Structural estimation of rational expectations models with recursive preferences

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Investors' recursive preferences in macrofinance

The bad news about long-run future consumption growth induces fear of stocks

Preferences over the timing in the temporal resolution of uncertainty
[Kreps and Porteus (1978); Dekel (1986); Chew and Epstein (1989)]

Chew-Dekel class of preferences

- Epstein and Zin (1989) is the most popular specification
- Routledge and Zin (2010)
- Backus et al. (2004)
- Campanale et al. (2010)
- ...

No arbitrage

$$\mathrm{E}\left[M_{t,t+1}\boldsymbol{R}_{t+1} - \boldsymbol{\imath}_{q+1} \middle| \mathcal{F}_t\right] = \boldsymbol{e}_{q+1}$$

$M_{t,t+1}$ = Stochastic Discount Factor (SDF) from t to $t+1$

$\boldsymbol{R}_{t+1}$ = vector of risk-free rate and cum-dividend excess returns from t to $t+1$

$\boldsymbol{\imath}_{q+1}$ = vector of ones

$\boldsymbol{e}_{q+1}$ = unit vector with one as first component

$\mathcal{F}_t$ = information set at t

Epstein and Zin (1989) specification

At t, the representative agent consumes C_t and enjoys utility V_t

$g_{t+1} \coloneqq \ln\left(C_{t+1}/C_t\right) \qquad v_t \coloneqq \left(V_t/C_t\right)^{1-1/\psi}$

$$v_t = 1-\beta+\beta\,\mathrm{E}\left[\exp\left[(1-\gamma)g_{t+1}\right]v_{t+1}^{\alpha}\Big|\mathcal{F}_t\right]^{1/\alpha} \qquad \alpha \coloneqq \frac{1-\gamma}{1-\frac{1}{\psi}}$$

"**subjective discount rate**" $\beta \in (0,1)$
"**risk aversion parameter**" $\gamma > 0$
"**elasticity of intertemporal substitution**" $\psi > 0$ $\qquad \boldsymbol{\theta} \coloneqq [\beta\ \gamma\ \psi]'$
$\mathcal{F}_t =$ agent's information at t

$$M_{t,t+1} = \beta e^{-\gamma g_{t+1}} \frac{v_{t+1}^{\alpha-1}}{\mathrm{E}\left[\exp\left[(1-\gamma)g_{t+1}\right]v_{t+1}^{\alpha}\Big|\mathcal{F}_t\right]^{1-1/\alpha}}$$

The (relative) magnitude of γ and ψ implies different attitudes toward (intertemporal) risk

Model estimation methods already proposed

v_{t+1} is **unobservable**

- Proxy (or model) the log return on the wealth portfolio r^A_{t+1}, since in complete markets

$$M_{t,t+1} = \exp\left[\alpha \ln(\beta) - \frac{\alpha}{\psi} g_{t+1} + (\alpha - 1) r^A_{t+1}\right]$$

  [Epstein and Zin (1991); Bansal and Yaron (2004); Bansal et al. (2007, 2016); Constantinides and Ghosh (2011); Grammig and Küchlin (2018); Meddahi and Tinang (2016)]

- Specify the dynamics of consumption growth through latent variables [Chen et al. (2013)]

joint hypothesis on preferences and

- proxy of r^A_{t+1} [$\sim$ Roll (1977)'s critique]
- macroeconomic model specification
- specification of consumption growth dynamics

Our estimation method . . .

