Revealing Features from Optimal Choice

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Abstract

Suppose that a decision maker (DM) uses *features* (namely, measurable characteristics) to describe alternatives. An analyst cannot observe directly which features of an alternative are considered by a DM, how the DM evaluates them, and which procedure the DM uses to make a choice. To what extent do choices reveal which features matter for the DM? We propose a "Pareto dominance" approach: the only assumption we make on the DM decision process is that the DM does not make choices that are Pareto dominated in the set of relevant features. In this framework, we characterise exactly which (collections of) pairs of a choice observation and a feasible set are informative about the features that the DM uses in her internal representation of alternatives.

JEL codes: D81, D83, D91

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1 Introduction

1.1 Motivation

Suppose that a decision maker (DM) uses *features* (namely, measurable characteristics) to describe alternatives. For instance, a consumer product can be described as

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a set of technical and economic characteristics and a political party in terms of its position on several issues. The number of conceivable features is typically large, and it is psychologically unrealistic to take for granted that the DM considers all of them. In fact, the subset of relevant features used in the DM's internal representation of the alternatives may be a very small proportion (as in Gabaix's [15] "sparsity" approach[1]).

Our DM focuses on a set of relevant features and uses them in some unobserved decision procedure to make a choice. What can we learn about the relevant features from the DM's choices? This identification question is important to understand the motivations behind consumption, investment, political decisions, and other activities. Crucially, we would like to obtain the identification with minimal assumptions on the aggregation/decision procedure used by the DM.

To sketch the idea we propose, suppose that there are only two possible features, representing the funding that political parties pledge for each of two issues, say the defence and the international aid budgets.[2] A platform for a party can then be completely described as a pair (x_1, x_2) of real numbers, measuring increases to defence and aid expenditure, respectively. Also, assume for the time being that each voter maximises some unknown *linear* function of the features (we will significantly relax this later). Three parties A, B and C propose, respectively, the following platforms:

$$
\begin{aligned}
x^A &= (5,0) \\
x^B &= (4,4) \\
x^C &= (0,5)\,.
\end{aligned}
$$

Can we tell which features are relevant for the supporters of each party? Even without making any assumption on whether either feature (if relevant) is valued positively or negatively, a vote for B entails that the voter considers *both* the defence and the aid budget. This is because if only defence had been relevant, then the voter would have favoured A if an increase of the defence budget was a positive feature, since x^A maximises the feature. And the voter would have favoured C

[1] Since "out of the thousands of variables that might be relevant, [the DM] takes into account only a few that are important enough to significantly change his decision."

[2] The example aims at simplicity rather than realism. As noted, we have in mind situations with a large number of features.

if that increase was a negative feature, since x^C minimises the feature. A symmetric reasoning excludes the case where only the aid budget is relevant.

On the other hand, a vote for A or for C does not help at all in identifying the relevant features. Both x^A and x^C can be seen as maximisers on the feasible set $X = \{x^A, x^B, x^C\}$ of some non-trivial weighted sum of *any* set of features, for some appropriate choice of weights. Observe that this indeterminacy also holds when a platform is chosen out of the reduced feasible set $\{x^A, x^B\}$.

What this example highlights is that *some* choices from *some* feasible sets are informative about which features are relevant for the DM, while other choices from the same feasible set or the same choices from subsets of the feasible sets are not. The aim of this paper is to develop the above observations into a general theory. We will fully characterise: (i) the structure of the set of possible relevant features; and (ii) which kind of pairs (X, x) of a feasible set and an observed (or assumed known) choice do reveal, partially or fully, the relevant features, under relatively weak assumptions on the DM's decision process.

1.2 The model

In our general analysis, a DM is simply viewed as an entity that takes features as an input to output a choice. We do not use preferences or other psychological variables as a primitive. Our only assumptions are that each feature counts either positively or negatively for the DM, and the following behavioural principle:

Admissibility: An alternative x is not chosen by the DM if there is a different alternative y that has weakly more of all positive relevant features, and weakly less of all negative relevant features.

In other words, we do not specify explicitly a set of admissible aggregation/decision procedures. Instead, we just assume (strong) Pareto optimal behaviour with respect to the relevant features, with the twist that neither the directions of improvement nor the dimensions in which Pareto optimality operates are known to the observer. This approach is in broad conceptual analogy with the "collective" multi-person household model initiated by Chiappori [11] and Apps and Rees [3], which reduces the complex problem of modelling a decision unit guided by several utility functions to the single requirement of Pareto optimality. In this analogy, the utility functions are reinterpreted intra-personally, as the motivations corresponding to the features. Also, recall that even when all features are considered and they are all positive, strongly Pareto optimal points would not be characterised by the maximisation of a linear objective. As recently shown by Che *et al.* [9], they may be rather associated with certain sequential maximisation or sequential Nash bargaining procedures, which again could be reinterpreted in an intrapersonal context as "behavioural" procedures.[3] In the extension of our model to multiple observations (Section 7), the admissibility assumption is compatible with violations of standard consistency restriction like WARP and SARP.[4]

We can now sharpen our identification question as follows. The *type* of the DM is the subset I of features that are relevant. Type I is *possible at* x if, for some hypothesis on the sign that the DM attaches to the features, we cannot find any other feasible alternative that has weakly more of the positive features in I and weakly less of the negative features in I. With this terminology, our question becomes whether, given a choice x, we can exclude at least some type. If so, then the type is *partially identified*. If we can actually exclude all types except one, then the type is *fully identified*.[5] These concepts are illustrated in Figure 1, which we will be repeatedly used in the paper. The feasible set is the convex set in blue. The only possible type at x considers both features (with feature 2 a good and feature 1 a bad). At w, which is "sub-optimal" in the whole feasible set, the type that only consider feature 1 as a positive is possible (and it is the only possible type). This type remains possible at y, but now an additional type, which considers both features (as positives), becomes possible. Since feature 2 can be both increased and decreased, type $\{2\}$ can surely improve and thus it is not possible. Finally, at z, all types are possible (considering features as negatives).

[3] In this vein, see e.g. De Clippel and Eliaz ([12]) for an intrapersonal bargaining model that explains behavioral "anomalies."

[4] Because our framework is quite general and can accommodate various specific decision models, we must remain agnostic on the interesting issue of which cognitive factor (such as attention or preference) determines the neglect of a feature.

[5] In some contexts it may be obvious whether some features (e.g., the price of a commodity) are positive or negative. As will be apparent, our analysis can be straightforwardly adapted to take into account these constraints (through restrictions on the object later called "evaluation function"), which facilitate identification.

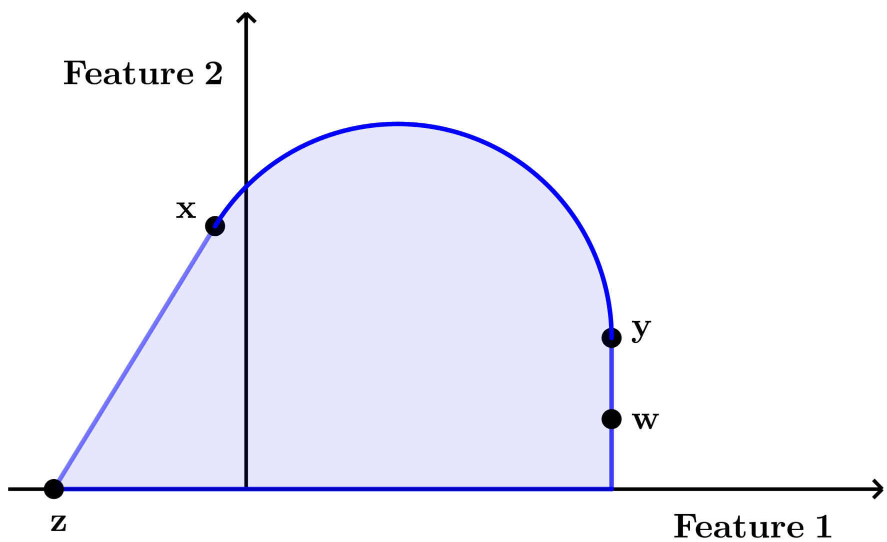


Figure 1: Possibility and identification: the type is fully identified at x and w, only partially identified at y, and not identified at z

1.3 The structure of possibility

We begin by investigating the general *structure* of the set of possible types. For example, suppose that the features are the expenditures on defence, international aid and health. Consider the following questions:

- Is there a conceivable observation (namely, a pair (X, x)) showing that the possible types of voter are exactly the following two: the one that considers only defence expenditure, and the one that considers expenditures on each one of the budgets?

- Suppose that an observation (X, x) shows that the type that considers only defence expenditure and the type that considers only health expenditures are possible. What can we conclude about the possibility of the other types?

We fully characterise the sets of possible types: given an observation, the set of types is closed under union; and conversely any set of types that is closed under union can be the set of possible types for *some* observation. This answers the questions above, and all similar ones.

1.4 The richness conditions for identification.

Next, we derive identification conditions. To understand the core insight, return to the initial example. We have seen that a choice between any pair of alternatives, such as x^B from $X = \{x^A, x^B\}$, could have been made by any type in the linear maximisation case, and this remains true with the more general definitions.[6] A choice like choicex^B is completely uninformative because there is only one feasible kind of "trade-off" between features when moving away from x^B: it is only possible to have a larger defence expenditure and a smaller aid expenditure, but not vice-versa. This is not enough to discriminate between types. A specific form of *richness* in feasible trade-offs is needed. Identifiability is equivalent to ensuring a sufficient variety of feasible trade-offs between features when moving away from the choice. For example, in $X = \{x^A, x^B, x^C\}$ changing the choice from x^B to x^A ($(4,4) \to (5,0)$) would entail diminishing expenditure on aid while increasing that on defence, while changing it from x^B to x^C ($(4,4) \to (0,5)$) would entail the opposite trade-off. The feasible set at x^B is "rich" and ensures identification. On the other hand, if the choice was x^A, changing it to x^B ($(5,0) \to (4,4)$) and changing it to x^C ($(5,0) \to (0,5)$) would entail the same kind of trade-off. The feasible set at x^A is not rich, and there is no identification.

While the finite case provides the intuition, the bulk of our analysis focuses on a general convex feasible set X.[7] In this case the richness requirement for partial identification can be sharply characterised in terms of the geometry of the feasible set around the observed choice, with two equivalent conditions.

The orthant condition says that the set of vectors of feasible directions towards other feasible points from the choice point x is not contained in an orthant.

The non-extremeness condition says that there is a selection of features such that x is not an "extreme" point in X , when X is regarded only in terms of those features.

It might appear at first sight that whenever a feasible set X looks sufficiently "like a segment" at the choice point x, i.e. the alternatives are sufficiently "sparse" around x, then (X, x) is not informative because there are few feasible directions at

[6]Indeed, x^B is admissible by assuming that the first feature is negative and is the only relevant one, making type $\{1\}$ possible; or that the second feature is positive and is the only relevant one, making type $\{2\}$ possible; or by making both the previous assumptions on positivity/negativity, with both features being relevant, making type $\{1, 2\}$ possible.

