constraint that already binds at q^c. The following corollary shows that this sitation closely follows the description above.

Corollary 1. *Consider the introduction of a borrowing constraint that already binds at y^c: $p^m(q^{min}) > \mu y^c$. Define y'' as the smallest $y \geq y^c$ for which in the right-sided neighborhood of y'', $M(q(y), (1-\mu)y)f(y)/g(q(y)) < \mu$ if that occurs, and as y^{max} otherwise. Then,*

- *The assignment rule $q(y)$ does not change;*
- *$p^{bc}(q(y)) < p^m(q(y))$, so that utility is higher for all households:*
  1. *$p^{bc}(q(y)) = \mu y$ for $y \in [y^c, y'']$;*
  2. *$p^{bc}(q(y))$ is described by $\pi^{bc}(q(y)) = M(q(y), y - p^{bc}(q(y)))$ with initial condition $p^{bc}(q(y'')) = \mu y''$, for $y \in [y'', y^*]$ if y^* exists, and for $y \in [y'', y^{max}]$ otherwise, in which y^* is the smallest $y > y''$ such that $p^{bc}(q(y)) = \mu y$.*
- *If y^* exists, the constraint binds the constrained price function again, and Proposition 1 applies.*

What happens if the mortgage qualification constraint is relaxed? Consider again the situation pictured in Figure 1. Households for whom the constraint is no longer binding will attempt to increase their housing consumption until the constraint binds again, or until they are on their housing demand function. However, if housing supply does not adjust, all households will stay in the same house, which will have become more expensive. If in the new situation the constraint is no longer binding for any household, the market returns to the equilibrium price function $p^m(q(y))$. If some households are still constrained, then in the new equilibrium the interval for which the constraint is binding will be a smaller. Prices will increase for all households with an income higher than y^*. The welfare of all these households will decrease, since their housing consumption and income do not change. However, the number of constrained households (for whom the borrowing constraint is binding) will be smaller than with the tighter constraint.

Note that the results of this section are sensitive to the assumption that all households have to borrow all the money needed for purchasing their houses. If some households with a given income experience a binding credit constraint while others own some wealth and are willing to invest it in their houses, the latter group may not experience a binding credit constraint while the former group does. In such a case the allocation of households over the housing stock will be affected by the borrowing constraint, as will be discussed below.

4.2 General borrowing constraints

In this subsection we consider a more general situation in which households can experience borrowing constraints of a general nature. The borrowing constraint is a household-specific maximum imposed on the purchase price of a house.[24] Since there is a one-to-one relationship between purchase price and user cost, we include this in our model as a maximum user cost ρ. We assume that ρ has positive support on $[\rho^{min}, \rho^{max}]$ for some $\rho^{min} \geq 0$ and $\rho^{max} \leq \infty$.

Households are characterized by their income and maximum user cost, an ordered pair (y, ρ). We denote the simultaneous distribution of income and maximum user cost as $F(y, \rho)$. $F(y, \rho)$ is thus the number of households with income at most equal to y who can bid at most ρ for a house. The corresponding density is $f(y, \rho)$. In the previous subsection we discussed a case in which ρ is a function of income, $\rho = \mu y$. This is a special case of the situation considered here, in which the density $f(y, \rho)$ is only positive for $\rho = \rho(y)$. The discussion that follows considers a different and much more general case in which $f(y, \rho)$ is a continuous function of its two arguments. Such a situation is compatible with the mortgage constraint discussed in the previous section if households may have wealth that can be used, possibly in addition to a mortgage loan to finance a house.

The ρ that is relevant for a particular household should be interpreted as the user cost that the household can afford. One relevant situation is that in which the mortgage qualification constraint of the previous section holds. Denoting the user cost permitted by the mortgage loan now as $\rho^l (= \rho^m(y)) = \mu y$, for households without

[24]This household-specific maximum can be conditional on the mortgage underwriting rules and the mortgage interest rate.

any wealth the relevant constraint is still $p \leq \mu y$. If the household has wealth that can be used to help finance the house, there is a second part of the user cost, to be denoted ρ^w which is a function of the household's wealth w. The constraint is now that the actual user cost p is at most equal to the sum $\rho = \rho^m + \rho^w$.

For simplicity, one may assume that user cost is proportional to the purchase price of a house, with the constant of proportionality equal to the mortgage interest rate, which is equal for all households. If it is further assumed that the opportunity cost of wealth is equal to this interest rate there is a conveniently simple relationship between the purchase price of a house and its user cost. The discussion below refers to this simplified case.

However, we note that the analysis is also relevant for other cases. For instance, if there is a down-payment constraint instead of a mortgage qualification constraint, the household must have enough wealth to pay a share σ that equals one minus the maximum loan-to-value ratio. Using the assumptions of the previous paragraph, the maximum user cost may be determined as follows. The down-payment constraint is: $\sigma P < w$, where P denotes the purchase price. Multiplication of both sides of the inequality by the mortgage interest rate and dividing by σ gives: $rP < rw/\sigma$. The left-hand side of this inequality is the user cost. The right hand side gives the maximum ρ of the user cost in this situation, which is now independent of income. This shows that the model discussed below is as relevant for situations with a down-payment constraint as with the mortgage qualification constraint of the previous section.