. . . for nonparametric time series models characterized by **conditional moment restrictions** that are functional equations solved by a **contraction mapping argument**

These restrictions are known up to

- an unknown Euclidean (finite-dimensional) parameter $\boldsymbol{\theta}$
- an unknown functional of the Euclidean parameter $v(\cdot;\boldsymbol{\theta})$
- the unknown transition density of the state variables f

The method that retains the entire original structure of the preferences and reduces misspecification risk

Starting point

Choose a few Markov weak stationary state variables $\boldsymbol{X}_t$ spanning $\mathcal{F}_t = \sigma(\boldsymbol{X}_t)$ *($v(\boldsymbol{X}_t; \boldsymbol{\theta}) \equiv v_t$ shows contraction [Hansen and Scheinkman (2012)])*

Estimate $\mathrm{E}[\cdot|\boldsymbol{X}_t]$ nonparametrically by $\hat{\mathrm{E}}[\cdot|\boldsymbol{X}_t]$

Step 1: admissible SDF's reconstruction

- Initiate the estimate of v_t at $\bar{v}$ *(either a constant or for simplified state variables dynamics)*
- Characterize SDF parameter space estimate $\hat{\Theta}_T$ by checking (*numerically*) which $\boldsymbol{\theta} = [\beta\ \gamma\ \psi]'$ and $\hat{v}_{[N]}(\cdot;\boldsymbol{\theta})$ allow for a contraction mapping

$$\hat{v}_{[i]}(\boldsymbol{X}_t;\boldsymbol{\theta}) = \begin{cases} 1-\beta+\beta\varphi_{[i-1]}(\boldsymbol{X}_t;\boldsymbol{\theta})^{\frac{1}{\alpha}}, & i \geqslant 1, \\ \bar{v}, & i = 1, \end{cases} \qquad i = 0, 1, \dots, N$$

$$\hat{\varphi}_{[i-1]}(\boldsymbol{X}_t;\boldsymbol{\theta}) := \hat{\mathrm{E}}\left[\exp\left[(1-\gamma)\, g(\boldsymbol{X}_{t+1}, \boldsymbol{X}_t)\right] \hat{v}_{[i-1]}(\boldsymbol{X}_{t+1};\boldsymbol{\theta})^{\alpha} \middle| \boldsymbol{X}_t\right]$$

- Construct an admissible SDF for each $\boldsymbol{\theta} \in \hat{\Theta}_T$:

$$\hat{m}_T(\boldsymbol{X}_{t+1}, \boldsymbol{X}_t;\boldsymbol{\theta}) := \beta \frac{\exp\left[-\gamma g(\boldsymbol{X}_{t+1}, \boldsymbol{X}_t)\right] \hat{v}_{[N]}(\boldsymbol{X}_{t+1};\boldsymbol{\theta})^{\alpha-1}}{\hat{\varphi}_{[N-1]}(\boldsymbol{X}_{t+1};\boldsymbol{\theta})^{1-\frac{1}{\alpha}}}$$

Step 2: true SDF estimation

Local GMM:

$$\hat{\boldsymbol{\theta}}_T := \underset{\boldsymbol{\theta} \in \hat{\Theta}_T}{\operatorname{argmin}} \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}(\boldsymbol{X}_t) \hat{\boldsymbol{e}}_T(\boldsymbol{X}_t; \boldsymbol{\theta})' \hat{\boldsymbol{\Omega}}_T^{-1}(\boldsymbol{X}_t; \boldsymbol{\theta}) \hat{\boldsymbol{e}}_T(\boldsymbol{X}_t; \boldsymbol{\theta}).$$

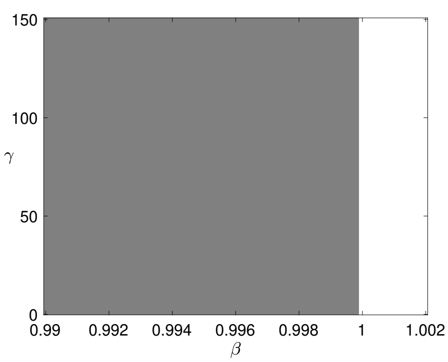
$\hat{\boldsymbol{e}}_T(\boldsymbol{X}_t; \boldsymbol{\theta})$ = empirical pricing errors vector [obtained by local-linear regressions]
$\hat{\boldsymbol{\Omega}}^{-1}(\boldsymbol{X}_t)$ = weighting matrix *(we consider the identity and the efficient matrix)*
$\mathbf{1}(\boldsymbol{X}_t)$ state variables' support trimmer [Tripathi and Kitamura (2003); Gagliardini and Ronchetti (2020)]