[7]E.g., in the initial political example, imagine that through a questionnaire the voter can choose out of all combinations of the platforms.

x. But this intuition is a trap: it is important to understand that the richness condition does *not* amount to placing bounds on a measure of sparsity around x. There are feasible sets that are arbitrarily sparse around the choice and yet still satisfy the richness condition.[8] This underscores the important point that in general *both the shape and the orientation* of the feasible set around x matter for identification.

The conditions for full identification are significantly more nuanced but, broadly speaking, they are of the same kind as the orthant condition for partial identification. The main change is that a *"contains an orthant"* type of requirement replaces the "not contained in an orthant" one. Interestingly, the dimensionality of the feasible set, which is not a significant aspect for partial identification, now turns out to be important. A corollary of our results provides answers to questions such as: "There are 100 possible features. Is there a type that we can in principle fully identify by observing the DM's choice from a feasible set that has dimension 99? What about with one of dimension 98?". (What do you think?).

When the convex feasible set X can specifically be obtained as the set of convex combinations of a finite number of points (i.e. a *polytope*), all possible types are "linear" (i.e., they maximise a linear objective function). As a consequence, a linear programming approach can be taken and the identifiability conditions can be expressed in a more operational way in terms of the structure of certain matrices. For the initial case of a finite number of points (without convex combinations), the conditions are essentially analogous.

1.5 Multiple observations

In practice, the analyst may observe more than a single choice event. Therefore, we extend our theory to the case where the data consist of multiple choice occasions. It turns out that most of the heavy-lifting is done by the single-observation analysis: our arguments can be easily adapted to the multi-observation environment. As we shall see, multiple observations not only increase, as expected, the identification power; but they also change in more subtle ways the nature of what

[8]For example, let X^ε be the convex hull of x^A, x^B and $y^\varepsilon = (5+\varepsilon, -\varepsilon)$, where ε is a small positive number. For any ε, any type that cares only about defence is not possible at x^A (which is dominated by y^ε or by x^B according to whether it is a positive or a negative feature), hence there is partial identification. Yet y^ε converges to x^A as ε goes to zero and X^ε can be made arbitrarily similar to the segment between x^A and x^B.

can be identified. Loosely speaking, the way identification power is increased is by enriching the feasible directions by *union*. In other words, the relevant object for the orthant inclusion/exclusion conditions described before becomes the union of the sets of feasible directions.

Beside its general value in solving the theoretical identifiability problem, we hope that the present work can be of use in other specific ways. For example, consider the *design* of experiments, political polls, market surveys and recommendation algorithms.[9] In all these cases the designer controls or influences the feasible set from which DMs choose. Our results are apt to guide the designer in tailoring the set of feasible choices to increase its informativeness.

2 Framework and definitions

Let $F = \{1, \ldots, N\}$ be the set of possible *features*. A DM chooses from a set $X \subseteq \mathbb{R}^N$ of feasible alternatives. An alternative $x = (x_1, ..., x_N) \in X$ is described by the amounts x_i, for all features $i \in F$, of that alternative.[10] Unless otherwise stated, the feasible set X is a nonempty closed bounded convex set of $\mathbb{R}^N$. In what follows we will denote the boundary of X by ∂X, the origin by $\mathbf{0}$, the vectors of ones by $\mathbf{1}$, and the unit vector with i^{th} component equal to one by $\mathbf{1}_i$.

We now introduce our core behavioural assumption.

Definition 1. An *evaluation function* is a function $e : F \to \{-1, 0, 1\}$ such that $e(i) \neq 0$ for some $i \in F$.

Definition 2. A point $x \in X$ is *e-admissible* if

$$(\forall i : e(i)\, y_i \geq e(i)\, x_i) \,\&\, (\exists i : e(i)\, y_i > e(i)\, x_i) \implies y \notin X.$$

The function e is interpreted as indicating whether or not a feature is ignored; and if not, whether it is valued positively or negatively. We have inbuilt in the definition the obvious assumption that not all features are ignored.

[9] On social media platforms algorithms learn what "features" consumers are interested in by presenting users with different "menus".

[10] Here, F is interpreted as the set of all conceivable features that could reasonably describe an alternative. Hence, the DM by assumption cannot use features that are not in F.

The DM's choice $x^* \in \partial X$ is observed. Our behavioural assumption is that there exists an (unobserved) evaluation function e such that the choice x^* is e-admissible.

A *type* is a subset $I \subseteq F$. When the type is a singleton we call it *elementary*. For an evaluation function e, we write $supp\,(e) = I$ to mean $e\,(i) \neq 0 \iff i \in I$.

Definition 3. A type $I \subseteq F$ is *possible* at $x^* \in X$ if there exists an evaluation function e such that:

(i) $supp\,(e) = I$;

(ii) x^* is e-admissible.

In this case, we also say that (e, I) is possible at x^*.

For a given vector $v \in \mathbb{R}^N$, we write $supp\,(v) = I$ to mean $v_i \neq 0 \iff i \in I$. Let $\langle .,. \rangle$ denote the inner product operation. We say that a type I is *linear* at x^* if there exists an $a \in \mathbb{R}^N$ such that $supp\,(a) = I$ and $\langle a, x^* \rangle \geq \langle a, x \rangle$ for all $x \in X$. In other words, a linear type at x^* is such that X is supported at x^* by a hyperplane whose non-zero coefficients correspond exactly with the features in I. In Figure 1, type $\{1\}$ is linear at y (maximising a linear function with direction $(1,0)$) but type $\{1,2\}$ is not, since no hyperplane with coefficients that are all non-zero supports the feasible set at y. The following Lemma, which will be repeatedly used later on, clarifies the relationship between linear and possible types:[11]

Lemma 1. *(i) If type I is linear at x^*, then I is possible at x^*. (ii) If type I is possible at $x^* \in X$, then there exists a type $J \subseteq I$ which is linear at x^*.*

Point y in Figure 1 demonstrates that the set of linear types can be a strict subset of the set of possible types.

Definition 4. The type is:

- *partially identified* at x^* if some type is possible at x^* and there exists a type that is not possible at x^*.
- *fully identified* at x^* if there exists exactly one type that is possible at x^*.
- *not identified* at x^* if all types are possible at x^*.

[11] All proofs are relegated to the Appendix.

2.1 Convex analysis

For the reader's convenience we gather here some standard facts and terminology used in the text. The *affine hull* $aff(S)$ (resp., *convex hull*) of a set $S \in \mathbb{R}^N$ is the smallest affine (resp., convex) set containing S, namely

$$aff(S) = \left\{ \sum_{i=1}^{k} \alpha_i x^i \,|\, k > 0,\, x^i \in S,\, \alpha_i \in \mathbb{R},\, \sum_{i=1}^{k} \alpha_i = 1 \right\},$$
$$conv(S) = \left\{ \sum_{i=1}^{k} \alpha_i x^i \,|\, k > 0,\, x^i \in S,\, \alpha_i \geq 0,\, \sum_{i=1}^{k} \alpha_i = 1 \right\}.$$

The *dimension* $\dim S$ of a set S is the dimension of its affine hull. The *relative interior* $ri\, S$ of a set S is the interior which results when S is regarded as a subset of its affine hull. For a convex set S in $\mathbb{R}^N$, $x \in S$ belongs to $ri\, S$ iff for all $y \in S$ there exists $\lambda > 1$ such that $\lambda x + (1-\lambda) y \in S$. The *face* E of a convex set S is a nonempty convex subset of S such that: if $x \in E$ and $x = \alpha y + (1-\alpha) z$ for some $y, z \in S$ and $\alpha \in (0,1)$, then $y, z \in E$ (i.e. no element of the face can be obtained as a convex combination of elements outside the face). A zero–dimensional face is an *extreme* point. That is, $x \in S$ is extreme if there exists no $y, z \in S \setminus \{x\}$ and $\alpha \in (0,1)$ such that $x = \alpha y + (1-\alpha) z$. A face E of S is *exposed* if there is a $w \in \mathbb{R}^N$ such that $E = \arg\max_{x \in S} \langle w, x \rangle$; if so we say that w exposes E. In particular, a point $x \in S$ is exposed if $x = \arg\max_{x \in S} \langle w, x \rangle$ for some $w \in \mathbb{R}^N$.

A *cone* is a set C such that if $x \in C$ and $\alpha \geq 0$ then $\alpha x \in C$. The *cone of feasible directions* of a set S at $x \in S$ is defined as

$$F_S(x) = \{\alpha (y - x) \,|\, y \in S,\, \alpha \geq 0\}.$$

This is the set of all directions in which it is possible to move away from x while remaining locally within the set S. From an economic perspective, the feasible directions tell us the feasible trade-offs between features.[12] The *tangent cone* at $x \in S$ is defined as $T_S(x) = cl\, F_S(x)$, where *cl* denotes the closure operator.[13] The

[12] A more general definition of the cone of feasible directions is $F_S(x) = \{y \in \mathbb{R}^N |\, \exists \varepsilon > 0 \text{ such that } x + \varepsilon y \in S\}$, but the two definitions coincide given that S is convex.

[13] See e.g. Rockafellar and Wets [18], Thm 6.9, for a proof that for convex sets more general definitions of the tangent cone reduce to this one.

tangent cone is a useful tool to describe the structure of a set S by means of the feasible directions *and* their limits. The *normal cone* of S at $x \in S$ is defined as

$$N_S(x) = \left\{ z \in \mathbb{R}^N | \langle z, y - x \rangle \leq 0, \forall y \in S \right\}.$$

Intuitively, the normal cone at the point $x \in S$ is the set of all vectors that make an acute angle with no vector from x to some point y in the feasible set. When $S \subseteq \mathbb{R}^N$ is a compact, convex set and $a \in \mathbb{R}^N$, the following fact holds:

Fact 1. $x^* \in \arg\max_{x \in S} \langle a, x \rangle$ *if and only if* $a \in N_S(x^*)$.

The *polar cone* S^0 of S is defined as

$$S^0 = \left\{ y \in \mathbb{R}^N | \langle y, x \rangle \leq 0, \forall x \in S \right\}$$

and its negative $-S^0$, which comprises all vectors that form non-obtuse angles with any vector in S, is called the *dual cone*. When S is convex, the normal and the tangent cones at x are polar to each other.

A *convex polytope* is a set $S = conv\left(x^1, ..., x^N\right)$ where the x^i are the *vertices*.[14] A *polyhedron* is a set $S = \{x \in \mathbb{R}^N \mid Bx \leq c\}$ where B is an $m \times N$-matrix and c is a vector in $\mathbb{R}^m$, with m finite. A polyhedron S is bounded if there is some $k > 0$ such that $||x|| \leq k$ for all $x \in S$, where $|| \cdot ||$ denotes the Euclidean norm. A set is a bounded polyhedron if and only if it is a convex polytope.