The supply side of the market is unchanged. The number of households assigned to a house must therefore be equal to the number of houses that is available. These households must have an income that is at least is as high as the critical level at which housing of the lowest quality is consumed and a maximum user cost that is larger than that of the lowest quality housing. There are thus potentially two groups of households demanding housing of minimum quality: (i) those with a maximum user cost $\rho = p(q^{min})$ and income $y \geq y^c$, and (ii) those with income $y = y^c$ and maximum user cost $\rho \geq p(q^{min})$:

$$F(y^{max}, \rho^{max}) - F(y^c, \rho^{max}) - F(y^{max}, p(q^{min})) + F(y^c, p(q^{min})) = S, \tag{17}$$

in which the last term shows up to avoid double-counting.

The value of $p(q^{min})$ is determined in the same way as before, namely by the condition in (11) that a household with the critical income must be indifferent between the housing of minimum quality and the outside option. Because housing is a normal good, $p(q^{min})$ is an increasing function of the critical income y^c, and the number of households with a maximum user cost above $p(q^{min})$ must thus be a decreasing function of y^c. It follows that the left-hand side of (17) is a decreasing function of y^c and that y^c is uniquely determined.

To trace out the user cost function we consider what happens at a combination of income y, quality q and an associated user cost $p(q)$. The idea is that all households with an income lower than y or a maximum user cost lower than $p(q)$ either have been assigned a house, or will not participate in the housing market. The supply of housing of quality q is $g(q)dq$ and this must be equal to the demand. Demand originates both from households experiencing borrowing constraints and from those who do not, so that

$$g(q)dq = f^{bc}(y, p(q))dp + f^{uc}(y, p(q))dy, \tag{18}$$

in which $f^{bc}(y, p(q))$ is the density of households who have not been assigned a house but are constrained at user cost $p(q)$,

$$f^{bc}(y, p(q)) = \int_{y}^{y^{max}} f(y, p(q))dy, \tag{19}$$

while $f^{uc}(y, p(q))$ is the density of unconstrained households choosing a house with quality q,

$$f^{uc}(y, p(q)) = \int_{p(q)}^{\rho^{max}} f(y, \rho)d\rho. \tag{20}$$

Figure 2 illustrates. The box indicates the combinations of income $y \in [y^{min}, y^{max}]$ and maximum user costs $\rho \in [\rho^{min}, \rho^{max}]$ for which the distribution $F(y, \rho)$ has positive support. The housing price is given as a function of income. It starts at the critical income and is shown until some higher y corresponding to housing demand q that commands price $p(q)$. The two narrow (blue) rectangles indicate the demand for housing at this point. The vertical one refers to unconstrained households, that is households with income y who are able to bid at least $p(q)$. The horizontal box refers to constrained households, who can just afford to bid $p(q)$ but cannot afford more expensive housing because of a borrowing constraint. Total demand for housing of quality q is equal to the number of households whose combinations of income and maximum loan belong to these two boxes.

Observe that unconstrained households will only choose the combination $(q, p(q))$ of housing quality and user cost if the first-order condition (12) holds. Using the definition of the marginal price $\pi(q) = dp/dq$, participation of unconstrained households at $(q, p(q))$ thus implies

$$dp = M(q, y - p(q))dq. \tag{21}$$

Substituting (21), we can rewrite (18) as

$$[g(q) - f^{bc}(y, p(q))M(q, y - p(q))]dq = f^{uc}(y, p(q))dy. \tag{22}$$

Since the right-hand side is non-negative, the expression in square brackets on the left-hand side must also be nonnegative. If this is the case, there are enough houses offering quality q available for all constrained and unconstrained households interested. This corresponds to a 'mixed' equilibrium in which a given type of housing is inhabited by both types of households, a situation that did not occur with the uniform mortgage qualification constraint studied in the previous subsection.