Data

- Quarterly U.S. data from 1952Q1 to 2019Q3
- Consumption growth, g_{t+1}, is proxied by the growth rate of the sum of the series *Personal Consumption Expenditures: Nondurables* and *Personal Consumption Expenditures: Services* (FRED).
- Six value-weighted Frama-French portfolios, two-way sort along size and book-to-market (Kenneth French).
- Risk-free rate, R^f_{t+1}: 3-month T-Bill (FRED).
- We adjust for inflation (PCEPI) and population growth (FRED).
- cay_t = Lettau and Ludvigson (2001) consumption-wealth ratio $\quad \boldsymbol{X}_t = [g_t \; cay_t]'$

(we also checked other test assets and predictors [corporate bond spread, labor income-to-consumption ratio, . . .])

Contraction of v



(a) $\psi = 1$ (b) $\psi \neq 1$

(Slow convergence for high beta and low alpha: truncated recursion at 20,000 iterations)

Estimates of the preference parameters

$\hat{\beta}_T$	0.989	0.987
	(0.967, 0.999)	(0.972, 1.00)
$\hat{\gamma}_T$	16.55	16.44
	(2.50, 23.94)	(1.85, 40.27)
$\hat{\psi}_T$	-	1.74
		(0.10, 2.66)
Estimation crit.	0.004	0.0039
	(0.002, 0.009)	(0.002, 0.009)

(Bootstrap 90% standard confidence intervals in parentheses)

"Reasonable" values for β, γ and ψ

Monte Carlo experiment I

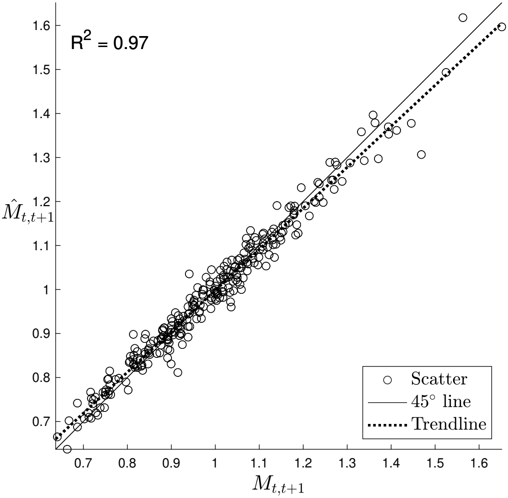
Bansal et al. (2012) Long-run risks model

Param.	**Value**	**Explanation**
β	0.9930	Subjective discount rate
γ	10.0000	Risk aversion
ψ	1 or 2	Elasticity of intertemporal substitution

$B = 1000$ simulated samples with length $T = 275$

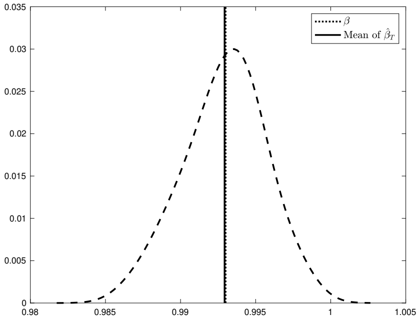
▸ Data Generating Process (Bansal-Yaron Long-run Risks model)

Monte Carlo experiment II

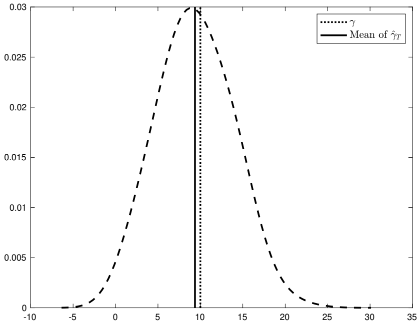


Reconstructed SDF against true SDF (case with $\psi = 2$)