The *polar set* of a set S is $S^* = \left\{y \in \mathbb{R}^N | \langle y, x \rangle \leq 1 \, \forall x \in S\right\}$. If S is a closed convex set such that $\mathbf{0} \in S$, then $(S^*)^* = S$. If S is a convex polytope and $\mathbf{0} \in int\, S$ then S^* is also a convex polytope and $\mathbf{0} \in int\, S^*$.

3 The structure of possibility

We first show that, when the feasible set is convex, any type can be made possible by some observation on the boundary, and conversely any observation on the boundary makes some type possible.

Proposition 1. *Let X be a convex compact subset of $\mathbb{R}^N$. Then:*

(i) For any type I there exists $x \in \partial X$ such that I is possible at x.

(ii) For any $x \in \partial X$ there exists a type I that is possible at x.

[14] When $S = \left(x^1, ..., x^N\right)$ we denote $conv\,(S)$ as $conv\left(x^1, ..., x^N\right)$.

The proof for part (i) is based on linear optimisation "discovering" the types. As noted previously, however, such a method will not necessarily discover *all* the types that are possible at each given observation using that same observation: it may be necessary to use a different observation. The content of the result from this perspective is that such a different observation always exists. To illustrate this fact, return once again to Figure 1: here, y does not maximise any linear objective in the direction $(1,1)$ but there is another feasible point where this direction is maximised. Convexity is essential: it is easy to construct a non-convex set with observations at which *no* type is possible.

Next, we deal with the question of which sets of possible types can in principle be inferred from the observation. For example, does a set of possible types have an interval structure, in the sense that if $I \subset J \subset K \subseteq F$ and types I and K are possible at some $x \in X$, then J must also be possible? The next example shows that this is not the case.

Example 1. (refer to Figure 2). Consider

$$X = conv\left((0,0,0),(1,0,0),(0,1,0),(0,0,1)\right),$$

and let $x^* = \left(\frac{1}{2},\frac{1}{2},0\right)$. Type $\{1,2,3\}$ is possible at x^* with $e = (1,1,-1)$. With $e = (0,0,-1)$, type $\{3\}$ is also possible at x^*. However type $\{1,3\}$ is not possible at x^*. It is is clear that this type can improve by moving from x^* along the pyramid base either toward the origin or toward the point with the highest first coordinate, $(1,0,0)$, according to whether the first feature counts as a negative or as a positive, respectively.

The main result of this section is a characterisation of the set of possible types. It shows that a collection of types is a collection of possible types for some pair (X, x^*) of feasible set and observation if, and only if, the collection is closed under union.

Theorem 1. *(i) For all $X \subseteq \mathbb{R}^N$ and $x^* \in X$, the set of possible types at x^* is closed under union. (ii) For any nonempty collection $\mathcal{I} \subseteq 2^F$ of types that is closed under union, there exists a convex polytope $X \subseteq \mathbb{R}^N$ and $x^* \in X$ such that the set of possible types at x^* is $\mathcal{I}$.*

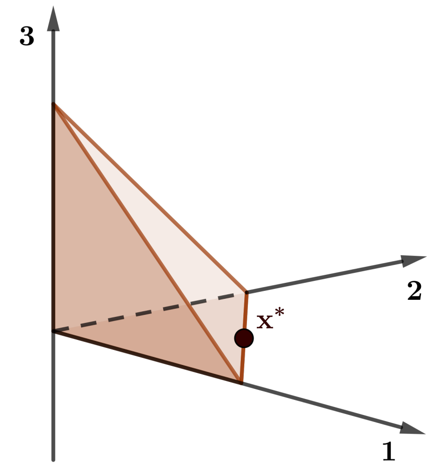


Figure 2: Figure for Example 1: types $\{3\}$ and $\{1, 2, 3\}$ are possible at x^*, while the "intermediate" type $\{1, 3\}$ is not.

4 Partial identification

Partial identification is important in many applications. Suppose for example that the features of a stock are its past returns. In such a financial context, it is useful information that the DM does not look beyond the past two years of history, even if we cannot tell whether both past years are considered, only one of them, or none.[15]

4.1 Characterisation

In this section we characterise partial identifiability from several angles. For our main result, we introduce a new definition:

Definition 5. A point $x \in X$ is *I-extreme*, for $I \subseteq F$, if there exist no $y, z \in X \setminus \{x\}$ and $\alpha \in (0, 1)$ such that $y_i \neq z_i$ and $x_i = \alpha y_i + (1 - \alpha) z_i$, for all $i \in I$.

Thus, a point is I-extreme if it is extreme in the projection of X on the subspace $\mathbb{R}^I \subseteq \mathbb{R}^N$. For example, in Figure 1, z is I-extreme for all $I \subseteq F$, while x is $\{1, 2\}$-extreme (or simply extreme) but neither $\{1\}$-extreme nor $\{2\}$-extreme.

[15]Here are some other examples of relevant partial identification: a policy maker ignores some specific subgroup in society; jurors include race as relevant feature in their judgement; investors/consumers take into account the ethical dimension of a product, like the conditions of the workforce and environmental damage, and in general the ESG score of a company.

Let $C\left(v^1,...,v^N\right) \subseteq \mathbb{R}^N$ denote the convex cone generated by a set of vectors $\left\{v^1,...,v^N\right\}$, that is, $x \in C\left(v^1,...,v^N\right)$ if and only if there exist numbers $\alpha_i \geq 0$ such that $x = \sum_i \alpha_i v^i$. Note this cone is closed. An *orthant* of $\mathbb{R}^N$ is a cone $C\left(v^1,...,v^N\right)$ for which, for all i, $v^i_j \in \{-1,1\}$ if $j = i$ and $v^i_j = 0$ if $j \neq i$.

The following result shows that partial identifiability is equivalent to the feasible directions at the observed choice not being contained in an orthant, and that this is in turn equivalent to the observed choice not being extreme for all types.

Theorem 2. *The following statements are equivalent, for $x^* \in \partial X$:*

(i) The type is partially identified at x^*.

(ii) For all orthants $C\left(v^1,...,v^N\right)$: $F_X\left(x^*\right) \nsubseteq C\left(v^1,...,v^N\right)$.

(iii) There exists an elementary type $\{i\} \subseteq F$ such that x^* is not $\{i\}$-extreme.

Condition (ii) captures two types of intuition regarding identifiability. This first is related to the *internal structure* -that is, independent of the coordinate system- of the feasible set at the observed point, while the second is related to the *orientation* of the feasible set in the coordinate space. Let's examine the first intuition. When X forms a "sharper" shape around x^*, it means that there are fewer feasible directions in which to move. As a consequence it is harder to improve, and then more types tend to become possible, undermining identification. Conversely, if X forms a "fatter" shape around x^*, so that there are more feasible directions in which to move, more types will become excluded, favouring identification. To make this argument formally, we express the idea of fatness through the relationship between the tangent cone at x^* and its dual cone $-T^0_X\left(x^*\right)$, and say that X is *fat at $x^* \in X$* if $T_X\left(x^*\right) \nsubseteq -T^0_X\left(x^*\right)$. If there exists $x \in T_X\left(x^*\right)$ for which $x \notin -T^0_X\left(x^*\right)$, it means that there exists a $y \in T_X\left(x^*\right)$ which forms an obtuse angle with x. In other, words, there is at least a direction in which the set $T_X\left(x^*\right)$ is fat (while it can be sharp in other directions).[16]

Corollary 1. *The type is partially identified at any x^* at which X is fat.*

[16]Note that the two-dimensional cases may be slightly misleading in this respect. In two dimensions, either $T_X\left(x^*\right) \subseteq -T^0_X\left(x^*\right)$ or $-T^0_X\left(x^*\right) \subseteq T_X\left(x^*\right)$, that is, either the set of feasible directions is unambiguously non-obtuse or it is unambiguously non-acute. But in higher dimensions, a cone may be thin in some directions and fat in others, so that neither of the dual containment relations holds.

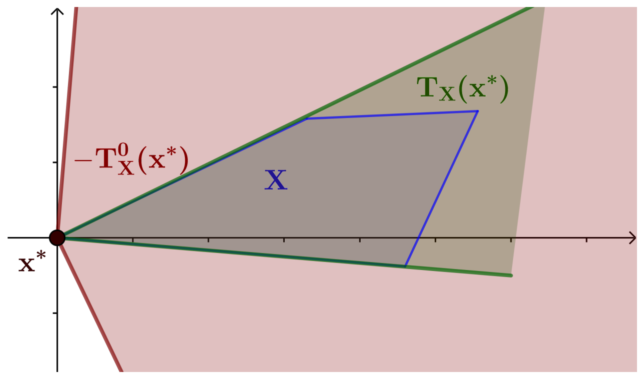


Figure 3: Sharpness with partial identification

But fatness is not necessary: even a sharp feasible set can be oriented in the right way for partial identification. See Figure 3, where type $\{2\}$ is not possible.

Therefore, shape can only be part of the story, and our conditions do not simply amount to bounds on the sparsity of alternatives around the choice. The orthant condition says that for identifiability the feasible directions vectors must straddle more than one orthant. This condition is thus in terms of a *relation* between a measure of sparsity around x^* and the coordinate system. While the requirement is favoured by fatter shapes around x^*, for a *given* geometric shape of the feasible set X, identification may depend on how X is oriented in the coordinate space. Figure 4 illustrates the identifying effect of rotation.

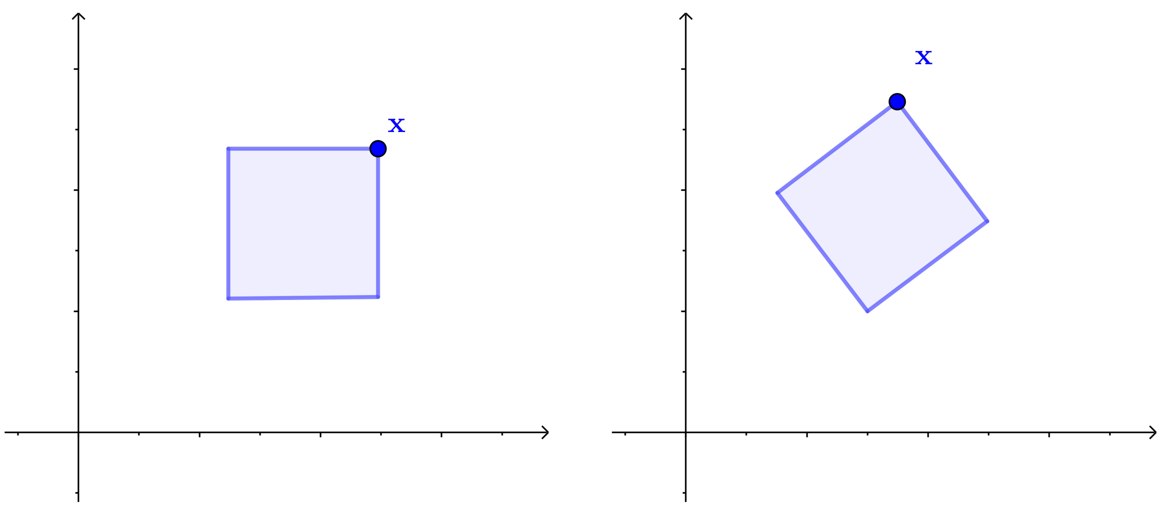


Figure 4: Identifying effect of rotation: the type is not identified at x in the feasible set on the left, but it becomes partially identified at x after the rotation on the right, where the horizontal type is not possible.

