Slow and Easy: a Theory of Browsing

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  - ✓ Not urgent
  - ✓ Attention may jump from item to item
  - ✓ Not known what options are available
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- ▷ This paper: information processing constraints

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- Example—new TV

TV-set	technology	sound	brand	screen
a	OLED	excellent	S	50”
b	OLED	good	P	50”
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- ▷ Main insights:
  - ✓ Consumer choice is not hard if use randomization and sacrifice the speed: logarithmic/linear complexity
  - ✓ System of attributes used to describe objects matters: languages that attain logarithmic/linear bounds in complexity
  - ✓ Simplest procedure: examine attributes sequentially, dismiss the item with positive probability if the attribute's value is bad

Model

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- Complete and transitive non-trivial preference $\succeq$ on A
- Nature chooses a non-empty menu $B \subseteq A$, unknown to the agent

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- ▷ Each time a random item is drawn according to the same distribution
  - ✓ Can encounter the same (or identical) item multiple times

Information Structures

- Language $Q = \{Q_i\}_{i \in N}$ is a collection of non-trivial binary partitions of A
  - ✓ Each partition maps to a binary property (attribute) of items
  - ✓ N—index set of partitions (attributes)
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- ▷ Example: $\{Q_1, Q_2\}$, where $Q_1 = \{\{a, b\}, \{c, d\}\}$, $Q_2 = \{\{a, c\}, \{b, d\}\}$
  - ✓ Language includes *“technology”* and *“sound”* attributes

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  - ✓ $\tau(s, v, j)$—probability to transition from s to v if attribute $\iota(s)$ has value j
- Each time a new alternative is drawn, state initializes at $s = 1$
  - ✓ Agent focuses on the current item, no recall of the past investigations
  - ✓ In the paper, we relax this assumption for part of the analysis

Formal Dynamics

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- ▷ TV-set example, language: $\{technology, sound\}$
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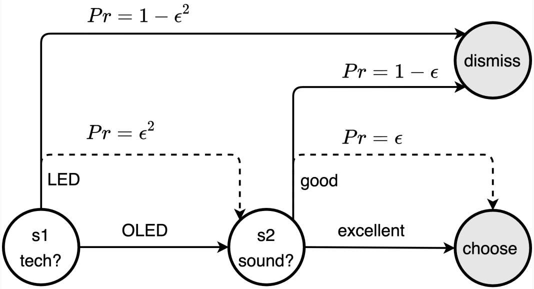

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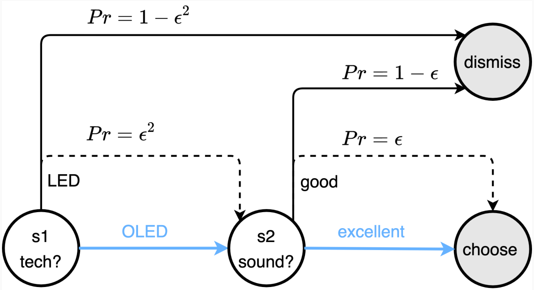

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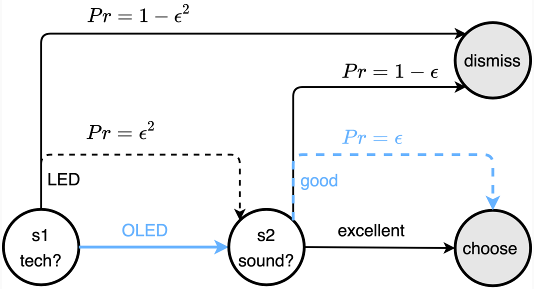

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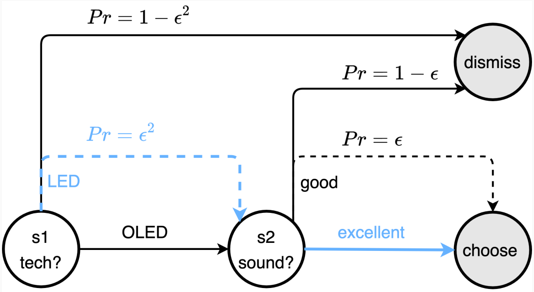

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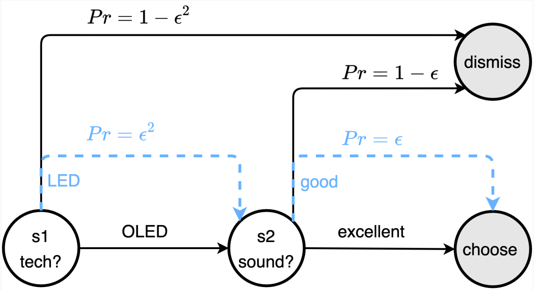

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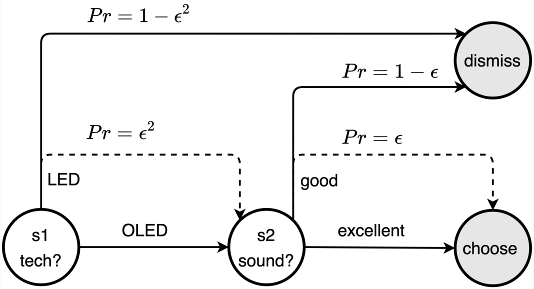

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- Imagine, the realized menu includes all but the best TV-set

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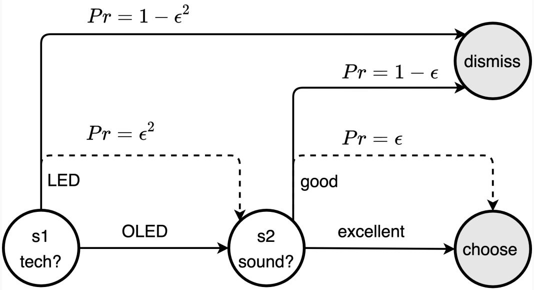

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- ▷ $\epsilon \longrightarrow 0$, optimal choice from any menu with probability 1

▸ Formal Stochastic Choice

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Definition. *A decision rule ψ solves the choice problem $(Q, \succeq)$ if*

$$\lim_{r \to \infty} \mathrm{Pr}\big(\text{choose } \succeq \text{-best item from menu } B\big) = 1 \qquad \forall B$$

Existence of a Solution

Proposition. *There exists a decision rule that solves the agent's choice problem if and only for any* $a, b \in A$*, if* $a \succ b$*, then* $a_i \neq b_i$ *for some* $i \in N$*.*

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- We consider languages that allow the agent to solve her choice problem
- Given the agent's language, what is the minimum amount of cognitive resources required to solve the choice problem?

