Sentiment, Mispricing and Excess Volatility in Presence of Institutional Investors*

Hervé Roche†
Universidad de Chile
Facultad de Economía y Negocios

Juan Sotes-Paladino‡
Universidad de los Andes, Chile
School of Business and Economics

August 2023

ABSTRACT

We study the equilibrium implications on asset prices of sentiment-driven investors' trading with benchmark-concerned institutions. In the model, both the (irrational) optimism of retail investors and the benchmarking concerns of institutions boost the aggregate demand for a stock. We show that the ensuing demand pressure has a depressing effect on the stock market price of risk but an ambiguous effect on volatility. The latter results from the combined effect of a benchmarking and a relative-wealth channels on the transmission of fundamental news to prices. In stark contrast with a well-known prediction of models with no institutions, the relative-wealth channel can induce a negative and asymmetric relation between investor sentiment and the stock return's excess volatility. It further creates novel countercyclical patterns in stock volatility that cannot be explained in the absence of investor sentiment. Our results have a number of implications for the interpretation of the empirically documented dynamics of mispricing and excess return volatility of financial assets.

Keywords: Sentiment, Excess Volatility, Mispricing, Benchmarking, Institutional Investors.

JEL Classification: G11, G12, G18, G41.

*We thank Antonio Gargano, Luis Goncalves-Pinto, Jan Hanousek (discussant), and Idan Hodor for helpful comments and suggestions. The paper has also benefited from comments made at presentations at the 2023 FMA European Conference and at the University of Chile (FEN). All errors are our own.

†E-mail: herve.roche@uai.cl.

‡Corresponding author. Monseñor Álvaro del Portillo 12455, Las Condes, RM, Chile. E-mail:jsotes@uandes.cl.

1 Introduction

A vast body of literature documents how different psychological biases and cognitive limits shape the investment behavior of individuals, introducing an element of irrationality, or "sentiment," in financial markets.[1] When these markets are further subject to frictions that limit the activity of the more rational investors, sentiment can cause systematic deviations of prices from fundamentals and excess return volatility.[2] In this sense, the observed tendency in recent years toward greater delegation of investment decisions to financial institutions ("institutionalization") might suggest that the effect of sentiment on mispricing should vanish as the tendency consolidates. The reason is that institutional investors are commonly viewed as sophisticated and less prone to committing systematic mistakes ("smart money").

However, recent theoretical and empirical findings depict a more nuanced landscape. First, Basak and Pavlova (2013) show that institutions' performance concerns relative to benchmark indexes ("benchmarking concerns") can result in upward pressure on the stock index and amplify the volatility of index stocks and the aggregate stock market. Second, DeVault et al. (2019) find that at least part of the demand shocks captured by sentiment metrics are not necessarily due to irrational beliefs, but rather reflect rational (e.g., risk management) decisions of institutions in response to their investment styles. Given these findings, can we expect institutional investors to correct or, on the contrary, worsen the financial distortions caused by sentiment?

In this paper, we tackle this question within a dynamic general equilibrium model of asset prices determination. In the model, risk-averse investors trade continuously and frictionlessly in a risky asset (a "stock") and a riskless asset ("cash") over a finite investment period. Investors belong to either of two classes: "retail" or "institutional." Retail investors have standard preferences, and can feature dogmatic optimistic or pessimistic beliefs about the stock's dividend growth rate. Thus, they can be subject to the type of "bullish" or "bearish" sentiment that has been typically associated with retail trading in empirical studies.[3] Institutional investors have otherwise identical preferences to retail investors except that their marginal utility of wealth is increasing in the level of a benchmark index. This follows Basak and Pavlova (2013)'s reduced-form approach to capturing the fact that, as agents for their delegating investors, institutional investors are typically evaluated

[1] For surveys of this literature see, e.g., Barberis and Thaler (2003); Hirshleifer (2015).

[2] See, e.g., Gromb and Vayanos (2010).

[3] See, among others, Kumar and Lee (2006), Barber and Odean (2008), Greenwood and Nagel (2009), Barber et al. (2009), Da et al. (2015).

(and compensated) in terms of both absolute and relative performance with respect to a benchmark portfolio. Unlike retail investors, they are rational in the sense of having the correct belief about the dividend growth rate.

We solve for the equilibrium in this economy and provide an explicit characterization of asset prices and portfolio allocations. Based on this characterization, we first compare the separate implications of sentiment versus benchmarking concerns on the equilibrium levels and dynamics of prices. Besides providing relevant reference cases against which to compare their joint effect, this analysis can inform the extent to which the two features can be observationally equivalent. Because both introduce a wedge in the demand for the stock relative to an otherwise identical rational non-institutional investor, bullish sentiment and benchmarking concerns exert a similar upward pressure on the stock price, with the effect increasing in the severity of the feature. The similarity is such that, for any given intensity of benchmarking concerns, the "index effect" that these concerns induce on the stock's price-dividend ratio at a point in time is equivalent to the effect of a given ("threshold") level of bullish sentiment across all aggregate wealth distributions.

When considered in isolation, both benchmarking concerns and bullish sentiment exacerbate stock return volatility, although to different extents. The reason is that both features lead to portfolio heterogeneity across investors, amplifying the effect of fundamental shocks on stock returns via a relative-wealth channel. According to this channel, positive (respectively, negative) shocks to fundamentals transmit to prices not only via higher (lower) expectations of future payoffs but also via a greater (lower) demand pressure, as a result of a wealth effect, from the now relatively wealthier (poorer) traders whose portfolio is overexposed to the shock—i.e., the institutions in one case, and the bullish retail investors in the other. The benchmark concerns of institutions, in addition, introduce a second volatility amplification channel, as positive (negative) shocks to prices further feed back into additional positive (negative) institutional demand for the stock to hedge relative performance risk. This exacerbates volatility beyond the levels induced by sentiment.

We then focus the analysis on our main case of interest, namely the pricing implications of sentiment-driven retail investors' trading with rational institutional investors. We are particularly interested in addressing two questions: (i) To what extent do rational but benchmarked-concerned institutions correct sentiment-induced mispricing? (ii) How does sentiment affect volatility in presence of institutions? While the mere addition of the effects of either type of feature on prices described above suggests an exacerbation of the associated distortions, the analysis of equilibrium reveals several surprising patterns.

First, whether greater institutionalization helps correct or distort sentiment-induced distortions on prices depends on the relative strength of retail sentiment vis-a-vis institutions' benchmarking concerns. For low levels of sentiment, even bullish retail investors choose to sell an increasing share of their stock holdings to the institutions as institutionalization grows. Because benchmark concerns increase their risk appetite, the institutions purchase these shares at increasingly higher prices, which worsens the stock overpricing as institutionalization raises. This result verifies the conjecture of DeVault et al. (2019) that the existence of sophisticated investors might push prices further away from their fundamental value than retail sentiment. However, the opposite happens for more severe levels of optimism, when retail sentiment leads to stronger demand for the stock than institutions' benchmarking concerns. In these situations, characterized by a low (potentially negative) stock risk premium, greater institutionalization can be accompanied by aggressive selling of the stock by the institutions, which helps to push the stock price closer to its fundamental value. Thus, institutions do help correct severe sentiment-induced overpricing that would otherwise result in financial "bubbles"—understood as a market with negative risk premium.

Second, when the trading counterparts of the irrational investors are institutions instead of non-institutional investors, bullish sentiment can actually *dampen* volatility. This result is in stark contrast with the one prevailing in a standard economy without institutions, where sentiment unambiguously creates excess volatility of the stock market (DeLong et al., 1990; Dumas et al., 2009). To understand this assume, for example, that retail investors are slightly to moderately optimistic about the stock's prospects. Instead of meeting the extra demand for the stock of the bullish investors (as rational non-institutional investors would in the standard economy), institutional investors demand more of the asset. They do so to the extent that, in equilibrium, none of the two investor types gets to lever up their portfolios to the desired (or to any) levels. Because both investor types end up having similar portfolios, shocks to fundamentals have no effect on the distribution of aggregate wealth, shutting down the relative-wealth amplification channel on volatility. The effect is such that, within this range of optimism, the stock excess volatility monotonically *falls* with investor sentiment. By comparison, this volatility increases monotonically within the same optimism range in the otherwise equivalent economy in which the rational counterparts of the sentiment investors are not institutions.

We further show that, by contrast, *bearish* sentiment always increases the stock return volatility in presence of rational institutions. Due to their benchmarking concerns, institutions are willing to buy more of the stock that bearish investors want to sell than equivalent non-institutional counterparts, amplifying differences in portfolio composition and the consequent relative-wealth channel thorough which fundamental shocks translate into return volatility. A consequence of these results is an asymmetric pattern whereby, relative to non-institutional rational investors, institutions reduce excess volatility in presence of bullish investors but increase it under widespread pessimism.

An analysis of the equilibrium dynamics of the model reveals additional novel patterns, as the distribution of aggregate wealth across investors responds endogenously to the arrival of cash flow news to determine prices and allocations. In the reference cases in which either bullish retail or rational institutional investors trade with rational non-institutional investors, wealth effects determine a cyclical pattern in price-dividend ratios and a countercyclical pattern in Sharpe ratios. These patterns are exacerbated in the second reference case, as rising prices can lead institutions to increase their stock demand, pushing prices even higher and Sharpe ratios even lower. The portfolio allocation of institutions additionally determines a cyclical pattern in the stock return volatility, as benchmarking concerns amplify the transmission of fundamental shocks to returns.

More surprisingly, the trading of the same rational institutions with bullish, instead of rational, retail investors can lead to the opposite, countercyclical pattern in return volatility. The switch in pattern follows from a switch in sign, in this case, of the relative-wealth channel on return volatility. Specifically, when retail sentiment is so high that its impact on the stock demand is stronger than the impact of institutions' benchmarking concerns, the retail investors are relatively overinvested in the stock. Positive fundamental news then make them relatively wealthier. The news also increase the index risk-hedging demand of institutions for the stock, leading to a sharp decline in the market price of risk. As their relative wealth increases and the market price of risk plummets, however, retail investors reduce the fraction of their wealth invested in the stock, inducing an increasing relative risk aversion pattern that is absent in partial equilibrium. The greater wealth of the retail investors implies that their lower demand prevails over the greater demand of institutions in the aggregate, making the relative-wealth channel now *reduce* (rather than amplify) the sensitivity of prices to fundamental news. The stronger the wealth effect, the steeper the fall in volatility.

The same negative impact of the relative-wealth channel on volatility can push volatility levels below those prevailing under either sentiment or benchmarking concerns alone. This result has important implications for the assessment of the effect of institutions, on the one hand, and of sentiment, on the other hand, on financial markets. First, it implies that, in presence of high sentiment, institutions can have a stronger depressing effect on volatility than equivalently rational but non-institutional peers, despite their benchmarking concerns. Second, in markets with high presence of institutional investors (high institutionalization), sentiment need not create "excess volatility" but substantially reduce it relative to the same highly institutionalized but fully rational market. This implication highlights the importance of distinguishing the degree of institutionalization of markets in empirical analyses that associate excess return volatility, as inferred from, e.g., volatility-ratio tests, to irrational behavior and mispricing (e.g., Shiller, 1979, 1981; Giglio and Kelly, 2018).

Our paper is related to two main strands of the literature. First, it is related to the literature on equilibrium pricing implications of institutional investors' incentives. Cuoco and Kaniel (2011) find that symmetric benchmark-adjusted compensation has a significant and unambiguous positive effect on the price of benchmark assets and a negative effect on their Sharpe ratios, while asymmetric schemes have a more ambiguous impact. Using a highly tractable model, Basak and Pavlova (2013), characterize explicitly the institutions' portfolios in response to benchmarking incentives and their effect on the prices, Sharpe ratios, return volatilities and correlations of benchmark versus non-benchmark assets. Several studies have built on this framework to rationalize observed asset pricing phenomena. Hong et al. (2014) use it to capture "status" (Keeping-Up-with-the-Joneses) concerns and explain the excessive trading of small local stocks and the trend-chasing behavior of individuals. Basak and Pavlova (2016) analyze the effect of the financialization of commodity futures markets on commodity futures prices, volatilities and correlations, as well as on the equity-commodity correlations. Buffa and Hodor (2022) study benchmark heterogeneity across asset managers to explain differences in the predictability of return comovement across cap-style and industry-sector portfolios. Hodor and Zapatero (2022) show that the interaction of the short investment horizons and benchmarking concerns of institutions can rationalize a downward-sloping term structure of risk premia. While accounting for wealth effects on portfolio allocations, these studies assume that all traders are rational, thus are not set up to assess the impact of sentiment on prices. Other studies in this literature do allow for the existence of irrational trading to explain how money managers subject to time-varying investors' flows (Vayanos and Woolley, 2013), or perceiving fees that depend on relative performance via an endogenous compensation contract (Buffa et al., 2019), can push prices away from fundamental value, as well as to study the effect of benchmarking concerns on information acquisition and market efficiency (Breugem and Buss, 2019; Sockin and Xiaolan, 2019). Because preferences are of the constant absolute risk aversion (CARA) type and the investment decisions of irrational traders are left out of the analysis (i.e., irrational trading is likened to "noise"), these studies abstract from the wealth effects of either institutions or sentiment investors that are key to our analysis.

Second, our paper is related to the literature that examines the impact of sentiment on prices in general equilibrium. Several studies show that different behavioral biases such as overconfidence (Daniel et al., 2001; Scheinkman and Xiong, 2003; Dumas et al., 2009), self-attribution bias (Daniel et al., 1998), extrapolative beliefs (Hong and Stein, 1999; Barberis and Shleifer, 2003; Barberis et al., 2015, 2018), among others, can lead to sentiment-like excess trading and have a significant impact on asset returns and volatility.[4] In modeling sentiment, we focus on the type of dogmatic beliefs conducive to irrational optimism or pessimism considered by, e.g., Kogan et al. (2006). As Martin and Papadimitriou (2022) point out, this type of beliefs is consistent with the evidence documented by Giglio et al. (2021) and Meeuwis et al. (2021) in portfolio choice contexts. Similarly to both Kogan et al. (2006) and Martin and Papadimitriou (2022), we account for risk aversion and endogenous wealth effects on portfolio decisions and prices. Unlike these authors, we assume that the trading counterpart of the sentiment-driven investors are financial institutions rather than otherwise identical direct investors.[5] We show that due to benchmarking concerns, these institutions can either exacerbate or correct the distortions associated to sentiment depending on the relative strength of the sentiment- versus the benchmark-driven demands for the assets.

