Testing Linear Cointegration Against Smooth Transition Cointegration

Oliver Stypka[1] Martin Wagner[2,3,4]

[1]Flossbach von Storch, Cologne
[2]Department of Economics, University of Klagenfurt
[3]Bank of Slovenia, Ljubljana
[4]Institute for Advanced Studies, Vienna

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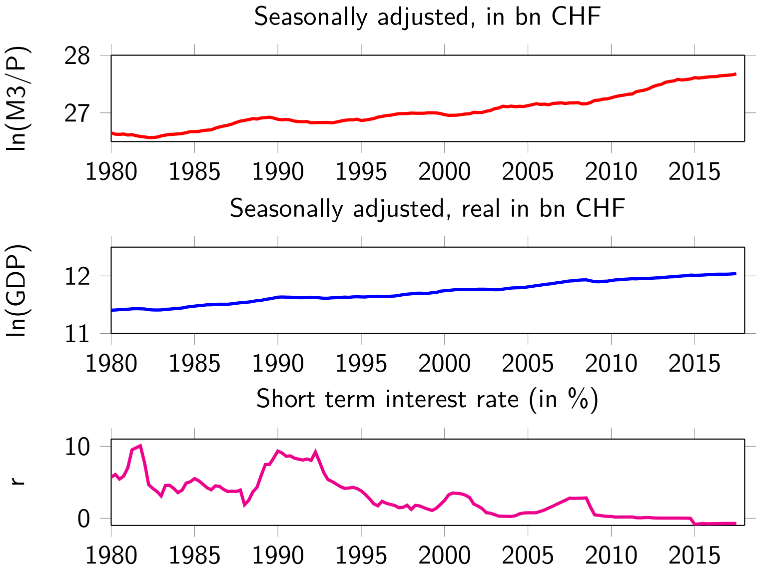
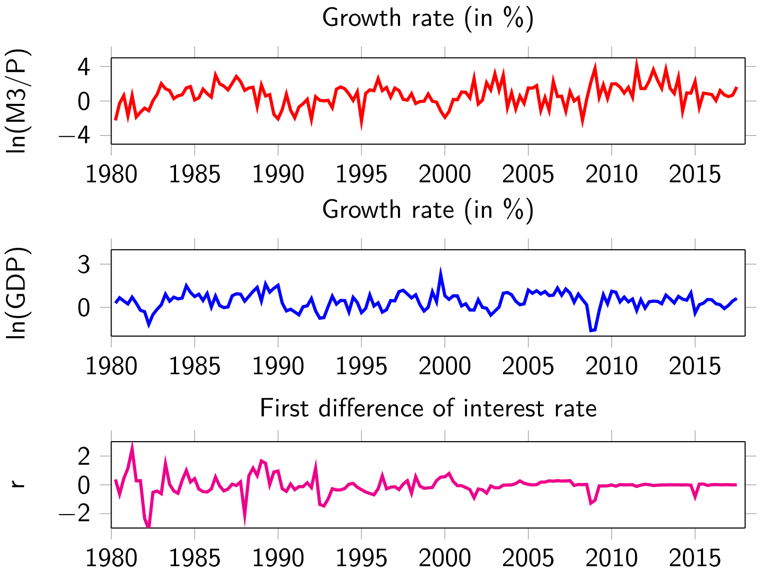
A Warning...

Overview

Motivation



Seasonally adjusted, in bn CHF
ln(M3/P)
28
27
1980
1985
1990
1995
2000
2005
2010
2015
Seasonally adjusted, real in bn CHF
ln(GDP)
12
11
1980
1985
1990
1995
2000
2005
2010
2015
Short term interest rate (in %)
r
10
0
1980
1985
1990
1995
2000
2005
2010
2015



Growth rate (in %)
ln(M3/P)
4
0
−4
1980
1985
1990
1995
2000
2005
2010
2015
Growth rate (in %)
ln(GDP)
3
0
1980
1985
1990
1995
2000
2005
2010
2015
First difference of interest rate
r
2
0
−2
1980
1985
1990
1995
2000
2005
2010
2015

$$\ln\left(\frac{M_t}{P_t}\right) = c + \delta t + \beta_1 \ln(Y_t) + \beta_2 r_t + u_t$$

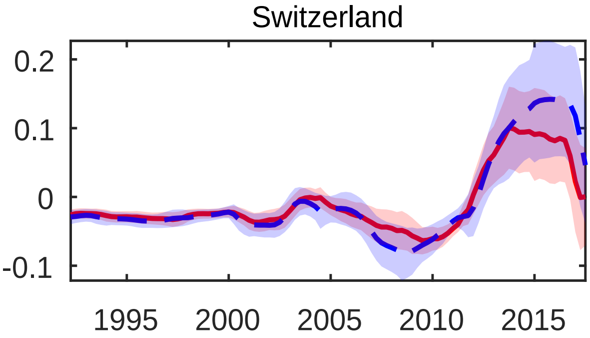


Figure: The red solid line displays the FM-OLS estimates and the blue dashed line displays the IM-OLS estimates for β_2. The corresponding 95% confidence bands are given by the red and blue shaded areas.

Motivation

Monitoring Euro Area Money Demand (1980/I–2017/III)

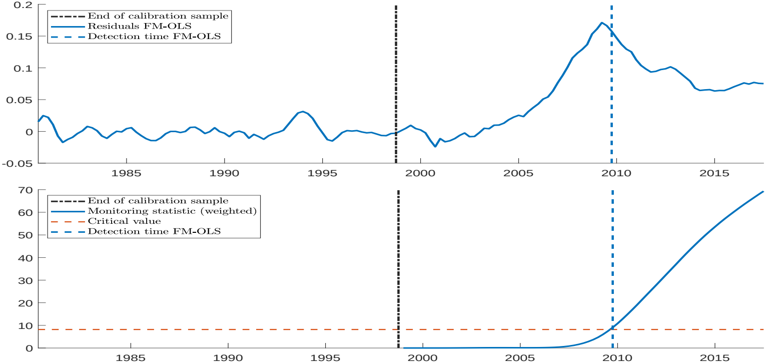


Figure: Residuals based on estimation until 1998IV. Detection time 2009IV.

Model and Theory

Linear Cointegrating Regression

$$y_t = D_t'\theta_D + \tilde{X}_t'\theta_X + u_t = Z_t'\theta + u_t,$$

where $Z_t := [D_t', \tilde{X}_t']'$, $\theta := [\theta_D', \theta_X']'$, with:

- deterministic regressors D_t,
- a non-cointegrated I(1) vector $\tilde{X}_t$,
- and a stationary error term u_t.

Linear cointegration may – by construction – be too restrictive:

- The parameters are assumed to be constant.
- The setting is linear in parameters and I(1) variables.

Smooth Transition Cointegration

For Simplified Presentation: Same Variables in Both Parts

Smooth Transition Cointegration

$$y_t = Z_t'\theta_L + Z_t'\theta_{NL} \times G(s_t, \theta_G) + u_t$$

with Z_t, u_t as above and:

- a smooth and bounded transition function $G(\cdot)$,
- and integrated variable or time trend as transition variable s_t.