Complexity Measures

In the paper:

- ▷ Memory load of a decision rule: $\mathcal{M}(\psi) = |S^o|$
  - ✓ Represents an "operational" memory required to implement the procedure
- ▷ Memory load of a language (given $\succeq$):

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  - ✓ Experimentally in Oprea (2020)
- Complexity (transitional) of a language (given $\succeq$):

$$\kappa_{\succeq}(Q) := \min_{\psi \text{ solves } (Q,\succeq)} \kappa(\psi)$$

Complexity of Languages for 4 Items and Strict Preference

Consider $A = \{a, b, c, d\}$, and $a \succ b \succ c \succ d$

	Language	Preference	$\mathcal{M}$	κ
Q	$\{\{a, b\}, \{c, d\}\},\ \{\{a, c\}, \{b, d\}\}$	$11 \succ 10 \succ 01 \succ 00$	2	6
R	$\{\{a, b\}, \{c, d\}\},\ \{\{a, d\}, \{b, c\}\}$	$11 \succ 10 \succ 00 \succ 01$	2	7
S	$\{\{a, c\}, \{b, d\}\},\ \{\{a, d\}, \{b, c\}\}$	$11 \succ 00 \succ 10 \succ 01$	3	9
T	$\{\{a\}, \{b, c, d\}\},\ \{\{b\}, \{a, c, d\}\},\ \{\{c\}, \{a, b, d\}\}$	$100 \succ 010 \succ 001 \succ 000$	3	9

Some details

Maximum Complexity

Theorem (Upper Bound). *If there are $k = |A|$ items, then for any $\succeq$:*

(i) For any language Q, $\kappa_{\succeq}(Q) \leq 3k - 3$;

(ii) There exists a language Q such that $\kappa_{\succeq}(Q) \geq k - 2$.

▸ Proof Idea for (i)

Minimum Complexity

Theorem (Lower Bound). *Let $\succeq$ have m indifference classes, then:*

(i) For any language Q, $\kappa_{\succeq}(Q) \geq 3\lceil \log_2 m \rceil$;

(ii) There exists a language Q such that $\kappa_{\succeq}(Q) = 3\lceil \log_2 m \rceil$;

(iii) If ψ *solves* $(Q, \succeq)$, *and* $\kappa(\psi) = 3\lceil \log_2 m \rceil$, *then* $\mathcal{M}(\psi)$ *is minimum among the rules that solve the choice problem* $(\widetilde{Q}, \succeq)$ *for any language* $\widetilde{Q}$,

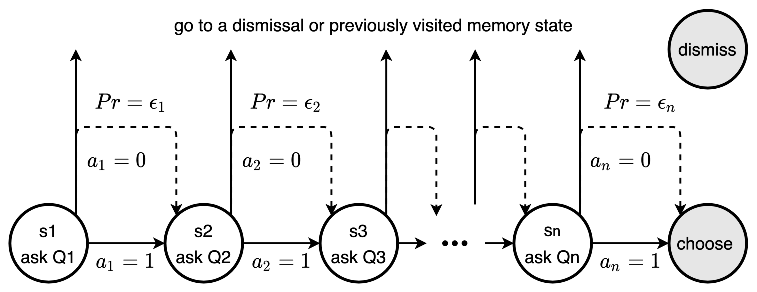
where $\lceil x \rceil$ *denotes the smallest natural number weakly greater than x.*

▸ Proof Idea for (i)

Simplest Languages and Decision Rules

Separable Decision Rules

- Consider the following decision rules with $\{\epsilon_i\}_{r=1,2,...} \in (0,1)$



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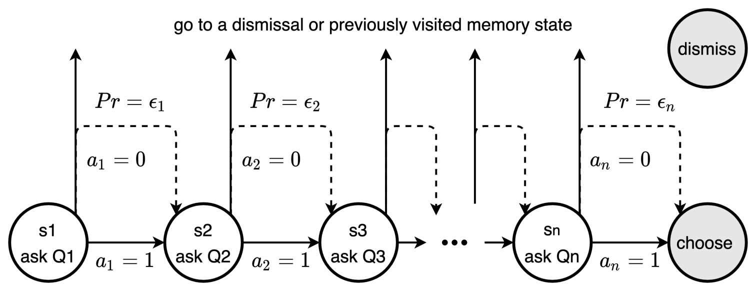

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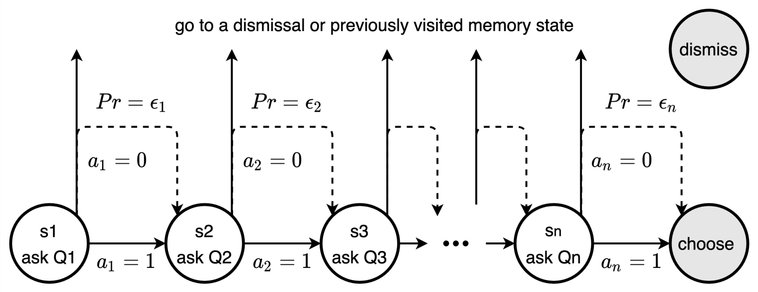

- Can enumerate attributes and attributes' values arbitrarily
- Call Ψ_n^+ the set of such rules with n memory states

Additive Utility

- Suppose the agent's language facilitates usage of an additive utility:

$$a \succ b \quad \Longrightarrow \quad \sum_{i \in N} \lambda_i a_i > \sum_{i \in N} \lambda_i b_i, \qquad (WLOG)\ \lambda_i \geq 0$$

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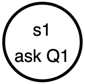

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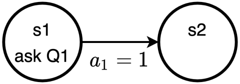

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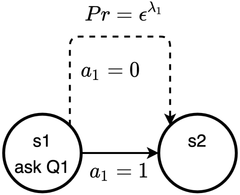

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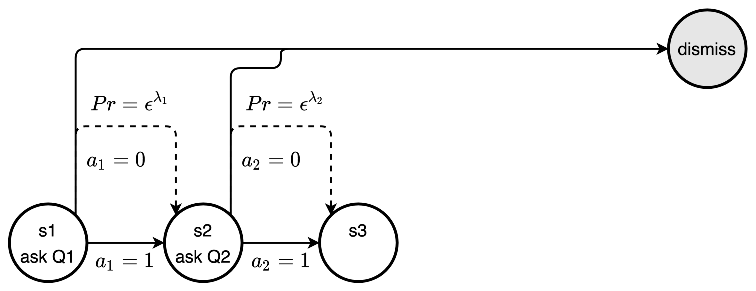

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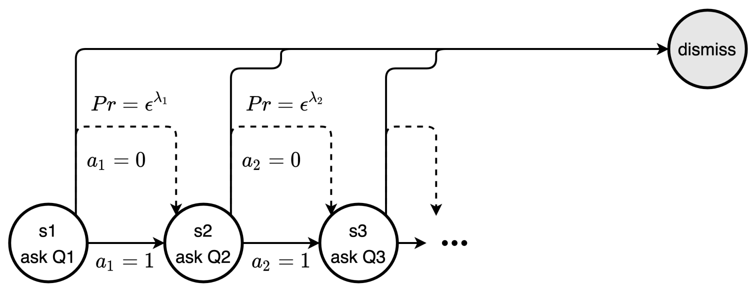
$$a \succ b \implies \sum_{i \in N} \lambda_i a_i > \sum_{i \in N} \lambda_i b_i, \qquad (WLOG)\ \lambda_i \geq 0$$