Condition (iii) can be better understood when related to the standard notions of extremeness and exposure. Recall that every exposed point is extreme, while the opposite does not necessarily hold (e.g., this is the case for points x and y in Figure 1). Extreme but not exposed points arise naturally in economic contexts, whenever a tradeoff between two features is linear on a range and nonlinear in a subsequent range. Think for example of nonlinear discounting beyond a threshold quantity, or of congestion or pollution costs starting to rise nonlinearly in the proximity of urban centres.

Corollary 2. *The type is partially identified at any $x^* \in \partial X$ that is not exposed.*

Since any exposed point is extreme, an immediate implication of Corollary 2 is that non-extremeness is a sufficient condition for partial identification. However, it is too strong to be also necessary. Theorem 2 fills the gap by devising a less stringent form of non-extremeness that is both necessary and sufficient for partial identification.[17]

Finally, the proof has also shown:

Corollary 3. *The type is partially identified at x^* if and only if there exists an elementary type $\{i\} \subseteq F$ that is not possible at x^*.*

This is interesting in two respects. First, since an elementary type is obviously linear, we see that partial identification is equivalent to the exclusion of some *linear* type. In other words, nothing would change in the statement of Theorem 2 if we had considered only a DM that is a linear maximiser. This will *not* hold for full identification. Second, the "only if" direction yields a simple test for (the lack of) partial identification, reducing from $2^N - 1$ to N the number of types to check in order to conclude that the type is not identified. In Section 6 we provide a concrete way to perform this check in terms of the properties of certain matrices, for the case where X is a convex polytope.

5 Full Identification

In this section we characterise full identifiability. As we have seen, for a convex feasible set some linear type is always possible at any point on the boundary. Ac-

[17] Note that the converse of the Corollary is not true. When X is the sphere $S = \{x \mid \sum_i x_i^2 \leq 1\}$, the type is *fully* identified at *any* exposed point (i.e. any point on the boundary). This will be implied by our later results on full identification.

cordingly, a suitable condition for full identifiability must achieve two aims: first, ensure that there exists only one possible linear type; and, second, exclude that any non-linear type is possible. We will accomplish this via a "contains an orthant" type of condition that is relatable to the "not contained in an orthant" condition for partial identifiability. However, in this case the statement needs to be somewhat more nuanced. Figure 5 begins to guide intuition by displaying some possible configurations of identification and non-identification for *linear* types. The feasible sets X are displayed in blue, the normal cones in purple and the cones of feasible directions, here equal to the tangent cones, in green. The observed choice x^* is always made to coincide with the origin. The key relation to note is the one between $T_X(x^*) = F_X(x^*)$ and the orthants contained in it. Clearly, unlike for partial identification, a simple containment relation will not work as a condition in this case.

For $1 \leq k \leq N$, define a *k-dimensional orthant of* $\mathbb{R}^N$ as a cone $C\left(v^{i_1}, \ldots, v^{i_k}\right) \subseteq \mathbb{R}^N$ such that $i_1, \ldots, i_k \in F$ are different indices and for all $n = 1, \ldots, k$, we have $v_j^{i_n} \in \{-1, 1\}$ if $j = i_n$ and $v_j^{i_n} = 0$ otherwise. For example, an N-dimensional orthant of $\mathbb{R}^N$ is a usual orthant, whereas a 1-dimensional orthant of $\mathbb{R}^N$ is a semi-axis.

Theorem 3. *The type is fully identified at $x^* \in \partial X$ if and only if either*

(i) $\dim X = N$ *and*

$$F_X(x^*) \supseteq \left\{x \in \mathbb{R}^N \mid \left(x_{i_1}, \ldots, x_{i_k}\right) \in K\right\}, \tag{1}$$

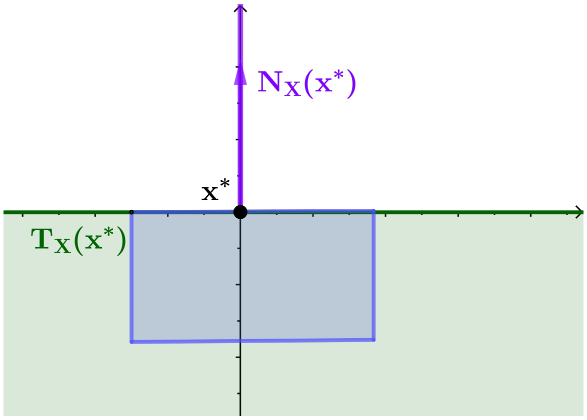
where $\{i_1, \ldots, i_k\}$ is the identified type and K is a closed convex cone in $\mathbb{R}^k$ such that int $K \cup \{0\}$ contains an orthant of $\mathbb{R}^k$,

or

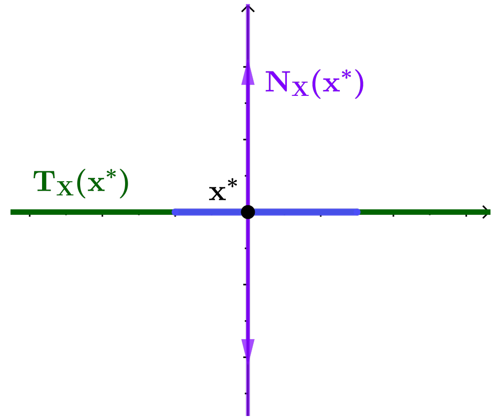
(ii) $\dim X = N - 1$ *and $x^* \in ri\, X$.*

We record separately an immediate but notable implication of the result, providing an absolute constraint on the feasible set for any hope of full identifiability at *some* observation:

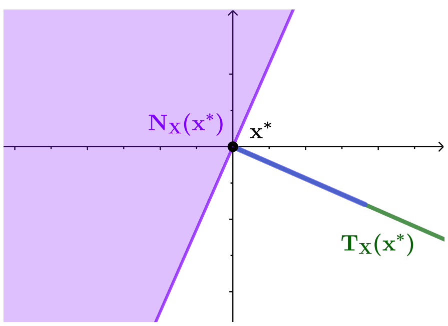
Corollary 4. *For all $x^* \in \partial X$, the type can be fully identified at x^* only if* $\dim X \geq N - 1$.



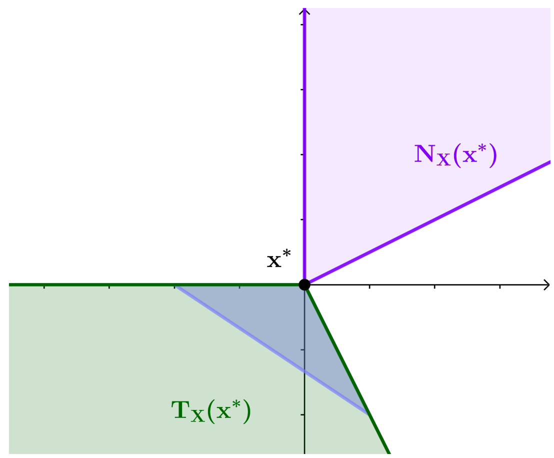
(a) Type $\{2\}$ is fully identified at x^*



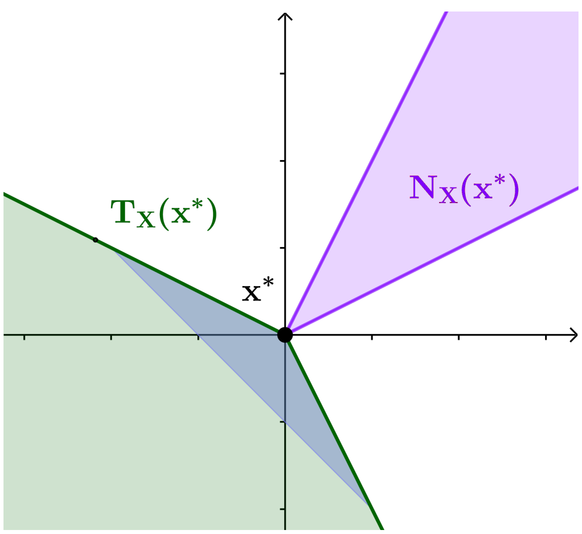
(b) Type $\{2\}$ is fully identified at x^*



(c) No type is identified, $T_X(x^*)$ contains **no** orthant



(d) Types $\{2\}$ and $\{1,2\}$ are possible, $T_X(x^*)$ **does** contain an orthant



(e) Type $\{1,2\}$ is fully identified

Figure 5: Orthant containment is not sufficient for identification.

Beside the reversal of the orthant containment relation, the condition of the theorem differs from the orthant condition for partial identifiability in some additional details. First, the contained orthant need not be full-dimensional. Second, what is contained needs to be "a bit more" than the orthant: it is in fact a cone whose *interior* in turn contains an orthant. Let us relate the statement to the cases in Figure 5. In the top left panel, $\dim X = 2$, K is the negative vertical semi-axis and $k = 1$. The interior of K, a cone in $\mathbb{R}^1$, is itself minus the origin. Therefore *int* K plus the origin contains an orthant in $\mathbb{R}^1$, which is again the negative vertical semi-axis. In the bottom left and the middle-right panels, $\dim X = 2$, K is the green cone and $k = 2$. The difference between the two panels is that in one case the interior of K (a cone in $\mathbb{R}^2$) plus the origin contains a two-dimensional orthant, while in the other case it does not. The remaining two panels illustrate the cases where the feasible set is not full dimensional (case (ii) of the statement). Here, $\dim X = 1$, and the difference between the two panels is whether or not x^* is in the relative interior of the feasible set. The statement takes care of all these situations, permitting exactly one linear type. In addition, thanks to the interiority aspect of the containment condition, it also excludes all situations where non-linear types are possible. This can be illustrated by the difference between points x and y in Figure 1 of the introduction. At y, $F_X(y)$ is the open half space to the left of the vertical axis, plus the negative part of this axis. It contains no K with the properties required in the statement, and the non-linear type $\{1, 2\}$ cannot be excluded. Around x the set has a similar shape as around y, but the orientation is different. $F_X(x)$ is the half-space to the right of the line through x and z, plus the half-line from x in the z direction. Clearly there exists a K containing the south-east orthant in its interior as required. Now type $\{1, 2\}$ is linear.

We conclude with an alternative characterisation in terms of normal cones, using similar arguments, where -at the cost of losing the analogy with partial identification- the dual task performed by the condition is made more explicit.

Theorem 4. *The type is fully identified at $x^* \in \partial X$ if and only if either*

(i) $N_X(x^*) \subseteq ri\, S \cup \{0\}$ for some k-dimensional orthant S of $\mathbb{R}^N$, $1 \leq k \leq N$, and $x^* \in ri\, Y$, where Y is a face of X parallel to $N - k$ coordinate axes with $\dim Y \geq N - k$.

or

(ii) $N_X(x^*)$ is a 1-dimensional subspace of $\mathbb{R}^N$.