Example: Logistic Transition Functions

$$\text{LSTR1: } G_1(s_t, \theta_G) = \frac{1}{1 + \exp(-\gamma(s_t - c))} - \frac{1}{2}, \text{ with } \gamma > 0$$

$$\text{LSTR2: } G_2(s_t, \theta_G) = \frac{1}{1 + \exp(-\gamma(s_t - c)^2)} - \frac{1}{2}, \text{ with } \gamma > 0$$

- For the transition variable s_t we consider two cases:

  (i) s_t is an element of $\tilde{X}_t$ or is an I(1) process not cointegrated with $\tilde{X}_t$,

  (ii) $s_t = t$.

- To have a unified notation we define:

$$X_t := \begin{cases} \tilde{X}_t & \text{in case } s_t \text{ is an element of } \tilde{X}_t \text{ or } s_t = t, \\ \begin{bmatrix} \tilde{X}_t \\ s_t \end{bmatrix} & \text{in case } s_t \text{ is I(1) and not cointegrated with } \tilde{X}_t. \end{cases}$$

  [Where in the second case s_t is ordered last w.l.o.g.]

- We furthermore define $v_t := \Delta X_t$ and denote its long-run variance as usual by Ω_{vv}.

Detailed Assumptions

- Testing linear cointegration against the alternative of smooth transition cointegration corresponds to testing:

$$H_0 : \theta_{NL} = 0 \quad \text{vs.} \quad H_1 : \theta_{NL} \neq 0.$$

- Under the null hypothesis of linear cointegration with $\gamma = 0$ some parameters are unidentified, e. g., for LSTR1:

$$y_t = Z_t' \theta_L + Z_t' \theta_{NL} \times \underbrace{\left(\frac{1}{1 + \exp\left(-0(s_t - c)\right)} - \frac{1}{2} \right)}_{=0} + u_t.$$

- This identification problem is tackled by using Taylor approximations of the transition function.

- A Taylor approximation of order n leads to a model of the form

$$y_t = Z_t'\beta_0 + \sum_{j=1}^{n} (Z_t s_t^j)'\beta_j + u_t^*.$$

- The null hypothesis of linearity of the cointegrating relationship is tested in this auxiliary regression by testing:

$$H_0 : [\beta_1', \ldots, \beta_n']' = 0 \quad \text{vs.} \quad H_1 : [\beta_1', \ldots, \beta_n']' \neq 0.$$

Problems

- The asymptotic analysis of LS estimators is complicated by the occurrence of terms of the form $X_t s_t^j$.
- Deriving consistency against fixed alternatives is non-trivial, both with standard and "Saikkonen-triangular array" asymptotics.

- As discussed in detail, e. g., in Luukkonen *et al.* (1988), there are situations in which a first order Taylor approximation leads to tests with trivial power.
- Consider the following smooth transition model with the "nonlinear" part only containing the intercept and with $s_t = x_t$:

$$y_t = \theta_1 + \theta_2 x_t + \theta_3 \times G(x_t, \theta_G) + u_t.$$

- A first order Taylor approximation leads to

$$y_t = \beta_1 + \beta_2 x_t + u_t^*,$$

and therefore tests based on this approximation have trivial power.

- In such cases higher order Taylor approximations, typically third order, are used.

- Another issue that requires some care is multi-collinearity of regressors in the Taylor approximation.
- First, consider $s_t = t$ and $D_t = (1, t, \ldots, t^{p-1})'$ with $p > 1$, then $D_t \otimes s_t = D_t \otimes t = (t, t^2, \ldots, t^p)'$.
- Clearly, for $p > 1$ at least the linear trend appears in D_t and $D_t \otimes s_t$.
- Second, if a constant is included (in the "linear" term) and s_t is already an element of the regressors $X_{t,L}$ the regressor s_t appears twice.
- This "multi-collinearity by construction" is easily overcome by excluding the corresponding regressor(s) in the Taylor approximation term(s).

OLS Asymptotics

Asymptotic Behavior of the "$X'u$-Term"

In case that s_t is an I(1) process:

$$T^{-\frac{j+2}{2}} \sum_{t=1}^{T} x_{t_i} s_t^j u_t \Rightarrow \int_0^1 B_{v_i}(r) B_s^j(r) dB_u(r) + j\Delta_{su} \int_0^1 B_{v_i}(r) B_s^{j-1}(r) dr + \Delta_{vu} \int_0^1 B_s^j(r) dr$$

In case that $s_t = t$:

$$T^{-(j+1)} \sum_{t=1}^{T} x_{t_i} t^j u_t \Rightarrow \int_0^1 B_{v_i}(r) r^j dB_u(r) + \Delta_{vu} \int_0^1 r^j dr$$

- The idea of FM-OLS is to correct for bias terms arising in the OLS limit and to correct for the correlation between X_t, s_t and u_t.
- The auxiliary model can be written in more compact form:

$$y_t = F_t'\beta + u_t^*,$$

with $F_t = [1, s_t, \dots, s_t^n]' \otimes Z_t$ and $\beta = [\beta_0', \dots, \beta_n']'$.

Fully Modified OLS

The FM-OLS estimator of β in the above model is given by

$$\hat{\beta}^+ = \left(\sum_{t=1}^{T} F_t F_t'\right)^{-1} \left(\sum_{t=1}^{T} F_t y_t^+ - M^*\right),$$

with $y_t^+ := y_t - v_t'\hat{\Omega}_{vv}^{-1}\hat{\Omega}_{vu}$ and model specific correction term M^*.

Integrated Modified OLS Estimation

Integrated Modified OLS

IM-OLS estimation is OLS estimation of the partial summed auxiliary model augmented by X_t, i. e.,

$$\begin{aligned} S_t^y &= S_t^{F\prime}\beta + X_t'\gamma + S_t^{u*}, \\ &= S_t^{\tilde{F}\prime}\beta_* + S_t^{u*}, \end{aligned}$$

where $S_t^y := \sum_{i=1}^t y_i$ and similarly for S_t^F and S_t^{u*}.

- Adding X_t "soaks up" all dynamic correlation between the regressors and the errors.
- Partial summation lets us get rid of "integrated$\times$stationary"-terms.
- For IM-OLS estimation no choices with respect to tuning parameters have to be made.
- By using properly modified residuals fixed-b inference is possible.

- As in Vogelsang and Wagner (2014) construct some additional regressors:

$$a_t := t \sum_{j=1}^{T} S_j^{\tilde{F}} - \sum_{j=1}^{t-1} \sum_{s=1}^{j} S_s^{\tilde{F}}, \qquad S_t^{\tilde{F}} := [S_t^{F\prime}, X_t^{\prime}]^{\prime}.$$

- The fixed-b long-run variance estimator is based on the residuals from the IM-OLS regression augmented by a_t:

$$S_t^y = S_t^{\tilde{F}\prime} \beta_* + a_t^{\prime} \kappa + S_t^{u*}.$$

- Denoting the residuals with $\tilde{S}_t^{u*}$ we use:

$$\hat{\omega}_{u\cdot v}^* := T^{-1} \sum_{i=2}^{T} \sum_{j=2}^{T} k\left(\frac{|i-j|}{M}\right) \Delta \tilde{S}_i^{u*} \Delta \tilde{S}_j^{u*}.$$

- The Wald-type test is based on FM-OLS estimation of:

$$y_t = Z_t'\beta_0 + Q_t'\beta_Q + u_t$$

- The null hypothesis is $H_0 : \beta_Q = 0$; and the corresponding test statistic is given by:

$$W_{FM} := \frac{\hat{\beta}_Q^{+\prime}(\tilde{Q}'\tilde{Q})\hat{\beta}_Q^{+}}{\hat{\omega}_{u \cdot v}},$$

  with $\tilde{Q} := Q - Z(Z'Z)^{-1}Z'Q$.