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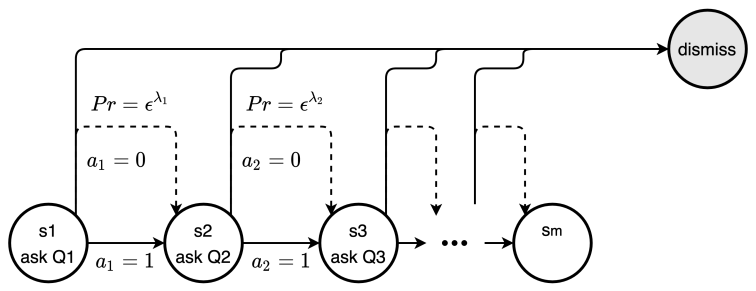

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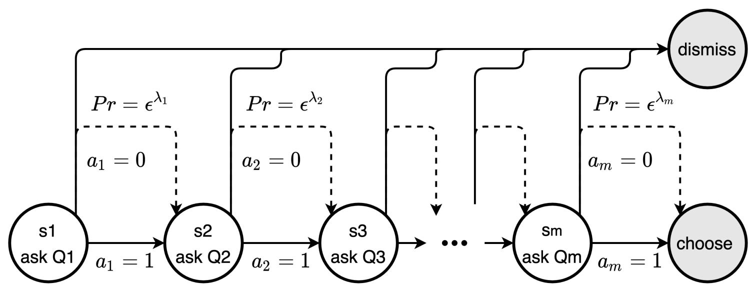

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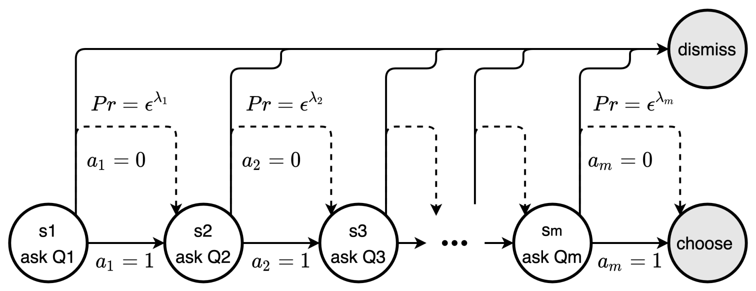

- Pr(choose item *a* during single investigation)$= (1-\eta)^{m-1} \cdot \epsilon^{\sum \lambda_i (1-a_i)}$

Adapted Languages

Definition. *Let* $\succeq$ *have* m *indifference classes. Language* Q *is adapted for* $\succeq$ *if there exists* $\lambda \in \mathbb{R}^N$ *such that:*

$$\begin{aligned} (i) \quad & a \succ b \quad \Longrightarrow \quad \sum_{i \in N} \lambda_i a_i > \sum_{i \in N} \lambda_i b_i \\ (ii) \quad & \big|\{i \in N | \lambda_i \neq 0\}\big| \; = \; \lceil \log_2 m \rceil \end{aligned}$$

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Proposition. *There exists an adapted language.*

▸ Proof

Remark. The utility function $u(a) = \sum_{i \in N} \lambda_i a_i$ induces a preference that might break ties in the original preference $\succeq$.

Proposition. *Let* $\succeq$ *have* m *indifference classes, then* Q *is adapted for* $\succeq$ *if and only if there exists* $\psi \in \Psi^{+}_{\lceil \log_2 m \rceil}$ *that solves* $(Q, \succeq)$.

Theorem (Simplest Languages). *Let $\succeq$ have m indifference classes, then:*

(i) If Q is adapted for $\succeq$, then $\kappa_{\succeq}(Q) = 3\lceil \log_2 m \rceil$;

(ii) If $(3/4) \cdot 2^n < m \leq 2^n$ for a natural n, then:

(a) $\kappa_{\succeq}(Q) = 3\lceil \log_2 m \rceil$ if and only if Q is adapted for $\succeq$;

(b) If ψ solves $(Q, \succeq)$, and $\kappa(\psi) = 3\lceil \log_2 m \rceil$, then $\psi \in \Psi^{+}_{\lceil \log_2 m \rceil}$.

▸ Proof Sketch

Literature Review and Conclusion

Literature Review

- **Optimal search:** Kohn and Shavell (1974); Weitzman (1979); Morgan and Manning (1985); Klabjan, Olszewski, and Wolinsky (2014); Sanjurjo (2017)
- **Memory-constrained search:** Dow (1991); Sanjurjo (2015), (2019)
- **Stochastic Browsing:** Cerreia-Vioglio, Maccheroni, Marinacci, Rustichini (2020), Rustichini (2020)
- **Hypothesis testing and learning with finite memory:** Cover (1969); Cover and Hellman (1970); Hellman and Cover (1970), (1971)
- **Automata and simple algorithms in Economics:** Abreu and Rubinstein (1988); Kalai and Stanford (1988); Banks and Sundaram (1990); Kalai and Solan (2003); Börgers and Morales (2004); Kocer (2010); Salant (2011); Mandler, Manzini, Mariotti (2012); Wilson (2014); Oprea (2020)

Conclusion

- Simple stochastic strategies achieve near optimality when time is not of the essence
- Descriptions that facilitate additive utility with few attributes are key for simplicity
- In the simplest procedures, "higher" memory state indicate higher quality of the item relative to the menu

Supplementary Slides

Maximum and Minimum Memory Load

Theorem (Upper Bound). *If there are $k = |A|$ items, then for any $\succeq$:*

(i) For any language Q, $\kappa_{\succeq}(Q) \leq k - 1$;

(ii) There exists a language Q such that $\kappa_{\succeq}(Q) \geq k/2 - 1$.

Theorem (Lower Bound). *Let $\succeq$ have m indifference classes, then:*

(i) For any language Q, $\kappa_{\succeq}(Q) \geq \lceil \log_2 m \rceil$;

(ii) There exists a language Q such that $\kappa_{\succeq}(Q) = \lceil \log_2 m \rceil$;

where $\lceil x \rceil$ denotes the smallest natural number weakly greater than x.

Extension: Relaxing Memory Initialization Assumption

A General Framework

- ▷ Baseline model: a state initializes at $s = 1$ with each new item

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A General Framework

- ▷ Baseline model: a state initializes at $s = 1$ with each new item
- ▷ General model: when a new item is drawn, the automaton transitions to a new state conditional on the previous state
- ▷ State space $S = S^o \cup \{choose\}$
- ▷ Specify probabilities:
  - ✓ To choose the current item, conditional on the current state and the learned attribute's value
  - ✓ To continue the investigation of the item and move to a memory state, conditional on the current state and the learned attribute's value
  - ✓ To dismiss the item, pick a new random item, and move to a memory state, conditional on the current state and the learned attribute's value
  - ✓ To move to a memory state, conditional on the current state and the event that a new item catches the agent's attention

Maximal Memory Load

Theorem (Upper Bound). *Consider a general model. Let k be the total number of items, then for any non-trivial $\succeq$:*

(i) For any language Q, $\mathcal{M}(Q) \leq k - 1$;

(ii) There exists a language Q such that $\mathcal{M}(Q) = k/2 - 1$.