The rest of the paper proceeds as follows. In Section 2 we describe the financial markets and investors' problems. In Section 3 we characterize explicitly the equilibrium prices and allocations in these markets. In Section 4 we compare the equilibrium in the general case in which sentiment investors trade with institutions to the reference cases in which only one of these features is present. In Section 5 we analyze the equilibrium in these markets when the distribution of wealth is endogenously determined. We discuss empirical implications of our findings in Section 6. Section 7 offers concluding remarks. We present all proofs in Appendix A.

2 Model

2.1 Economic Setting

We consider a pure exchange economy with finite horizon T populated by two classes of traders, retail and institutional investors, each of which in principle can exhibit irrational sentiment (optimism or pessimism) about the economy's driving fundamentals.

[4]For a comprehensive survey of asset pricing models based on psychological considerations, see Barberis (2018).

[5]Similarly, Krishnamurthy and Li (2021) study the effect of sentiment on financial crises in presence of a financial intermediaries.

Financial Market. The financial market consists of a single risky security (a stock market portfolio); one share of the stock is available for trading. The stock only pays a dividend at the final time T. Let S and D denote the stock and dividend (cash flow) processes, respectively. For simplicity, we assume that the process D follows a geometric Brownian motion, i.e.

$$dD_t = D_t(\mu dt + \sigma dB_t),$$

where μ is the mean dividend growth rate, σ is the dividend volatility, and dB_t are the increments of the standard Wiener (cash flow "news") process under (the true) probability measure $\mathbb{P}$.

In addition, a zero coupon bond is available in zero net supply. The zero coupon delivers a sure payment of one at time T; following Kogan et al. (2006), we use the bond as the numeraire so its price is always equal to one.

Investor Preferences. Agents derive utility from their terminal wealth. Following Basak and Pavlova (2013), there are two classes of investors: retail (R) and institutional (I). Retail investors have standard logarithmic utility, i.e.

$$u_R(W_T^R) = \log W_T^R.$$

Institutional investors have otherwise identical preferences to retail investors except that their utility is scaled by the value of a benchmark index Y:

$$u_I(W_T^I) = (1 - \upsilon + \upsilon Y_T)\log W_T^I, \qquad \upsilon \in [0,1), \tag{1}$$

where without loss of generality (in a single-stock economy) we let the benchmark index coincide with the stock market, i.e., we set $Y = S$. In the sequel, we show that a time-t measure of the strength of the institutional investor's benchmarking concern is:

$$q_t \triangleq \frac{\upsilon D_t e^{\mu(T-t)}}{1 - \upsilon + \upsilon D_t e^{\mu(T-t)}}, \qquad q_t \in [0,1),$$

which depends positively on the benchmark weight υ in I's utility, the level of dividends D_t, and the remaining time horizon $T - t$.

Specification (1) follows Basak and Pavlova (2013)'s reduced-form approach to capturing the fact that, as agents for their delegating investors, institutional investors are typically evaluated (and compensated) in terms of both absolute and relative performance with respect to a benchmark portfolio, so their marginal utility is increasing in the level of this benchmark. The specification can also capture relative performance concerns facing, e.g., status-conscious investors (Hong et al., 2014).

Investor Beliefs. For $k \in \{R, I\}$, investor k believes that the mean growth rate of the dividend process D is constant and equal to μ^k. Investor k's beliefs are represented by an exponential martingale ξ^k whose evolution under $\mathbb{P}$ is given by

$$d\xi_t^k = \xi_t^k \delta^k \sigma dB_t,$$

where $\delta^k \triangleq (\mu^k - \mu)/\sigma^2$ is the "optimism" in investor's k beliefs, and $\xi_0^k = 1$. ξ_T^k is the Radon-Nikodym derivative of the probability measure $\mathbb{P}^k$ with respect to $\mathbb{P}$, the probability measure under which the dividend mean growth rate is equal to μ^k. Under $\mathbb{P}^k$, the evolution of the dividend process D is given by

$$dD_t = D_t \left((\mu + \sigma^2 \delta^k) dt + \sigma dB_t^k \right),$$

where $dB_t^k = dB_t - \sigma \delta^k dt$ is the increment of a standard Wiener process under $\mathbb{P}^k$. Finally, we assume that investors agree to disagree, which reflects each class of investors' degree of overconfidence in its judgments. In the sequel, E_t^k denotes the conditional expectation at time t under investor k's beliefs.

Under $\mathbb{P}$, the dynamics of the stock price are given by

$$dS_t = S_t(\mu_{S,t} dt + \sigma_{S,t} dB_t).$$

At time t, investor k decides the fraction $\theta_t^k \in \mathbb{R}$ of her portfolio to allocate in the stock, with the remaining fraction $1 - \theta_t^k$ allocated in the bond. At time 0, and without loss of generality, investor k is endowed with a fraction λ^k of the stock share (with $\lambda^I + \lambda^R = 1$) and no bond. At time t, investor k's budget constraint is given by

$$dW_t^k = \theta_t^k W_t^k (\mu_{S,t}^k dt + \sigma_{S,t} dB_t^k), \tag{2}$$

where $\mu_{S,t}^k = \mu_{S,t} - \sigma_{S,t} \sigma \delta^k$, and $W_0^k = \lambda^k S_0$.

In most of the empirical and theoretical discussion on the topic (see references in Section 1), sentiment-driven trading is associated to retail investors. Thus, for comparability and without loss of generality, we assume that retail investors can display sentiment but institutional investors are fully rational ($\delta^I = 0, \xi_t^I = 1$ for all $t \in [0, T]$) throughout the rest of the analysis.

2.2 Portfolio Problem and Equilibrium Definition

At time t, investor k maximizes her lifetime utility of wealth

$$\begin{aligned} J_k(W_t^k) &= \max_{\theta^k} \; E_t^k[u_k(W_T^k)] \\ &= \max_{\theta^k} \; \frac{1}{\xi_t^k} E_t[\xi_T^k u_k(W_T^k)], \end{aligned}$$

subject to the budget constraint (2).

Clearly, markets are dynamically complete. This implies the existence of a unique state price density process π with $\mathbb{P}$-dynamics:

$$d\pi_t = -\kappa_t \pi_t dB_t,$$

where κ denotes the (endogenously determined) stock market price of risk.

We define equilibrium in these markets in the usual way, as consisting of a set of portfolio allocations and asset prices such that: (i) the individual portfolio allocations of the retail and institutional investors are optimal, and (ii) bond and stock markets clear. We start by examining the equilibria under different reference economies.

3 Equilibrium Characterization

We start by characterizing the stock's equilibrium price-dividend ratio and price of risk in this economy:

Proposition 1. *The time-t equilibrium price-dividend ratio and market price of risk are given by:*

$$S_t/D_t = \overline{(S/D)}_t \frac{1}{\varpi_t^I\big(1-\gamma(T-t)q_t\big) + (1-\varpi_t^I)\big(1-\gamma(\delta^R(T-t))\big)}, \tag{3}$$

$$\kappa_t = \bar{\kappa}\left(1 - \frac{\varpi_t^I(1-\gamma(T-t))q_t + (1-\varpi_t^I)\big(1-\gamma(\delta^R(T-t))\big)\delta^R}{\varpi_t^I\big(1-\gamma(T-t)q_t\big) + (1-\varpi_t^I)\big(1-\gamma(\delta^R(T-t))\big)}\right), \tag{4}$$

where $\gamma(x) \triangleq 1 - e^{-\sigma^2 x}$ ($\gamma(x) < 1, \gamma'(x) > 0$), and $\overline{(S/D)}$ and $\bar{\kappa}$ are the equilibrium price-dividend ratio and market price of risk in the standard ("STD") economy with no sentiment ($\delta^R = 0$) or *institutional investors* $(\upsilon = 0)$ *given by:*

$$\overline{(S/D)}_t \triangleq (S_t/D_t)|_{\delta^R=0,\upsilon=0} = e^{(\mu-\sigma^2)(T-t)},$$
$$\bar{\kappa} \triangleq \kappa_t|_{\delta^R=0,\upsilon=0} = \sigma.$$

Both greater optimism $\delta^R > 0$ and benchmarking concerns $q > 0$ lead to higher market valuations S_t/D_t in excess of fundamental values $\overline{(S/D)}_t$, with pessimism $(\delta^R < 0)$ creating the opposite effect. The higher (respectively, lower) prices translate into lower (higher) market prices of risk κ_t, reducing (increasing) the appeal of the stock in the portfolio allocation problem of investors and restoring the market equilibrium between the increased (reduced) demand and supply. Thus, the introduction of either optimistic (pessimistic) or institutional investors to an otherwise standard economy induces asset "overvaluation" ("undervaluation") from the perspective of their rational non-institutional trading counterparts. The severity of this mispricing increases with sentiment or the intensity of benchmarking concerns.

To see how the pricing effects of sentiment and benchmarking relate to the demand for the stock of the retail and institutional investors, we next characterize their equilibrium portfolio allocations and the stock return volatility:

Proposition 2. *The time-t portfolio weights in the stock of the retail and institutional investors are:*

$$\theta_t^R = \frac{\kappa_t}{\sigma_{S,t}} + \frac{\sigma}{\sigma_{S,t}}\delta^R, \tag{5}$$
$$\theta_t^I = \frac{\kappa_t}{\sigma_{S,t}} + \frac{\sigma}{\sigma_{S,t}}q_t, \tag{6}$$

so that the leverage $(\theta_t^R - 1)\varpi_t^R$ *of the retail investors is:*

$$(\theta_t^R - 1)\varpi_t^R = \varpi_t^I(1-\varpi_t^I)\frac{\sigma}{\sigma_{S,t}}(\delta^R - q_t). \tag{7}$$

The equilibrium stock return volatility is:

$$\sigma_{S,t} = \bar{\sigma}_S\left(1 + \varpi_t^I\frac{\gamma(T-t)q_t(1-q_t) + (1-\varpi_t^I)\big(\gamma(\delta^R(T-t)) - \gamma(T-t)q_t\big)(\delta^R - q_t)}{\varpi_t^I\big(1-\gamma(T-t)q_t\big) + (1-\varpi_t^I)\big(1-\gamma(\delta^R(T-t))\big)}\right) \geq \bar{\sigma}_S, \tag{8}$$

where $\bar{\sigma}_S = \sigma$ *is the equilibrium price-dividend ratio and market price of risk in the STD economy with no sentiment* $(\delta^R = 0)$ *or institutional investors* $(\upsilon = 0)$.

Equation (7) shows that the strength of retail investors' stock demand relative to the stock demand of institutions, $\delta^R - q_t$, indicates whether the time-t stock allocation in R-investors' portfolio is levered ($\delta^R - q_t > 0$) or not ($\delta^R - q_t < 0$). In an all-rational investor economy with institutions we have $\delta^R - q_t = -q_t < 0$, so the retail investors always lend money to the institutions. Proposition 2 shows that this situation need not hold in the presence of sentiment, as the leverage of retail investors is ultimately determined by their degree of optimism δ^R relative to the strength q_t of the benchmarking concerns of the institutional investors. The latter term is always smaller than 1, so the impact of benchmarking concerns is somehow limited. In particular, if the retail investors are sufficiently bullish on the stock's prospects so that $\delta^R > 1$, institutional investors become lenders and retail investors borrowers.

Whereas according to Eq. (8) both sentiment ($\delta^R \neq 0$) and benchmarking concerns ($q > 0$) create "excess volatility" with respect to the STD case (as $\sigma_{S,t} > \bar{\sigma}_S$ in both cases), the contribution of each of these features to this result is not entirely obvious from this expression. To assess these contributions, we decompose the stock return volatility into a fundamental, a benchmarking, and a relative-wealth components. Specifically, let us formally write:

$$
\begin{aligned}
dq_t/q_t &= \mu_{q,t}dt + \sigma_{q,t}dB_t, \\
d\varpi_t^I/\varpi_t^I &= \mu_{\varpi^I,t}dt + \sigma_{\varpi^I,t}dB_t.
\end{aligned}
$$

Further letting $\varepsilon_{S,t}^x = \frac{\partial S_t}{\partial x_t} \times \frac{x_t}{S_t}$ denote the elasticity of the stock price with respect to x at time t, we have the following:

Lemma 1. *The equilibrium stock return volatility can be decomposed as:*

$$
\sigma_{S,t} = \varepsilon_{S,t}^D \sigma_{D,t} + \varepsilon_{S,t}^q \sigma_{q,t} + \varepsilon_{S,t}^{\varpi^I} \sigma_{\varpi^I,t}, \tag{9}
$$

where $\sigma_{q,t} = (1 - q_t)\sigma$, $\sigma_{\varpi^I,t} = -(1 - \varpi_t^I)(\delta^R - q_t)\sigma$, *and*

$$
\begin{aligned}
\varepsilon_{S,t}^D &= 1, \\
\varepsilon_{S,t}^q &= \frac{\varpi_t^I \gamma(T-t)q_t}{\varpi_t^I\big(1 - \gamma(T-t)q_t\big) + (1 - \varpi_t^I)\big(1 - \gamma(\delta^R(T-t))\big)} > 0, && (10) \\
\varepsilon_{S,t}^{\varpi^I} &= \frac{\gamma(T-t)q_t - \gamma(\delta^R(T-t))}{\varpi_t^I\big(1 - \gamma(T-t)q_t\big) + (1 - \varpi_t^I)\big(1 - \gamma(\delta^R(T-t))\big)} \varpi_t^I. && (11)
\end{aligned}
$$

Thus, the excess volatility ratio is:

$$EVR_t \triangleq \sigma_{S,t}/\bar{\sigma}_S - 1 = \Psi_{q,t} + \Psi_{\varpi^I,t} \geq 0, \tag{12}$$

where:

$$\Psi_{q,t} = \varepsilon^{q}_{S,t}(1-q_t) > 0, \tag{13}$$

$$\Psi_{\varpi^I,t} = -\varepsilon^{\varpi^I}_{S,t}(1-\varpi^I_t)(\delta^R - q_t). \tag{14}$$

The first term in (9) is the direct effect of fundamental news on return volatility. It reflects the fact that positive (respectively, negative) cash flow news signal a greater (smaller) terminal dividend D_T, so the stock price S must thus adjust proportionally to reflect investors' updated expectations.

The second and third terms are the indirect impact of these fundamental news on stock return volatility via the changes they induce in, respectively, the institutions' benchmarking intensity and relative wealth share (i.e., the level of institutionalization), holding each other constant. Since these indirect impacts are the drivers of the excess volatility in this economy relative to the STD economy, we interpret them as the "benchmarking," Ψ_q, and "relative-wealth", Ψ_{ϖ^I}, propagation channels of fundamental shocks to excess volatility.