- The conditional long-run variance estimator used is given by:

$$\hat{\omega}_{u \cdot v} := \hat{\Omega}_{uu} - \hat{\Omega}_{uv}\hat{\Omega}_{vv}^{-1}\hat{\Omega}_{vu},$$

  using the OLS residuals of the above Taylor approximation and $v_t = \Delta X_t$.

- The starting point is FM-OLS estimation of the null model:

$$y_t = Z_t'\beta_0 + u_t$$

- The resulting FM-OLS residuals $\hat{u}_t^+ := y_t^+ - Z_t'\hat{\beta}_0^+$ are then used as dependent variable in:

$$\hat{u}_t^+ = \tilde{Q}_t'\beta_{\tilde{Q}} + \psi_t,$$

with $\tilde{Q} := Q - Z(Z'Z)^{-1}Z'Q$.

- The parameter $\beta_{\tilde{Q}}$ needs to be estimated with a suitable correction to FM-OLS; similar to Wagner and Hong (2016, Proposition 4).
- This results in the test statistic:

$$LM_{FM} := \frac{\hat{\beta}_{\tilde{Q}}^{+\prime}(\tilde{Q}'\tilde{Q})\hat{\beta}_{\tilde{Q}}^{+}}{\tilde{\omega}_{u\cdot v}},$$

with $\tilde{\omega}_{u\cdot v}$ based on the residuals from the linear (null) model.

- For IM-OLS testing is based on the equation:

$$S_t^y = S_t^{Z\prime}\beta_0 + S_t^{Q\prime}\beta_Q + X_t'\gamma + S_t^u,$$

with the null being again $H_0 : \beta_Q = 0$, which is now part of a "bigger" parameter vector $\boldsymbol{\beta}_*$, i.e.,

$$\beta_Q = R_Q\boldsymbol{\beta}_* = \begin{bmatrix} 0 & I_{\dim(\beta_Q)} & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_Q \\ \gamma \end{bmatrix} = 0.$$

- Since we use OLS in a linear regression model, Wald- and LM-type tests only differ by the variance estimator chosen, and we end up with:

$$\{W, LM, Fb\}_{IM} := \frac{\hat{\beta}_{Q,*}' \left(R_Q \hat{V}_{IM} R_Q'\right)^{-1} \hat{\beta}_{Q,*}}{\omega_{u\cdot v}},$$

with $\omega_{u\cdot v} \in \{\hat{\omega}_{u\cdot v}, \tilde{\omega}_{u\cdot v}, \hat{\omega}_{u\cdot v}^*\}$ and $\hat{V}_{IM}$ an estimator of the "$X'X$"-part of the estimator variance.

Proposition

Under the null hypothesis of linear cointegration it holds that:

$$W_{\mathrm{FM}}, LM_{\mathrm{FM}}, W_{\mathrm{IM}}, LM_{\mathrm{IM}} \xrightarrow{d} \chi^2_q,$$

with $q = \dim(\beta_Q)$ depending on the model and the Taylor approximation order.

Proposition

If $M = bT$ with $b \in (0, 1]$ being held fixed as $T \to \infty$, then it holds for the fixed-b test statistic under the null hypothesis that:

$$Fb_{\mathrm{IM}} \xrightarrow{d} \frac{\chi_q^2}{N(\tilde{P}^*)},$$

with χ_q^2 independent of $N(\tilde{P}^*)$, where $N(\cdot)$ is a function of (a function of) standard Wiener processes $\tilde{P}$ that depends upon bandwidth and kernel function.

PROPOSITION

Under the alternative hypothesis of smooth transition cointegration:

$$y_t = Z_t'\theta_L + Z_t'\theta_{NL} \times G(s_{t,T}, \theta_G) + u_t,$$

with $\theta_{NL} \neq 0$, $\gamma \neq 0$ and $s_{t,T} := \frac{T_0}{T} t$ for time as transition variable and $s_{t,T} := \sqrt{\frac{T_0}{T}} s_t$ otherwise it holds that:

$$LM_{\mathrm{FM}}, LM_{\mathrm{IM}} = O_{\mathbb{P}}(T/M_T),$$

with M_T denoting the bandwidth used for long-run covariance estimation.

Finite Sample Performance

Finite Sample Performance

Simulation Design: Size

Under the null hypothesis we generate data according to:

$$y_t = \theta_0 + \theta_1 x_{1t} + \theta_2 x_{2t} + u_t,$$

with the errors u_t and $v_t = \Delta x_t$ generated as:

$$\begin{aligned} u_t &= \rho_1 u_{t-1} + \varepsilon_t + \rho_2(e_{1t} + e_{2t}), \quad u_0 = 0, \\ v_{it} &= e_{it} + 0.5 e_{i,t-1}, \quad i = 1, 2, \end{aligned}$$

with $(\varepsilon_t, e_{1t}, e_{2t})' \sim \mathcal{N}(0, I_3)$.

- ρ_1 controls the level of serial correlation in the error term u_t, and ρ_2 controls regressor endogeneity.
- The parameter values are set to $\theta_0 = \theta_1 = \theta_2 = 1$.
- $T \in \{100, 200, 500\}$ and $\rho_1 = \rho_2 \in \{0, 0.3, 0.6, 0.8\}$.
- The number of replications is 5,000 in all cases and all tests are carried out at the nominal 5% level.
- We use the Bartlett kernel and the Andrews (1991) bandwidth.

Empirical Null Rejection Probabilities

Transition variable $s_t = x_{2t}$

		D-OLS		FM-OLS		IM-OLS		
T	ρ_1, ρ_2	$W_{D,AIC}$	$W_{D,BIC}$	W_{FM}	LM_{FM}	W_{IM}	LM_{IM}	Fb_{IM}
Panel A: First order Taylor approximation ($n = 1$)								
100	.0	.0872	.0770	.1460	.0582	.1168	.0542	.0548
	.3	.1474	.1392	.1540	.0672	.1526	.0800	.1304
	.6	.2348	.1822	.1826	.0586	.1976	.1006	.2502
	.8	.4256	.2630	.2536	.0610	.3028	.1544	.5190
200	.0	.0660	.0654	.1100	.0530	.0974	.0598	.0532
	.3	.1116	.1112	.1260	.0636	.1164	.0780	.0936
	.6	.1690	.1506	.1538	.0574	.1454	.0892	.1394
	.8	.2742	.1948	.1940	.0498	.1960	.1020	.2918
Panel B: Third order Taylor approximation ($n = 3$)								
100	.0	.1756	.1530	.3136	.0884	.2426	.0372	.0704
	.3	.2588	.2380	.3006	.0578	.3128	.0690	.2350
	.6	.4146	.3056	.3080	.0312	.3868	.0736	.6072
	.8	.6482	.3916	.3874	.0678	.5590	.1110	.9030
200	.0	.0964	.0884	.2058	.0566	.1658	.0386	.0538
	.3	.1784	.1742	.2096	.0510	.2178	.0686	.1346
	.6	.2764	.2452	.2358	.0274	.2770	.0742	.3202
	.8	.4490	.3126	.3022	.0286	.3976	.0720	.6646