Minimal Memory Load

Theorem (Lower Bound). *Consider a general model. Let $m \geq 2$ be the total number of indifference classes of $\succeq$, then:*

(i) For any language Q, $\mathcal{M}(Q) \geq \lceil \log_2 m \rceil$;

(ii) There exists a language Q such that $\mathcal{M}(Q) = \lceil \log_2 m \rceil$.

If Preference is Strict, a Language May Require $k-1$ Memory States

- Let $A = \{a^1, ..., a^k\}$, $a^1 \succ ... \succ a^k$
- Consider $Q = \{Q_1, ..., Q_{k-1}\}$ with $Q_l = \{\{a^l\}, \{a^1, ..., a^{l-1}, a^{l+1}, ..., a^k\}\}$
- Need at least $k-1$ attributes to differentiate any pair of items

Proof Ideas

Lower Bound in Transitional Complexity—Simple Paths

- Focus on simple paths from $s = 1$ to $s = choose$
- Item-dependent probability that the path occurs
- For $a \in A$, $\omega(a)$— the highest probability among all simple paths

Lemma. *A decision rule solves the choice problem if and only if:*

(i) $a \succ b$ *implies* $\omega(b)/\omega(a) \longrightarrow 0$ *for all* $a, b \in A$;

(ii) $\omega(a) > 0$ *for all* $a \in A$.

- Similar to "Z-tree" technique in Kandori, Mailath, Rob (1993)

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Strong and Weak Transitions

- Strong link $(s, v, j) \in \mathcal{T}$ if $\lim \tau(s, v, j) > 0$
- Weak link $(s, v, j) \in \mathcal{T}$ if $\lim \tau(s, v, j) = 0$

Lemma. *If the decision rule solves the choice problem, then highest-probability paths for different alternatives use different sets of weak links.*

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Lower Bound in Transitional Complexity—Proof Idea

- Let ψ solves $(Q, \succeq)$ with k items, $n = \lceil \log_2 k \rceil$
- ψ should have at least $2n$ strong links
  - ✓ At least n attributes should be examined in n states
  - ✓ Each state has at least 2 outgoing strong links
- ψ should have at least n weak links
  - ✓ Each item maps to a distinct set of weak links
  - ✓ Hence $2^{\#\text{weak links}} \geq k$
- The total number of links in ψ is at least $2n + n$, i.e. $\kappa(Q) \geq 3n$
- If $\kappa(\psi) = 3n$, there are exactly $2n$ strong and n weak links

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Memory Load—a Rough Complexity Measure

	Language	Preference	Memory load
Q	$\{\{a,b\},\{c,d\}\},\ \{\{a,c\},\{b,d\}\}$	$11 \succ 10 \succ 01 \succ 00$	$\mathcal{M}(Q) = 2$
Q^*	$\{\{a,b\},\{c,d\}\},\ \{\{a,d\},\{b,c\}\}$	$11 \succ 10 \succ 00 \succ 01$	$\mathcal{M}(Q^*) = 2$
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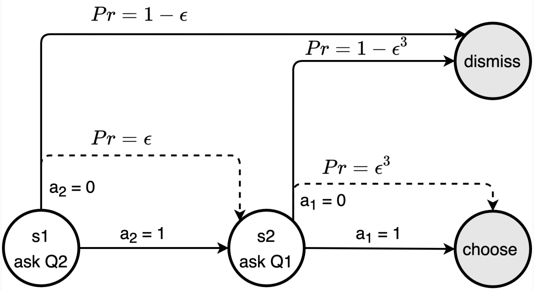

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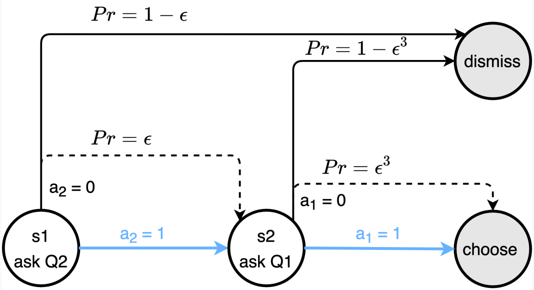

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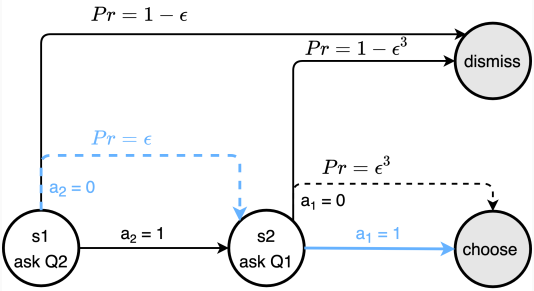

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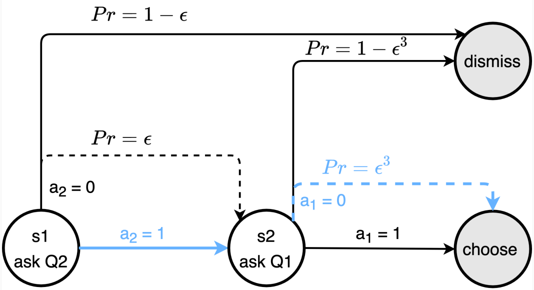

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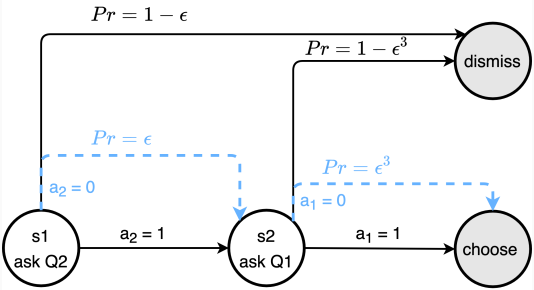

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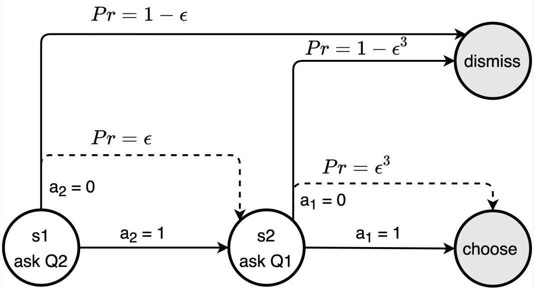