Benchmarking concerns create a positive feedback from prices to the stock demand. In an economy with institutional investors, the higher (lower) price stemming from investors' updated expectations after positive (negative) cash flow news raises (depresses) the institution's benchmarking-related demand for the stock in order to keep up with the benchmark. Thus, the aggregate demand and the price for the stock changes more than in the STD economy in response to the same cash flow news, amplifying the sensitivity of prices to dividend shocks. It is easy to see from Lemma 1 that, for a given benchmarking intensity q, the benchmarking channel is not only positive but also increasing in the degree of optimism δ^R of the retail investors. Thus, the amplification of excess volatility induced by institutions is always greater when trading with optimist rather than rational retail counterparts.

The relative-wealth channel arises endogenously in equilibrium whenever $0 < \varpi^I_t < 1$, i.e., whenever no investor type absorbs the entire economy. In this case, differences in the portfolio compositions of institutional versus retail investors lead to differences in the dynamics of their relative wealth. To the extent that changes in wealth translate to changes in stock demands (as is the case with log preferences), the aggregate wealth distribution becomes a stochastic (state) variable whose volatility adds fundamental risk to the stock relative to the STD case.

One can check from Eq. (11) that the relative-wealth elasticity of stock prices $\varepsilon_S^{\varpi^I}$ decreases, while the relative demand strength $\delta^R - q_t$ increases, with this degree of optimism. Moreover, each of these quantities can be positive or negative depending on how optimistic the retail investors are. Thus, it is in principle possible that, in trading with institutions, bullish investors attenuate the relative wealth-induced excess volatility of stock returns relative to their rational retail counterparts. In Section 4.2.2 below we provide the conditions under which this possibility arises.

4 Analysis of Equilibrium

4.1 Reference Economies

To further isolate the channels through which the presence of both retail sentiment and institutions' benchmarking concerns affect financial markets equilibrium, in this section we examine two relevant reference economies: the Basak and Pavlova (2013)'s setting featuring institutional investors but no sentiment (BP), and the economy featuring sentiment but no institutions (SENT). The proofs for all results in this section are given in Appendix A as special cases of the results in Section 3.

4.1.1 Benchmarking Concerns and No Sentiment (BP)

Basak and Pavlova (2013) introduce heterogeneity across investor types in a STD economy ($\delta^R = 0, \upsilon = 0$) by including positive benchmarking concerns ($0 < \upsilon < 1$) in the objective function (1) of the institutions.[6] The authors show that these benchmarking concerns induce an extra demand for the stock that raises the price-dividend ratio above and depresses the stock market price of risk below the levels prevailing in the STD economy, such that:

$$(S/D)_t^{BP} \triangleq (S_t/D_t)|_{\delta^R=0} = \overline{(S/D)}_t \frac{1}{1-\gamma(T-t)\varpi_t^I q_t} \geq \overline{(S/D)}_t, \tag{15}$$

$$\kappa_t^{BP} \triangleq \kappa_t|_{\delta^R=0} = \bar{\kappa}\left(1 - \frac{(1-\gamma(T-t))\varpi_t^I q_t}{1-\gamma(T-t)\varpi_t^I q_t}\right) \leq \bar{\kappa}. \tag{16}$$

For both $(S/D)_t^{BP}$ and κ_t^{BP}, the differences from their equilibrium values in the STD economy $\overline{(S/D)}_t$ and $\bar{\kappa}$ increase with the "benchmarked wealth" $\varpi_t^I q_t$, which we identify with the product of the fraction of aggregate wealth in I's hands, ϖ_t^I, and the intensity of their benchmarking concerns,

[6] In this section we only highlight the aspects of these authors' analysis that are most relevant for our purposes.

q_t. Basak and Pavlova (2013) refer to the upward pressure on prices and depressing effect on market price of risk resulting from benchmarking concerns as an "index effect."

These authors show that the presence of institutions further increases the stock return volatility relative to the STD economy. The effect is an increasing function of the benchmarked wealth $\varpi_t^I q_t$:

$$\sigma_{S,t}^{BP} \triangleq \sigma_{S,t}|_{\delta^R=0} = \bar{\sigma}_S \left(1 + \gamma(T-t)\frac{\varpi_t^I q_t(1-\varpi_t^I q_t)}{1-\gamma(T-t)\varpi_t^I q_t}\right) \geq \bar{\sigma}_S. \tag{17}$$

Using our results from Section 3, we can decompose the associated excess volatility ratio, EVR^{BP}, into its benchmarking and relative-wealth shock propagation channels as $EVR_t^{BP} = \Psi_{q,t}^{BP} + \Psi_{\varpi^I,t}^{BP}$, with:

$$\begin{aligned} \Psi_{q,t}^{BP} &= \frac{\gamma(T-t)\varpi_t^I q_t}{1-\gamma(T-t)\varpi_t^I q_t}(1-q_t) > 0, \\ \Psi_{\varpi^I,t}^{BP} &= \frac{\gamma(T-t)\varpi_t^I(1-\varpi_t^I)q_t^2}{1-\gamma(T-t)\varpi_t^I q_t} > 0. \end{aligned}$$

The positive sign of Ψ_q^{BP} is expected from our more general result (13), and its expression implies that the amplifying effect on return volatility of the institutions' benchmarking concerns rises with the extent of institutionalization ϖ_t^I of the economy, the more so the more intense benchmarking concerns q are.

The positive sign of $\Psi_{\varpi^I}^{BP}$ implies that, in presence of institutions but no sentiment, relative wealth effects also exacerbate the response of stock return to fundamental shocks. Intuitively, a positive (negative) dividend shock makes institutions, who overweight the stock in their portfolios, relatively wealthier (poorer). Confronted with a higher (lower) wealth, the positive sensitivity of their demand to wealth leads institutions to demand more (less) of the stock, pushing its price even higher (lower).

4.1.2 Sentiment and No Benchmarking Concerns (SENT)

In this economy, sentiment-driven (either optimistic or pessimistic) retail investors trade in the stock and the bond alongside identical but rational investors. The specialization of our framework to this case ($\delta^R \neq 0$ and $\upsilon = 0$) resembles the setup of Kogan et al. (2006) with log preferences, and is formally equivalent to a model of differences of opinion (e.g., Panageas, 2020) in which one of the two investors classes has the correct prior about the dividend growth rate μ.

Sentiment introduces a wedge between the demands for the stock of irrational and rational investors, with optimistic investors ($\delta^R > 0$) overweighting and pessimistic investors ($\delta^R < 0$) underweighting the stock in their portfolios. The following result summarizes the impact of the ensuing pressure on prices:

Lemma 2. *In presence of sentiment and absence of institutional investors, the stock market price-dividend ratio and price of risk are*

$$(S/D)_t^{SE} \triangleq (S_t/D_t)|_{\upsilon=0} = \overline{(S/D)}_t \frac{1}{1-\varpi_t^R \gamma(\delta^R(T-t))}, \tag{18}$$

$$\kappa_t^{SE} \triangleq \kappa_t|_{\upsilon=0} = \bar{\kappa}\left(1 - \frac{\varpi_t^R\big(1-\gamma(\delta^R(T-t))\big)\delta^R}{1-\varpi_t^R \gamma(\delta^R(T-t))}\right). \tag{19}$$

Thus, the price-dividend ratio rises above (respectively, falls below) the corresponding ratio $\overline{(S/D)}$ *in the STD economy when sentiment investors are optimistic (pessimistic), with the difference* $|(S/D)_t^{SE} - \overline{(S/D)}|$ *increasing in sentiment* $|\delta^R|$. *Similarly, the market price of risk under optimistic (respectively, pessimistic) sentiment falls below (rises above) its equilibrium value* $\bar{\kappa}$ *in the STD economy, with the difference* $|\kappa_t^{SE} - \bar{\kappa}|$ *increasing in sentiment.*

Importantly, sentiment also creates "excess volatility" in stock returns:

Lemma 3. *In presence of sentiment and absence of institutional investors, the stock return volatility and excess volatility ratio are:*

$$\sigma_{S,t}^{SE} \triangleq \sigma_{S,t}|_{\upsilon=0} = \bar{\sigma}_S\left(1 + \frac{\varpi_t^R(1-\varpi_t^R)\gamma(\delta^R(T-t))}{1-\varpi_t^R\gamma(\delta^R(T-t))}\delta^R\right) \geq \bar{\sigma}_S, \tag{20}$$

$$EVR_t^{SE} \triangleq EVR_t|_{\upsilon=0} = \Psi_{\varpi^I,t}^{SE} = \frac{\varpi_t^R(1-\varpi_t^R)\gamma(\delta^R(T-t))}{1-\varpi_t^R\gamma(\delta^R(T-t))}\delta^R \geq 0. \tag{21}$$

Thus, under heterogeneity in investor types $(0 < \varpi_t^R < 1)$, *both positive (optimism) and negative (pessimism) sentiment exacerbate the stock return volatility relative to the STD economy* $(EVR_t^{SE} > 0)$. *Moreover,* $\sigma_{S,t}^{SE}$ (EVR_t^{SE}) *is increasing in sentiment for a fixed wealth distribution and reverts back to the STD equilibrium value* $\bar{\sigma}_S$ (0) *when all investors are sentiment driven* $(\varpi_t^R = 1)$.

Whenever sentiment-driven investors trade in the stock with rational investors, the positive impact of sentiment on volatility arises not only under optimism but also under pessimism, and increases monotonically and symmetrically with the level of (positive or negative) sentiment.[7]

The excess volatility of stock returns in the SENT economy is purely a relative-wealth effect. Similarly to the institutional investor in the BP economy, a sentiment-driven *optimistic* investor (a

[7] This effect is robust to more general constant relative risk aversion (CRRA) preferences, with its intensity decreasing in the coefficient of relative risk aversion.

rational investor in the case of *pessimistic* sentiment) is overexposed to the stock compared to the other investor type. A positive dividend shock makes the investor type that is overexposed to the stock relatively wealthier. The wealthier investor then demands more of the stock, pushing its price even higher. Conversely, a negative dividend shock makes the same investor relatively poorer, which impacts negatively on her demand for the stock and, as a result, on the stock price. In this sense, the presence of sentiment in an all-retail economy *always* exacerbates return volatility with respect to the STD case, as does the presence of institutions in the BP setting. This positive relationship between sentiment and excess volatility is consistent with the prediction of earlier models of noise trading and sentiment risk (DeLong et al., 1990; Dumas et al., 2009).

4.1.3 Comparison of the Pure Effects of Benchmarking vs. Sentiment

The similarities in effects on prices and return dynamics of benchmarking concerns (section 4.1.1) and sentiment (Section 4.1.2) raise the question of how the asset pricing implications of these features compare. One approach to address this question is to make the two features quantitatively comparable and examine the resulting similarities and differences in equilibrium values. We have shown that, in the SENT economy, the stock price-dividend ratio, market price of risk and volatility change monotonically with the degree of optimism δ^R. Thus, keeping all other parameters and the horizon $T-t$ fixed, it is possible to find the value $\delta^R = \check{\delta}_t^R$ such that the equilibrium values of any one of these variables in the SENT and BP economies coincide. We can then compare the equilibrium effect of benchmarking concerns and sentiment in the other endogenous variables.[8] The following result shows that the level of optimism that equates price-dividend ratios across the two economies is independent from the distribution of wealth ϖ_t^R across the agents:

Lemma 4. *The degree of optimism $\check{\delta}_t^R$ that leads to identical price-dividend ratios for the stock in the BP and SENT economies at any given horizon $T-t$ equals:*

$$0 < \check{\delta}_t^R = \frac{\log(1-\gamma(T-t)q_t)}{\log(1-\gamma(T-t))} < q_t. \tag{22}$$

At this level of optimism, $\kappa_t^{BP} \geq \kappa_t^{SE}$, and $\sigma_{S,t}^{BP} \geq \sigma_{S,t}^{SE}$.

Lemma 4 shows that, for any parameterization of the BP economy, there is a level of optimism in

---

[8] In this exercise, we assign the distribution weights ϖ^R and $1-\varpi^R$ in the SENT economy to, respectively, the rational and irrational investors. Thus, the irrational investor has the same weight as the institutional investor of the BP economy ($\varpi^I = 1-\varpi^R$).

the SENT economy that creates the same upward shift in price-dividend ratios across *all* distributions of aggregate wealth as the "index effect" identified by Basak and Pavlova (2013). Moreover, at this level of optimism, the index effect on the market price of risk and the stock volatility is always greater than the effect of sentiment.

Fig. 1 illustrates the equilibrium initial price-dividend ratios, market prices of risk and volatilities under the degree of optimism $\check{\delta}_t^R$, where for comparability the rest of the model parameters follow the baseline parameterization of Basak and Pavlova (2013)'s single-stock economy. Equilibrium values are plotted as a function of the share of aggregate wealth $1-\varpi_t^R$ of the institutional or the sentiment-prone (optimistic) retail investors, respectively, in the BP and SENT economies. The

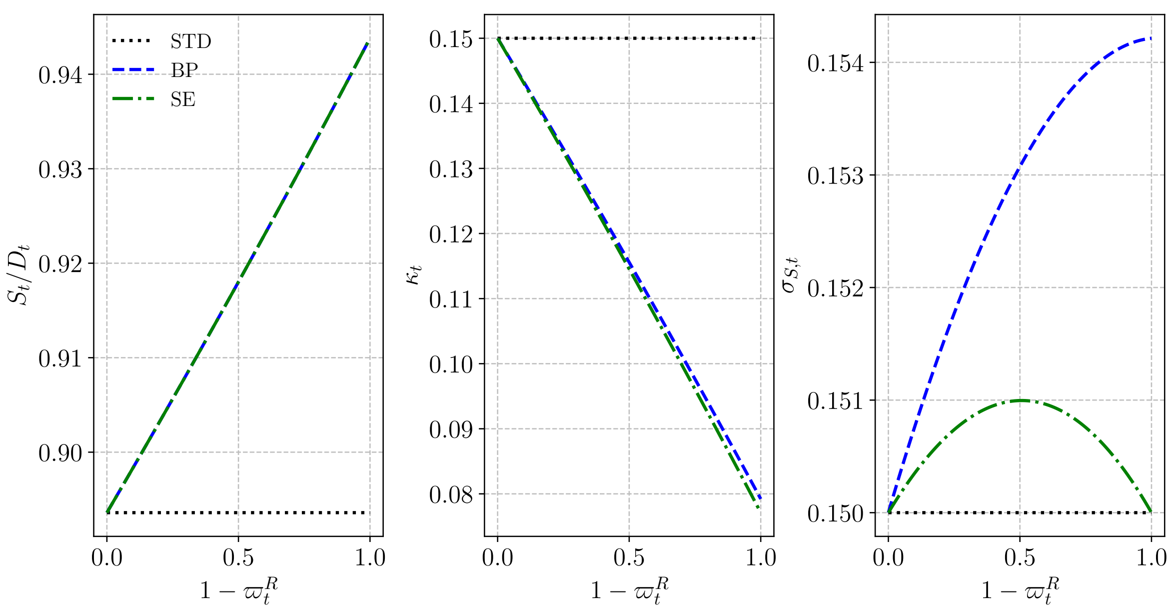

Figure 1: **Equilibrium under the reference economies**

This figure plots the equilibrium price-dividend ratio (leftmost panel), market price of risk (center panel) and stock return volatility (rightmost panel) under the STD (dotted black line), BP (dashed blue line), and SENT (dash-and-dot green line) economies. In all cases, equilibrium values are depicted as a function of the share of aggregate wealth of either I-investors (BP case) or sentiment R-investors (SENT case). Across all graphs, $\delta^R = \check{\delta}_0^R = 0.486$. The rest of the model parameters follow the parameterization in Basak and Pavlova (2013): $\mu = 0, \sigma = 0.15, t = 0, T = 5, D_0 = 1, \upsilon = 0.5$.

stock overvaluation relative to the STD economy rises with the share of wealth of the institutions (BP case) or the sentiment investors (SENT case) in each of the economies. Accordingly, equilibrium stock market prices of risk follow very similar decreasing patterns across the two economies, with values being (slightly) lower in the SENT economy in line with Lemma 4.