Empirical Null Rejection Probabilities

Transition variable $s_t = t$

		D-OLS		FM-OLS		IM-OLS		
T	ρ_1, ρ_2	$W_{D,AIC}$	$W_{D,BIC}$	W_{FM}	LM_{FM}	W_{IM}	LM_{IM}	Fb_{IM}
Panel A: First order Taylor approximation ($n = 1$)								
100	.0	.1168	.1014	.1260	.0542	.0926	.0546	.0592
	.3	.1982	.1876	.2448	.1254	.1784	.1110	.1676
	.6	.4100	.3312	.4622	.2344	.2824	.1346	.3676
	.8	.7182	.5972	.6778	.3822	.5170	.2628	.7334
200	.0	.0598	.0568	.0592	.0476	.0544	.0512	.0498
	.3	.1002	.0984	.1054	.0584	.0940	.0730	.0672
	.6	.1788	.1750	.1568	.0460	.1024	.0740	.1060
	.8	.3448	.3530	.2686	.0288	.1242	.0666	.1946
Panel B: Third order Taylor approximation ($n = 3$)								
100	.0	.3040	.2620	.3352	.0808	.2404	.0498	.0862
	.3	.4734	.4444	.5250	.2542	.4406	.1048	.3308
	.6	.8300	.7438	.8578	.4638	.6844	.0932	.8158
	.8	.9794	.9434	.9704	.6338	.9312	.2102	.9884
200	.0	.1352	.1272	.1650	.0616	.1146	.0486	.0588
	.3	.2914	.2854	.4182	.2038	.2960	.0968	.1664
	.6	.6146	.5542	.7010	.3656	.4432	.0732	.4900
	.8	.9202	.8804	.9130	.5344	.7738	.1712	.8800

For the alternative we use the following DGP:

$$y_t = Z_t'\theta_L + Z_t'\theta_{NL} \times G(s_t, \gamma, c) + u_t,$$

with $Z_t = [1, x_t']'$ and errors u_t, $v_t = \Delta x_t$ as generated for the null.

- The parameter values are set again to $\theta_L = [1, 1, 1]'$.
- As transition function we consider $G(\cdot) \in \{G_1(\cdot), G_2(\cdot)\}$ and transition variable $s_t \in \{x_{2t}, t\}$.
- We consider location parameter $c = 0$ for $s_t = x_{2t}$ and $c = T/2$ for $s_t = t$ and use the scaling parameters $\gamma \in \{0.01, 0.1, 1, 10\}$.
- We consider a grid of (including the null) 21 points for $\theta_{NL} := \kappa\theta_L$, with values for κ chosen from the interval $[0, 2]$ on an equidistant grid with mesh 0.1.

Finite Sample Performance

Size-Corrected Power: LSTR1 with $s_t = x_{2t}$

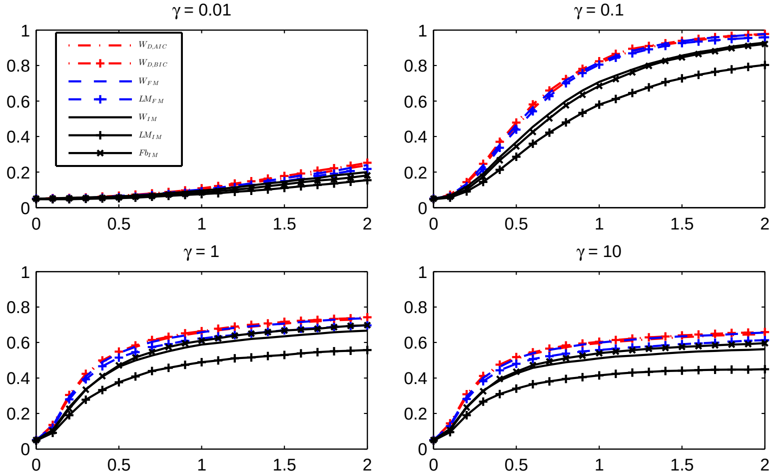


Figure: Size-corrected power for $T = 100$, Taylor approximation of order $q = 1$ and $\rho_1 = \rho_2 = 0.3$.

Finite Sample Performance

Size-Corrected Power: LSTR2 with $s_t = x_{2t}$

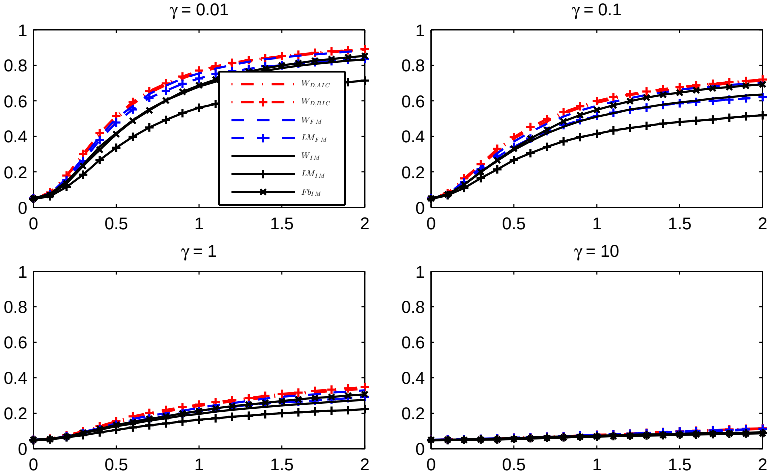


Figure: Size-corrected power for $T = 100$, Taylor approximation of order $q = 1$ and $\rho_1 = \rho_2 = 0.3$.

Finite Sample Performance

Size-Corrected Power: LSTR2 with $s_t = x_{2t}$

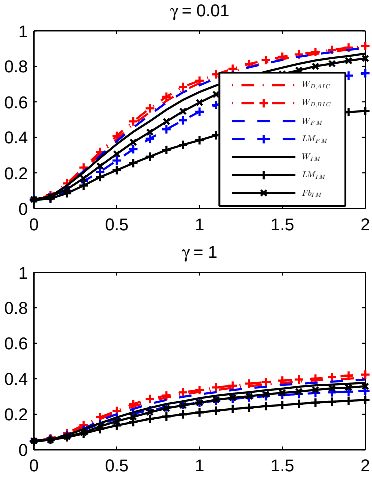


Figure: Size-corrected power for $T = 100$, Taylor approximation of order $q = 3$ and $\rho_1 = \rho_2 = 0.3$.

Finite Sample Performance

Size-Corrected Power: LSTR1 with $s_t = t$

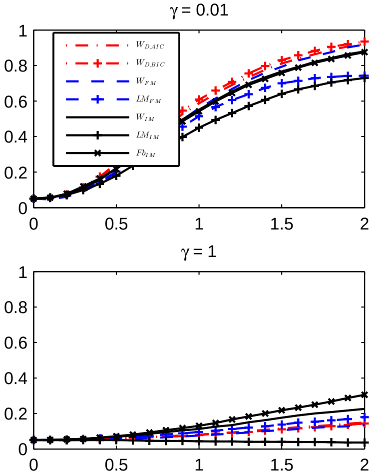


Figure: Size-corrected power for $T = 100$, Taylor approximation of order $q = 1$ and $\rho_1 = \rho_2 = 0.3$.