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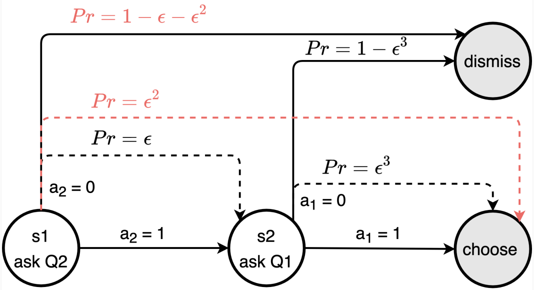

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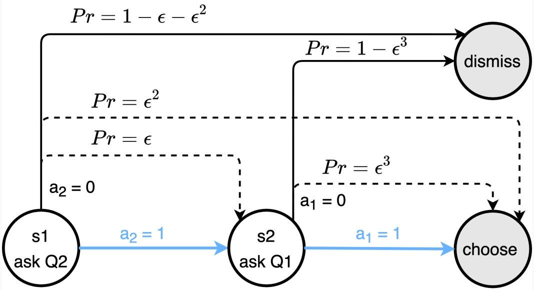

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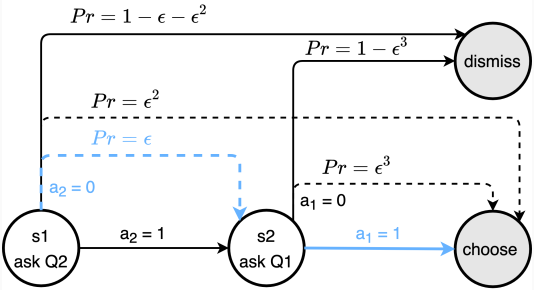

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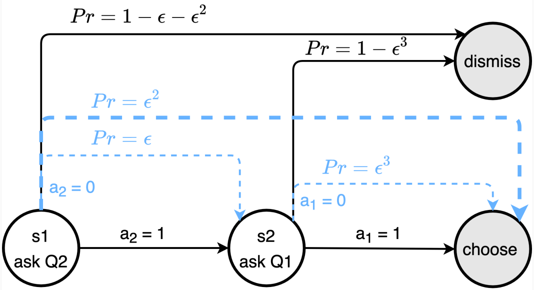

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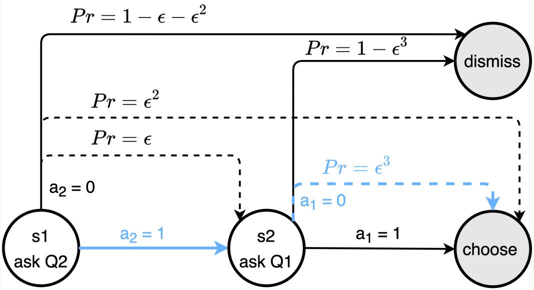

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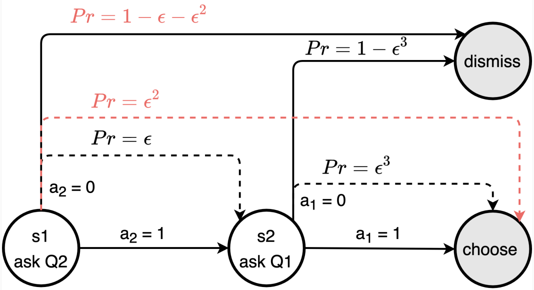

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Dynamics (Baseline Model)

- Markov Chain $\boldsymbol{Y} = (Y_1, Y_2, ...)$ with realizations $(y_1, y_2, ...)$
- Interpretation: $y_t = (a, s) \in A \times \left(S^o \cup \{choose\}\right)$
- Starting state: $Pr\left(Y_1 = (a, s)\right) = \rho^B(a) \cdot \delta_1^s$
- Transitional probabilities

$$
\begin{aligned}
Pr\Big(Y_t = (a, s) \;\Big|\; Y_{t-1} = (b, v)\Big) \quad &= \quad (1-\eta) \cdot \delta_b^a \cdot \tau\big(v, s, b_{\iota(v)}\big) + \\
&\quad\;\; (1-\eta) \cdot \tau\big(v, dismiss, b_{\iota(v)}\big) \cdot \rho^B(a) \cdot \delta_1^s + \\
&\quad\;\; \big[1 - \tau\big(v, choose, b_{\iota(v)}\big)\big] \cdot \eta \cdot \rho^B(a) \cdot \delta_1^s \\
Pr\Big(Y_t = (a, choose) \;\Big|\; Y_{t-1} = (b, v)\Big) \quad &= \quad \tau\big(v, choose, b_{\iota(v)}\big) \cdot \delta_b^a \\
Pr\Big(Y_t = (a, s) \;\Big|\; Y_{t-1} = (b, choose)\Big) \quad &= \quad \delta_b^a \cdot \delta_{choose}^s
\end{aligned}
$$

- Where $\rho^B(a)$ is the probability to draw item a from menu B

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Stochastic Choice

- $\rho^B(b)$—probability to draw item b from menu B
- $q(b)$—probability to choose item b during a single investigation
- $p^B(b)$—probability to choose item b from menu B

Lemma (Generalized Luce Rule).

$$p^B(a) \;=\; \frac{\rho^B(a)\cdot q(a)}{\sum_{b\in B}\rho^B(b)\cdot q(b)}$$

with the convention that $p^B(a)=0$ *if the denominator assumes value zero.*

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Intuition for the Upper Bound

- Design an automaton that maps each item $a \in A$ to a unique probability ϵ_a of choosing this item during a single investigation

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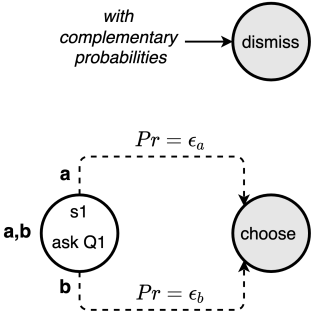
- $f(2) = 1$



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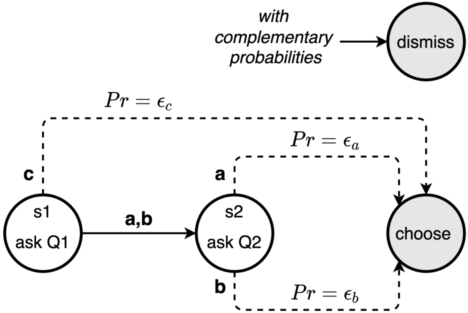
- $f(k+1) = 1 + f(k) = 1 + k - 1 = k$