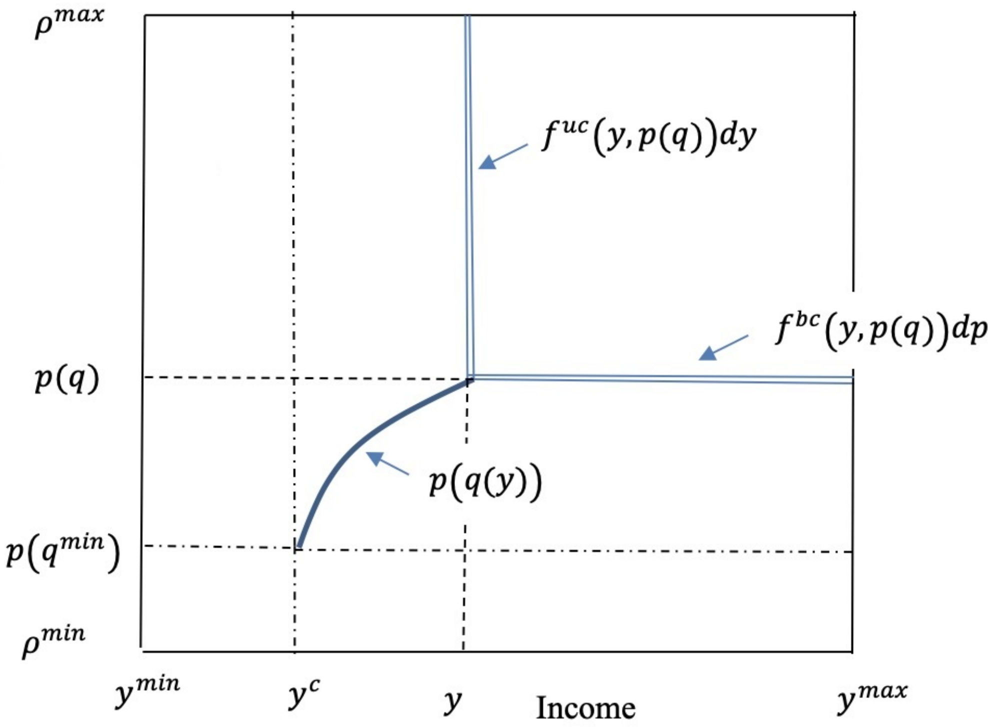

Figure 2: Constrained and unconstrained housing demand

If the expression in square brackets on the left-hand side of (22) is negative, (21) does not hold at $(q, p(q))$ and a mixed equilibrium is not feasible. In this situation, there are so many households with a binding borrowing constraint at $p(q)$ that they consume all housing with quality q and nothing is left for unconstrained households. The second term on the right-hand side of (18) thus disappears ($f^{uc} = 0$), and instead we have

$$g(q)dq = f^{bc}(y, p(q))dp. \tag{23}$$

The slope of the user cost function will now be determined by the densities of housing and of borrowing-constrained households in such a way that all constrained households are exactly on their constraint:

$$\pi(q)(= \frac{dp}{dq}) = \frac{g(q)}{f^{bc}(y, p(q))}. \tag{24}$$

Note that (23) allows the density of borrowing-constrained households to be larger than the density of housing. If many borrowing-constrained households are clustered in a particular price-quality range, they may occupy all housing for a range of qualities, as happened in the previous subsection. Unconstrained households will be put 'on hold' until all the constrained households have been served. Constrained households can be put on hold, because constrained household that are assigned to q no longer compete for higher-quality houses. Because constrained households are forced to accept lower housing consumption, unconstrained households are enabled to consume housing of higher quality relative to the unconstrained equilibrium. If there is a quality range where all houses are occupied by borrowing-constrained households, the marginal price is also lower than the marginal willingness to pay for unconstrained households interested in this housing.[25]

As will be clear by now, the equilibrium with credit constraints in the model of the present subsection differs substantially from that in the previous subsection with a uniform mortgage qualification constraint. In particular, the assignment rule in the constrained equilibrium now differs from that in the unconstrained one. Households experiencing a binding borrowing constraint will in general consume less housing than unconstrained households with the same income level. Some of them may even be pushed out of the housing market, while other households with lower incomes but less tight borrowing constraints will be able to enter. However, with the generalized borrowing constraints of the present subsection it is still true that - relative to the corresponding equilibrium without constraints, house prices will be lower.

The key observation here is that the income of unconstrained households demanding housing of any quality q will never be higher than in the unconstrained equilibrium and will be lower if some borrowing constraints are binding at prices below $p(q)$. To see this, consider first the critical income. This will never be higher than in the unconstrained equilibrium and therefore $p(q^{min})$ will never be higher than in the unconstrained equilibrium. If the critical income is lower in the constrained equilibrium, the price for housing of minimum quality will also be lower. Moreover, the marginal price at $p(q^{min})$ will also be lower. In the present version of the model, constrained households will consume less housing than unconstrained households with the same income level. This implies that unconstrained households will at least consume the same housing quality than in the unconstrained equilibrium and thus that their marginal willingness to pay for housing is lower than in the unconstrained equilibrium. Houses of all qualities will be inhabited by households that have at most the same income as in the unconstrained equilibrium. Hence the marginal price of housing will be at most equal to that in the unconstrained equilibrium. Since we have already drawn a similar conclusion for the price level of housing of minimum quality, it follows that for all levels of housing quality the price will at most be equal to that in the unconstrained equilibrium. If some households are driven out of the market by the borrowing constraints, the equilibrium housing price will be lower for all quality levels. If this is not the case, but some households are forced to accept a lower housing quality by the borrowing constraints, then the prices for all higher quality levels are lower than they would be in the unconstrained equilibrium.

On the other hand, it is easy to verify that the price of housing of quality q or lower will not be affected by the presence of binding borrowing constraints that

[25] To see this, note that in this range the expression in square brackets on the left-hand side of (22) is negative implying that $\frac{g(q)}{f^{bc}(y,p(q))} < M(q, y - p(q))$ and use (24).

refer exclusively to households assigned to houses of higher quality. However, this appears to be an exceptional situation because binding borrowing constraints push households toward lower quality housing and should therefore be expected to concentrate constrained demand at the low-quality range. We may conclude that the presence of borrowing constraints that are binding for some households will never result in higher house prices for unconstrained households and will result in strictly lower (total and marginal) house prices, and therefore increased housing consumption for unconstrained households with actual user costs higher than the maximum user cost of some constrained other households.[26] The first conclusion is in line with the analysis of the previous subsection, but the second is a substantial deviation. The benefits of the unconstrained households are related to the lower housing consumption of the constrained households, which suggests that the constraints reduce utility for constrained households.