Finite Sample Performance

Size-Corrected Power: LSTR2 with $s_t = t$

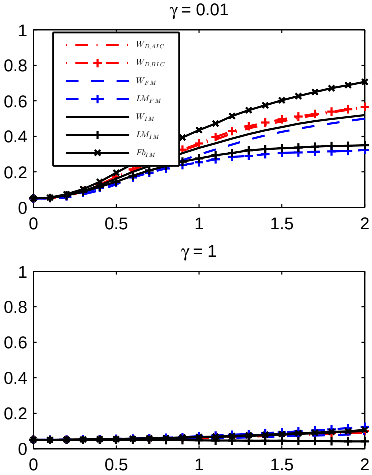


Figure: Size-corrected power for $T = 100$, Taylor approximation of order $q = 1$ and $\rho_1 = \rho_2 = 0.3$.

Illustration with Long-Run Money Demand

We consider the simple long-run money demand equation:

$$\ln\left(\frac{M_t}{P_t}\right) = c + \delta t + \beta_1 \ln(Y_t) + \beta_2 r_t + u_t,$$

with (non-cointegrated) I(1) processes $\ln(Y_t)$ and r_t.

- M_t is given by M_3.
- P_t is the consumer price index.
- Y_t is real gross domestic product.
- r_t is a 3-month interest rate.

Variable	Description	Source
Y_t	Gross Domestic Product, Expenditure Approach, Chained Volume Estimates, National Currency, Quarterly Levels, Seasonally Adjusted, National Reference Year, Reference Period: $2015-16$.	OECD
r_t	Nominal Short-Term Interest Rate, Per Cent per Annum, Quarterly.	OECD
M_t	Broad Money (M3), Seasonally Adjusted, National Currency, Quarterly.	FRED
P_t	Consumer Price Index (CPI), All Items, Reference Period: $2015-16=100$, Quarterly.	OECD

- All variables quarterly, seasonally adjusted with country specific starting points and last observation 2017/III.
- Australia, Canada, Czech Republic, Denmark, Israel, New Zealand, Norway, South Korea, Sweden, Switzerland, UK, USA and Euro Area

Long-Run Money Demand

Augmented Dickey-Fuller and Phillips-Perron Tests

	ln(M3/P)			ln(GDP)			Interest Rate		
	ADF	PP	PP(fb)	ADF	PP	PP(fb)	ADF	PP	PP(fb)
AUS	-1.76	-1.64	-1.61	-2.90	**-3.74**	**-3.55**	-2.92	-2.83	-2.84
CAN	-2.54	-2.08	-1.81	-2.17	-2.35	-2.38	-3.36	-2.82	-2.80
CHE	-1.59	-1.31	-1.05	-2.68	-2.58	-2.63	-2.95	-2.98	-2.97
CZE	-2.32	-2.19	-2.25	-2.28	-1.86	-1.83	-1.96	-1.75	-1.70
DEN	-2.25	-2.25	-2.26	-2.47	-1.99	-1.97	-2.62	-2.64	-2.66
ISR	-2.62	-2.59	-2.57	-2.67	-2.76	-2.70	-2.23	-2.47	-2.46
KOR	**-3.61**	**-4.14**	**-4.49**	-1.73	-1.56	-1.77	-2.70	-2.97	-2.95
NZL	-3.10	-2.93	-2.90	-2.11	-2.26	-2.26	-2.81	**-4.56**	**-4.62**
NOR	-1.93	-1.66	-1.57	-3.16	-3.16	**-3.45**	-2.82	-3.25	**-3.22**
SWE	-2.19	-2.01	-1.88	-1.81	-1.99	-2.01	**-4.33**	-2.91	**-3.06**
UK	-2.09	-1.08	-0.70	-1.49	-1.77	-1.71	**-3.57**	-2.72	-2.72
USA	-0.72	-0.94	-0.92	-1.32	-1.61	-1.60	-3.24	-2.96	-2.94
EA	-2.16	-1.18	-0.79	-2.18	-1.71	-1.81	-3.02	-2.79	-2.88

Table: Bold entries indicate rejection at the 5% level. PP(fb) denotes the one-step version of the Vogelsang and Wagner (2013) test.

	Shin Test			PU Test
	D-OLS	FM-OLS	IM-OLS	
AUS	**0.1968**	**0.1670**	**0.0726**	7.7920
CAN	**0.2249**	**0.1139**	**0.0593**	5.2664
CHE	**0.6087**	**0.6440**	**0.1707**	17.2928
CZE	**0.2113**	**0.1479**	**0.0736**	19.6403
DNK	**0.1111**	0.0848	0.0551	3.1702
ISR	**0.2498**	**0.1533**	0.0461	3.0915
KOR	**0.2835**	**0.2270**	**0.0855**	28.5091
NZL	**0.1112**	**0.1015**	**0.0528**	10.1979
NOR	**0.1430**	**0.1351**	**0.0640**	11.7283
SWE	0.0725	0.0572	**0.0555**	5.1052
UK	**0.5892**	**0.4693**	**0.2325**	6.3602
USA	0.0975	0.0418	0.0338	7.2720
EA	**0.1642**	0.0698	0.0430	8.2181

Table: Results of the cointegration test by Shin (1994) and the no-cointegration test of Phillips and Ouliaris (1990) for the linear regression using Andrews (1991) bandwidth and the Bartlett kernel. Bold entries indicate rejection at the 5% level.

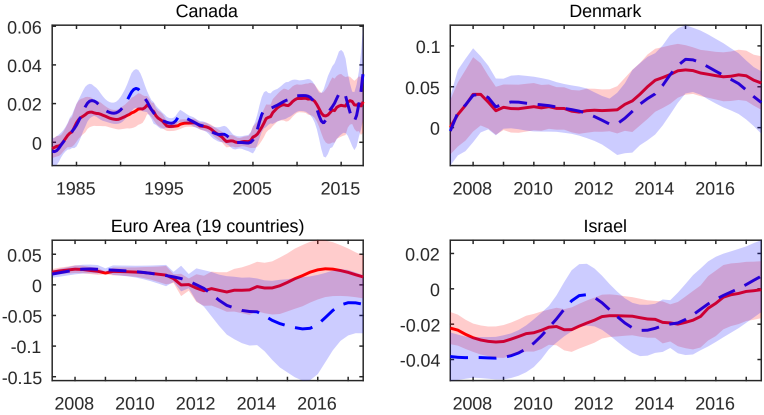


Figure: The red solid line displays the FM-OLS estimates and the blue dashed line displays the IM-OLS estimates for β_2. The corresponding 95% confidence bands are given by the red and blue shaded areas.