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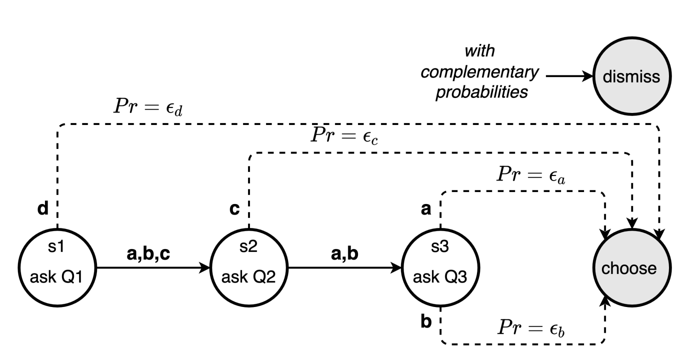
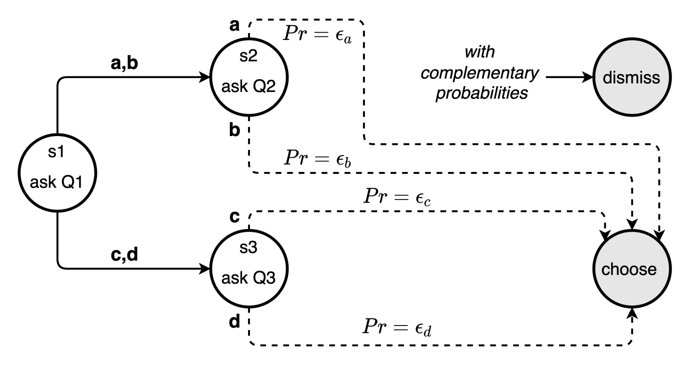
- Show by induction that $f(k) = k - 1$ states are sufficient
- $f(k+1) = 1 + f(k) = 1 + k - 1 = k$



Intuition for the Upper Bound

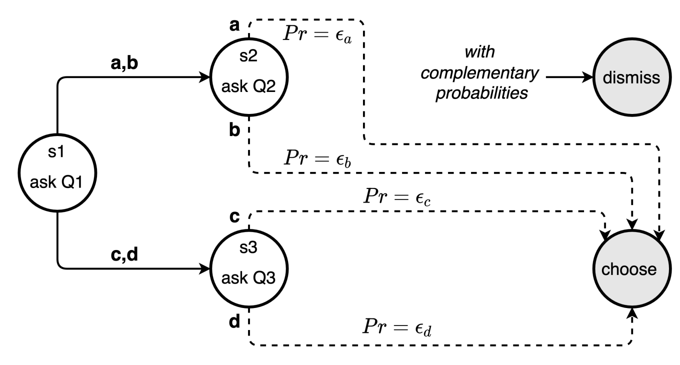
- Design an automaton that maps each item $a \in A$ to a unique probability ϵ_a of choosing this item during a single investigation
- Show by induction that $f(k) = k - 1$ states are sufficient
- $f(k_1 + k_2) = 1 + f(k_1) + f(k_2) = k_1 + k_2 - 1$



Back

Intuition for the Upper Bound

- Design an automaton that maps each item $a \in A$ to a unique probability ϵ_a of choosing this item during a single investigation
- Show by induction that $f(k) = k - 1$ states are sufficient
- Pick sequences $\{\epsilon_a\}_{r=1,2,..}$ for $a \in A$ that solve the choice problem



Existence of Adapted Languages

- WLOG, $\succeq$ is strict:
- Adapted language for k items:

$$(i) \quad a \succ b \implies \sum_{i \in N} \lambda_i a_i > \sum_{i \in N} \lambda_i b_i$$

$$(ii) \quad \left|\{i \in N | \lambda_i \neq 0\}\right| = \lceil \log_2 k \rceil$$

Proof 1:

- Augment the set of items to make $|A| = 2^n$, where $n = \lceil \log_2 k \rceil$
- Consider some collection $\lambda_i > 0$, $i \in \{1, ..., n\}$
- Utility $u(a) = \sum_i \lambda_i a_i$ induces a (strict) preference on vectors of attributes
- Label items in set A accordingly, get an adapted language

Back

Existence of Adapted Languages

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$$(ii) \quad \left|\{i \in N | \lambda_i \neq 0\}\right| = \lceil \log_2 k \rceil$$

Proof 2:

- Example: consider $a \succ b \succ c \succ d \succ e \succ f \succ g \succ h$
- Language $Q = \{Q_1, Q_2, Q_3\}$
  - ✓ Q_1 : $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, e, f, g, h$
  - ✓ Q_2 : $\boldsymbol{a}, \boldsymbol{b}, c, d, \boldsymbol{e}, \boldsymbol{f}, g, h$
  - ✓ Q_3 : $\boldsymbol{a}, b, \boldsymbol{c}, d, \boldsymbol{e}, f, \boldsymbol{g}, h$
- Linear utility: $u(x) = 2^2 \cdot x_1 + 2^1 \cdot x_2 + 2^0 \cdot x_3 = 4x_1 + 2x_2 + x_3$

Back

Lower Bound Characterization Theorem-proof idea for (ii.a)

Theorem (Simplest Languages). *Let $\succeq$ have m indifference classes, then:*

(i) If Q is adapted for $\succeq$, then $\kappa_{\succeq}(Q) = 3\lceil \log_2 m \rceil$;

(ii) If $(3/4) \cdot 2^n < m \leq 2^n$ for a natural n, then:

(a) $\kappa_{\succeq}(Q) = 3\lceil \log_2 m \rceil$ if and only if Q is adapted for $\succeq$;

(b) If ψ solves $(Q, \succeq)$, and $\kappa(\psi) = 3\lceil \log_2 m \rceil$, then $\psi \in \Psi^+_{\lceil \log_2 m \rceil}$.

Recall **Proposition:** *Let $\succeq$ have m indifference classes, then Q is adapted for $\succeq$ if and only if there exists $\psi \in \Psi^+_{\lceil \log_2 m \rceil}$ that solves $(Q, \succeq)$.*

Want to prove that when $(3/4) \cdot 2^n < k \leq 2^n$, if ψ solves the choice problem and $\kappa(\psi) \leq 3\lceil \log_2 m \rceil$, then $\psi \in \Psi^+_{3\lceil \log_2 m \rceil}$

Back to the Theorem

Lower Bound Characterization Theorem: Proof Sketch (1)

- ▷ For each item a, consider a highest-probability path from $s = 1$ to $s = choose$
- ▷ Say that $(s, v, j) \in \mathcal{T}$ is a weak link, if $\lim \tau_r(s, v, j) \longrightarrow 0$, otherwise it is a strong link

Lemma. *If the decision rule solves the choice problem, then highest-probability paths for different alternatives use different sets of weak links.*

Lemma. *If ψ solves choice problem with m items, and $\kappa(\psi) = 3\lceil \log_2 k \rceil$, then ψ has n states, $2n$ strong, and n weak links, where $n = \lceil \log_2 k \rceil$.*

Back to the Theorem

Characterization Theorem: Sketch of the Proof (2)

▷ A simple path contains at most 1 link outgoing from a given state

Lemma. *Let the total number of items be k, $n = \lceil \log_2 k \rceil$, and $k > (3/4) \cdot 2^n$. If ψ solves the choice problem and $\kappa(\psi) = 3n$, then for each pair of weak links there is a highest-probability path that use both these links.*