The right-most panel offers a quantitative illustration of the difference, following Lemma 4, in the amplification effects of benchmarking concerns and sentiment on stock return volatility. When the share of the economy in the institutional or the sentiment-driven investor type is less than half, excess volatility increases with this share in both the BP and SENT cases. However, it increases more rapidly in the BP case, causing the effect of benchmarking on stock volatility to be substantially larger than the effect of sentiment. The more marked difference between the two cases appears when the institutional and the sentiment-driven investor type, depending on the case, becomes large enough to encompass more than half of aggregate wealth.

The different impact of the two features on excess volatility can be traced back to a comparison of the benchmarking and relative-wealth channels in each of the two economies. In particular, the benchmarking channel is absent in the SENT economy, whereas the relative-wealth channel is present, and has a positive effect, in both settings. In addition, it can be shown that whenever the share of wealth of the retail rational investors ϖ_t^R is small enough (with $\varpi_t^R < 0.5$ being sufficient), the magnitude of the relative-wealth channel is larger, i.e., changes in relative wealth lead to greater volatility of stock returns in the BP economy than in the SENT economy.[9]

4.2 General Case: Interaction of Benchmarking and Sentiment

We have shown that the trading of either irrationally optimistic or institutional investors with otherwise equivalent rational non-institutional investors have similar boosting effects on stock prices and return volatilities, even if their precise quantitative impact on the latter can differ (sometimes substantially). Considering these similarities and differences, how do optimistic and institutional investors trade with each other and what are the implications on prices? To answer these questions, in this section we analyze the equilibrium under the general ("GE") case in which sentiment retail investors trade alongside institutional investors.

4.2.1 Effect of institutionalization

DeVault et al. (2019) conjecture that the existence of sophisticated investors need not help prices converge to, and might actually make them deviate even more from, their fundamental value. We

[9]More precisely,

$$\Psi^{BP}_{\varpi^I,t} > \Psi^{SE}_{\varpi^I,t} \Leftrightarrow \varpi_t^R < \frac{1 - \frac{\log(1-\gamma(T-t))}{\log(1-\gamma(T-t)q_t)}(1-\gamma(T-t)q_t)}{1 + \frac{\log(1-\gamma(T-t))}{\log(1-\gamma(T-t)q_t)}} < 0.5.$$

examine this conjecture within our setup by studying whether the introduction of institutions to (i.e., the institutionalization of) a market populated by optimistic retail investors exacerbates or, on the contrary, helps correct overpricing.

To this aim, we analyze how the price-dividend ratio S_t/D_t changes in Eq. (3) as the share ϖ_t^I of aggregate wealth in I's hands increases, for different levels of sentiment of the R investors. We find that for high enough levels of sentiment, a greater level of institutionalization of markets always helps *correct* overpricing:

Lemma 5. *In an economy populated by irrational retail and rational institutional investors, whether a higher level of institutionalization decreases, does not change, or increases the stock price-dividend ratio depends on whether the level of bullish sentiment δ^R exceeds, equals, or falls below the threshold $\check{\delta}_t^R$ that equalizes price-dividend ratios in the BP and SENT economies.*

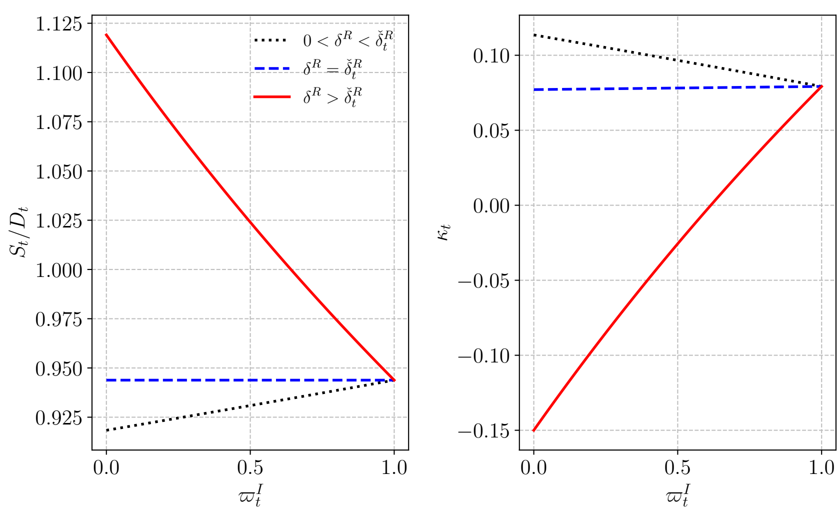

Figure 2: **Equilibrium prices in the GE case under mild to high levels of optimism**
This figure plots the equilibrium price-dividend ratio (left panel) and market price of risk (right panel) under the GE economy for three levels of optimism δ^R of the R-investors: mild optimism ($0 < \delta^R < \check{\delta}_t^R$, dotted black line), middle-ranged optimism ($\delta^R = \check{\delta}_t^R$, dashed blue line), and high optimism ($\delta^R > \check{\delta}_t^R$, solid red line). In all cases, equilibrium values are depicted as a function of the share of aggregate wealth of I-investors. Model parameters are as in Fig. 1.

Whether greater institutionalization helps correct or distort sentiment-induced distortions on prices depends on the relative strength of retail sentiment vis-a-vis institutions' benchmarking concern. Figure 2, which plots the equilibrium stock price-dividend ratios and market prices of risk as a function of the share of institutional investors in aggregate wealth, and Figure 3, which illustrates the associated optimal asset allocations, offer intuition for this result. For low levels of sentiment

($\delta^R < \check{\delta}_t^R$), even bullish retail investors choose to sell an increasing fraction of their stock holdings to the institutions as the level of institutionalization rises. Because benchmark concerns increase their risk appetite, the institutions purchase these shares at increasingly higher price-dividend ratios, determining a pattern of stock overpricing that worsens with institutionalization. This result verifies the conjecture of DeVault et al. (2019) that the existence of sophisticated investors might push prices further away from their fundamental value than the presence of sentiment-driven retail investors.

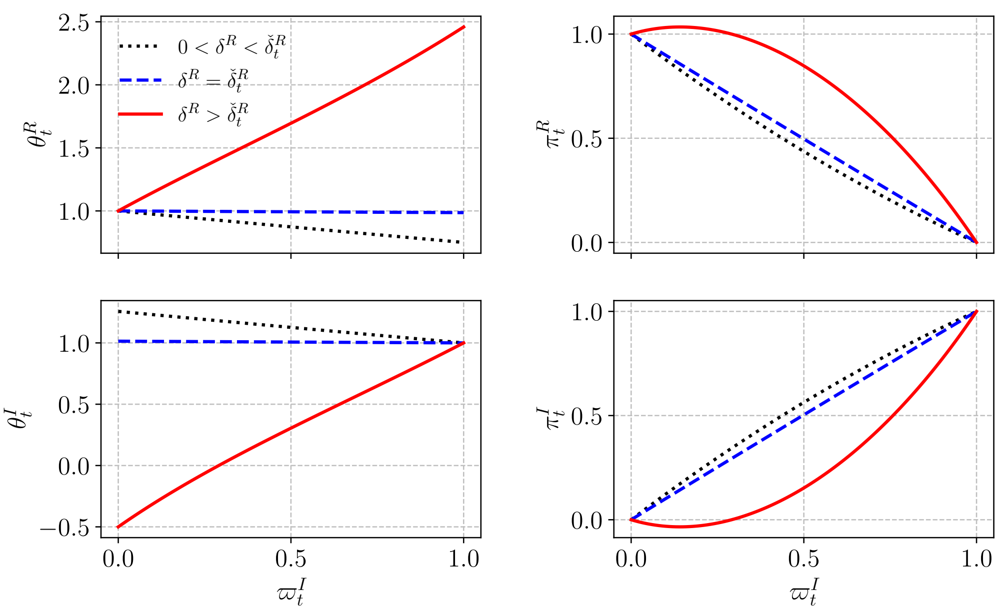

Figure 3: **Equilibrium allocations in the GE case under mild to high levels of optimism**
This figure plots the equilibrium weights (left panels) and number of shares $\pi_t = \theta_t^k W_t^k / S_t, k \in \{I, R\}$ of the stock (right panels) in the portfolios of bullish retail (top panels) and rational institutional (bottom panels) investors under the equilibrium cases illustrated in Fig. 2: mild optimism ($0 < \delta^R < \check{\delta}_t^R$, dotted black line), middle-ranged optimism ($\delta^R = \check{\delta}_t^R$, dashed blue line), and high optimism ($\delta^R > \check{\delta}_t^R$, solid red line). In all cases, equilibrium values are depicted as a function of the share of aggregate wealth of I-investors. Model parameters are as in Fig. 1.

However, the opposite holds for higher levels of optimism ($\delta^R > \check{\delta}_t^R$), when retail bullish sentiment leads to stronger demand for the stock than institutions' benchmarking concerns. Such strong bullish sentiment can lead to severe levels of overvaluation and a negative market risk premium, akin to a financial "bubble," under low levels of institutionalization. When the risk premium is low enough, however, rational institutions, no matter how concerned about their benchmark, will find it optimal to reduce their portfolio allocation in the stock. As institutionalization grows, aggressive selling by the institutions pushes the stock price closer to its fundamental value (see the STD case in Fig. 1) and eventually reverses it to levels consistent with a positive risk premium. Importantly, the threshold that separates "low" from "high" sentiment, $\check{\delta}_t^R$, is invariant to the level of institutionalization and coincides with the threshold level of bullish sentiment in which the "pure" effects of sentiment on the stock's price dividend ratio is identical to the effect of benchmarking in an all-rational-investors economy (BP).

4.2.2 Excess volatility

The simple addition of the effects of benchmarking concerns (Section 4.1.1) and sentiment (Section 4.1.2) on excess volatility may suggest that, in presence of both bullish R-investors and institutional I-investors, the stock return volatility must rise beyond the BP and SENT levels. The following result indicates that this intuition need not always hold:

Proposition 3. *In presence of institutional* $(\upsilon > 0)$ *and irrational retail* $(\delta^R \neq 0)$ *investors, there exists a unique degree of optimism* $\hat{\delta}^R(D_t, \varpi_t^I, T-t) > \check{\delta}_t^R > 0$ *such that:*

$$\frac{\partial \sigma_{S,t}}{\partial \delta^R} \begin{cases} > 0, & \delta^R > \hat{\delta}^R(D_t, \varpi_t^I, T-t) \\ = 0, & \delta^R = \hat{\delta}^R(D_t, \varpi_t^I, T-t) \\ < 0, & \delta^R < \hat{\delta}^R(D_t, \varpi_t^I, T-t) \end{cases} .$$

This implies, in particular, that for $0 < \delta^R < \check{\delta}_t^R$ *the effect of optimistic sentiment is to* reduce *the stock return volatility across all wealth distributions* ϖ_t^I *relative to the BP case.*

In contrast to a standard (STD) economy, the introduction of sentiment investors to an otherwise all-rational (retail plus institutional) investor economy can actually *dampen* the excess volatility of the stock market. Figure 4, which illustrates the equilibrium stock return volatility across different levels of optimism of the R-investors, makes it clear that excess volatility can be significantly lower in the GE case than in the BP case.

Proposition 3 shows that within a range of no-to-moderate optimism $0 < \delta^R < \check{\delta}_t^R$ the stock excess volatility *monotonically falls* with investor sentiment (red solid line in Fig. 4) across all aggregate wealth distributions. By comparison, notice that, in line with the commonly held view that increasing sentiment translates into greater excess volatility, the stock excess volatility does increase monotonically within the same range of optimism in the otherwise equivalent SENT economy in which the rational investors are non-institutional (green dash-and-dot line).

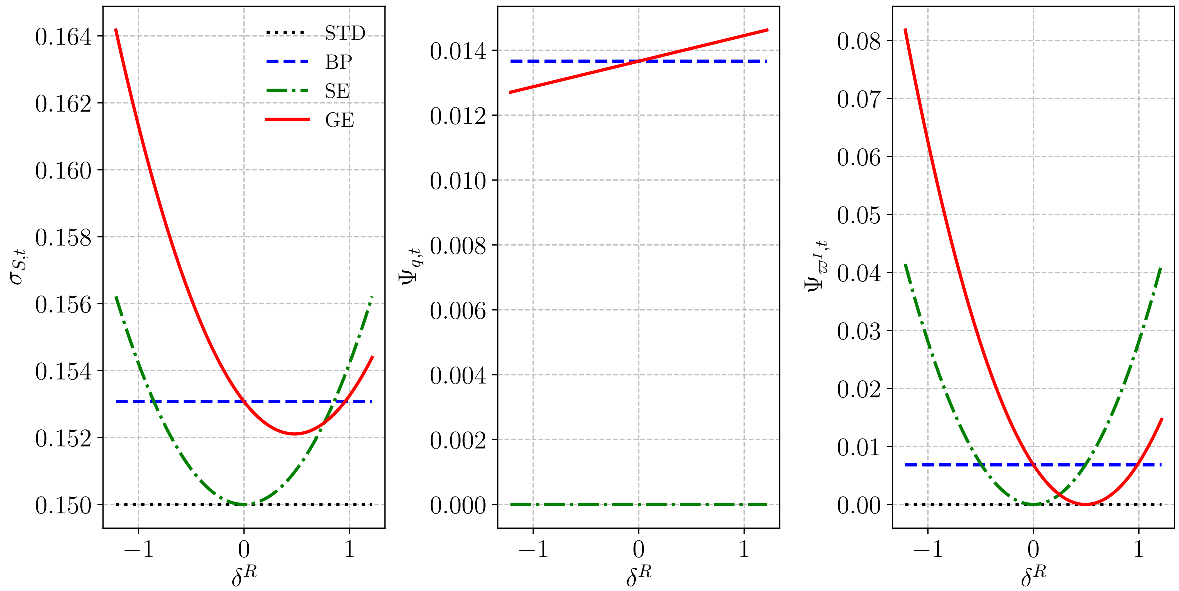

Figure 4: **Sentiment and equilibrium volatility**

The figure plots the stock return volatility (leftmost panel), and the benchmarking (center panel) and relative-wealth (rightmost panel) shock propagation channels on excess volatility, under the STD (dotted black line), BP (dashed blue line), SENT (dash-and-dot green line), and GE (red solid line) economies. In all cases, equilibrium values are depicted as a function of the degree of optimism of the sentiment retail investors, for $\varpi_t^I = 0.5$. The rest of the model parameters are as in Fig. 1.