In this characterisation, the part asserting that the normal cone is contained in an orthant guarantees the uniqueness of the possible linear types. On the other hand, the interiority condition on the face excludes the possibility of any non-linear type. Again, refer to Figure 1, where the partially identified point y does not meet the interiority condition whereas the fully identified point w does.

6 The linear case

6.1 Polytopes

In this section we focus on the case where the feasible set is a convex polytope. This is a useful particular case to consider: many situations of economic interests are described as linear programming problems, and of course this type of set arises from the convexification of a finite set of feasible alternatives. Furthermore, in this environment the set of possible types at some observation x coincides with the set of linear types at x. This makes it possible to offer a different perspective on the issue of identifiability.

Let X be a non-empty convex polytope in $\mathbb{R}^N$ defined by:

$$X = \{x \in \mathbb{R}^N \mid Bx \leq c\},$$

where B is an $m \times N$-matrix and c is a vector in $\mathbb{R}^m$. From now on we will denote a^I, $I \subseteq F$, vectors $a \in \mathbb{R}^N$ for which $supp\,(a) = I$.

The first result formalises the previous assertion about linear types:

Lemma 2. *If X is a convex polytope, type I is possible at $x^* \in \partial X$ if and only if I is linear.*

Let $B^{(j)}$ be the j-row of matrix B. We say that constraint j is *active* at $x^* \in X$ if

$$B^{(j)} x^* = c_j.$$

Denote by $\bar{B}\,(x^*)$ the matrix of constraints that are active at x^*. The next result says that type I being possible is equivalent to the existence of a coefficient vector a^I that is in the convex cone generated by the constraints that are active at x^*:

Theorem 5. *Type I is possible at $x^* \in \partial X$ if and only if there exist $a^I \in \mathbb{R}^N$ and $y \geq 0$ such that $a^I = \bar{B}^T\,(x^*)\,y$.*

The previous result applies to any x^* on the boundary of the convex polytope X. In the remainder of this section, we will focus on the important special case when x^* is a vertex of X at the intersection of N hyperplanes with linearly independent normal vectors, i.e., a non-degenerate basic feasible solution of a linear program. In that case, $\bar{B}(x^*)$ is an invertible $N \times N$ matrix, which allows for a simple criterion to tell whether or not type I is possible.

Theorem 6. *Suppose that $x^* \in \partial X$ and $\bar{B}(x^*)$ is an invertible $N \times N$ matrix. Then type I is possible at x^* if and only if there exists $a^I \in \mathbb{R}^N$ such that*

$$D(x^*)\, a^I \geq 0 \tag{2}$$

for $D(x^*) = \left(\bar{B}^T(x^*)\right)^{-1}$.

To get a better understanding of condition (2), it may be beneficial to prove it directly. To do this, we consider vectors at x^* in the directions of the adjacent vertices and check if any component of a^I improves. The intuition is that type I is possible if and only if there is no way to improve the value of $\langle a^I, x \rangle$ by moving from x^* along a vector d in the direction of a vertex adjacent to x^*, i.e. when $x = x^* + \lambda d,\ \lambda > 0$.

Corollary 5. *The type is partially identified at x^* if and only if there is a column in $D(x^*)$ that contains both a positive and a negative entry.*

Observe that Corollary 5 is the analog, for the special case of polytopes, of Corollary 3 for general convex sets. It provides an operational test for the impossibility of an elementary type.

For the next statement, we assume, as before, that x^* is a vertex of X at the intersection of N hyperplanes with linearly independent normal vectors.

Corollary 6. *The full type is always possible at any $x^* \in \partial X$ such that $\bar{B}(x^*)$ is an invertible $N \times N$ matrix. Hence, only the full type can be fully identified at such x^*.*

6.2 Finite sets

Our analysis has focused on convex sets. In some cases, the analysis for non-convex sets does not require fundamental changes. Take for example the case in which the feasible set consists of a finite number of alternatives, represented by vectors $x^1, \ldots, x^k \in \mathbb{R}^N$. Conditions similar to those obtained previously can be derived in this environment, too. Let $\hat{X} = [x^1 \ldots x^k]$ be the corresponding $N \times k$ matrix. Let $\mathbf{1}_k$ denote the $k-$dimensional vector of 1s.

Proposition 2. *Type I is possible at $x^* \in X$ such that $x^* \notin int\, conv(X)$ if and only if there exists $a^I \in \mathbb{R}^N \setminus \{0\}$ such that*

$$E(x^*)\, a^I \geq 0 \tag{3}$$

for the $k \times N$ matrix $E(x^*) = \left((x^*)\mathbf{1}_k^T - \hat{X}\right)^T$.

Note the similarity between (3) and (2) for polytopes. Similarly to Corollary 5, we have now the following:

Corollary 7. *With X finite, the type is partially identified at $x^* \in X$ such that $x^* \notin int\, conv(X)$ if and only if there is a column in $E(x^*)$ that contains both a positive and a negative entry.*

If i is the column with the opposite sign entries, then type $\{i\}$ can improve irrespective of whether his evaluation of the relevant feature is positive or negative. Theorem 7 then shows that, just like for a general convex set, the type is partially identified at x^* if and only if there is a type $\{i\}$ that is not possible at x^*. What is more, the sign condition is also related to the orthant condition (ii) of Theorem 2. As we noted, the fact that the set of feasible directions straddles two different orthants can be interpreted as the existence of at least two feasible rates of exchange attaching opposite signs to the values of some feature. But the sign condition of the finite case says exactly this, for movements from the chosen alternatives to two different ones.

Finally, observe that the caveat in the statement, $x^* \notin int\, conv(X)$, is necessary in a finite environment. This is because some interior points may be chosen by a non-linear type. Here, as well as in other non-convex cases, more radical changes might be required. This may be an interesting topic for future research.

7 Multiple observations

We have dealt with identification from a single choice event. In practice, the analyst may observe more than a single choice event. Intuitively, this will increase identification power. Our analysis for a single observation provides immediate implications on the way this increase comes about. Suppose that the data are a finite set of pairs of feasible set and choice observation, $\mathcal{O} = \{(X_t, x^t)\}_{t \in T}$, where T is a finite index set. A type I is possible at $\mathcal{O}$ if there exists an evaluation function e such that (e, I) is possible at each (X_t, x^t). The notions of partial and full identification at $\mathcal{O}$ are adapted in the obvious way.

One important initial caveat is that there is no analog of Proposition 1 here, as the various observations might "contradict" each other and make each type impossible. Therefore, the model is falsifiable by multiple observations. For example, suppose that for two observations the feasible set is the same and coincides with the one depicted in Figure 1. If the observations are x and y, then no single type could have produced them. In fact, the only type that is possible at each observations taken on its own is $\{1, 2\}$, but in one case the evaluation function must treat feature 1 as a negative while in the other case it must treat it as a positive. No common evaluation function serves the purpose of explaining the observations together.

On the other hand, if the observations were y from the feasible set in Figure 1 (where types $\{1, 2\}$ and $\{1\}$ are possible) and x from the right hand panel of Figure 4 (where types $\{1, 2\}$ and $\{2\}$ are possible), then type $\{1, 2\}$ would be fully identified, since on both occasions the features that make type $\{1, 2\}$ possible are positives.

Taking into account this caveat, the following results formalise how multiple observations sharpen identifiability.

Theorem 7. *Suppose that some type is possible at $\mathcal{O}$. The type is partially identified if and only if there is no orthant K such that $\bigcup_t F(x^t) \subseteq K$.*

For full identification, the following result extends Theorem 3 to multiple observations.

Theorem 8. *Suppose that some type is possible at $\mathcal{O}$ and that all feasible sets X_t are full-*

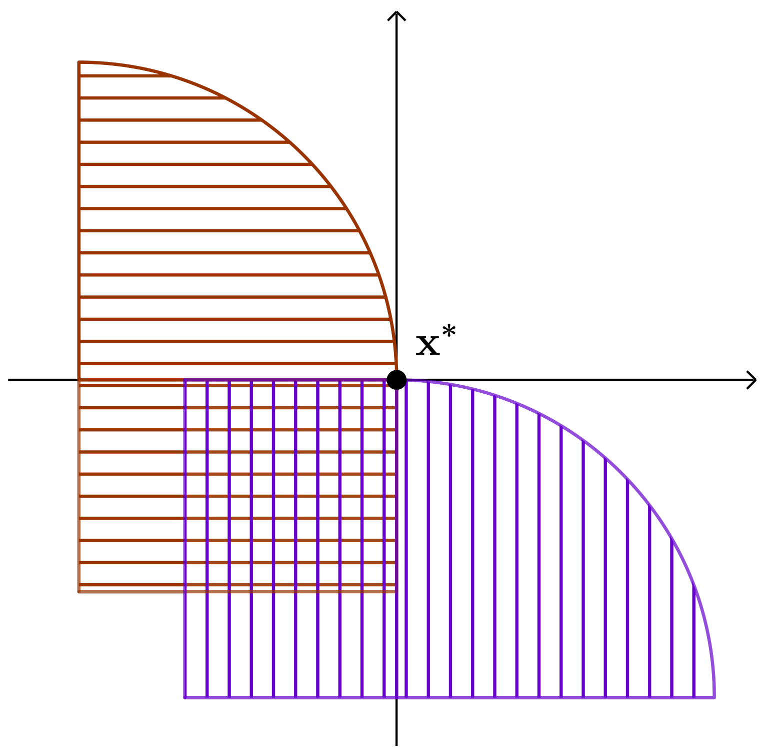


Figure 6: Type $\{1,2\}$, which is non-linear (and thus not fully identifiable) at x^* in each observation taken on its own, is the type that is uniquely identified by the two observations taken together.

dimensional. The type is fully identified if

$$\bigcup_t F\left(x^t\right) \supseteq \left\{x \in \mathbb{R}^N \mid \left(x_{i_1}, \ldots, x_{i_k}\right) \in K\right\}, \tag{4}$$

where $\{i_1, \ldots, i_k\}$ *is the identified type and* K *is a closed convex cone in* $\mathbb{R}^k$ *such that* $int\, K \cup \{0\}$ *contains an orthant of* $\mathbb{R}^k$*. The condition is necessary with* $\bigcup_t F\left(x^t\right)$ *replaced by* $conv(\bigcup_t F\left(x^t\right))$*.*

An interesting difference with the single observation case -where the uniquely identified type must be linear- is that with multiple observations the uniquely identified type can be non-linear at *each* observation in $\mathcal{O}$. The example in Figure 6 illustrates.