Consider a borrowing-constrained household. The house they would have preferred in the situation without borrowing constraints is no longer available to them, notwithstanding the lower price. Instead, they had to accept a lower level of housing services at a total price that is low enough for their marginal willingness to pay for housing to exceed the actual marginal price. Their utility will certainly be lower than they could have reached at the currently prevailing user cost function (should their borrowing constraint be relaxed while those of all others remained in place), but it is unclear if their utility is also lower than in the situation without borrowing constraints for any household. If their constraint is mild, in the sense that housing consumption is modestly reduced, the net effect may be positive, as is always the case with a uniform mortgage qualification constraint, but if the constraint forces them to reduce their housing consumption substantially, it will be negative. Hence the welfare effect of the borrowing constraints on the constrained households is ambiguous in the situation studied in the present subsection.

5 Buy-to-let investors

Let us consider what happens if buy-to-let investors enter the market. We assume that buy-to-let investors do not experience borrowing constraints. They have access to capital that allows them to buy any house, but they need a return that exceeds γ. As before, household utility only depends on q and c, not on tenure type.

If buy-to-let investors buy houses at the prevailing market price $P^m(q)$ when no borrowing constraints are present, they will be able to let these house against a rent $p^m(q)$, which offers the investor a return of exactly γ. Because this return is not sufficiently attractive to trigger investments in the housing market, buy-to-let

[26] Note that these constrained households may either consume lower quality housing or may have been pushed out of the market because they are not even allowed to occupy housing of minimum quality.

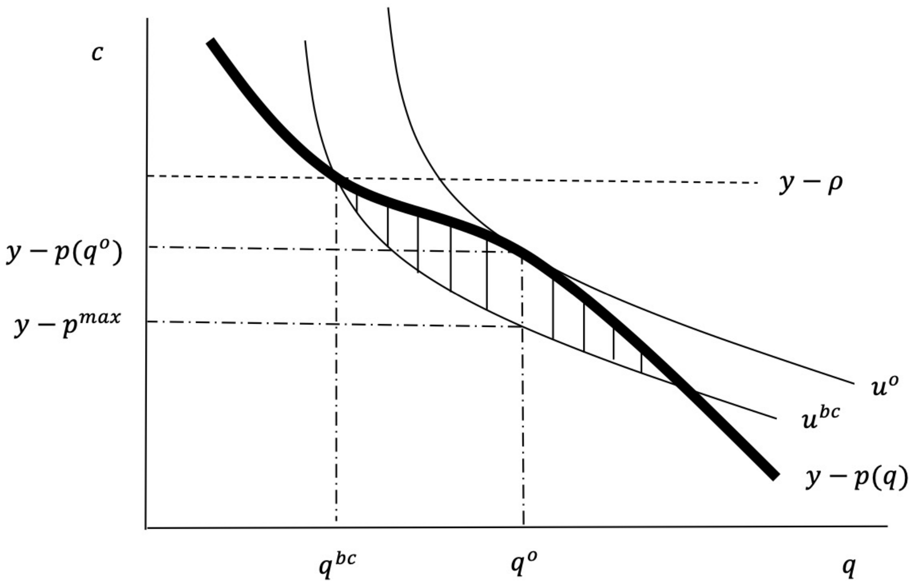

Figure 3: Borrowing constraints and profitable buy-to-let

investors will not enter a market like the one described in Section 3.3.

However, the situation is different when some households experience a binding borrowing constraint. As we have seen above, households restricted by a borrowing constraint have a marginal willingness to pay for housing that exceeds the marginal price. Then there are households willing to pay more than the user cost $p^{bc}(q)$ as rent if this offers them the possibility to consume more housing than they are able to do in owner-occupied housing with the borrowing constraint present. Hence the return of the buy-to-let investor will be higher than γ.

The model thus predicts that buy-to-let investors will enter the market when borrowing constraints are binding for some households. We argue below that this unleashes an arbitrage process that ends when user costs are equal to the user costs in the equilibrium without borrowing constraints.

Consider the situation in Figure 3, in which the prevailing user costs $p(q)$ are affected by the presence of borrowing constraints. The bold line indicates the budget constraint of a household, that is the difference between income and the user cost of housing, $y - p^{bc}(q)$. The household also faces a maximum user cost ρ, and the resulting borrowing constraint is indicated by the dashed line. As a result, the highest utility that the household can achieve is u^{bc} and it chooses to consume quality q^{bc}.

However, in the absence of a binding borrowing constraint, but assuming for now that prices remain fixed, the household would choose housing consumption q^o. A single buy-to-let investor could then purchase a house of quality q^0 at the prevailing market price, finance it with user cost $p(q^o)$ and offer it to the borrowing-constrained household at a rent between $p(q^o)$ and p^{max}. This investment would offer the investor a return above γ and it would give the household the possibility to increase its utility by consuming the quality it would consume without borrowing constraints. In fact, any offer to rent a house implying that the household reaches a combination of housing and other consumption somewhere in the shaded area means a possibility to improve utility relative to the present state of constrained owner-occupied housing consumption for the household.

However, competition between buy-to-let investors would drive up prices. While price increases only contribute to the incentives to invest in the housing market, they may affect the quality of housing that is available to a household, irrespective of whether it is owning or renting. The following proposition characterizes the equilibrium with buy-to-let investment.