Monte Carlo experiment III



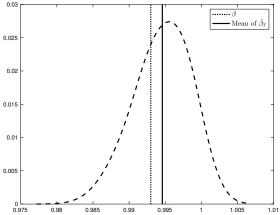
(c) Distribution of $\hat{\beta}_T$



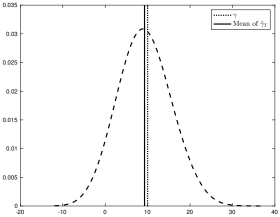
(d) Distribution of $\hat{\gamma}_T$

(case with $\psi = 1$)

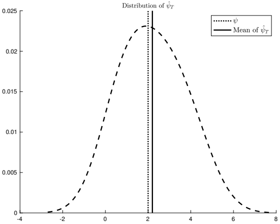
Monte Carlo experiment IV



(e) Distribution of $\hat{\beta}_T$



(f) Distribution of $\hat{\gamma}_T$



(g) Distribution of $\hat{\psi}_T$

(case with $\psi = 2$)

Conclusion

We propose a novel estimation method for nonparametric Markovian time series models characterized by **conditional moment restrictions** that are functional equations solved by a **contraction mapping argument**, such as recursive preference models

Estimates of Epstein-Zin preferences for U.S. equity and T-Bill markets during 1952-2019:
$\hat{\beta}_T \approx 0.99 \quad \hat{\gamma}_T \approx 16 \quad \hat{\psi}_T \approx 1.7$

Adaptable to models with other Chew-Dekel preferences (e.g., semi-weighted utilities, generalized disappointment aversion, . . .) and other structural models

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Thank you for your attention

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Definition of Epstein-Zin preferences

Definition

$$F(x,y) = \left((1-\beta)x^{1-\frac{1}{\psi}} + \beta y^{1-\frac{1}{\psi}}\right)^{\frac{1}{1-\frac{1}{\psi}}}, \tag{1a}$$

$$\mathcal{R}_t(x_{t+1}) = G^{-1}\left(\mathrm{E}_t\left[G(x_{t+1})\right]\right), \tag{1b}$$

$$G(x) = \frac{x^{1-\gamma}}{1-\gamma}, \tag{1c}$$

Go back to introduction

Weak instruments

- Within asset pricing, we often use moment conditions of the form

$$\mathrm{E}[\boldsymbol{h}(\boldsymbol{X}_t;\boldsymbol{\theta}) \otimes \boldsymbol{Z}_t] = 0, \tag{2}$$

  where
  - $\boldsymbol{h}$ is a criterion function of
    - variables collected in the vector $\boldsymbol{X}_t$;
    - the parameters in the vector $\boldsymbol{\theta} \in \Theta$.
  - $\boldsymbol{Z}_t$ is a vector of instruments.
- In asset pricing, $\boldsymbol{Z}_t$ often correlates "weakly" with $\boldsymbol{X}_t$, e.g. see Stock & Wright (2000, Ectra).
- Result: large standard errors for the estimator $\hat{\boldsymbol{\theta}}$.
- Directly addressed by various authors, e.g. Kleibergen and Zhan (2020); Manresa et al. (2017); Stock and Wright (2000); Yogo (2004).
- ... but standard errors remain large.
- We minimize a criterion function based on kernel regressions to estimate the parameters (no need for instruments).

Go back to estimation and testing

Consumption growth close to one

- C_{t+1}/C_t small and not so volatile, i.e. $C_{t+1}/C_t \sim 1$.
- The parameter ψ mainly regulates how strong the interest rate (i.e. risk-free rate) responds to changes in consumption growth; plausible values are between 0.5 and 4, and 1.5 is often used in calibrated models.
- Consequently, the term

$$\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}} \tag{3}$$

  It is also close to one for a wide range of plausible values for ψ.
- E.g. $1.001^{1/0.5}$ and $1.001^{1/1.5}$ not so different.
- Consequently, large standard errors for estimates of ψ, e.g. see Chen et al. (2013); Constantinides and Ghosh (2011); Grammig and Küchlin (2018).