The economic intuition for these effects can be traced back to the decomposition of excess volatility into the benchmarking and relative-wealth shock propagation channels. For $0 < \delta^R < \check{\delta}_t^R$, the benchmarking effect on the stock demand is stronger than the sentiment effect ($\delta^R < q_t$) and creates heterogeneity in the portfolio holdings across the two investor types. As δ^R rises, however, the gap between the two stock demands shrinks until disappearing at $\delta^R = q_t = 0$ ($\delta^R \approx \check{\delta}_t^R$). At this point, none of the two investor types gets to lever up their portfolios as much as they would if trading with either rational retail (for the case of I-investors) or rational non-institutional (for the case of sentiment-driven R-investors) counterparts. As the heterogeneity in portfolio allocations across investors disappears the relative-wealth channel shuts down, while the magnitude of the benchmarking channel keeps rising with the degree of optimism (see Section 3). Since their combined effect on the stock return excess volatility falls according to Proposition 3, it must be that within this range of optimism the magnitude of the relative wealth channels falls more rapidly than the benchmarking channel rises. As optimism rises and the demand strength of the sentiment-driven investors exceeds that of the institutional investors ($\delta^R > q_t$), the difference $\delta^R - \check{\delta}_t^R$ turns positive and increases the differences in the portfolios across investor types that activate the relative-wealth channel's positive effect on return volatility. Since the benchmarking channel still rises with $\delta^R - q_t$, the combined effect of both channels on excess volatility increases with δ^R, explaining the pattern of excess increasing stock return volatility at high levels of optimism ($\delta^R > \hat{\delta}^R$).

Note that, by contrast, in presence of rational institutions *bearish* sentiment always increases the stock return volatility. Due to their benchmarking concerns, institutions are willing to buy more of the stock shares that bearish investors sell than equivalent non-institutional counterparts. This creates greater differences in portfolio composition across investors, making the relative-wealth channel amplify return volatility. A consequence of these results is an asymmetric pattern of excess volatility whereby, relative to non-institutional rational investors, institutions attenuate excess volatility in presence of bullish sentiment but exacerbate it under widespread pessimism.

5 Dynamic effects

As the economy unfolds over time, the distribution of aggregate wealth across investor types changes endogenously in response to the arrival of positive and negative cash flow news and determines equilibrium prices and allocations. To characterize these dynamics, we fix the initial stock share endowments λ^I and λ^R of the I and R investors to λ and $1-\lambda$, respectively, and solve for the corresponding time-t aggregate wealth shares to obtain the following:

Lemma 6. *Given initial wealth distribution $\varpi_0^I = \lambda$ and $\varpi_0^R = 1-\lambda$ for, respectively, the I and R investors, the equilibrium market price of risk, price-dividend ratio, and return volatility are given by Eqs.* (4)*,* (3)*, and* (8)*, while the equilibrium portfolio allocations to the stock and the optimal borrowing are given by Eqs.* (5) *and* (7)*, for:*

$$\varpi_t^I = \frac{\lambda}{\lambda + \left(1 - \gamma(\frac{1}{2}\delta^R(\delta^R - 1)t)\right)\left(\frac{q_t}{q_0}\right)^{\delta^R}\left(\frac{1-q_t}{1-q_0}\right)^{1-\delta^R}(1-\lambda)}, \tag{23}$$

$$\varpi_t^R = 1 - \varpi_t^I. \tag{24}$$

Moreover, whether positive cash flow news decrease *the institutional investors' share of aggregate* *wealth depends on whether the sentiment retail investors are sufficiently optimistic, as given by:*

$$\frac{\partial(\varpi_t^I)}{\partial D_t}\begin{cases} < 0, & \delta^R > q_t \\ = 0, & \delta^R = q_t \\ > 0, & \delta^R < q_t \end{cases}. \tag{25}$$

To provide intuition for this result, we first proceed as in Section 4.1.3 and compare the equilibria under the BP and SENT economies where the only non-standard features are, respectively, the benchmark concerns of I investors or the sentiment of R investors. We do so by referring to Fig. 5, which illustrates the interim equilibrium as of $t = 1$ (for $T = 5$) under the parameterization of Fig. 1, conditional on different realizations of the cash flow news D_t.

Based on the analysis above, one would expect similarities and differences between the effects on sentiment, on the one hand, and benchmark concerns, on the other, on equilibrium configurations. Lemma 6 indicates that good (bad) cash flow news, as represented by values of $dD_t > 0$ (< 0), make the institutional investors in the BP case, and the optimistic investors in the SENT case, wealthier (poorer), thus increasing (decreasing) the aggregate stock demand and, accordingly, the stock's price-dividend ratio. These wealth effects lead to a countercyclical behavior for the stock's Sharpe ratio (market price of risk). The SENT case shows that the presence of wealth effects is enough to create this pattern, even with the fixed level of optimism of our setup. The BP case shows that benchmarking concerns, through their interaction with wealth effects, further exacerbate it.

Where the BP and SENT cases differ, once again, is in the impact of cash flow news on return volatility. Indeed, the stock's excess volatility rises substantially in the BP economy but stays approximately constant in the SENT economy as cash flow news and the relative wealth of either the I-investor or the optimistic R-investor, respectively, rises. As we saw before, this occurs because positive news increase the stock market price which, along with the associated wealth effects, leads I-investors to demand more of the stock. To attenuate this demand, volatility must necessarily rise. Note that this implies a *cyclical* pattern of volatility over a wide range of positive cash flow news in the BP economy.

The different effect of wealth on the stock demand or, equivalently, on the demand for leverage of institutional vs. sentiment-driven investors, can be understood by reference to the differences in portfolio allocations across the BP and SENT economies, as illustrated in Fig. 6. For $D_t < 2$, the fraction of wealth allocated to the stock (leftmost bottom panel) monotonically increases among I investors (dashed blue line), but decreases instead among R investors, with D_t. The allocations

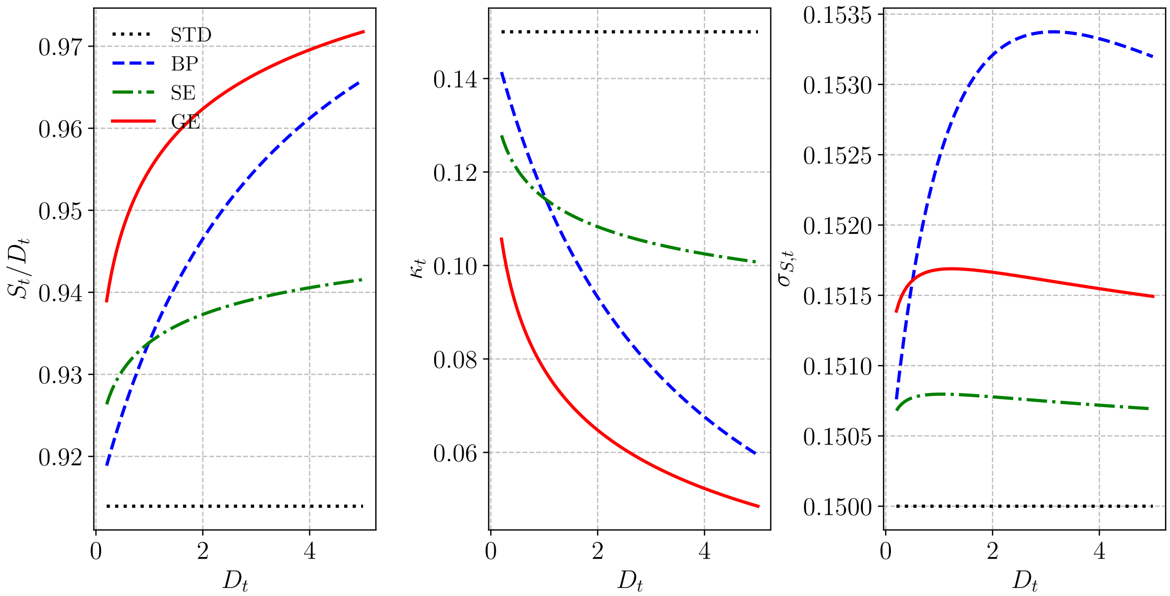

Figure 5: **Interim equilibrium in the GE and reference economies**

This figure plots the equilibrium price-dividend ratio (leftmost panel), market price of risk (center panel) and stock return volatility (rightmost panel) under the STD (dotted black line), BP (dashed blue line), SENT (dash-and-dot green line), and GE (red solid line) economies. In all cases, equilibrium values are depicted as a function of cash flows D_t as of $t = 1$, for a fixed initial share of aggregate wealth $\varpi_0^I = 0.5$. The rest of the model parameters are as in Fig.1.

determine opposite leverage patterns across the two investor types— increasing for I investors, decreasing for R investors.[10]

These trading patterns highlight an interesting contrast: even though their intrinsic (log) preferences display constant relative risk aversion (CRRA), the optimistic investors behave as if their relative risk aversion *increased* with wealth instead (IRRA preferences) by decreasing the fraction of their portfolio allocated to the stock in response to positive wealth shocks. This apparent contradiction is purely an equilibrium outcome: as their wealth increases, the increased demand of optimistic investors pushes the stock price higher and the Sharpe ratio lower, worsening the risk-return tradeoff that the stock offers. This effect is absent in partial equilibrium, and in the current context implies that the optimists' demand for the stock grows less than proportionally with their wealth, leading to the observed pattern in portfolio weights.

[10]To facilitate comparison with the BP case, the portfolio of the sentiment-driven retail investors in the SENT economy is identified with the superscript "I" and plotted in the bottom panels in Fig. 6, while the superscript "R" (and the top row of panels) is reserved for their rational retail counterparts.

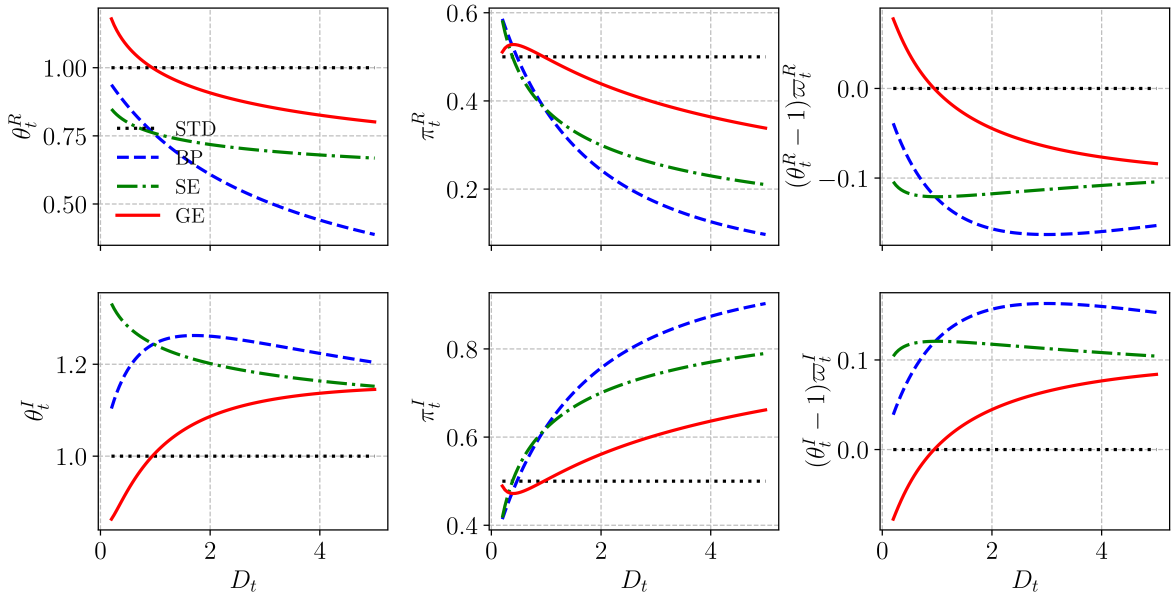

Figure 6: **Interim equilibrium portfolio allocations under the GE and reference economies**

This figure plots the equilibrium weights (leftmost panels) and number of shares $\pi_t = \theta_t^k W_t^k / S_t, k \in \{I, R\}$ of the stock (center panels) in the portfolios, and the leverage (rightmost panels), of bullish retail (top panels) and rational institutional (bottom panels) investors. Depicted cases correspond to the STD (dotted black line), BP (dashed blue line), SENT (dash-and-dot green line), and GE (red solid line) economies. Across panels, equilibrium values are plotted against cash flows D_t as of $t = 1$, for a fixed initial share of aggregate wealth $\varpi_0^I = 0.5$. Model parameters are as in Fig.5.

What happens when both the sentiment-driven and the institutional investors trade with each other? As in our characterization of equilibrium of Section 3, the GE case of Fig. 5 shows that the effect on prices and Sharpe ratios is as anticipated: At all levels of cash flows D_t, the former increase, while the latter falls, beyond the levels in the benchmark BP and SENT economies.

The effect on volatility, once again, is less obvious: In markets with predominantly good news ($D_t > 1$), the introduction of optimistic investors to a BP economy not only reduces excess volatility substantially at all D_t levels, but it also turns the cyclical pattern into a flat or decreasing pattern. A comparison of the top and bottom rows of Fig. 6 reveals that under these patterns of volatility and Sharpe ratios the institutional investors purchase increasingly more shares of the stock from the optimistic investors, which they finance by borrowing increasing amounts of money. The result represents a divorce in the patterns of volatility and institutions' leverage, both of which are tightly related in the BP economy, when the trading counterparts of the institutions are not rational but sentiment-driven retail investors.