Proposition 2. *The assignment, user cost function and welfare in the equilibrium with borrowing constraints and buy-to-let investment are equal to the assignment, user cost function and welfare of the equilibrium without borrowing constraints.*

Proof. The investor charging the highest rent will be able to pay most to obtain the house, so competition ensures that in equilibrium buy-to-let investors charge the highest rent that renters would still choose to pay. Because households do not care about tenure type and investors can arbitrage away the impact of borrowing constraints, the highest rent equals the highest user cost that households would be willing to pay. The highest user cost will be paid by the household with the highest marginal willingness to pay. As a result, the equilibrium assignment follows the ranking of Lemma 1 and the user cost function follows the differential equation of 2. Because the assignemt and the user cost function are identical, welfare is also identical to the equilibrium without borrowing constraints. □

The activity of buy-to-let investors thus drives up housing prices from a situation of borrowing constraints. The prices of houses for which demand was depressed by borrowing constraints increase until the possibility of profit-making buy-to-let activities has disappeared. In this situation, all households who were initially borrowing-constrained avoid the implied restriction by moving to rental housing. The allocation of housing over households is identical to that in a pure owner-occupied market without borrowing constraints as a result of the arbitrage of buy-to-rent investors.[27]

To see how this this equilibrium with buy-to-let investment plays out in Figure 3, first consider the case of uniform borrowing constraints as considered in Section 4.1. In this case, we know from Proposition 1 that the assignment is not affected by the presence of borrowing constraints, but that prices are lower for households with income levels higher than the income level at which the borrowing constraints starts to bind, compared to the situation without borrowing constraints. Buy-to-let investment thus does not affect the assignment in the equilibrium with borrowing constraints either, and only drives up prices.

[27]In a model in which borrowing constraints have macro-prudential benefits, such constraints may nevertheless be useful as they protect households (and banks) against the risks associated with mortgage default, at least to the extent that buy-to-let investors are better able to carry these risks.

In the equilibrium with buy-to-let investment, the household in Figure 3 would thus still consume q^{bc} and would not choose to rent q^0 or any other housing quality. User costs and rents would be higher, so that the budget constraint would lie below the bold line. At q^{bc}, this lower budget constraint would be tangent to an indifference curve corresponding to a lower utility than u^{bc}. Indeed, uniform borrowing constraints increase utility for all constrained households and those with higher incomes, and this utility gain is lost when buy-to-let investment drives up prices.

In the case of general borrowing constraints as considered in Section 4.2, constrained households may suffer from the presence of borrowing constraints. If this is the case, then poorer, unconstrained households consume the housing quality they would otherwise have consumed. Then buy-to-let investment would allow a household consuming q^{bc} to increase its housing consumption, for instance to q^0. Prices would still increase, so that u^0 would not be attainable, but some utility below u^0, and above u^{bc} if borrowing constraints decreased their utility, would be. Of course, the poorer, unconstrained households would then not be willing to pay the higher user costs, and would choiose to consume lower-quality housing, losing the utility gain that resulted from borrowing constraints.

6 Conclusion

This paper provides an analysis of the interaction between binding borrowing constraints and buy-to-let investment behavior in the context of urban housing markets where the housing stock can be considered as given, at least in the short run. In such a market positive income shocks make housing more expensive, even if population size remains unchanged. With high house prices, borrowing constraints may become binding for many low income and/or low wealth households. Although naive households may thus dislike such restrictions, they imply lower house prices and it is shown that in a simple benchmark case their impact on utility is positive or zero, implying a Pareto-improvement. In a more general setting, borrowing constraints still decrease house prices, and improve welfare for all unrestricted households, while they may decrease welfare of the most restricted households.

Binding borrowing restrictions open up possibilities for profitable arbitrage by buy-to-let investors. By offering the houses preferred by the restricted households as rental housing, they allow them to reach the same level of housing consumption as in the case without borrowing constraints, albeit at a higher price than is relevant in the situation when borrowing constraints are binding. In the equilibrium with free entry of buy-to-let investors the allocation of housing over households is the same as in the situation without borrowing constraints.

These conclusions have relevance for housing markets in large cities like Amsterdam where buy-to-let investors have been very active since the recovery of the housing market from the Global Financial Crisis and the ensuing euro crisis. It has recently been argued that house prices in the Netherlands are stronger associated with borrowing constraints than with housing shortages, which may be interpreted as pointing to the relevance of the assignment model. Although buy-to-let investors were initially welcomed as offering renters more possibilities on the Amsterdam housing market, where social housing is only available for low-income households and private renting was until recently mostly used by expats, a much more critical attitude has emerged in recent years when the strong growth of the private rental resulted in a decline of the share of owner-occupied housing. Our analysis suggests that the suspicion that buy-to-let investors drive up prices and let houses to households that would have been owner-occupiers, had they not been restricted by borrowing constraints, has merit. Indeed, the equilibrium with borrowing constraint is better for all except the most tightly restricted would-be owners.

On the other hand, our analysis differs from the opinion of many housing market watchers in the relatively positive verdict on borrowing constraints. In addition to protecting the homeowners against taking too much risk, they also help to mitigate the level of house prices in a market with fixed supply. It is true that the unrestricted households benefit most from this effect, because for restricted households there are not only lower house prices but – except in the benchmark case with uniform mortgage qualification constraints – often also lower housing consumption, so that the net impact on their utility is ambiguous. For those experiencing the tightest borrowing restrictions the net effect is likely negative. Given the negative sentiment about buy-to-let investors, it is paradoxical that the impact of their behavior is unambiguously positive for exactly those households, while the less severely or unrestricted households loose welfare because of buy-to-let activity.