Go back to estimation and testing

Nadaraya-Watson regression

- Non-parametric regression equation (where $\boldsymbol{X}_t$ is one-dimensional):

$$\mathrm{E}[y|x] = g(x) = \int y \frac{f(y,x)}{x} dy. \tag{4}$$

- We estimate $g(x)$ by

$$\hat{g}(x) := \frac{\sum_{i=1}^{N} K\left(\frac{X_i - x}{h_N}\right) Y_i}{\sum_{j=1}^{N} K\left(\frac{X_j - x}{h_N}\right)} = \sum_{i=1}^{N} w(X_i, x; h_N) Y_i, \tag{5}$$

where

$$w(X_i, x, h_N) := \frac{K\left(\frac{X_i - x}{h_N}\right)}{\sum_{j=1}^{N} K\left(\frac{X_j - x}{h_N}\right)}. \tag{6}$$

- We want w to be large (small) when $|X_i - x|$ is small (large).
- We use $K(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$.
- Pick h_N to minimize errors.

Recursion

Silverman bandwidth

The Silverman (1986) bandwidth is

$$h_T = 0.9 \times \min\left\{\sigma_G, \frac{r_q(G)}{1.34}\right\} T^{-1/5}, \tag{7}$$

where σ_G is the standard deviation of G and $r_q(G)$ the interquartile range of G. Go back

- We have a contraction mapping of $v(\boldsymbol{X}_t;\boldsymbol{\theta})$ w.r.t. $v(\boldsymbol{X}_{t+1};\boldsymbol{\theta})$ (Hansen and Scheinkman, 2012, PNAS).
- We compute $\forall\boldsymbol{\theta}\in\Theta$ and $\forall\boldsymbol{x}\in\mathcal{X}$ iterating over $i=0,1,\dots,N$:

$$v_{[i]}\left(\boldsymbol{x};\boldsymbol{\theta}\right)=\begin{cases}1-\beta+\beta\varphi_{[i-1]}\left(\boldsymbol{x};\boldsymbol{\theta}\right)^{\frac{1}{\alpha}}, & i\geqslant 1,\\ \bar{v}, & i=0,\end{cases}$$

where

$$\varphi_{[i-1]}\left(\boldsymbol{X}_t;\boldsymbol{\theta}\right):=\mathrm{E}\left[\exp\left[\left(1-\gamma\right)g\left(\boldsymbol{X}_{t+1},\boldsymbol{X}_t\right)\right]v_{[i-1]}\left(\boldsymbol{X}_{t+1};\boldsymbol{\theta}\right)^{\alpha}\middle|\boldsymbol{X}_t\right].$$

- We estimate $\varphi_{[i-1]}(\boldsymbol{x};\boldsymbol{\theta})$ by a Nadaraya-Watson (kernel) regression: $\hat{\varphi}_{T,[i-1]}\left(\boldsymbol{x};\boldsymbol{\theta}\right)$.
- We iteratively compute $\hat{v}_{T,[i]}\left(\boldsymbol{x};\boldsymbol{\theta}\right)=1-\beta+\beta\hat{\varphi}_{T,[i-1]}\left(\boldsymbol{x};\boldsymbol{\theta}\right)$ until convergence.
- Then, we estimate the SDF by

$$\hat{m}_T(\boldsymbol{X}_{t+1},\boldsymbol{X}_t;\boldsymbol{\theta}):=\beta\frac{\exp\left[-\gamma g\left(\boldsymbol{X}_{t+1},\boldsymbol{X}_t\right)\right]\hat{v}_{T,[N]}\left(\boldsymbol{X}_{t+1};\boldsymbol{\theta}\right)^{\alpha-1}}{\hat{\varphi}_{T,[N-1]}\left(\boldsymbol{X}_{t+1};\boldsymbol{\theta}\right)^{1-\frac{1}{\alpha}}},$$