Corollary. *Let the total number of items be k, $n = \lceil \log_2 k \rceil$, and $k > (3/4) \cdot 2^n$. If ψ solves the choice problem and $\kappa(\psi) = 3n$, then in every state, ψ has exactly one outgoing weak link and exactly two outgoing strong links.*

▸ Back to the Theorem

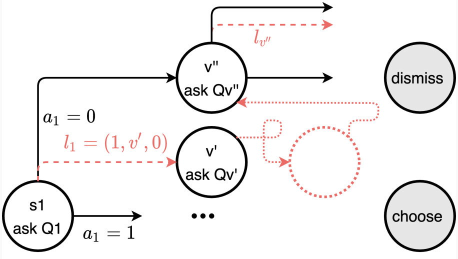
Characterization Theorem: Sketch of the Proof (3)

- ▷ WLOG attribute $s \in \{1, ..., n\}$ is investigated in state s.
- ▷ WLOG, for each state s:
  - ✓ $\tau(s, v, 1) = 1$ for some v
  - ✓ $\tau(s, v', 0) = \epsilon_s$ and $\tau(s, v'', 0) = 1 - \epsilon_s$ for some v', v'', and $\epsilon_s \longrightarrow 0$
- ▷ Recall: to show that $\psi \in \Psi_n^+$, we need to show additionally that there is a labeling of the states such that in the formula above:
  - ✓ $v = v' = s + 1$, where state $n + 1$ denotes *choose*
  - ✓ $v'' \in \{1, .., s\} \cup \{dismiss\}$
- ▷ Idea: use induction in n, where $n = \lceil \log_2 k \rceil$, k is the number of items, and condition $k > (3/4) \cdot 2^n$ holds
  - ✓ Induction base: $n = 1$, straightforward
  - ✓ Induction step?

▸ Back to the Theorem

Characterization Theorem: Sketch of the Proof (4)

- Consider $s = 1$, have $\tau(1, v, 1) = 1$, $\tau(1, v', 0) = \epsilon_1$, $\tau(1, v'', 0) = 1 - \epsilon_1$
- $v \notin \{1, \mathit{choose}, \mathit{dismiss}\}$, since more than 1 item has $a_1 = 1$
- $v' \notin \{1, \mathit{choose}, \mathit{dismiss}\}$, $v'' \neq \mathit{choose}$; otherwise, no more than $2^{n-1} + 1 \leq (3/4) \cdot 2^n$ different subsets of weak links used
- Towards a contradiction, assume $v'' \notin \{1, \mathit{dismiss}\}$



- Highest-probability path cannot include both weak links l_1 and $l_{v''}$, in contradiction

Characterization Theorem: Sketch of the Proof (5)

- ▷ We know: $\tau(1, v, 1) = 1$, $\tau(1, v', 0) = \epsilon_1$, $\tau(1, v'', 0) = 1 - \epsilon_1$
  - ✓ $v, v' \notin \{1, choose, dismiss\}$, $v'' \in \{1, dismiss\}$
- ▷ At least one of the two statements should hold:
  - ✓ $\left|\{a \in A | a_i = 1\}\right| > (3/4) \cdot 2^{n-1}$
  - ✓ $\left|\{a \in A | a_i = 0\}\right| > (3/4) \cdot 2^{n-1}$
- ▷ Let $\left|\{a \in A | a_i = 1\}\right| > (3/4) \cdot 2^{n-1}$, consider rule ψ':
  - ✓ Delete state $s = 1$ in rule ψ and its outgoing links
  - ✓ Redirect each link that ends at $s = 1$ in ψ to $s = dismiss$ in ψ'
  - ✓ Make state v the first state in ψ'
- ▷ ψ' solves the problem constrained to items $\{a \in A | a_i = 1\}$
  - ✓ $\kappa(\psi') \leq 3n - 3$
  - ✓ Use induction assumption to find configuration of links outgoing from all other states except of $s = 1$ in ψ

▸ Back to the Theorem

Characterization Theorem: Sketch of the Proof (6)

- Last statement to prove: that $v' = v$.

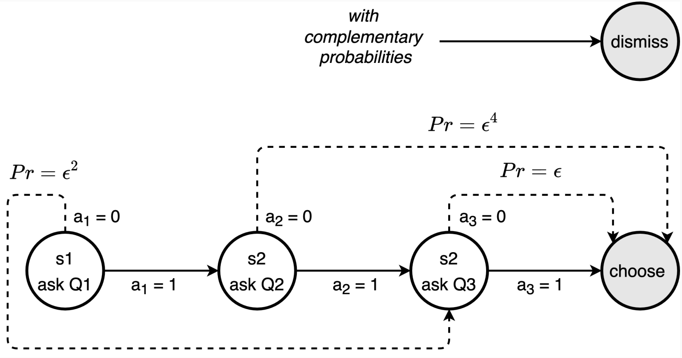
- Assume $v' \neq v$, then weak link $(1, v', 0)$ and weak link l_v, outgoing from state v, cannot be in the same highest-probability path, contradiction

- Similar arguments work if $\left|\{a \in A | a_i = 0\}\right| > (3/4) \cdot 2^{n-1}$
  - ✓ Note that $\left|\{a \in A | a_i = 0\}\right| \leq (1/2) \cdot 2^n$
  - ✓ Hence $\left|\{a \in A | a_i = 0\}\right| > (1/4) \cdot 2^n$
  - ✓ If $v \neq v'$, a weak link outgoing from v' is not used in any highest-probability paths for items with $a_1 = 1$
  - ✓ Thus, no more than $(1/4) \cdot 2^n$ sets of weak links used in highest-probability paths for items with $a_1 = 1$, contradiction

Back to the Theorem

(Counter) Example

- ▷ $k = 5$, so $n = \lceil \log_2 5 \rceil = 3$, $k = 5 \leq (3/4) \cdot 2^3 = 6$
- ▷ 111 $\succ$ 110 $\succ$ 011 $\succ$ 000 $\succ$ 100



(Counter) Example 2

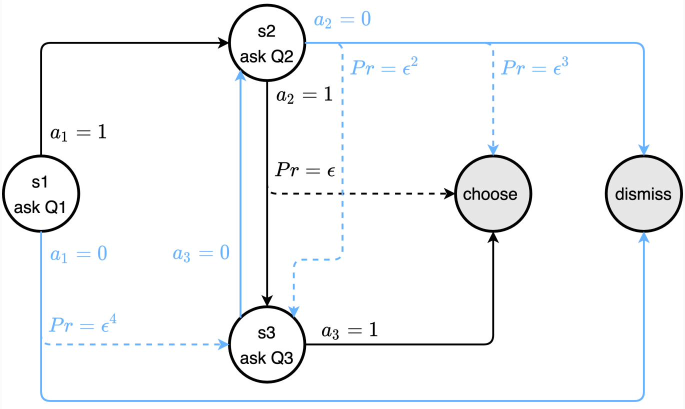
- $111 \succ 110 \succ 101 \succ 100 \succ 001 \succ 010 \succ 000$