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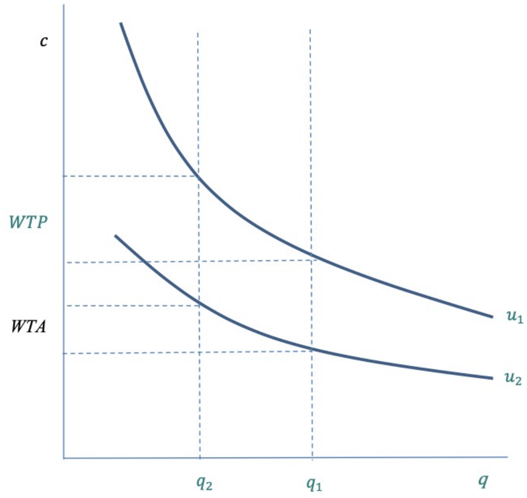

Figure A1: Illustration of switching houses

Appendix

A.1 Higher income households consume more housing

In the assignment model the marginal price of housing is not given. To show that housing consumption is still increasing in income, recall that normality of housing implies that the marginal willingness to pay for housing $M(q, c)$ is increasing in c for given q. Consider two households, 1 and 2, with incomes y_1 and y_2, $y_1 > y_2$ and housing consumption q and q_2, $q_1 < q_2$. It will be shown that both households can benefit from switching houses. Household 1 reaches a higher utility, u_1, than household 2, u_2. The willingness to pay of household 1 for the larger house can be written as:

$$WTP = \int_{q_1}^{q_2} M(q, c(q, u_1))dq \tag{A1.1}$$

where $c(q, u_1)$ denotes the value of other consumption that keeps the household on its initial indifference curve when housing consumption is q. Similarly, we can write the minimum required compensation (willingness to accept) of household 2 for the smaller house as:

$$WTA = \int_{q_1}^{q_2} M(q, c(q, u_2))dq \tag{A1.2}$$

Where the interpretation of $c(q, u_2)$ is analogous. Since $M(q, c(q, u_1)) > M(q, c(q, u_2))$ for all q, we must have $WTP > WTA$, which implies that both households can reach a higher utility level if they switch houses. Figure A1 illustrates.

A.2 Derivation of equation (13)

Note that the slope of the housing price function is always equal to the marginal willingness to pay of the households that have been assigned to the houses with the quality considered, see (11). Starting from the total derivative of the marginal rate of substitution, we can therefore derive:

$$
\begin{aligned}
d\pi &= \frac{\partial M}{\partial q}dq + \frac{\partial M}{\partial (y-p)}d(y-p) \\
&= \frac{\partial M}{\partial q}dq + \frac{\partial M}{\partial (y-p)}(dy - \pi dq) \\
&= [\frac{\partial M}{\partial q} - \pi\frac{\partial M}{\partial (y-p)}]dq + \frac{\partial M}{\partial (y-p)}dy \\
&= [\frac{\partial M}{\partial q} - \pi\frac{\partial M}{\partial (y-p)}]dq + \frac{\partial M}{\partial (y-p)}\frac{g(q)}{f(y)}dq.
\end{aligned}
\tag{A2.1}
$$

The right-hand side of the first line is the total derivative of the marginal willingness to pay. The second line uses the definition of the marginal price of housing. The third line is a rearrangement of terms. The fourth line uses the assignment rule (10). Re-writing the last line gives:

$$
\frac{d\pi}{dq} = [\frac{\partial M}{\partial q} - \pi\frac{\partial M}{\partial (y-p)}] + \frac{\partial M}{\partial (y-p)}\frac{g(q)}{f(y)}
\tag{A2.2}
$$

It is now easy to verify that (13) holds if $d\pi/dq = 0$.

A.3 The second-order condition

The first-order condition (12) requires that the slop of the budget line is equal to that of the indifference curve. The second-order condition requires that a move along the budget line starting from the point where the first-order condition is satisfied results in a lower utility. This is the case if the budget line is locally less convex than the indifference curve. We show here that this is always the case if the assignment rule is followed.

By the definition of the marginal willingness to pay, it must be true on an indifference curve that:

$$dc = -M dq.$$

Now how M changes along an indifference curve:

$$
\begin{aligned}
dM &= \frac{\partial M}{\partial q}dq + \frac{\partial M}{\partial (y-p)}dc \\
&= \frac{\partial M}{\partial q}dq - \frac{\partial M}{\partial (y-p)}M dq \\
&= [\frac{\partial M}{\partial q} - \pi\frac{\partial M}{\partial (y-p)}]dq.
\end{aligned}
\tag{A3.1}
$$

The first line is the total derivative of M, now with the notation c for $y-p(q)$. The second line imposes a move along the indifference curve and the third line uses the equality between the marginal price and the marginal willingness to pay for housing. The expression in square brackets in (A3.1) is the second derivative of the indifference curve with reversed sign.