which is the empirical counterpart of the model SDF $M_{t,t+1}$

Nadaraya-Watson regression

What are reasonable values for β, γ, and ψ? I

- β defines "current relative valuation placed on receiving a good at an earlier date compared with receiving it later."
  - E.g. you receive a candy bar today or, say, one month from today; the second option is β times less worth than the first option
  - Therefore, it is intimately linked to the average gross risk-free rate: $R^f \approx 1/\beta$
  - R^f is close to 1.01 monthly
  - Quarterly β in [0.98 : 0.99] is plausible
- γ defines willingness to substitute consumption across states of nature:
  - If γ is low, you do not mind if your consumption level falls significantly when a recession hits.
  - Therefore, γ links consumption growth to expected excess returns $\mathrm{E}_t[R_{j,t+1} - R^f_{t+1}]$.
  - Calibrated models suggest that γ is between 5 and 15, e.g. see Bansal & Yaron (2004, JF) or Mehra & Prescott (1985, JME).
- ψ defines willingness to substitute consumption across time:
  - Measures response of risk-free rate to change in consumption growth:

$$\psi \approx -\frac{(C_{t+1}/C_t)d(C_t/C_{t+1})}{dR^f_{t+1}/R^f_{t+1}}.$$

What are reasonable values for β, γ, and ψ? II

- In the data, this response is rather small, e.g. see Yogo (2004);
- That is, plausible values of ψ are between 0.5 and 2.

What are reasonable values for β, γ, and ψ? III

- In the context of Epstein-Zin preferences, we can also look at this problem as follows:
  - Suppose that consumption growth and returns, $R_{j,t+1}$, are homoskedastic and jointly lognormal with variances σ_C^2 and σ_j^2, then we can write (see Campbell, 2018):

$$r_{t+1}^f = -\ln(\beta) + \frac{1}{\psi} \mathrm{E}_t \left[\Delta \ln(C_{t+1})\right] - \frac{\theta}{2\psi^2}\sigma_C^2 + \frac{\theta - 1}{2}\sigma_w^2,$$

$$\mathrm{E}_t \left[r_{j,t+1} - r_{t+1}^f\right] = \frac{\theta}{\psi}\sigma_{jC} + (1-\theta)\sigma_{jA} - \frac{\sigma_j^2}{2},$$

  where $\theta := (1-\gamma)/(1-1/\psi)$, $r_{j,t+1}$ is the log-return on a risky asset j, and σ_{jA} and σ_{jA} are the covariance terms between $r_{j,t+1}$ and, respectively, $\Delta \ln(C_{t+1})$ and $\ln(R_{t+1}^A)$.
  - These equations also suggest that for quarterly values, we have β very close to 0.99, γ between 2 and 20, and ψ around 1.5.

Go back to point estimates

Regularization

- The concentration ratio of any p.d. matrix $\boldsymbol{Z}$ is defined as:

$$\mathrm{CR}\left(\boldsymbol{Z}\right) := \sqrt{\frac{\lambda_{\max}\left(\boldsymbol{Z}\right)}{\lambda_{\min}\left(\boldsymbol{Z}\right)}}, \tag{8}$$

  where $\lambda_{\max}\left(\boldsymbol{Z}\right)$ is the largest eigenvalue of $\boldsymbol{Z}$ and $\lambda_{\min}\left(\boldsymbol{Z}\right)$ is the smallest eigenvalue of $\boldsymbol{Z}$.
- We inflate the diagonal elements of $\boldsymbol{Z}$ by $\bar{\epsilon} > 0$.
- We note that $\mathrm{CR}\left(\boldsymbol{Z} + \bar{\epsilon}\boldsymbol{I}\right) < \mathrm{CR}\left(\boldsymbol{Z}\right)$.
- We find the smallest $\bar{\epsilon} > 0$ such that

$$\bar{\epsilon} = \underset{\epsilon}{\operatorname{argmin}} \frac{1}{T} \sum_{t=1}^{T} \left[15 - \mathrm{CR}\left(\hat{\boldsymbol{\Omega}}_T(\boldsymbol{X}_t)^{-1} + \epsilon\boldsymbol{I}\right)\right]^2. \tag{9}$$