Long-Run Money Demand

Moving Window Estimation – Coefficient to Interest Rate

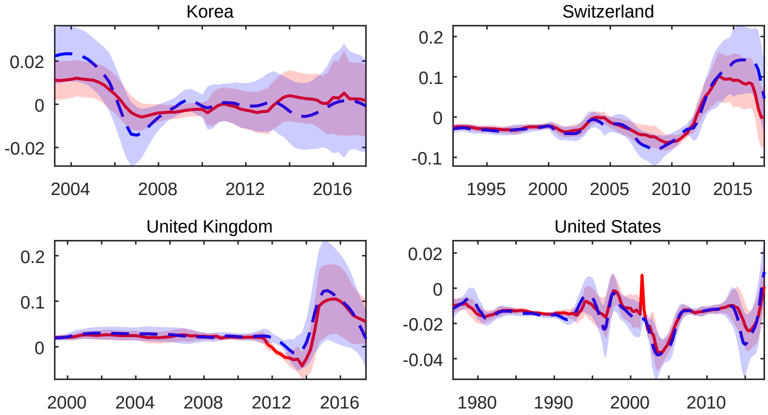


Figure: The red solid line displays the FM-OLS estimates and the blue dashed line displays the IM-OLS estimates for β_2. The corresponding 95% confidence bands are given by the red and blue shaded areas.

Long-Run Money Demand: $s_t = r_t$

$s_t = r_t$		D-OLS		FM-OLS		IM-OLS		
	Start	$W_{D,AIC}$	$W_{D,BIC}$	W_{FM}	LM_{FM}	W_{IM}	LM_{IM}	Fb_{IM}
Panel A: First Order Taylor Approximation ($n = 1$)								
CAN	1970Q1	**36.15**	**35.45**	**87.49**	**9.91**	**55.65**	**7.84**	**375.95**
CHE	1980Q1	**13.23**	**13.23**	**11.52**	**13.55**	**18.64**	**31.89**	**615.11**
DEN	1995Q1	**8.83**	**8.83**	**19.24**	**7.82**	**18.75**	**9.11**	**105.36**
ISR	1995Q1	**20.10**	**20.10**	**27.44**	4.62	**22.77**	**10.55**	**169.55**
KOR	1991Q1	**208.03**	**191.31**	**202.21**	**31.86**	**128.72**	**16.17**	**504.30**
UK	1987Q1	5.55	4.28	5.10	**55.05**	**13.28**	**59.11**	**814.81**
USA	1964Q1	**18.83**	**18.83**	**12.65**	6.37	**20.25**	**9.76**	**760.90**
EA	1995Q1	**36.79**	**34.95**	**25.59**	3.74	**26.31**	**8.54**	**1058.57**
Panel B: Third Order Taylor Approximation ($n = 3$)								
CAN	1970Q1	**40.95**	**40.95**	**106.06**	**30.41**	**114.92**	14.78	**4786.17**
CHE	1980Q1	**106.90**	**106.90**	**132.06**	**39.87**	**153.55**	**42.09**	**6411.93**
DEN	1995Q1	**33.45**	**33.45**	**28.08**	15.58	**34.38**	12.87	**299.86**
ISR	1995Q1	**344.29**	**344.29**	**453.20**	**33.11**	**405.33**	**26.91**	**1579.14**
KOR	1991Q1	**602.69**	**602.69**	**623.73**	**39.45**	**545.76**	**25.71**	**2085.52**
UK	1987Q1	**175.69**	**271.37**	**271.63**	**285.23**	**277.68**	**70.45**	**2968.68**
USA	1964Q1	**23.19**	**23.19**	**61.97**	**17.32**	**86.37**	13.94	**2951.75**
EA	1995Q1	**398.03**	**129.51**	**103.86**	15.02	**112.53**	11.80	**4735.91**

Table: Bold numbers indicate rejection at the 5% level. For the standard tests the corresponding critical values of the χ^2_3- and χ^2_9-distribution are given by 7.81 and 16.92.

Long-Run Money Demand: $s_t = t$

$s_t = t$		D-OLS		FM-OLS		IM-OLS		
	Start	$W_{D,AIC}$	$W_{D,BIC}$	W_{FM}	LM_{FM}	W_{IM}	LM_{IM}	Fb_{IM}
Panel A: First Order Taylor Approximation ($n = 1$)								
CAN	1970Q1	**49.04**	**54.99**	**126.11**	5.98	**87.54**	**9.96**	**673.71**
CHE	1980Q1	**18.20**	**18.20**	**32.26**	**33.34**	**40.60**	**34.88**	**483.29**
DEN	1995Q1	**22.68**	**22.68**	**23.98**	6.39	**34.03**	**11.65**	**153.86**
ISR	1995Q1	**41.06**	**41.06**	**49.23**	6.98	**44.79**	**14.72**	**375.84**
KOR	1991Q1	**288.21**	**288.21**	**289.45**	**20.67**	**171.98**	**17.10**	**525.56**
UK	1987Q1	**22.24**	4.44	3.53	**37.03**	**11.71**	**49.54**	**917.39**
USA	1964Q1	**24.90**	**24.90**	**13.74**	1.58	**16.28**	**9.30**	**618.54**
EA	1995Q1	**50.95**	**68.41**	**36.19**	4.27	**34.30**	**8.38**	**2035.80**
Panel B: Third Order Taylor Approximation ($n = 3$)								
CAN	1970Q1	**135.49**	**138.93**	**253.43**	**33.96**	**242.83**	15.97	**6252.21**
CHE	1980Q1	**1271.84**	**256.79**	–	**64.73**	**358.60**	**41.50**	**6332.94**
DEN	1995Q1	**71.59**	**71.59**	**84.76**	**22.35**	**97.21**	13.22	**357.12**
ISR	1995Q1	**529.44**	**529.44**	**183.54**	**40.04**	**232.44**	**23.63**	**1334.70**
KOR	1991Q1	**662.97**	**662.97**	**623.19**	**44.82**	**530.29**	**24.85**	**1563.99**
UK	1987Q1	**242.11**	**349.44**	**292.65**	**322.31**	**316.10**	**69.50**	**2420.62**
USA	1964Q1	**163.31**	**163.31**	**433.14**	10.58	**381.73**	15.02	**6729.98**
EA	1995Q1	**355.05**	**355.05**	**208.76**	**25.15**	**164.12**	11.15	**5223.22**

Table: Bold numbers indicate rejection at the 5% level. For the standard tests the corresponding critical values of the χ^2_3- and χ^2_9-distribution are given by 7.81 and 16.92.

- A large amount of rejections throughout across n and s_t.

Panel A: First Order Taylor Approximation ($n = 1$)	
$s_t = r_t$	Canada, Denmark, South Korea, Switzerland
$s_t = t$	South Korea, Switzerland
Panel B: Third Order Taylor Approximation ($n = 3$)	
$s_t \in \{r_t, t\}$	Israel, South Korea, Switzerland, United Kingdom

TABLE: List of countries with rejections throughout.

Summary and Conclusions

Summary and Conclusions

- We have provided tests for the null of linear cointegration against the alternative of smooth transition cointegration.
- The tests are based on FM-OLS and IM-OLS estimators considered for this type of Taylor approximation polynomial.
- We face some limitations in the setting, both with respect to X_t and also s_t.
- Roughly, the LM-tests perform better than the Wald-tests.
- The next step is to develop FM- and IM-type estimation for smooth transition cointegration models.