(Counter) Example 2

- $111 \succ 110 \succ 101 \succ 100 \succ 001 \succ 010 \succ 000$

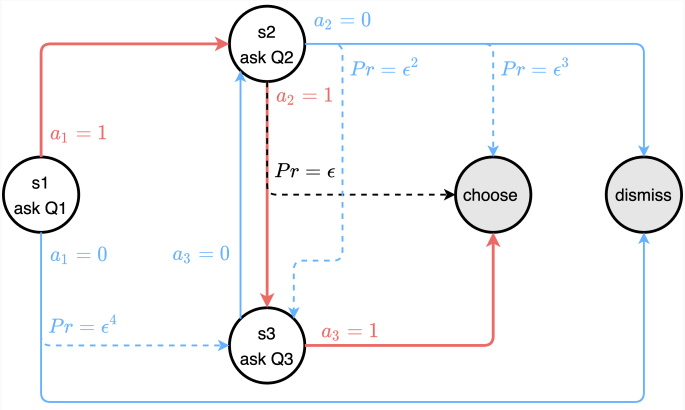
(Counter) Example 2

- $111 \succ 110 \succ 101 \succ 100 \succ 001 \succ 010 \succ 000$



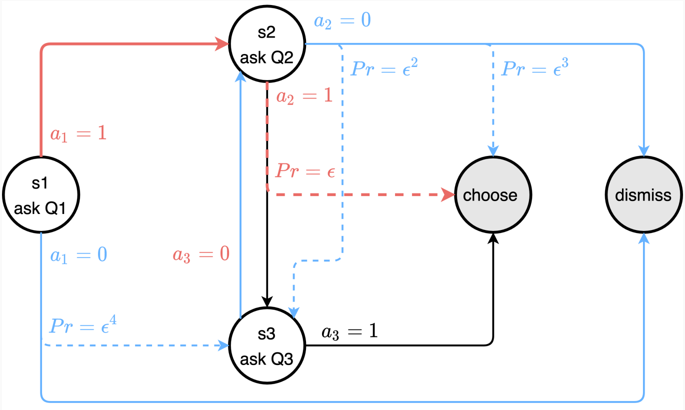
(Counter) Example 2

▷ $\color{red}{111} \succ 110 \succ 101 \succ 100 \succ 001 \succ 010 \succ 000$



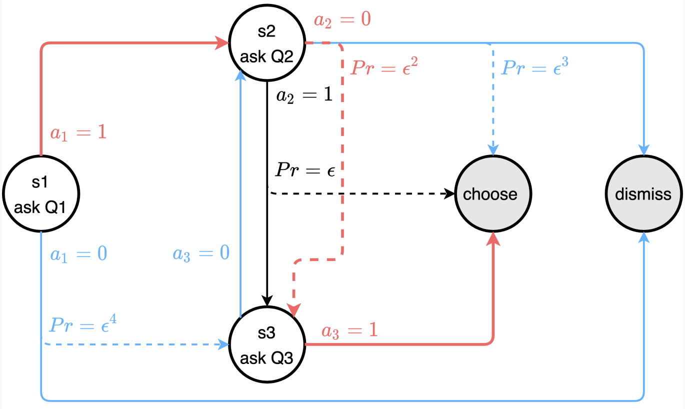
(Counter) Example 2

- $111 \succ 110 \succ 101 \succ 100 \succ 001 \succ 010 \succ 000$



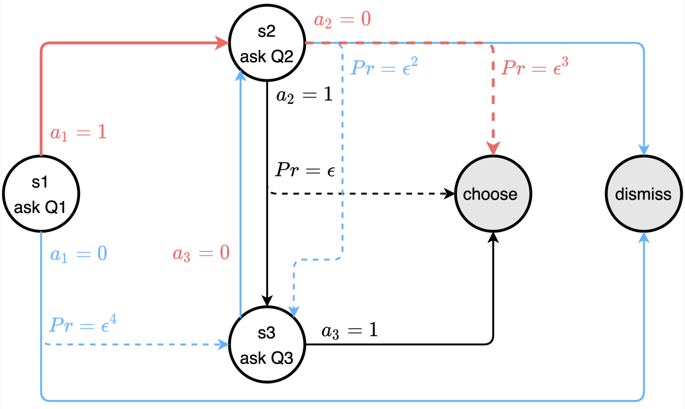
(Counter) Example 2

- $111 \succ 110 \succ 101 \succ 100 \succ 001 \succ 010 \succ 000$



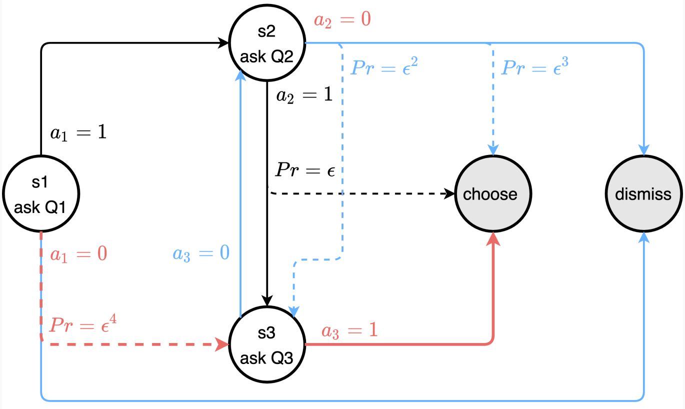
(Counter) Example 2

- ▷ 111 $\succ$ 110 $\succ$ 101 $\succ$ 100 $\succ$ 001 $\succ$ 010 $\succ$ 000



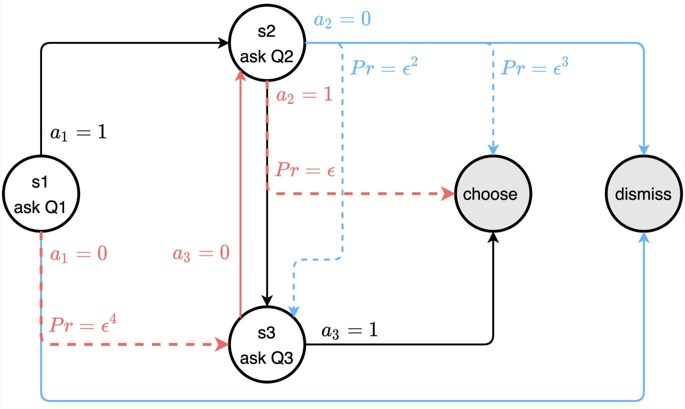
(Counter) Example 2

▷ 111 $\succ$ 110 $\succ$ 101 $\succ$ 100 $\succ$ 001 $\succ$ 010 $\succ$ 000



(Counter) Example 2

- $\triangleright$ $111 \succ 110 \succ 101 \succ 100 \succ 001 \succ$ **010** $\succ 000$



(Counter) Example 2

▷ $111 \succ 110 \succ 101 \succ 100 \succ 001 \succ 010 \succ 000$