- Note that $\hat{\boldsymbol{\Omega}}_T(\boldsymbol{X}_t)^{-1}$ is an "*uninverted*" matrix, e.g.

$$\hat{\boldsymbol{\Omega}}_T(\boldsymbol{X}_t)^{-1} = \sum_{i=1}^{T-1} w(\boldsymbol{X}_t, \boldsymbol{X}_i; h_T)\boldsymbol{R}_{i+1}\boldsymbol{R}_{i+1}'. \tag{10}$$

The data generating process (DGP)

- The model is presented in log form.
- Small caps denote logs, e.g. $g_t = \ln(G_t)$, we write:

$$\mathrm{E}_t \left[\exp \left\{ \underbrace{\theta \ln(\beta) - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r^a_{t+1}}_{\ln\left(\beta^\theta G_{t+1}^{-\frac{\theta}{\psi}} (R^A_{t+1})^{\theta - 1}\right)} + r_{j,t+1} \right\} \right] = 1. \tag{11}$$

- Bansal & Yaron (2004, JF) postulate:

$$g_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}, \tag{12a}$$

$$g_{d,t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}, \tag{12b}$$

$$x_{t+1} = \rho x_t + \varphi_\epsilon \sigma_t \epsilon_{t+1}, \tag{12c}$$

$$\sigma^2_{t+1} = \bar{\sigma}^2 + \nu_1(\sigma^2_t - \bar{\sigma}^2) + \sigma_w w_{t+1}, \tag{12d}$$

- Note that x_t and σ^2_t are the state variables, i.e. $\boldsymbol{X}_t = (x_t, \sigma^2_t)'$.

Go back to Monte Carlo experiment

- R^A_{t+1} is the return on the "consumption claim" with price P_t.
- The log of price-consumption ratio $z_t = \ln(P_t/C_t)$ obeys:

$$z_t = A_0 + A_1 x_t + A_2 \sigma^2_t, \tag{13}$$

and using a standard Campbell-Shiller (1988) decomposition, we have that:

$$r^a_{t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}, \tag{14}$$

where $\kappa_1 = \exp(\bar{z}) / (1 + \exp(\bar{z}))$ and $\kappa_0 = -\ln(\kappa_1) - (1 - \kappa_1)\ln(1/\kappa_1 - 1)$, with $\bar{z} = E[z_t]$.

- A_0, A_1, and A_2 are functions of the model's parameters:
  - Substitute r^a_{t+1} and z_t into (11) and set $r^a_{t+1} = r_{j,t+1}$;
  - Solve for A_0, A_1, and A_2 so that (11) holds.

Solutions for A_0, A_1, and A_2

Go back to Monte Carlo experiment

- R^M_{t+1} is the return on a market portfolio with price P^M_t.
- The market portfolio yields a dividend stream that grows with the rate $g_{d,t}$.
- The log of price-dividend ratio $z_{m,t}$ obeys

$$z_{m,t+1} = A_{0,m} + A_{1,m}x_{t+1} + A_{2,m}\sigma^2_{t+1}, \tag{15}$$

  and:

$$r^m_{t+1} = \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + g_{d,t+1}, \tag{16}$$

  where $\kappa_{1,m} = \exp(\bar{z}_m)/\left(1 + \exp(\bar{z}_m)\right)$ and
  $\kappa_0 = -\ln(\kappa_{1,m}) - (1 - \kappa_{1,m})\ln(1/\kappa_{1,m} - 1)$, with $\bar{z}_m = E[z_{m,t}]$.
- $A_{1,m}$, $A_{1,m}$, and $A_{2,m}$ are functions of the model's parameters:
  - Substitute r^m_{t+1} and $z_{m,t}$ into (11) and set $r^m_{t+1} = r_{j,t+1}$;
  - Solve for $A_{1,m}$, $A_{1,m}$, and $A_{2,m}$ so that (11) holds.