Some References

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Assumptions

- Let $\{\Delta X_t\}_{t\in\mathbb{Z}} = \{v_t\}_{t\in\mathbb{Z}}$ and denote with $\{\xi_t\}_{t\in\mathbb{Z}} = \{[u_t, v_t'])'\}_{t\in\mathbb{Z}}$ the process generated by:

$$\xi_t = C(L)\xi_t = \sum_{j=0}^{\infty} C_j \xi_{t-j}^0,$$

  with $\sum_{j=1}^{\infty} j||C_j|| < \infty$ and $\det(C(1)) \neq 0$.

- The process $\{\xi_t^0\}_{t\in\mathbb{Z}}$ is a strictly stationary and ergodic martingale difference sequence (MDS) with natural filtration $\mathcal{F}_t = \sigma(\{\xi_s^0\}_{-\infty}^t)$.

- Moreover, we assume a positive definite covariance matrix $\Sigma_{\xi^0\xi^0}$ and $\sup_{t\in\mathbb{Z}} \mathbb{E}[\|\xi_t^0\|^r|\mathcal{F}_{t-1}] < \infty$ a.s. for some $r > 4$.

For the deterministic component we assume that there exists a sequence of $p \times p$ scaling matrices $A_D = A_D(T)$ and a p-dimensional vector of càdlàg functions $D(s)$, with $0 < \int_0^s D(z)D(z)'dz < \infty$ for $0 < s \leq 1$, such that for $0 \leq s \leq 1$ it holds that:

$$\lim_{T \to \infty} T^{1/2} A_D D_{[sT]} = D(s).$$

[For the leading case of polynomial time trends, the deterministic component has the form $D_t = [1, t, t^2, \ldots, t^{q-1}]'$ with $G_D = \mathrm{diag}(T^{-1/2}, T^{-3/2}, T^{-5/2}, \ldots, T^{-(q-1/2)})$ and $D(s) = [1, s, s^2, \ldots, s^{q-1}]'$.]

Assumptions

Kernel and Bandwidth

The kernel function $k(\cdot)$ satisfies:

1. $k(0) = 1,\ k(\cdot)$ is continuous at 0 and $\bar{k}(0) := \sup_{x \geq 0} |k(x)| < \infty$
2. $\int_0^\infty \bar{k}(x)dx < \infty$, where $\bar{k}(x) = \sup_{y \geq x} |k(y)|$

The bandwidth satisfies $M_T \to \infty$ with $\lim_{T \to \infty} (M_T^{-1} + T^{-1/2} M_T) = 0$.

Assumptions

Transition Function

- The transition function is given by

$$G(s_t, \theta_G) := G_*(h(s_t, \theta_G)),$$

where

$$h(s_t, \theta_G) := \gamma \prod_{i=1}^{n} (s_t - c_i),$$

with $c_n \geq \ldots \geq c_1$, $\gamma > 0$.

- The function $G_*(\cdot) \colon \mathbb{R} \mapsto \mathbb{R}$ is n-times continuously differentiable in an open interval including zero with $G_*(0) = 0$ and bounded.

- With respect to the derivatives we assume that:

$$\frac{\partial G_*(s)}{\partial s}|_{s=0} \neq 0 \quad \text{and} \quad \frac{\partial^n G_*(s)}{\partial^n s}|_{s=0} \neq 0.$$

Regression with Integrated Variables

$$y_t = x_t\beta + u_t,\ x_t = x_{t-1} + v_t,\ \hat{\beta} - \beta = \left(\sum_{t=1}^{T} x_t^2\right)^{-1} \sum_{t=1}^{T} x_t u_t$$

$$T\left(\hat{\beta} - \beta\right) \Rightarrow \left(\int_0^1 B_v^2(r)dr\right)^{-1} \left(\int_0^1 B_v(r)dB_u(r) + \Delta_{vu}\right),$$

$$\text{with } \Delta_{vu} := \sum_{j=0}^{\infty} \mathbb{E} v_{t-j} u_t \quad [\mathbb{E} x_t u_t = \mathbb{E}(\sum_{j=0}^{t-1} v_{t-j}) u_t]$$

$$\frac{1}{\sqrt{T}} x_{\lfloor rT \rfloor} = \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor rT \rfloor} v_t \Rightarrow B_v(r) = \Omega_{vv}^{1/2} W_v(r)$$

$$\frac{1}{T^2} \sum_{t=1}^{T} x_t^2 = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{x_t}{\sqrt{T}}\right)^2 \Rightarrow \int_0^1 B_v^2(r)dr$$

$$\frac{1}{T} \sum_{t=1}^{T} x_t u_t = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(\frac{x_t}{\sqrt{T}}\right) u_t \Rightarrow \int_0^1 B_v(r)dB_u(r) + \Delta_{vu}$$

$y_t = x_t\beta + u_t,\ \hat{\beta}^+ := \left(\sum_{t=1}^T x_t^2\right)^{-1}\left(\sum_{t=1}^T x_t y_t^+ - \hat{\Delta}_{vu}^+ T\right),$
$y_t^+ := y_t - v_t\hat{\Omega}_{vv}^{-1}\hat{\Omega}_{vu},\ \hat{\Delta}_{vu}^+ := \hat{\Delta}_{vu} - \hat{\Delta}_{vv}\hat{\Omega}_{vv}^{-1}\hat{\Omega}_{vu}$

$$
\begin{aligned}
T\left(\hat{\beta}^+ - \beta\right) &= \left(\frac{1}{T^2}\sum_{t=1}^T x_t^2\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^T x_t u_t^+ - \hat{\Delta}_{vu}^+\right) \\
&= \left(\frac{1}{T^2}\sum_{t=1}^T x_t^2\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^T x_t u_t - \frac{1}{T}\sum_{t=1}^T x_t v_t\hat{\Omega}_{vv}^{-1}\hat{\Omega}_{vu} - \hat{\Delta}_{vu}^+\right) \\
&\Rightarrow \left(\int_0^1 B_v^2(r)dr\right)^{-1}\left(\int_0^1 B_v(r)dB_u(r) + \Delta_{vu} - \int_0^1 B_v(r)dB_v(r)\Omega_{vv}^{-1}\Omega_{vu}\right. \\
&\qquad \left. - \Delta_{vv}\Omega_{vv}^{-1}\Omega_{vu} - \Delta_{vu}^+\right) \\
&= \left(\int_0^1 B_v^2(r)dr\right)^{-1}\int_0^1 B_v(r)dB_{u\cdot v}(r), \quad B_{u\cdot v}(\cdot) := B_u(\cdot) - B_v(\cdot)\Omega_{vv}^{-1}\Omega_{vu}
\end{aligned}
$$

Some Asymptotic Results

FM-OLS Limiting Distribution

Proposition

Under the assumptions given in the paper it holds under the null hypothesis, with $\beta_0 = [\beta_0', 0', \dots, 0']'$, that:

$$A^{-1}\left(\hat{\beta}^{+} - \beta_0\right) \xrightarrow{d} \left(\int_0^1 J(r)J(r)'dr\right)^{-1} \int_0^1 J(r)dB_{u \cdot v}(r),$$

with $B_{u \cdot v}(r) := B_u(r) - B_v(r)'\Omega_{vv}^{-1}\Omega_{vu}$, A the scaling matrix, and

$$J(r) := \begin{cases} \mathbf{B}_s^{(0,n)}(r) \otimes \begin{bmatrix} D(r) \\ B_v(r) \end{bmatrix} & \text{in case (i)} \\ \mathbf{r}^{(0,n)} \otimes \begin{bmatrix} D(r) \\ B_v(r) \end{bmatrix} & \text{in case (ii)} \end{cases}$$

where $\mathbf{B}_s^{(0,n)}(r) := [1, B_s(r), \dots, B_s^n(r)]'$ and $\mathbf{r}^{(0,n)} := [1, r, \dots, r^n]'$.