The results for the linear and finite cases are also easily extended. For example, Theorem 6 is extended as follows. Suppose that for all $t \in T$, $x^t \in \partial X_t$ and $\bar{B}_t\left(x^t\right)$ is an invertible $N \times N$ matrix. Then type I is possible at $\mathcal{O}$ if and only if there exists

$a^I \in \mathbb{R}^N$ such that[18]

$$\mathbf{DA^I} \geq 0$$

for the $N \times N$ matrix $D_t(x^*) = \left(\bar{B}_t^{\,T}(x^t)\right)^{-1}$,

$$\mathbf{D} = \left[D_1(x^1)\ldots D_{|T|}(x^{|T|})\right],$$

and

$$\mathbf{A^I} = \begin{bmatrix} a^I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a^I \end{bmatrix}.$$

Note that $\mathbf{D}$ is $N \times N \cdot |T|$ and $\mathbf{A^I}$ is $N \cdot |T| \times |T|$. Other results are extended similarly.

8 Related literature

The classical view of alternatives as bundles of characteristics demanded by a consumer is due to Lancaster [17]. In his analysis, these characteristics are objectively known and are the same for all consumers. This assumption is problematic, since different consumers may well focus on different characteristics and the analyst may find it hard to tell which characteristic any given consumer deems relevant.[19] Much of the literature in this vein has relied on specific functional forms to identify preferences. More recently, Blow, Browning and Crawford [7] have pioneered a non-parametric, "revealed preference" type of analysis in such characteristics-based models. They characterise exactly which types of market choices by hetero-

[18]Notation $M \geq 0$ means here that each element of matrix M is non-negative.

[19]The issues that motivated Lancaster's "new approach", such as the evaluation of new goods and understanding the complementarities between them, are still very relevant. Ingenious solutions must be provided ad hoc (e.g. Gentzkow [16]) when utility is defined on goods instead of features because, as Lancaster puts it, "there is no reason except "tastes"" why even wood and bread should not be close substitutes. Instead, the fact that objects such as of bread and wood are described very dissimilarly makes their non-substitutability intuitive within the theory. The problem, addressed by our analysis, is to turn such descriptions into observables.

geneous consumers are consistent with the model. At a broad conceptual level, our analysis is related to this approach, although the development is markedly different in that we take a more abstract, choice-theoretic standpoint. For example, we don't assume that the feasible set is determined by competitive budgets, nor that a utility function is being maximised. From this more abstract perspective, the very recent work by Allen and Rehbeck [1] appears to be first to introduce *attribute variation* (as opposed to choice set variation) in *stochastic* choice. Using this type of variation, they characterise a large class of models in which the desirability of different attributes is captured by a utility index. They show in particular that the class of strict perturbed utility models (strict PUMs) is completely characterised by a form of multivariate law of demand on utility indices and choice probabilities.

One natural reason why DMs may (differentially) neglect features is bounded rationality and specifically limited attention. In this vein, the already mentioned Gabaix [15] studies cognitively constrained agents who vastly simplify reality by efficiently allocating limited attention to different features. This leads them to perform "sparse optimisation", namely optimisation where only a few of the potentially relevant features are "switched on". Sparse econometric models capture a similar insight in an effort to make good predictions with only a small number of features playing a critical role. In such cases our view of a DM is close to that of Gabaix, in that we admit DMs who may switch off many features. However, we remain agnostic on whether or not a discrepancy between the analyst's and the agent's description of the alternatives is due to bounded rationality. For instance, a voter who focuses on a single political issue may well be expressing a political attitude rather than a cognitive limitation. Also, sparsity is not an *intrinsic* feature of our analysis and we admit DMs who consider many or even all features. Our framework fits Gabaix's [15] approach and the literature it has spawned; that of Demuynck and Seel [13], who study consumers who focus their attention on a subset of the goods; and that of Chetty, Looney and Kroft [10], who study consumers who may ignore the feature "sales tax" when making a purchase. Note that our approach differs from that taken in Cerigioni and Galperti [8], who look at how the order of presentation/framing of attributes influences agent's evaluation of a good.

The theoretical study of surveys is gaining increasing traction in economics, and we have mentioned that one potential application of our framework is to the design of surveys for identification purposes. A recent example of this interest is Apesteguia and Ballester [2], who consider the complementary issue of the *rationalisation* of a DM's survey responses.[20]

At the technical level, our model presents points of contact with Che *et al.* [9]. Their paper elegantly resolves the long standing question about the connection between linear maximisation and the Pareto optimal points of a convex set (also dealt with in the already mentioned classic paper by Arrow, Barankin and Blackwell [4]). They characterise the strong Pareto optimal frontier in terms of *sequential exposure,* namely the iterated application of linear maximisation to expose the face to whose interior a Pareto optimal point belongs. As we have noted at several places, some of the complexity in our characterisations comes precisely from the existence of points that are not (directly) exposed. Our example in Figure 1 is similar to the "canonical" example in [9], the main difference being that in our approach the directions of improvements are unknown. While we don't make use of their sequential exposure characterisation, it might be interesting, in future work, to distinguish types of different "orders". For example, at the choice point y in Figure 1 we might say that type $\{1\}$ is first-order, as its choice can be exposed in one-step by linear maximisation, whereas type $\{1,2\}$ is "second-order", since its choice can only be exposed in two steps of linear maximisation. That said, one of the main issues for identification in our paper, the orientation of the set, is not relevant at all in [9].

9 Concluding remarks: stochastic choice

We have studied in Section 7 how multiple observations increase the identifying power. For future work, however, we are interested in a more radical extension of the framework. Suppose that we observe a *distribution* p of choices with a finite support $x^1, \ldots, x^k \in X$. Then, if for every $j = 1, \ldots, k$, the decision-maker's choice is fully identified at x^j, then we can recover the distribution of types completely. But, even if the decision-maker's type is only partially identified for some j, we can still recover the distribution of types up to a certain limit, as explained below.

[20] Their concept of rationalisability for binary responses is that the DM's opinion can be expressed as a point on the real line in such a way that the DM endorses the questions that are closely aligned with the opinion.

Let $\mathcal{I} = 2^F \setminus \{\emptyset\}$ be the set of all types. For $j = 1, \ldots, k$, let $\Pi(j) \subseteq \mathcal{I}$ be the set of possible types at x^j. In particular, if the DM's type is fully identified at x^j, then $\Pi(j)$ consists of a single element. If the DM's type is not identified at x^j, then $\Pi(j) = \mathcal{I}$.

Let $\mathbb{P}$ be the underlying distribution of types, $\sum_{I \in \mathcal{I}} \mathbb{P}(I) = 1$. For any $I \in \mathcal{I}$, the lower and the upper estimates of $\mathbb{P}(I)$ are given by

$$\mathbb{P}_*(I) = \sum_{j:\, \Pi(j) = \{I\}} p(x^j), \qquad \mathbb{P}^*(I) = \sum_{j:\, \Pi(j) \supseteq \{I\}} p(x^j)$$

The lower estimate $\mathbb{P}_*(I)$ takes into account cases where I is fully identified. In contrast, the upper estimate $\mathbb{P}^*(I)$ takes into account cases where I is a possible type. Following the same logic for a set of types $T \subseteq \mathcal{I}$, the lower and the upper estimates of probability $\mathbb{P}(T)$ are given by

$$\mathbb{P}_*(T) = \sum_{j:\, \Pi(j) \subseteq T} p(x^j), \qquad \mathbb{P}^*(T) = \sum_{j:\, \Pi(j) \cap T \neq \emptyset} p(x^j).$$

Note that $\mathbb{P}^*(T) = 1 - \mathbb{P}_*(T)$, which implies that $\mathbb{P}_*$ alone is sufficient to characterize the set of possible distributions of types. These are distributions $\mathbb{P}$ on $\mathcal{I}$ such that

$$\mathbb{P}(T) \geq \mathbb{P}_*(T) \tag{5}$$

for any $T \subseteq \mathcal{I}$. Thus, even if the decision-maker's type is not completely identifiable for some x^j, we can still recover the true distribution of types up to the set (5).[21]

Assuming more stochastic observations, from several feasible sets, should help us to refine the set of possible type distributions.

[21] Technically, the true distribution of types $\mathbb{P}$ belongs to the *core* of the belief function $\mathbb{P}_*$. Note that $\mathbb{P}_*$ is in general not a probability distribution because it might not be additive. The belief function $\mathbb{P}_*$ is characterized by $2^{|\mathcal{I}|} - 2$ numbers corresponding to all non-trivial subsets of types. A more compact representation of the core of $\mathbb{P}_*$ requiring $|\mathcal{I}| - 1$ numbers is given by its centroid, which is the probability distribution in the center of the core. Such compact representation is valid under the principle of insufficient reason (Eichberger and Pasichnichenko [14]) with respect to the types in $\Pi(j)$.

Appendix A. Proofs

Lemma 1

(i) Suppose that there exists $a \in \mathbb{R}^N$ such that $supp\,(a) = I$ and $\langle a, x^* \rangle \geq \langle a, x \rangle$ for all $x \in X$. Define an evaluation function e by setting $e\,(i) = 0 \iff a_i = 0$, $e\,(i) = 1 \iff a_i > 0$, and $e\,(i) = -1 \iff a_i < 0$. Suppose that for some $x \in X$, for all i, $e\,(i)\,x_i \geq e\,(i)\,x_i^*$. Then it must be $e\,(i)\,x_i = e\,(i)\,x_i^*$ for all i, in view of x^* being a maximiser of $\langle a, . \rangle$ on X. This means that x^* is e-admissible, and therefore I is possible at x^*.

(ii) Let I be possible at $x^* \in X$. First, consider the case where $I = F$. Since $x^* \in \partial X$, by the supporting hyperplane theorem, there exists a non-zero $a \in \mathbb{R}^N$ such that $\langle a, x^* \rangle \geq \langle a, y \rangle$, for all $y \in X$. Then, the type $J = supp(a)$ is linear at x^* and satisfies $J \subseteq F$, where $J \subset F$ if $a_i = 0$ for some $i \in I$.

Next, suppose that $I \subset F$. In this case, consider the projection of all vectors $y \in X$ on the subspace V generated by $\{\mathbf{1}_i\}_{i \in I}$.[22] Let X_V denote this projection of all $y \in X$ on V. Since X is a compact, convex set, so is X_V. Observe that the projection x_V^* of x^* on V lies in ∂X_V, where ∂ denotes the boundary of X_V in the subspace V. To see this, suppose that $x_V^* \notin \partial X_V$. Then, for any evaluation function e with $e(i) \neq 0$, for all $i \in I$, there exists $y \in X_V$ with $e\,(i)\,y_i > e\,(i)\,(x_V^*)_i$, for all $i \in I$. But, this implies that I is not possible at $x^* \in X$ such that we have arrived at our desired contradiction. It follows that $x_V^* \in \partial X_V$. Therefore, by the supporting hyperplane theorem applied in V, there exists a non-zero $a \in \mathbb{R}^N$ such that $a_i = 0$, for all $i \in F \setminus I$, and $\langle a, x_V^* \rangle \geq \langle a, y \rangle$, for all $y \in X_V$. Clearly, it also holds that $\langle a, x^* \rangle \geq \langle a, y \rangle$, for all $y \in X$. Hence, the type $J = supp(a)$ is linear at x^* and satisfies $\emptyset \neq J \subseteq I$.