In Appendix A2 we considered the second derivative of the equilibrium housing price function, $d\pi/dq$, also with reversed sign, see (A2.1). The second-order condition is satisfied if the budget line is less convex than the indifference curve in the optimum, that is if:

$$-\frac{d\pi}{dq} \leq -\frac{dM}{dq_{u,constant}} \tag{A3.2}$$

Comparison of the two equations makes clear that this is equivalent $\frac{\partial M}{\partial(y-p)} \frac{g(q)}{f(y)} > 0$, which is true because housing is normal and demand is only expressed for existing housing by existing households, implying that both $g(q)$ and $f(y)$ are positive. Condition (A3.2) can alternatively be formulated as:

$$\frac{d\pi}{dq} > [\frac{\partial M}{\partial q} - \pi \frac{\partial M}{\partial(y-p)}]. \tag{A3.3}$$

Consider the housing of minimal quality q^{min}. For any given price $p(q^{min})$ we set, there will (within limits imposed by the income distribution and tastes) be critical incomes for both groups at which households are indifferent between the outside option and living in the housing of minimal quality. However, the marginal willingness to pay for housing will differ. Allocate the housing to the group with the highest marginal willingness to pay for housing. We can then use the method proposed in section 3.2 to trace out the housing price function for this group. At some quality it may be the case that there are households of the other group opting for such housing. The marginal willingness to pay for housing of both groups is identical and if the relevant indifference curves are locally similar so that (A3.3) is satisfied, there will be a segment of mixed housing occupation.

B. Taste differences

The assumption that all households have the same tastes appears unrealistic and we will therefore in this section consider what changes in the model if we allow heterogeneity in tastes. More specifically, we assume that there are $n \geq 2$ groups of households. Within each group all households have the same tastes, but their incomes differ. We use a superfix $i = 1...n$ to refer to groups. Incomes belong to group-specific intervals $[y^{(i,min)}, y^{(i,max)}]$ and the income distributions $F^i(y^i)$ are differentiable and increasing on that interval. The total number of households in all groups exceeds the number of houses. Moreover, the housing distribution $G(q)$ is differentiable and increasing on that same interval.

To provide intuition for the results that follow, it is helpful to recall the familiar Rosen (1974) analysis in which households that differ in tastes or incomes choose an equilibrium while taking a hedonic price function as given. Different households in general choose different positions. Equilibrium requires that all varieties present in the market will be chosen by some households and that for each variety there is equilibrium between supply and demand. The situation considered here is a simplified one in which there is only one continuous characteristic of the heterogeneous good while the supply of each variety is given. It seems plausible, and is confirmed by the analysis that follows, that for each group of households the equilibrium will be similar to that studied in the previous section: housing consumption and expenditure are increasing in income and the marginal willingness to pay for housing is equal to its marginal price. The main issue that arises is that the curvature of the one-dimensional hedonic price function may be such that the implied budget constraint is more convex that the indifference curve for one or more groups of households. If this happens, there are 'holes' in the housing consumption of such groups in the sense that one or more housing quality intervals are skipped by the members of such a group. That is, housing consumption of the households in this group is not everywhere continuous in income, but may have 'jumps' at a limited number of incomes.

To provide a formal analysis, start by noting that in market equilibrium the housing price function $p(q)$ must still be increasing in q and that in all groups households with higher incomes consume more housing. The reasons are identical to the case with homogeneous households. This implies that for each group i households with incomes below a critical income level $y^{(i,c)}$ will choose the outside option, while housing consumption is strictly increasing in income for those realizing at least this critical income level. The critical income level may be identical to the minimum income in the group.

An equilibrium requires the existence of a housing price function $p(q)$ defined on the interval $[q^{min}, q^{max}]$ that is taken as given by all households. Moreover, all households maximize their utility subject to the budget constraint (2) that incorporates this price function. To have all houses filled, every q in the relevant interval must be the optimal choice for households in at least one group. That is, there must be at least one combination (i, y) with $y \in [y^{(i,min)}, y^{(i,max)}]$ for which q is the optimal choice. To have all households allocated, there must be intervals of housing qualities $[q^{(i,j,min)}, q^{(i,j,max)}]$ with $q^{(i,j,min)} \geq q^{min}$ and $q^{max} \leq q^{max}, j = 1...J(i)$, $J(i) \geq 1$, and $q^{(i,j,max)} < q^{(i,j+1,min)}, j = 1...J(i) - 1$ if $J(i) \geq 2$, such that households of type i choose houses of every quality q inside these intervals. Every $q \in [q^{min}, q^{max}]$ must belong to at least one such interval, possibly at its border.

The price function must be differentiable on every interval $[q^{(i,j,min)}, q^{(i,j,max)}]$ by the argument used in the analysis of the market with homogeneous households. On such intervals the marginal price must always be equal to the marginal willingness to pay of the households of the group to which this interval refers. The assumed properties of the utility function then ensure that the price function will be differentiable on such intervals. The equilibrium price function will hence be differentiable almost everywhere, that is except on a set of measure 0.

It is important to observe that not only the necessary first-order condition has to be satisfied, but also the sufficient second-order condition. The second-order condition requires that the budget line is less convex than the indifference curve at the point where both are tangent. Formally, the condition, which is discussed in Appendix A.3, requires:[28]

$$\frac{d\pi}{dq} \geq [\frac{\partial M^i}{\partial q} - \pi \frac{\partial M^i}{\partial (y-p)}] \quad (25)$$

If the first-order condition is satisfied, but the second-order condition fails, households express no demand for housing of quality q but they may demand housing of lower or higher quality. This is the reason why households of a particular group may demand no housing in specific quality intervals.