Correction Terms for FM-OLS Based Tests

The correction term $M^* := [M_0^{*\prime}, M_1^{*\prime}, \ldots, M_n^{*\prime}]'$ depends on the approximation order and transition variable and is given by:

$$M_j^* := \begin{cases} \begin{bmatrix} j\hat{\Delta}_{su}^{+} \sum_{t=1}^{T} D_t s_t^{j-1} \\ \hat{\Delta}_{vu}^{+} \sum_{t=1}^{T} s_t^{j} + j\hat{\Delta}_{su}^{+} \sum_{t=1}^{T} X_t s_t^{j-1} \end{bmatrix} & \text{in case (i)} \\ \begin{bmatrix} 0_p \\ \hat{\Delta}_{vu}^{+} \sum_{t=1}^{T} t^{j} \end{bmatrix} & \text{in case (ii)} \end{cases}$$

IM-OLS Limiting Distribution

Proposition

Under the assumptions given in the paper it holds under the null hypothesis, with $\boldsymbol{\beta}_{*,0} := [\boldsymbol{\beta}_0', (\Omega_{vv}^{-1}\Omega_{vu})']'$, that:

$$\tilde{A}^{-1}\left(\hat{\boldsymbol{\beta}}_* - \boldsymbol{\beta}_{*,0}\right) \xrightarrow{d} \left(\int_0^1 f(r)f(r)'dr\right)^{-1}\int_0^1 f(r)B_{u\cdot v}(r)dr$$
$$= \left(\int_0^1 f(r)f(r)'dr\right)^{-1}\int_0^1 [F(1) - F(r)]dB_{u\cdot v}(r),$$

where

$$f(r) := \begin{bmatrix} \int_0^r J(s)ds \\ B_v(r) \end{bmatrix}, \quad F(r) := \int_0^r f(s)ds$$

and $J(r)$ as defined before.

Fixed-b Inference

Fixed-b Inference: Simple Example I

- Consider a simple "almost standard" (i.e. HAC) regression:

$$y_t = x_t\beta + u_t,$$

with $T^{-1}\sum_{t=1}^{\lfloor rT\rfloor} x_t^2 \rightarrow rQ$, $Q > 0$ and $z_t = x_t u_t$ such that:

$$\frac{1}{T^{1/2}}\sum_{t=1}^{\lfloor rT\rfloor} z_t \Rightarrow \omega^{1/2}W(r).$$

- Then: $\sqrt{T}\left(\hat{\beta}-\beta\right) \Rightarrow \mathcal{N}(0, \omega Q^{-2})$.
- With a consistent estimator $\hat{\omega} \rightarrow \omega$ it follows that:

$$t_\beta = \frac{\hat{\beta}-\beta_0}{\sqrt{\hat{V}ar(\hat{\beta})}} = \frac{\hat{\beta}-\beta_0}{\hat{\omega}^{1/2}\hat{Q}^{-1}} \Rightarrow \mathcal{N}(0,1).$$

- Using a consistent estimator $\hat{\omega} = \hat{\Gamma}_0 + 2\sum_{j=1}^{T-1} k(j/M)\hat{\Gamma}_j$, with $\hat{\Gamma}_j = T^{-1}\sum_{t=j+1}^{T} \hat{z}_t\hat{z}_{t-j}$ and $\hat{z}_t = x_t\hat{u}_t$, "hides" finite sample effects of kernel function $k(\cdot)$ and bandwidth M.

Fixed-b Inference: Simple Example II

- Consider a bandwidth proportional to sample size, i.e. $M = bT$.
- Then under appropriate assumptions it holds that $\hat{\omega} \Rightarrow \omega P(b,k)$, where $P(b,k)$ is a function of $W(r)$ that depends upon bandwidth b and kernel function $k(\cdot)$.
- This leads to a fixed-b limit distribution of the t-statistic of the form:

$$t_\beta \Rightarrow \frac{W(1)}{P(b,k)}$$

- See, e. g., Kiefer and Vogelsang (2005).
- Critical values can be tabulated for (a grid of) values of b and different kernel functions $k(\cdot)$.

Long-Run Money Demand

Long-Run Money Demand

Residuals from Linear Cointegrating Regression

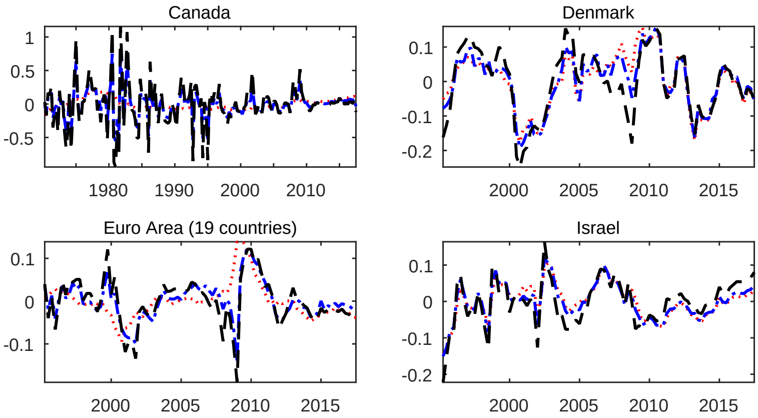

Figure: The red dotted line shows the D-OLS residuals, the blue dashed dotted line the FM-OLS residuals and the black dashed line the IM-OLS residuals.

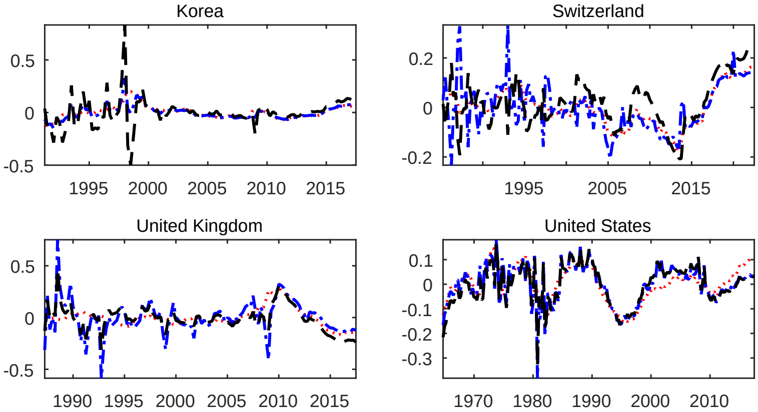

Figure: The red dotted line shows the D-OLS residuals, the blue dashed dotted line the FM-OLS residuals and the black dashed line the IM-OLS residuals.