Proposition 1

(i) Recall first that under the assumptions of the statement, for any non-zero $a \in \mathbb{R}^N$, there exists $x\,(a) \in \partial X$ such that $\langle a, x\,(a) \rangle \geq \langle a, y \rangle$ for all $y \in X$.[23] Thus,

[22] The projection of $y \in \mathbb{R}^N$ on the subspace V generated by $\{\mathbf{1}_i\}_{i \in I}$ is $z \in \mathbb{R}^N$ such that $z_i = y_i$ for $i \in I$ and $z_i = 0$ otherwise.

[23] We give the proof of this well-known fact here for completeness. Clearly, the function $f : X \to \mathbb{R}^N$, defined by $f(x) = \langle a, x \rangle$, $a \in \mathbb{R}^N$ is continuous. By Weierstrass Theorem, this implies that f has a maximiser $x\,(a) \in X$. If $x\,(a)$ is in the interior of X, then it is easy to show that $a = 0$. Since

choosing a such that $supp\,(a) = I$, I is linear at $x\,(a)$. The conclusion follows from Lemma 1. (ii) By convexity, there exists a non-zero $a \in \mathbb{R}^N$ such that $\langle a, x\rangle \geq \langle a, y\rangle$ for all $y \in X$, and again Lemma 1 yields the conclusion with $I = supp\,(a)$.

Theorem 1

(i) Let types I and J be possible at x^*. We will show that $(e, I \cup J)$ is also possible at x^* with

$$e(i) = \begin{cases} e_I\,(i) & \text{for } i \in I \\ e_J\,(i) & \text{for } i \in J \setminus I \\ 0 & \text{otherwise} \end{cases}$$

Note that by definition $supp(e) = I \cup J$. Assume that $x \in \mathbb{R}^N$, $e(i)x_i \geq e(i)x_i^*$ for all $i \in I \cup J$, and $e(\hat{i})x_{\hat{i}} > e(\hat{i})x_{\hat{i}}^*$ for some $\hat{i} \in I \cup J$. We must show that $x \notin X$. The fact that (e_I, I) is possible implies that $x \notin X$ or $e(i)x_i = e(i)x_i^*$ for all $i \in I$. In the second case, we have $x_i = x_i^*$ for all $i \in I$, which implies $e_J(i)x_i = e_J(i)x_i^*$ for all $i \in I \cap J$. Hence, we have $\hat{i} \in J \setminus I$ and $e_J(i)x_i \geq e_J(i)x_i^*$ for all $i \in J$. Since (e_J, J) is possible, this implies $x \notin X$. Therefore, $I \cup J$ is a possible type.

(ii)[24] Let $\mathcal{I} \subseteq 2^F$ be a nonempty collection of types that is closed under union. For each type $I \in \mathcal{I}$, let $\mathbf{1}_I \in \mathbb{R}^N$ be the vector whose i^{th} component is 1 if $i \in I$ and 0 otherwise (using the simplified notation $\mathbf{1}_i$ when $I = \{i\}$), and let $x_I = \frac{1}{|I|}\mathbf{1}_I$ (observe that $\langle x_I, \mathbf{1}\rangle = 1$). Also, let $x^i = \frac{1}{2}\mathbf{1}_i$. Note that, by the closure under union of $\mathcal{I}$, $supp\,(x) \in \mathcal{I}$ for all x in the convex hull $F^{\mathcal{I}}$ of the vectors x_I. We are going to construct the convex polytope X of the statement in various steps. First, let D be the downward closed[25] and convex hull of the vectors x_I such that $I \in \mathcal{I}$ and of the vectors $\left\{x^i\right\}_{i \in F}$. Next, to obtain a bounded set, let $Y = \{x \in D \,|\, x \geq -\mathbf{1}\}$. Note that the vectors x_I maximise on Y the linear function $\langle ., \mathbf{1}\rangle$ and $F^{\mathcal{I}}$ forms a face of Y. Moreover $\mathbf{0} \in int\,Y$ and Y is a convex polytope. It follows that the polar set $Y^* = \left\{x \in \mathbb{R}^N |\, \langle x, y\rangle \leq 1\, \forall y \in Y\right\}$ is also a convex polytope with $\mathbf{0} \in int\,Y^*$. Also, since $\langle \mathbf{1}, y\rangle \leq 1$ for all $y \in Y$, $\mathbf{1} \in Y^*$.

Now we set $X = Y^*$, and show that the set of types that are possible at $x^* = \mathbf{1}$

this case is excluded, $x\,(a) \in \partial X$.

[24] We are grateful to a referee for suggesting the argument in this proof.

[25] The downward closure of a set S is the set $\cup_{x \in X} \left\{y \in \mathbb{R}^N |\, y \leq x\right\}$.

is $\mathcal{I}$. Take $I \in \mathcal{I}$. We show that (e, I) is possible at $\mathbf{1}$ with $e(i) = 1$ for all $i \in I$. In fact, if there is an x such that $x_i \geq 1$ for all $i \in I$ with $x_i > 1$ for some $i \in I$, then $\langle x, \mathbf{1}_I \rangle > \langle \mathbf{1}, \mathbf{1}_I \rangle$, which implies $\left\langle x, \frac{1}{|I|}\mathbf{1}_I \right\rangle > 1$, so that $x \notin X$.

Conversely, suppose that a type J is possible at $\mathbf{1}$. As will be shown later (Lemma 2), for a convex polytope a type that is possible at some x is linear at x. Therefore, $\mathbf{1}$ maximises some linear function $\langle c, . \rangle$, with $supp\,(c) = J$ and $\langle c, \mathbf{1} \rangle = 1$. Note that $c \in Y$. Since $\langle y, \mathbf{1} \rangle < 1$ for all $y \in Y$ that are not in $F^{\mathcal{I}}$, $c \in F^{\mathcal{I}}$. Therefore, $J \in \mathcal{I}$.

Theorem 2

$(i) \implies (ii)$: Suppose that $C\left(v^1, ..., v^N\right)$ is an orthant such that $F_X\left(x^*\right) \subseteq C\left(v^1, ..., v^N\right)$. Since $C\left(v^1, ..., v^N\right)$ is closed, we also have $T_X\left(x^*\right) \subseteq C\left(v^1, ..., v^N\right)$. Hence $C^0\left(v^1, ..., v^N\right) \subseteq T^0_X\left(x^*\right)$ (recall that S^0 denotes the polar cone of a set S), that is, $C\left(-v^1, ..., -v^N\right) \subseteq N_X\left(x^*\right)$. In particular, for any $I \subseteq F$, $a^I = \sum_{i \in I}(-v^i) \in N_X\left(x^*\right)$. Then $\left\langle a^I, x^* \right\rangle \geq \left\langle a^I, x \right\rangle$ for all $x \in X$. By the first part of Lemma 1, type I is possible at x^*. Thus any type I is possible at x^* and the type is not identified.

$(ii) \implies (iii)$: Suppose there is no orthant that contains $F_X(x^*)$. Then, there exist $v, w \in F_X(x^*)$ such that $v_i > 0$ and $w_i < 0$ for some $i \in \{1, \ldots, N\}$. Hence, $y_i > x^*_i > z_i$ for some $y, z \in X$. This implies that $x^*_i = \alpha y_i + (1-\alpha) z_i$ for some $\alpha \in (0, 1)$. Therefore, x^* is not $\{i\}$-extreme. Additionally, $y_i > x^*_i > z_i$ implies that for any evaluation function e with $supp(e) = \{i\}$, x^* is not e-admissible at x^* ($y_i > x^*_i$ excludes $e(i) = 1$, and $x^*_i > z_i$ excludes $e(i) = -1$). Therefore, type $\{i\}$ is not possible at x^*.

$(iii) \implies (i)$: If x^* is not $\{i\}$-extreme, then there exist $y, z \in X \setminus \{x^*\}$ and $\alpha \in (0, 1)$ such that $x^*_i = \alpha y_i + (1-\alpha) z_i$ with $y_i \neq z_i$. Then either $y_i > x^*_i > z_i$ or $z_i > x^*_i > y_i$ holds. In any case, x^* is not e-admissible for any evaluation function e such that $supp(e) = \{i\}$. Therefore, type $\{i\}$ is not possible at x^* and the type is partially identified at x^*.

Corollary 1

Suppose that $T_X\left(x^*\right)$ is contained in an orthant $C\left(v^1, ..., v^N\right)$. Then $\left\langle x, v^i \right\rangle \geq 0$ for each $x \in T_X\left(x^*\right)$ and each basis vector v^i. It follows that $v^i \in -T^0_X\left(x^*\right)$ for all v^i, and therefore $T_X\left(x^*\right) \subseteq C\left(v^1, ..., v^N\right) \subseteq -T^0_X\left(x^*\right)$. Hence, if X is fat at x^*, $T_X\left(x^*\right)$

cannot be contained in an orthant, and we can apply part (ii) of Theorem 2 to prove the statement.

Corollary 2

Suppose that the type is not identified at x^*. Then in particular all elementary types $I = \{i\}$ are possible at x^*. The second part of Lemma 1 implies that there exists $a^i = (0, ..., 0, a_i, 0, ..., 0)$, $a_i \neq 0$, such that $\langle a^i, x^* \rangle \geq \langle a^i, x \rangle$ for all $x \in X \setminus \{x^*\}$ and $i \in F$. It must be $\langle a^i, x^* \rangle > \langle a^i, x \rangle$ for some i, for otherwise $a_i x_i = a_i x_i^*$ for all i and $x = x^*$. Hence, for all $x \in X \setminus \{x^*\}$,

$$\left\langle \sum_i a^i, x^* \right\rangle = \sum_i \left\langle a^i, x^* \right\rangle > \sum_i \left\langle a^i, x \right\rangle = \left\langle \sum_i a^i, x \right\rangle, \tag{6}$$

so that x^* is exposed.

Theorem 3

Sufficiency. For condition (i), suppose that (1) holds and let $I = \{i_1, \ldots, i_k\}$. We will show that any type $I' \neq I$ is not possible.

First, let I' be such that there exists $i' \in I'$ such that $i' \notin I$, and let e be an evaluation function for type I'. It follows from (1) that both $x^* + \varepsilon \mathbf{1}_{i'}$ and $x^* - \varepsilon \mathbf{1}_{i'}$ are in X for some $\varepsilon > 0$. If I' is possible, then either $e(i') = 1$ or $e(i') = -1$. Clearly, x^* is not e-admissible in the first case because $x^* + \varepsilon \mathbf{1}_{i'} \in X$, and it is not e-admissible in the second case because $x^* - \varepsilon \mathbf{1}_{i'} \in X$. Hence, I' is not possible.

Now assume there exists $I' \subset I$ that is possible with some evaluation function e. Assume also that $I = F$. The proof for $I \neq F$ is essentially identical but requires additional notation. Since $int\, K \cup \{0\}$ contains an orthant of $\mathbb{R}^N$, the next lemma shows that the projection of K on the subspace V generated by $\{\mathbf{1}_{i'}\}_{i' \in I'}$ coincides with the whole subspace V.