The economic intuition for this result is as follows. Comparing the effects of negative versus positive cash flow news we see that volatility *falls* (slightly) in both cases. The level of volatility is still higher than in the SENT economy, and lower (except at very low levels of D_t) than in the BP economy, implying that the amplification effect through the benchmarking channel is still active. What this volatility pattern means then is that changes in this amplification effect following the cash flow news are being offset by changes in the opposite direction of the relative-wealth channel. Given that aggregate wealth is evenly split at $t = 0$ in this illustration, the changes in interim relative wealth associated with these cash flow news must be small or nil. Then it must be that the higher demand for the stock of I-investors in response to, e.g., a positive cash flow news, is being met by a lower demand of the R-investors. A comparison of the solid red and dash-and-dot green lines in the top left panel of Fig. 6 shows that this is indeed the case, as the weight of the stock in the portfolio of the optimistic R-investor falls more steeply when trading with a rational I-investor (red line) than with a rational R-investor (green line).

But why does this happen? Looking at Fig. 5, we see that as D_t rises the market price of risk in the GE case falls more steeply (because of the effect of benchmarking) than in the SENT case, justifying the greater fall in R's stock demand in the former economy. Something similar happens with negative cash flow news: the demand of the optimistic R-investors increases more when trading with rational I-investors than when trading with otherwise equivalent R-investors, because the benchmarking concerns induce the former to reduce their position in the stock very aggressively, leading to a steeper increase in the market price of risk.

How different can the patterns in volatility be in the GE case compared to the SENT and BP reference cases? In particular, can volatility be countercyclical over a "reasonable" range of the cash flows D_t? This feature would be somehow desirably from an empirical point of view, given the typically countercyclical pattern of volatility that the literature (see, e.g., Mele, 2007) has documented.

Fig. 7 illustrates the interim equilibrium at $t = 1$ for high levels of optimism of the sentiment-driven investor, under an otherwise identical parameterization as in Fig. 5.[11] Two features of the volatility pattern in the GE case, in the comparison with either the SENT or BP cases, are particularly surprising:

[11] More precisely, $\delta^R = 1$. At this level of optimism, the demand for the stock of the I investors cannot dominate the demand of optimistic R investors ($\delta^R \geq q_t$), which according to Lemma 6 leads positive cash flow news to increase the latter's share of wealth.

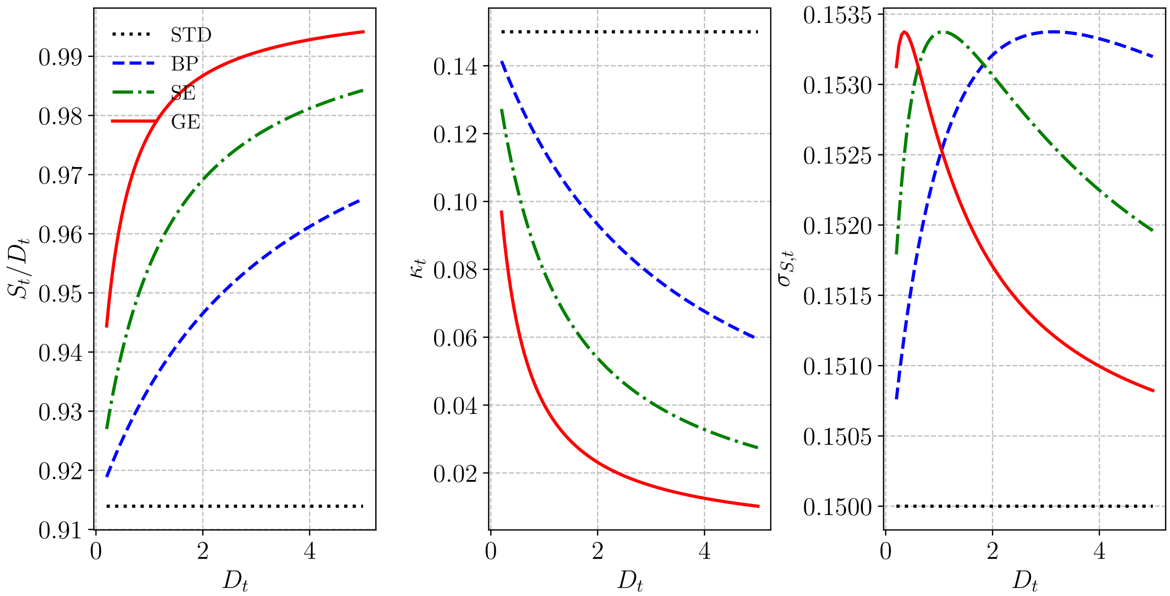

Figure 7: **Interim equilibrium: higher optimism**

This figure plots the equilibrium price-dividend ratio (leftmost panel), market price of risk (center panel) and stock return volatility (rightmost panel) under the STD (dotted black line), BP (dashed blue line), and SENT (dash-and-dot green line) economies, for relatively high level of optimism $\delta^R = 1$. In all cases, equilibrium values are depicted as a function of cash flows D_t as of $t = 1$, for a fixed initial share of aggregate wealth $\varpi_0^I = 0.5$. The rest of the model parameters are as in Fig.1.

(i) Volatility is highly countercyclical over almost the entire range of D_t, even when it is cyclical in the reference SENT (for $D_t < 1$) and BP (for $D_t \leq 3$) economies.

(ii) Volatility can be lower not only than in the case of an economy with all rational investors in which some are institutional (BP), but also than in an economy where there are only retail investors but some are optimistic (SENT).

The intuition for this result is as follows. With high optimism ($\delta^R > q_t$), according to Propositon 2 the R-investors are relatively overinvested in the stock compared to the I-investors. Following Lemma 6, positive cash flow news make R-investors relatively wealthier, and I-investors relatively poorer. As R-investors become wealthier, however, their "equilibrium-induced" increasing relative risk aversion leads them to reduce the fraction of wealth invested in the stock. The positive cash flow news still lead I-investors to increase the weight of their portfolio allocated in the stock. Because R-investors become relatively wealthier, however, their reduced demand prevails in the aggregate, inducing the relative-wealth channel to *reduce* the sensitivity of prices to CF news. Remember that, by contrast, the relative-wealth channel always increases this sensitivity in the BP and SENT economies. The negative impact of the relative-wealth channel on volatility is specific to the GE case and explains both the *level* effect (volatility is lower in the GE case than in either the BP or SENT cases), and the *slope* effect (volatility is countercyclical). Because stronger wealth effects lead to steeper falls in volatility, both effects are exacerbated at higher levels of optimism.

6 Empirical Implications

Our findings offer several insights on the role of institutional investors in financial markets, and the impact of sentiment trading on asset prices. First, the positive but limited influence that benchmarking concerns have on the demand of rational institutional investors, relative to the demand of bullish retail investors, implies that institutions are unlikely to help correct situations of low to moderate asset overpricing, but can exert a significant correcting force for more severe overpricing levels. In other words, while arbitrarily high levels of overpricing can be consistent with sufficiently high levels of irrational sentiment, they are not easily rationalized by reference solely to the benchmarking concerns of rational institutional investors.[12]

Second, the attenuating impact that changes in the wealth of sentiment-driven relative to institutional investors have on return volatility implies that, in presence of high sentiment, rational institutions can have a stronger depressing effect on volatility than similar non-institutional peers. It also implies that in markets with high presence of institutional investors, sentiment need not create "excess volatility" but substantially reduce it relative to the case of no sentiment. This implication highlights the importance of distinguishing the degree of institutionalization of markets in empirical analyses that associate excess return volatility, as inferred from, e.g., volatility-ratio tests, to irrational behavior and mispricing. In particular, it implies that in markets with significant presence of institutions and high price-dividend ratios, the observation of *low* return volatility will be more indicative than *high* volatility of bullish sentiment.

Lastly, our results on the endogenous interaction of changes in the distribution of wealth and fundamentals can help reconcile existing models of institutional asset pricing with the empirically documented countercyclical pattern of volatility in the stock market returns.

[12] Sotes-Paladino and Zapatero (2019) show that option-like, benchmark-linked incentives can induce rational institutional investors to overinvest in overpriced, "bubble" securities, but mostly when these securities do not belong to the benchmark.

7 Conclusion

Despite the significant trend toward the portfolio delegation of households to institutional managers in recent years, most studies on the effect of sentiment-driven trading on asset prices assume that the other side of the trade is taken by rational direct investors. Similarly, against the abundant evidence on irrational trading by individual investors, most of the asset pricing literature that considers the effect of institutions assumes that non-institutional counterparts are fully rational. In this paper, we account for the simultaneous presence of both institutions and sentiment-driven retail trading in financial markets to find several novel equilibrium patterns.

First, for low to moderate levels of optimism institutions can exacerbate, while for greater optimism they help correct the overpricing of the stock market that bullish sentiment induces. The result highlights the often overlooked fact that the benchmarking-related demand for a benchmark stock, thus its pressure on the stock's price, is always positive but bounded. By contrast, the non-benchmark-related (mean-variance) demand for the same stock is unbounded and, in presence of sentiment-induced overpricing, can have the opposite sign, potentially leading to an overall negative (and large) price pressure.

Second, the joint effect of sentiment and benchmarking concerns on the stock volatility can be radically different from the addition of the effects stemming from either feature in isolation. When the trading counterparts are standard rational investors, both sentiment and benchmarking concerns introduce portfolio heterogeneity across investor types and create a relative-wealth channel through which the transmission of fundamental shocks to asset prices is always amplified. The benchmarking concerns of institutions induce an additional positive feedback effect from prices to demand that further amplifies the impact of fundamental shocks on stock returns. When bullish sentiment and institutional investors trade with each other, the latter channel can only amplify the stock return variation in response to fundamental shocks. By contrast, the relative-wealth channel can instead attenuate this volatility.

This attenuation effect has rich implications for the level of volatility in financial markets and its dynamics over the business cycle. It can push volatility levels below those prevailing under either sentiment or benchmarking concerns. This result implies that rational institutions can have a stronger depressing effect on volatility in presence of high sentiment than similar non-institutional peers. It also implies that in markets with high presence of institutional investors sentiment need not create "excess volatility" relative to the case of no sentiment. The attenuation effect of the relative-wealth channel on stock return volatility induced by the trading of rational institutions with bullish, instead of rational, retail investors can lead to countercyclical pattern in return volatility, a pattern that is largely consistent with the existing empirical evidence.

Our results have a number of implications for the ongoing debate around the role of institutional investors in financial markets, as well as for the empirical inference of sentiment from market determined variables such as prices and volatility. Importantly, it implies that both the impact of the trend towards a greater institutionalization of markets in the correction of the distortions created by sentiment, and the degree to which sentiment distort prices and volatility in the first place, are not linear but result from a complex interaction between sentiment and benchmarking effects.

References

Barber, B. M., and T. Odean. 2008. All that Glitters: The Effect of Attention on the Buying Behavior of Individual and Institutional Investors. *Review of Financial Studies* 21:785–818.

Barber, B. M., T. Odean, and L. Zheng. 2009. Do Retail Trades Move Markets? *Review of Financial Studies* 22:151–186.

Barberis, N. 2018. Psychology-Based Models of Asset Prices and Trading Volume. *Handbook of Behavioral Economics* 1:79–175.

Barberis, N., R. Greenwood, L. Jin, and A. Shleifer. 2015. X-CAPM: an extrapolative capital asset pricing model. *Journal of Financial Economics* 115:1–24.

Barberis, N., R. Greenwood, L. Jin, and A. Shleifer. 2018. Extrapolation and bubbles. *Journal of Financial Economics* 129:203–227.

Barberis, N., and A. Shleifer. 2003. Style Investing. *Journal of Financial Economics* 68:161–199.

Barberis, N., and R. Thaler. 2003. A Survey of Behavioral Finance. *Handbook of the Economics of Finance* 1:1053–1128.

Basak, S., and A. Pavlova. 2013. Asset Prices and Institutional Investors. *American Economic Review* 103:1728–1758.

Basak, S., and A. Pavlova. 2016. A Model of Financialization of Commodities. *Journal of Finance* 71:1511–1556.

Breugem, M., and A. Buss. 2019. Institutional Investors and Information Acquisition: Implications for Asset Prices and Informational Efficiency. *Review of Financial Studies* 32:2260–2301.

Buffa, A., D. Vayanos, and P. Woolley. 2019. Asset Management Contracts and Equilibrium Prices. Boston U. School of Management Research Paper No. 2492529.

Buffa, A. M., and I. Hodor. 2022. Institutional Investors, Heterogeneous Benchmarks and the Comovement of Asset Prices. Working Paper.

Cuoco, D., and R. Kaniel. 2011. Equilibrium Prices in the presence of Delegated Portfolio Management. *Journal of Financial Economics* 101:264–296.

Da, Z., J. Engelberg, and P. Gao. 2015. The Sum of All FEARS Investor Sentiment and Asset Prices. *Review of Financial Studies* 28:1–32.

Daniel, K., D. Hirshleifer, and A. Subrahmanyam. 1998. Investor Psychology and Security Market Under- and Overreactions. *Journal of Finance* 53:1839–1885.

Daniel, K., D. Hirshleifer, and A. Subrahmanyam. 2001. Overconfidence, Arbitrage, and Equilibrium Asset Pricing. *Journal of Finance* 56:921–965.

DeLong, J. B., A. Shleifer, L. H. Summers, and R. J. Waldman. 1990. The Survival of Noise Traders in Financial Markets. *Journal of Business* 64:1–19.

DeVault, L., R. Sias, and L. Starks. 2019. Sentiment Metrics and Investor Demand. *Journal of Finance* 74:985–1024.

Dumas, B., A. Kurshev, and R. Uppal. 2009. Equilibrium Portfolio Strategies in the Presence of Sentiment Risk and Excess Volatility. *Journal of Finance* 64:579–629.

Giglio, S., and B. Kelly. 2018. Excess Volatility: Beyond Discount Rates. *Quarterly Journal of Economics* 133:71–127.

Giglio, S., M. Maggiori, J. Stroebel, and S. Utkus. 2021. Five Facts about Beliefs and Portfolios. *American Economic Review* 111:1481–1522.

Greenwood, R., and S. Nagel. 2009. Inexperienced Investors and Bubbles. *Journal of Financial Economics* 93:239–258.

Gromb, D., and D. Vayanos. 2010. Limits of Arbitrage: The State of the Theory. *Annual Review of Financial Economics* 2:251–275.

Hirshleifer, D. 2015. Behavioral Finance. *Annual Review of Financial Economics* 7:133–159.

Hodor, I., and F. Zapatero. 2022. Asset Pricing Implications of Heterogeneous Investment Horizons. Working Paper.

Hong, H., W. Jiang, N. Wang, and B. Zhao. 2014. Trading for Status. *Review of Financial Studies* 27:3171–3212.