Solutions for $A_{1,m}$, $A_{1,m}$, and $A_{2,m}$

Go back to Monte Carlo experiment

- To obtain the risk-free rate, r^f_{t+1}, substitute $r^f_{t+1} = r_{j,t+1}$ into (11):

$$r^f_{t+1} = -\ln(\beta) + \frac{1}{\psi}\,\mathrm{E}_t\left[g_{t+1}\right] + \frac{1-\theta}{\theta}\,\mathrm{E}_t\left[r^a_{t+1} - r^f_{t+1}\right] - \frac{1}{2\theta}\,\mathrm{var}_t\left(m_{t,t+1}\right). \quad (17)$$

- Then, we use

$$\mathrm{E}_t\left[g_{t+1}\right] = \mu + x_t, \quad (18)$$

$$\mathrm{E}_t\left[r^a_{t+1} - r^f_{t+1}\right] = -\lambda_{m,\eta}\sigma_t^2 + \lambda_{m,\epsilon}\kappa_1 A_1\varphi_\epsilon\sigma_t^2 + \kappa_1 A_2\lambda_{m,w}\sigma_w^2 - \frac{1}{2}\mathrm{var}_t(r^a_{t+1}), \quad (19)$$

$$\mathrm{var}_t(r^a_{t+1}) = \left[1 - (\kappa_1 A_1\varphi_\epsilon)^2\right]\sigma_t^2 + (A_2\kappa_1\sigma_w)^2, \quad (20)$$

$$\mathrm{var}_t\left(m_{t,t+1}\right) = \left(\lambda^2_{m,\eta} + \lambda^2_{m,\epsilon}\right)\sigma_t^2 + \lambda^2_{m,w}\sigma_w^2, \quad (21)$$

and use these results to simulate r^f_{t+1}.

Auxiliary parameters $\lambda_{m,\eta}$, $\lambda_{m,\epsilon}$, $\lambda_{m,w}$, $\beta_{m,\epsilon}$, and $\beta_{m,w}$

Parameters

We find that

$$A_0 = \frac{1}{1-\kappa_1}\left[\ln(\beta) + \kappa_0 + \left(1 - \frac{1}{\psi}\right)\mu + \kappa_1 A_2 (1-\nu_1)\bar{\sigma}^2 + \frac{\theta}{2}\left(\kappa_1 A_2 \sigma_w\right)^2\right],$$

$$A_1 = \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho}, \quad A_2 = \frac{\frac{1}{2}\left[\left(\theta - \frac{\theta}{\psi}\right)^2 + (\theta A_1 \kappa_1 \varphi_\epsilon)^2\right]}{\theta(1-\kappa_1\nu_1)},$$

and

$$A_{0,m} = \frac{\begin{pmatrix} \theta \ln(\beta) + (\theta-1)\kappa_0 + \kappa_{0,m} + (\theta-1)(\kappa_1 - 1)A_0 + \left(\theta - 1 - \frac{\theta}{\psi}\right)\mu + \mu_d \\ + \left[(\theta-1)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right](1-\nu_1)\bar{\sigma}^2 + \frac{1}{2}\left[(\theta-1)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right]^2 \sigma_w^2 \end{pmatrix}}{1-\kappa_{1,m}},$$

$$A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1-\kappa_{1,m}\rho}, \quad A_{2,m} = \frac{(1-\theta)(1-\kappa_1\nu_1)A_2 + \frac{1}{2}H}{(1-\kappa_{1,m}\nu_1)},$$

with

$$H := \lambda_{m,\eta}^2 + (\beta_{m,\epsilon} - \lambda_{m,\epsilon})^2 + \varphi_d^2,$$

$$\beta_{m,\epsilon} := \kappa_{1,m} A_{1,m} \varphi_\epsilon, \qquad \beta_{m,w} := \kappa_{1,m} A_{2,m},$$

$$\lambda_{m,\epsilon} := (1-\theta)\kappa_1 A_1 \varphi_\epsilon, \quad \lambda_{m,w} := (1-\theta) A_2 \kappa_1, \quad \lambda_{m,\eta} := -\gamma.$$

Go back to Monte Carlo experiment