The assignment of households to housing is described by a generalization of (10):

$$g(q)dq = \sum_i \delta^i(q) f^i(y) dy^i \quad (26)$$

In this equation $\delta^i(q)$ is a 0-1 variable indicating that households of type i chose houses with quality q. Hence $\delta^i(q) = 1$ if the first-order condition is satisfied for an income $y \varepsilon [y^{i,min}, y^{i,max}]$ for which the second-order condtion is also satisfied.

Since there may be households of more than one group demanding housing of quality q, we cannot use this equation to determine the relationship between changes in housing consumption and income of specific groups. However, we can find an alternative way of doing so by considering the change in the first-order condition (12) that occurs if we move to a slightly higher housing quality. If households of group i continue to express demand at this higher quality, $d\pi/dq = dM^i/dq$ if we move along the housing price function. Elaboration of this condition gives:

$$\frac{d\pi}{dq} = [\frac{\partial M^i}{\partial q} - \pi \frac{\partial M^i}{\partial (y-p)}] + \frac{\partial M^i}{\partial (y-p)} \frac{dy^i}{dq} \quad (27)$$

This equation implies that $dy^i/dq > 0$ if the second-order condition is satisfied. (27) gives a relationship between the change in the marginal price of housing and the change in the incomes of households of group i assigned to the housing concerned. We can solve this equation for dy^i/dq and substitute the result into (26). This gives us an expression for the change in the marginal price of housing:

$$\frac{d\pi}{dq} = \frac{\sum_i \delta^i(q) f^i(y^i) (\frac{\partial M^i}{\partial (y^i - p)})^{-1} [\frac{\partial M^i}{\partial q} - \pi \frac{\partial M^i}{\partial (y^i - p)}] + g(q)}{\sum_i \delta^i(q) f^i(y^i) (\frac{\partial M^i}{\partial (y^i - p)})^{-1}} \quad (28)$$

Equations (27) and (28) give us the possibility to trace out the housing price function and the allocation of households from a given starting point. To find a suitable starting point, consider an arbitrary price $p(q^{min})$ for housing of minimum quality and check if there are incomes of households in the various groups for which indifference between the outside option and the housing of minimum quality obtains at the chosen price, that is if (12) is satisfied. This is not necessarily the case. The price may be too high for a specific group in the sense that even for households with

[28]If the q is at the border of the interval for which group i expresses housing demand, $d\pi/dq$ should be interpreted as the limit of $d\pi/dq$ when housing consumption approaches the border from inside the interval.

the maximum income the outside option is preferred. The price may also be too low in the sense that even the household with the minimum income prefers the housing of minimal quality to the outside alternative. If there is such a critical income at which the indifference occurs, then all households in de group with a higher income will also disregard the outside option in equilibrium. This means that we can determine a minimum number of households that will participate in the market for any chosen price $p(q^{min})$: the sum of all households with at least the critical income over all groups. This minimum demand is a continuous function of the price $p(q^{min})$. If we assume that all households prefer housing of minimum quality to the outside option if the price is sufficiently low, minimum demand can take on every value between 0 and the total number of households.

Now suppose that we choose the price $p^*(q^{min})$ so, that the minimum number of participants in the market equals the number of available houses. The marginal willingness to pay for housing at q^{min} will in general differ between the groups. Allocate the houses of minimum quality to the group with the lowest marginal willingness to pay and start tracing out the housing price function as in the previous section.[29] At each value of q, check if there are households belonging to a different group for whom the first- and second-order conditions for a utility maximum are satisfied. If so switch to the allocation process associated with (25)-(27) outlined above.

This algorithm will lead to an equilibrium if the household groups that start participating in the market at higher quality do so at the income levels at which they are indifferent between the outside option and housing of minimum quality at the price $p(q^{min})$. However, this will not necessarily be the case. We should expect the first households of such groups to start participating at a lower income. The reason is that the marginal willingness to pay for housing of households of this second group who are indifferent between housing of minimum quality and the outside option is higher than the marginal price of housing. This means that it is possible that the actual demand for housing at the chosen price exceeds the number of available houses. If so, we must increase the price $p^*(q^{min})$. We repeat this process until convergence occurs.

It may be noted that two different aspects of heterogeneity are important in this analysis. One is that households realize the same marginal utility of housing at different income levels, the other is that the convexity of the indifference curve (the first derivative of the marginal willingness to pay for housing) may also be different. It is possible that the first aspect is relevant, while the second is not. This situation occurs if the indifference curves of households belonging to different groups are parallel to each other, that is if $u^i(c, q) = u(\theta^i + c, q)$. Two households belonging to groups g and g' will then choose exactly the same location on the hedonic price function if the difference between their incomes is equal to the difference between their θ s.

[29] Allocation of these households to any other group will result in bunching of the group with the lowest marginal willingness to pay at the minimum quality, which is incompatible with equilibrium.

When a particular value of housing consumption q is chosen by households of group i with income y, that same value of housing consumption will also be chosen by households of group j with income $y - \theta^i + \theta^j$. Note that the share of households of the two groups may still vary over housing qualities, depending on their income distributions. For instance, with a linear demand function for housing,

$$q = \alpha^i + \beta\pi + \gamma y^v, \tag{29}$$

differences in the constant term α^g have a similar impact on the demand for housing as differences in income. Note that the share of households of the two groups living in housing of a particular quality may still vary over housing qualities, depending on their income distributions.