Hong, H., and J. C. Stein. 1999. A unified theory of underreaction, momentum trading, and overreaction in asset markets. *Journal of Finance* 54:2143–2184.

Kogan, L., S. A. Ross, J. Wang, and M. M. Westerfield. 2006. The Price Impact and Survival of Irrational Traders. *Journal of Finance* 61:195–229.

Krishnamurthy, A., and W. Li. 2021. Dissecting Mechanisms of Financial Crises: Intermediation and Sentiment. Working Paper.

Kumar, A., and C. Lee. 2006. Retail Investor Sentiment and Return Comovements. *Journal of Finance* 61:2451–2486.

Martin, I., and D. Papadimitriou. 2022. Sentiment and Speculation in a Market with Heterogeneous Beliefs. *American Economic Review* 112:2465–2517.

Meeuwis, M., J. A. Parker, A. Schoar, and D. I. Simester. 2021. Belief Disagreement and Portfolio Choice. Forthcoming, *Journal of Finance*.

Mele, A. 2007. Asymmetric stock market volatility and the cyclical behavior of expected returns. *Journal of Financial Economics* 86:446–478.

Panageas, S. 2020. The Implications of Heterogeneity and Inequality for Asset Pricing: A Survey. *Foundations and Trends in Finance* 12:199–275.

Scheinkman, J. A., and W. Xiong. 2003. Overconfidence and Speculative Bubbles. *Journal of Political Economy* 111:1183–1219.

Shiller, R. J. 1979. The Volatility of Long-Term Interest Rates and Expectations Models of the Term Structure. *Journal of Political Economy* 87:1190–1219.

Shiller, R. J. 1981. Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends? *American Economic Review* 71:421–436.

Sockin, M., and M. Z. Xiaolan. 2019. Delegated Learning in Asset Management. Working Paper.

Sotes-Paladino, J. M., and F. Zapatero. 2019. Riding the Bubble with Convex Incentives. *Review of Financial Studies* 32:1416–1459.

Vayanos, D., and P. Woolley. 2013. An Institutional Theory of Momentum and Reversal. *Review of Financial Studies* 26:1087–1145.

Appendix

A Proofs

Proof of Proposition 1. Market completeness allows us to rewrite the investors' optimization problems as:

$$\begin{array}{ll} \max_{W_T^k} & E_t[\xi_T^k U_k(W_T^k)] \\ \text{s.t.} & E_t[\pi_T W_T^k] \leq W_0^k. \end{array}$$

The first order conditions are given by

$$\begin{aligned} \frac{\xi_T^R}{W_T^R} &= \psi_R \pi_T \\ \frac{\xi_T^I Y_T}{W_T^I} &= \psi_I \pi_T, \end{aligned}$$

where ψ_R and ψ_I are the Lagrange multipliers associated with the retail and institutional investors' budget constraints respectively. Using the fact that $W_T^R + W_T^I = S_T = D_T$, we find that

$$\begin{aligned} \pi_t &= E_t[\pi_T] \\ &= \frac{1}{\psi_R} E_t\left[\frac{\xi_T^R}{D_T}\right] + \frac{1}{\psi_I} E_t\left[\frac{\xi_T^I Y_T}{D_T}\right]. \end{aligned}$$

It follows that

$$\begin{aligned} W_t^R &= \frac{1}{\pi_t} E_t\left[\pi_T W_T^R\right] \\ &= \frac{\xi_t^R}{\psi_R \pi_t}. \end{aligned} \tag{26}$$

and similarly

$$\begin{aligned} W_t^I &= \frac{1}{\psi_I \pi_t} E_t\left[\xi_T^I Y_T\right] \\ &= \frac{\xi_t^I}{\psi_I \pi_t}\left(1 - \upsilon + \upsilon D_t e^{\mu(T-t)}\right). \end{aligned} \tag{27}$$

Finally let ϖ_t^R and ϖ_t^I denote the shares of wealth of the retail and institutional investors, respectively, so that $\varpi_t^R + \varpi_t^I = 1$. Note that

$$\frac{\xi_t^R \psi_I}{\xi_t^I \psi_R} = \frac{\varpi_t^R}{\varpi_t^I}\left(1 - \upsilon + \upsilon D_t e^{\mu(T-t)}\right).$$

The state price density is given by

$$\pi_t = \frac{\xi_t^I}{\psi_I D_t}\left[\frac{\xi_t^R \psi_I}{\xi_t^I \psi_R} e^{-(\mu+\sigma^2(\delta^R-1))(T-t)} + \upsilon D_t + (1-\upsilon) e^{-(\mu-\sigma^2)(T-t)}\right].$$

Writing $d\pi_t = -\pi_t \kappa_t dB_t$ and applying Ito's lemma, we find that the market price of risk κ is given by:

$$
\begin{aligned}
\kappa_t &= \sigma\left(1 - \frac{\delta^R \varpi_t^R \big(1-\upsilon+\upsilon D_t e^{\mu(T-t)}\big) e^{-(\mu+\sigma^2(\delta^R-1))(T-t)} + \varpi_t^I \upsilon D_t}{\varpi_t^R \big(1-\upsilon+\upsilon D_t e^{\mu(T-t)}\big) e^{-(\mu+\sigma^2(\delta^R-1))(T-t)} + \varpi_t^I \big(\upsilon D_t + (1-\upsilon) e^{-(\mu-\sigma^2)(T-t)}\big)}\right) \\
&= \sigma\left(1 - \frac{\delta^R \varpi_t^R e^{-(\mu+\sigma^2(\delta^R-1))(T-t)} + \varpi_t^I q_t e^{-\mu(T-t)}}{\varpi_t^R e^{-(\mu+\sigma^2(\delta^R-1))(T-t)} + \varpi_t^I \frac{\upsilon D_t + (1-\upsilon)e^{-(\mu-\sigma^2)(T-t)}}{1-\upsilon+\upsilon D_t e^{\mu(T-t)}}}\right) \\
&= \sigma\left(1 - \frac{\delta^R \varpi_t^R e^{-(\mu+\sigma^2(\delta^R-1))(T-t)} + \varpi_t^I q_t e^{-\mu(T-t)}}{\varpi_t^R e^{-(\mu+\sigma^2(\delta^R-1))(T-t)} + \varpi_t^I e^{-(\mu-\sigma^2)(T-t)}\big(1-q_t(1-e^{-\sigma^2(T-t)})\big)}\right) \\
&= \sigma\left(1 - \frac{\delta^R \varpi_t^R e^{-\sigma^2\delta^R(T-t)} + \varpi_t^I q_t e^{-\sigma^2(T-t)}}{\varpi_t^R e^{-\sigma^2\delta^R(T-t)} + \varpi_t^I \big(1-q_t(1-e^{-\sigma^2(T-t)})\big)}\right) \\
&= \sigma\left(1 - \frac{\delta^R(1-\varpi_t^I)\big(1-\gamma(\delta^R(T-t))\big) + \big(1-\gamma(T-t)\big)\varpi_t^I q_t}{(1-\varpi_t^I)\big(1-\gamma(\delta^R(T-t))\big) + \varpi_t^I\big(1-q_t\gamma(T-t)\big)}\right),
\end{aligned}
$$

where we used to fact that $q_t = \frac{\upsilon D_t e^{\mu(T-t)}}{1-\upsilon+\upsilon D_t e^{\mu(T-t)}}$ and $\gamma(\tau) = 1-e^{-\sigma^2\tau}$. Observe that in absence of institutional investors, i.e., $\upsilon = 0$, we simply have

$$\kappa_t \triangleq \kappa_t^{SE} = \sigma(1-\delta^R).$$

The case $\delta^R = 0$ defines the BP economy of Section 4.1.1, and we have

$$\kappa_t^{BP} = \sigma\left(1 - \frac{\big(1-\gamma(T-t)\big)\varpi_t^I q_t}{1-\varpi_t^I + \varpi_t^I\big(1-q_t\gamma(T-t)\big)}\right)$$

Comparing the market price of risk given in relation (4) with its equilibrium value in an economy where there is no institutional investor, i.e., $\upsilon = 0$ given in relation (A), it is easy to verify that $\kappa_t < \kappa_t^{SE}$ whenever

$$\delta^R - q_t < q_t \frac{1-(1-q_t)\gamma(T-t)}{1-q_t\gamma(T-t)}.$$

The equilibrium stock price over dividend ratio is given by

$$
\begin{aligned}
S_t/D_t &= (W_t^R + W_t^I)/D_t \\
&= \frac{1}{\varpi_t^R e^{-(\mu+\sigma^2(\delta^R-1))(T-t)} + \varpi_t^I \frac{\upsilon D_t + (1-\upsilon)e^{-(\mu-\sigma^2)(T-t)}}{1-\upsilon+\upsilon D_t e^{\mu(T-t)}}} \\
&= \frac{e^{(\mu-\sigma^2)(T-t)}}{\varpi_t^R + \varpi_t^I \frac{\upsilon D_t e^{(\mu-\sigma^2)(T-t)} + (1-\upsilon)}{1-\upsilon+\upsilon D_t e^{\mu(T-t)}}} \\
&= \overline{(S/D)}_t \frac{1}{\varpi_t^R\big(1-\gamma(\delta^R(T-t))\big) + \varpi_t^I\big(1-\gamma(T-t)q_t\big)},
\end{aligned}
$$

where we used the definition of q_t, the fact that $\varpi_t^R = 1-\varpi_t^I$ as well as $1-\upsilon = \frac{1-q_t}{q_t}\upsilon D_t e^{\mu(T-t)}$ and $\gamma(T-t) = 1-e^{-\sigma^2(T-t)}$. □

Proof of Proposition 2. We use relations (26) and (27) to apply Ito's lemma and we identify the diffusion terms with those given by budget constraints (2). This leads to

$$
\begin{aligned}
\theta_t^R \sigma_{S,t} &= \kappa_t + \sigma\delta^R \\
\theta_t^I \sigma_{S,t} &= \kappa_t + \sigma q_t.
\end{aligned}
$$

Then, since $\varpi_t^R \theta_t^R + \varpi_t^I \theta_t^I = 1$, we observe that leverage $(\theta_t^R - 1)\varpi_t^R$ is given by:

$$(\theta_t^R - 1)\varpi_t^R = \varpi_t^R \varpi_t^I \frac{\sigma}{\sigma_{S,t}} (\delta^R - q_t).$$

From $\varpi_t^R \theta_t^R + \varpi_t^I \theta_t^I = 1$, we also obtain that

$$\begin{aligned}
\sigma_{S,t} &= \kappa_t + (1-\varpi_t^I)\sigma\delta^R + \varpi_t^I \sigma q_t \\
&= \sigma \left(1 - \frac{\delta^R (1-\varpi_t^I) e^{-\sigma^2 \delta^R (T-t)} + \big(1-\gamma(T-t)\big)\varpi_t^I q_t}{(1-\varpi_t^I) e^{-\sigma^2 \delta^R (T-t)} + \varpi_t^I \big(1 - q_t \gamma(T-t)\big)} + \varpi_t^R \delta^R + \varpi_t^I q_t \right) \\
&= \bar{\sigma}_S \left(1 + \varpi_t^I \frac{\gamma(T-t) q_t (1-q_t) + (1-\varpi_t^I)\big(1 - \gamma(T-t) q_t - \big(1 - \gamma(\delta^R (T-t))\big)\big)(\delta^R - q_t)}{(1-\varpi_t^I)\big(1-\gamma(\delta^R(T-t))\big) + \varpi_t^I \big(1 - \gamma(T-t) q_t\big)} \right).
\end{aligned}$$

If $\upsilon = 0$, then $q_t = 0, \delta^R - q_t = \delta^R$, so we have

$$\sigma_{S,t} = \bar{\sigma}_S \left(1 + \delta^R \varpi_t^I (1-\varpi_t^I) \frac{\gamma(\delta^R(T-t))}{(1-\varpi_t^I)\big(1-\gamma(\delta^R(T-t))\big) + \varpi_t^I} \right)$$

It is easy to verify that in this case the volatility is increasing in the degree of optimism δ^R.

Finally, we show that $\sigma_{S,t} \geq \overline{\sigma}_S$: Given relation (8) in the paper, it is enough to show that

$$\gamma(T-t) q_t (1-q_t) + \big(\gamma(\delta^R(T-t)) - \gamma(T-t) q_t\big)(\delta^R - q_t) \geq 0$$

The first term is always positive and the second term is negative iff $\check{\delta}^R \leq \delta^R \leq q_t$, where $\check{\delta}^R$ is defined in (22). For $0 < a \leq x \leq 1$, define auxiliary function φ_a with

$$\varphi_a(x) = \gamma(T-t)x + \gamma(a(T-t))(a-x) - \gamma(T-t)ax.$$

φ_a is linear in x with $\varphi_a(a) = \gamma(T-t)a(1-a)$ and $\varphi_a(1) = (1-a)\big(\gamma(T-t) - \gamma(a(T-t))\big) > 0$ as $a \leq 1$ and function γ is increasing. We conclude that φ_a is non-negative. $\square$

Proof of Lemma 1. Recall that

$$S_t/D_t = \overline{(S/D)}_t \frac{1}{\varpi_t^I \big(1 - \gamma(T-t) q_t\big) + (1-\varpi_t^I)\big(1 - \gamma(\delta^R(T-t))\big)},$$

and $q_t = \frac{\upsilon D_t e^{\mu(T-t)}}{1-\upsilon+\upsilon D_t e^{\mu(T-t)}}$. Let us formally write

$$dq_t/q_t = \mu_{q_t} dt + \sigma_{q_t} dB_t,$$

and observe that by Ito's lemma $\sigma_{q_t} = \frac{\frac{\partial q_t}{\partial D_t}}{q_t} D_t \sigma = (1-q_t)\sigma$. Then, we have the following stock volatility decomposition

$$\sigma_{S,t} = \varepsilon_{S,t}^D \sigma + \varepsilon_{S,t}^q \sigma_{q_t} + \varepsilon_{S,t}^{\varpi^I} \sigma_{\varpi_t^I}, \tag{28}$$

where $\varepsilon^x_{S,t} = \frac{\partial S_t}{\partial x_t} \times \frac{x_t}{S_t}$ denotes the elasticity of the stock price with respect to variable x at time t, and

$$\begin{aligned}
\varepsilon^D_{S,t} &= 1,\\
\varepsilon^q_{S,t} &= \frac{\varpi^I_t \gamma(T-t)q_t}{\varpi^I_t\big(1-\gamma(T-t)q_t\big) + (1-\varpi^I_t)\big(1-\gamma(\delta^R(T-t))\big)} > 0,\\
\varepsilon^{\varpi^I}_{S,t} &= \frac{\gamma(T-t)q_t - \gamma(\delta^R(T-t))}{\varpi^I_t\big(1-\gamma(T-t)q_t\big) + (1-\varpi^I_t)\big(1-\gamma(\delta^R(T-t))\big)}\varpi^I_t.
\end{aligned}$$

Given definition (12), expressions (13) and (14) then follow in a straightforward way from decomposition (9).
□

Proof of Lemma 2. This is a special case of the proof of Proposition 1 when $\upsilon = 0$. □

Proof of Lemma 3. This is a special case of the proof of Proposition 2 when $\upsilon = 0$. □

Proof of Lemma 4. Replacing variable ϖ^R_t by variable ϖ^I_t in relation (18), the price-dividend ratios in the BP and in the SENT economies are equal if and only if

$$\frac{1}{1-\varpi^I_t(1-e^{-\sigma^2\delta^R(T-t)})} = \frac{1}{1-\gamma(T-t)\varpi^I_t q_t},$$

or, equivalently, $\gamma(T-t)q_t = 1-e^{-\sigma^2\delta^R(T-t)}$, i.e., $\delta^R_t = \frac{\log(1-\gamma(T-t)q_t)}{\log(1-\gamma(T-t))} \triangleq \check{\delta}^R_t$.

To show that $\check{\delta}^R < q_t$, i.e., that $\log[1-\gamma(T-t)q_t] < q_t \log[1-\gamma(T-t)]$ define, for $(a,x) \in (0,1)^2$, the auxiliary function φ_a, with $\varphi_a(x) = \log[1-ax] - x\log[1-a]$. Observe that $\varphi_a''(x) = -a^2/(1-ax)^2 < 0$, so that φ_a is concave with $\varphi_a(0) = \varphi_a(1) = 0$, so φ_a must be positive on $[0,1]$. We conclude that $\check{\delta}^R < q_t$.

Next, we show that $\kappa^{BP}_t > \kappa^{SE}_t|_{\delta^R=\check{\delta}^R}$. When $\delta^R = \check{\delta}^R$, we have $\gamma(T-t)q_t = \gamma(\delta^R(T-t))$, and replacing ϖ^R_t by ϖ^I_t in relation (19), we find that

$$\kappa^{SE}_t = \overline{\kappa}\left(1 - \frac{\varpi^I_t(1-\gamma(T-t)q_t)\delta^R}{1-\varpi^I_t\gamma(T-t)q_t}\right).$$

Thus $\kappa^{BP}_t \geq \kappa^{SE}_t$ iff

$$(1-\gamma(T-t)q_t)\delta^R \geq (1-\gamma(T-t))q_t,$$

or equivalently

$$\frac{\log[1-\gamma(T-t)q_t]}{\log[1-\gamma(T-t)]} \geq \frac{(1-\gamma(T-t))q_t}{1-\gamma(T-t)q_t}.$$

For $a \in (0,1)$, define auxiliary function φ_a with

$$\varphi_a(x) = (1-ax)\log(1-ax) - (1-a)x\log(1-a),$$

for $x \in [0,1]$. Observe that $\varphi_a''(x) = \frac{a^2}{1-ax} > 0$. Since $\varphi_a(0) = \varphi_a(1) = 0$ and φ_a is convex, it must be the case that φ_a is negative on $[0,1]$. Since $\log[1-\gamma(T-t)] < 0$, we conclude that we always have $\kappa^{BP}_t \geq \kappa^{SE}_t|_{\delta^R=\check{\delta}^R}$.

Finally, we show that $\sigma^{BP}_{S,t} \geq \sigma^{SE}_{S,t}|_{\delta^R=\check{\delta}^R}$. When $\delta^R = \check{\delta}^R$ we have, replacing ϖ^R_t by ϖ^I_t in relation (20),

$$\sigma^{SE}_{S,t}|_{\delta^R=\check{\delta}^R} = \overline{\sigma}\left(1 + \frac{\varpi^I_t(1-\varpi^I_t)\gamma(T-t)q_t}{1-\varpi^I_t\gamma(T-t)q_t}\frac{\log[1-\gamma(T-t)q_t]}{\log[1-\gamma(T-t)]}\right)$$

Thus $\sigma^{BP}_{S,t} \geq \sigma^{SE}_{S,t}$ iff

$$(1-\varpi^I_t)\frac{\log[1-\gamma(T-t)q_t]}{\log[1-\gamma(T-t)]} < 1 - \varpi^I_t q_t.$$

It is easy to verify that auxiliary function φ, with $\varphi(x) = -1 - \varpi x + (1-\varpi)\frac{\log[1-ax]}{\log[1-a]}$ and $(a, \varpi) \in (0,1)^2$, is increasing on $[0,1]$ and that $\varphi(1) = 0$. Thus, φ is negative on $[0,1]$. We conclude that when $\delta^R = \check{\delta}^R$, we always have $\sigma^{BP}_{S,t} \geq \sigma^{SE}_{S,t}$. $\square$

Proof of Lemma 5. To account for the presence of rational institutions, we let $\upsilon > 0$. Replacing ϖ^R_t by $1 - \varpi^I_t$ in Eq. (3) and taking the partial derivative of S_t/D_t with respect to ϖ^I_t, we obtain the condition

$$\frac{\partial(S_t/D_t)}{\partial \varpi^I_t} < 0 \text{ iff } e^{-\sigma^2\delta^R(T-t)} < 1 - \gamma(T-t)q_t,$$

or, equivalently, iff $\delta^R > \log\big(1-\gamma(T-t)q_t\big)/\log\big(1-\gamma(T-t)\big) = \check{\delta}^R_t$. To sum up, we have

$$\frac{\partial(S_t/D_t)}{\partial \varpi^I_t} \begin{cases} > 0, & \delta^R < \check{\delta}^R_t \\ = 0, & \delta^R = \check{\delta}^R_t \\ < 0, & \delta^R > \check{\delta}^R_t \end{cases}.$$

$\square$

Proof of Proposition 3. Recall that

$$\begin{aligned} \sigma_{S,t} &= \sigma\left(1 - \frac{\delta^R\varpi^R_t e^{-\sigma^2\delta^R(T-t)} + \varpi^I_t q_t\big(1-\gamma(T-t)\big)}{\varpi^R_t e^{-\sigma^2\delta^R(T-t)} + \varpi^I_t\big(1-q_t\gamma(T-t)\big)} + \varpi^R_t(\delta^R - q_t) + q_t\right) \\ &= \sigma\left(1 + \varpi^I_t q_t + \varpi^I_t\Big(-x - \frac{q_t\big(1-\gamma(T-t)\big) - x\big(1-q_t\gamma(T-t)\big)}{\varpi^R_t e^{-\sigma^2\delta^R(T-t)} + \varpi^I_t\big(1-q_t\gamma(T-t)\big)}\Big)\right) \\ &= \sigma\left(1 + \varpi^I_t q_t + \varpi^I_t\Big(-x + \frac{x-a}{\varpi^I_t + b\varpi^R_t e^{-\theta x}}\Big)\right) \end{aligned}$$

with

$$\begin{aligned} x &= \delta^R \\ a &= \frac{q_t\big(1-\gamma(T-t)\big)}{1-q_t\gamma(T-t)} \\ b &= \frac{1}{1-q_t\gamma(T-t)} \\ \theta &= \sigma^2(T-t). \end{aligned}$$

Then, define

$$\varphi(x) = -x + \frac{x-a}{\varpi^I_t + b\varpi^R_t e^{-\theta x}}.$$

We have .

$$\varphi'(x) = -1 + \frac{(\varpi^I_t + b\varpi^R_t e^{-\theta x}) + \theta(x-a)b\varpi^R_t e^{-\theta x}}{(\varpi^I_t + b\varpi^R_t e^{-\theta x})^2}.$$

Set $z = \varpi^I_t + b\varpi^R_t e^{-\theta x}$ so that

$$\varphi'(x) = \frac{z-\varpi^I_t}{z^2}\Big(\frac{(1-z)z}{z-\varpi^I_t} - \log(z-\varpi^I_t) - a\theta + \log(b\varpi^R_t)\Big).$$

Finally, consider $\psi(z) = \frac{(1-z)z}{z-\varpi_t^I} - \log(z - \varpi_t^I) - a\theta + \log(b\varpi_t^R)$ with $z > \varpi_t^I$. We have

$$\psi'(z) = -1 - \frac{\varpi_t^I \varpi_t^R}{(z - \varpi_t^I)^2} - \frac{1}{z - \varpi_t^I} < 0.$$

Then, note that $\lim_{z\to(\varpi_t^I)^+} \psi(z) = \infty$ and $\lim_{z\to\infty} \psi(z) = -\infty$. It follows that there is a unique z^* such that $\psi(z^*) = 0$ and $\psi > 0$ (respectively, < 0) on (ϖ_t^I, z^*) (resp., (z^*, ∞)). Then, define

$$x^* = -\frac{1}{\theta} \log \frac{z^* - \varpi_t^I}{b\varpi_t^R}.$$

Since variable z is decreasing in variable x, we deduce that $\varphi'(x) > 0$ (resp., < 0) iff $x > x^*$ (resp., $x < x^*$).

Finally observe that when $\delta^R = \check{\delta}^R$, we have $x = \check{x}$ and $\check{x}$ is such that $b = e^{\theta\check{x}}$, which implies that the corresponding value of z denotes $\check{z}$ is such that $\check{z} = 1$. Then, $\psi(1) = -a\theta + \log b$. Then, set $s = 1 - e^{-\theta} \in (0, 1)$ and observe that $a = \frac{q_t(1-s)}{1-q_t s}$ and $b = \frac{1}{1-q_t s}$, so that

$$\psi(1) = q_t(1-s)\log(1-s) - (1-q_t s)\log(1-q_t s).$$

Then recall that $q_t \in [0, 1]$ and notice that $g(y) = y \log y$ is a convex function so that

$$g(q_t(1-s) + 1 - q_t) < q_t g(1-s) + (1-q_t)g(1).$$

As $g(1) = 0$, we obtain that $\psi(1) > 0$. Thus, we must have $\check{z} < z^*$, which implies that for all $x < \check{x}$, i.e., $\delta^R < \check{\delta}_t^R$, as δ^R increases, the stock volatility decreases for any wealth distribution. $\square$

Proof of Lemma 6. Recall that

$$\frac{\xi_t^R \psi_I}{\xi_t^I \psi_R} = \frac{\varpi_t^R}{\varpi_t^I}\left(1 - \upsilon + \upsilon D_t e^{\mu(T-t)}\right).$$

It follows that

$$\frac{\varpi_0^R}{\varpi_0^I} e^{-\frac{1}{2}\sigma^2(\delta^R)^2 t + \sigma\delta^R B_t} = \frac{\varpi_t^R}{\varpi_t^I} \frac{1 - \upsilon + \upsilon D_t e^{\mu(T-t)}}{1 - \upsilon + \upsilon D_0 e^{\mu T}}.$$

Then, as $D_t = D_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$, given $D_0 > 0$, we find that

$$\frac{\varpi_0^R}{\varpi_0^I} e^{-\frac{1}{2}\sigma^2(\delta^R)^2 t - (\mu - \frac{\sigma^2}{2})\delta^R t} \left(D_t/D_0\right)^{\delta^R} = \frac{\varpi_t^R}{\varpi_t^I} \frac{1 - \upsilon + \upsilon D_t e^{\mu(T-t)}}{1 - \upsilon + \upsilon D_0 e^{\mu T}},$$

i.e.,

$$\frac{\varpi_0^R}{\varpi_0^I} e^{-\frac{1}{2}\sigma^2(\delta^R)^2 t - (\mu - \frac{\sigma^2}{2})\delta^R t} \left(D_t/D_0\right)^{\delta^R} = \frac{\varpi_t^R}{\varpi_t^I} \frac{1 - \upsilon + \upsilon D_t e^{\mu(T-t)}}{1 - \upsilon + \upsilon D_0 e^{\mu T}},$$

i.e.,

$$\varpi_t^I = \frac{\varpi_0^I}{(1 - \varpi_0^I)\frac{1-\upsilon+\upsilon D_0 e^{\mu T}}{1-\upsilon+\upsilon D_t e^{\mu(T-t)}} \left(D_t/D_0\right)^{\delta^R} e^{-\frac{1}{2}\sigma^2(\delta^R)^2 t - (\mu - \frac{\sigma^2}{2})\delta^R t} + \varpi_0^I}.$$

Then, from the definition of q_t, one can check that

$$\begin{aligned} D_t/D_0 &= \frac{q_t}{1-q_t}\frac{1-q_0}{q_0} e^{\mu t}, \\ \frac{1 - \upsilon + \upsilon D_0 e^{\mu T}}{1 - \upsilon + \upsilon D_t e^{\mu(T-t)}} &= \frac{q_t}{q_0}\frac{D_0}{D_t} e^{\mu t} = \frac{1-q_t}{1-q_0}. \end{aligned}$$

It follows that

$$\begin{aligned}\varpi_t^I &= \frac{\varpi_0^I}{\varpi_0^I + (1-\varpi_0^I)\frac{1-q_t}{1-q_0}\left(\frac{q_t}{1-q_t}\frac{1-q_0}{q_0}e^{\mu t}\right)^{\delta^R} e^{-\frac{1}{2}\sigma^2(\delta^R)^2 t-(\mu-\frac{\sigma^2}{2})\delta^R t}} \\ &= \frac{\varpi_0^I}{\varpi_0^I + (1-\varpi_0^I)\left(\frac{q_t}{q_0}\right)^{\delta^R}\left(\frac{1-q_t}{1-q_0}\right)^{1-\delta^R} e^{-\frac{1}{2}\sigma^2\delta^R(\delta^R-1)t}}.\end{aligned}$$

Note that when $\delta^R = 0$ (BP), the expression is much simpler as we get

$$\varpi_t^I = \frac{\varpi_0^I}{\varpi_0^I + (1-\varpi_0^I)\left(\frac{1-q_t}{1-q_0}\right)}.$$

Finally, since $\frac{\partial q_t}{\partial D_t} > 0$, we find that ϖ_t^I is increasing (decreasing) in cash flows D iff auxiliary function φ is decreasing (increasing) where $\varphi(q) = q^{\delta^R}(1-q)^{1-\delta^R}$. φ is a smooth function and

$$\begin{aligned}\varphi'(q) &= q^{\delta^R-1}(1-q)^{-\delta^R}\left(\delta^R(1-q)-(1-\delta^R)q\right) \\ &= q^{\delta^R-1}(1-q)^{-\delta^R}(\delta^R-q).\end{aligned}$$

To sum up we have

$$\frac{\partial(\varpi_t^I)}{\partial D_t}\begin{cases} >0, & \delta^R < q_t \\ =0, & \delta^R = q_t \\ <0, & \delta^R > q_t\end{cases}.$$

□