Lemma 3. *Let $C\left(v^1, \ldots, v^N\right)$ be an orthant of $\mathbb{R}^N$ and K be a closed convex cone such that $C\left(v^1, \ldots, v^N\right) \subset int\, K \cup \{0\}$. Then, for any $I' \subset F$, the projection of K on the subspace V generated by $\{\mathbf{1}_{i'}\}_{i' \in I'}$ coincides with V.*

Proof. Because K is a convex cone, we only need to show that $\mathbf{1}_{i'}$ and $-\mathbf{1}_{i'}$ belong to the projection for all $i' \in I'$. One of the two vectors $\mathbf{1}_{i'}$ and $-\mathbf{1}_{i'}$ lies in $C\left(v^1, \ldots, v^N\right)$

and, hence, also in K by definition, so that it belongs to the projection. Without loss of generality, let this vector be $-\mathbf{1}_{i'}$. As for the other vector $\mathbf{1}_{i'}$, take any $j \notin I'$. By the condition of the lemma, there is a neighbourhood of v^j that is contained in K. This implies that there exists an $\varepsilon > 0$ such that $v^j + \varepsilon \mathbf{1}_{i'} \in K$. But then, the vector $d = \frac{1}{\varepsilon}(v^j + \varepsilon \mathbf{1}_{i'})$ is also in K. The projection of d on V is $\mathbf{1}_{i'}$. $\square$

It follows from Lemma 3 that there exists $d \in K$ such that its projection $\bar{d}$ on V coincides with the evaluation function e, i.e., $\bar{d}_i = e(i)$ for all $i \in F$. Since by condition (1) $x^* + \varepsilon d \in X$ for some $\varepsilon > 0$, this implies that x^* is not e-admissible for I'. Therefore, I is the only possible type.

Sufficiency of (ii) is proven similarly. Note that $\dim X = N - 1$ implies that there is a unique (up to a non-zero scalar multiplication) vector $w \in \mathbb{R}^N$ that is orthogonal to all vectors in $F_X(x^*)$. Since w exposes X, we have $x^* \in \arg\max_{x \in X} \langle w, x \rangle$ and type $I = supp(w)$ is possible by Lemma 1. Let $I' \subseteq F$ be such that there exists $i' \in I'$ such that $i' \notin I$. Since $x^* \in ri\, X$, it must be that $x^* + e(i')\varepsilon \mathbf{1}_{i'} \in X$ for some $\varepsilon > 0$. But then x^* is not e-admissible for I'. Assume now that $I' \subset I$. Since $x^* \in ri\, X$, the projection of $F_X(x^*)$ on the subspace V generated by $\{\mathbf{1}_{i'}\}_{i' \in I'}$ coincides with V. Hence, there exists $d \in F_X(x^*)$ such that its projection on V is e. Since $x^* + \varepsilon d \in X$ for some $\varepsilon > 0$, this implies that x^* is not e-admissible for I'. Hence, such I' is not possible either.

Necessity. Suppose that the type is fully identified at $x^* \in \partial X$. In particular, this means that if $x^* \in \arg\max_{x \in X} \langle v, x \rangle$ and $x^* \in \arg\max_{x \in X} \langle w, x \rangle$ for $v, w \in \mathbb{R}^N$, then $supp(v) = supp(w)$. In the following two lemmas, we will use convex analysis to explore the restrictions this imposes on the structure of the normal and tangent cones at x^*.

Lemma 4. *The type is fully identified at $x^* \in \partial X$ only if either*

(i) $N_X(x^*) \subseteq ri\, C\left(v^{i_1}, \ldots, v^{i_k}\right) \cup \{0\}$ *for some k-dimensional orthant $C\left(v^{i_1}, \ldots, v^{i_k}\right)$ of $\mathbb{R}^N$, $1 \leq k \leq N$,*

or

(ii) $N_X(x^*)$ *is a 1-dimensional subspace of $\mathbb{R}^N$.*

Proof. Note that for any nonzero vector $x \in \mathbb{R}^N$, there exist a unique $1 \leq k \leq N$ and a unique k-dimensional orthant $C\left(v^{i_1}, \ldots, v^{i_k}\right)$ such that $x \in ri\, C\left(v^{i_1}, \ldots, v^{i_k}\right)$.

It follows from the definition of full identification and lemma 1 that, since the type is fully identified, all non-zero vectors $a \in N_X(x^*)$ must share the same set of indices i with non-zero entries a_i. Thus, all non-zero vectors of $N_X(x^*)$ must belong to the relative interiors of the orthants of the same dimensionality k. If they all belong to the same k-dimensional orthant, then (i) holds. Otherwise, assume that there exist $x, y \in N_X(x^*) \setminus \{0\}$ in different k-dimensional orthants. Since x and y share the same set of indices i with non-zero entries, this implies that $x_i > 0$ and $y_i < 0$ for some $i = 1, \ldots, N$. But then, we have both $\alpha x_i + (1-\alpha) y_i = 0$ for some $\alpha \in (0,1)$, and $\alpha x + (1-\alpha) y \in N_X(x^*)$ by convexity. This contradicts the assumption of full identification unless $\alpha x + (1-\alpha) y = 0$. Thus, if x and y belong to different k-dimensional orthants, then they belong to the same 1-dimensional subspace. Since for a non-zero vector, there is only one 1-dimensional subspace containing it, condition (ii) holds. The lemma is proved. □

Lemma 5. *The type is fully identified at $x^* \in \partial X$ only if either*

(i) $\dim X = N$ *and*

$$T_X(x^*) = \left\{ x \in \mathbb{R}^N \mid \left(x_{i_1}, \ldots, x_{i_k}\right) \in K_T \right\}, \tag{7}$$

where $\{i_1, \ldots, i_k\}$ is the identified type and K_T is a closed convex cone in $\mathbb{R}^k$ such that int $K_T \cup \{0\}$ contains an orthant of $\mathbb{R}^k$,

or

(ii) $\dim X = N - 1$ *and $x^* \in ri\, X$.*

Proof. First, consider the case $\dim X = N$. Let $I = \{i_1, \ldots, i_k\}$ be the identified type. Note that for any $y \in \mathbb{R}^N$, there exist unique y_I and $y_{F \setminus I}$ such that $y = y_I + y_{F \setminus I}$, where $(y_I)_i = 0$ for all $i \in F \setminus I$ and $(y_{F \setminus I})_i = 0$ for all $i \in I$. Since I is the fully identified type, we have $\left\langle y_{F \setminus I}, x \right\rangle = 0$ for all $x \in N_X(x^*)$. Hence, $y \in T_X(x^*)$ if and only if $\langle y_I, x \rangle \leq 0$ for all $x \in N_X(x^*)$. Clearly, such y_I's form a closed convex cone which we denote $T_{X,I}$.

We will show next that only condition (i) of Lemma 4 is possible in the case $\dim X = N$. To the contrary, assume that $N_X(x^*)$ is a 1-dimensional subspace of $\mathbb{R}^N$. But then, because $N_X(x^*)$ and $T_X(x^*)$ are polar and $x, -x \in N_X(x^*)$ for some $x \neq 0$, we have $\langle y, x \rangle \leq 0$ and $\langle y, -x \rangle \leq 0$ for any $y \in T_X(x^*)$, which implies

$\langle y, x \rangle = 0$. Thus, $T_X(x^*)$ is contained in the orthogonal complement[26] of $N_X(x^*)$ and, therefore, $\dim T_X(x^*) \le N-1$. This contradicts the assumption $\dim X = N$. Thus, by Lemma 4, all the non-zero vectors of $N_X(x^*)$ are contained in the relative interior of a single k-dimensional orthant $C(v^{i_1}, \dots, v^{i_k})$ of $\mathbb{R}^N$.

We will use the result of the previous paragraph to show that $ri\, T_{X,I} \cup \{0\}$ contains a k-dimensional orthant. Indeed, for any non-zero $x \in N_X(x^*)$, we can write $x = \sum_{n=1}^{k} \alpha_{i_n} v^{i_n}$, where $\alpha_{i_n} > 0$ for all $n = 1, \dots, k$, so that $\left\langle -v^{i_n}, x \right\rangle = -\alpha_{i_n} < 0$. Hence, $\langle y, x \rangle < 0$ for any non-zero $y \in C(-v^{i_1}, \dots, -v^{i_k})$ and $x \in N_X(x^*)$, which implies that all the non-zero vectors of $C(-v^{i_1}, \dots, -v^{i_k})$ are contained in $ri\, T_{X,I}$. Then, the representation (7) follows.

Consider now the case $\dim X \le N-1$. Note that there exists $y \in \mathbb{R}^N$ such that $y \ne 0$ and $\langle y, x - x^* \rangle = 0$ for all $x \in X$. This implies that both $y, -y \in N_X(x^*)$, so that only condition (ii) of Lemma 4 remains possible. Thus, $\dim X \le N-2$ is incompatible with full identification, since there exist at least two linearly independent vectors in $N_X(x^*)$ in this case. As for $\dim X = N-1$, observe that when x^* is on the relative boundary of X, there exists a supporting hyperplane containing x^* with the normal z in the same subspace. Since $z \in N_X(x^*)$ and also $y \in N_X(x^*)$ for some $y \ne 0$ such that $\langle y, z \rangle = 0$, condition (ii) of Lemma 4 is violated. Therefore, $x^* \in ri\, X$. The lemma is proved. □

To conclude the proof of the theorem, it is left to show that (7) implies (1) when the possible type is unique. Because the tangent cone is the closure of the cone of feasible directions, this does not follow automatically.

Let $C(v^1, \dots, v^k)$ be the orthant of $\mathbb{R}^k$ that is contained in $int\, K_T \cup \{0\}$, i.e., for any $n = \overline{1,k}$, we have $v^n \in \mathbb{R}^k$ and either $v^n = \mathbf{1}_n$ or $v^n = -\mathbf{1}_n$. For all $n = \overline{1,k}$, let

$$\hat{v}^n = v^n - \varepsilon \sum_{\substack{n' = \overline{1,k} \\ n' \ne n}} v^{n'} \tag{8}$$

(an example is shown in Figure 7). Since $C(v^1, \dots, v^k)$ is in $int\, K_T \cup \{0\}$, there exists $\varepsilon > 0$ such that $C(\hat{v}^1, \dots, \hat{v}^k)$ is in $int\, K_T \cup \{0\}$ too. Fix such ε and define K by

$$K = C(\hat{v}^1, \dots, \hat{v}^k).$$

[26] The orthogonal complement $V^\perp$ of a subspace V of $\mathbb{R}^N$ is the subspace of all vectors $w \in \mathbb{R}^N$ that are orthogonal to all vectors $v \in V$, that is $V^\perp = \left\{ w \in \mathbb{R}^N \,|\, \langle w, v \rangle = 0 \ \forall v \in V \right\}$.

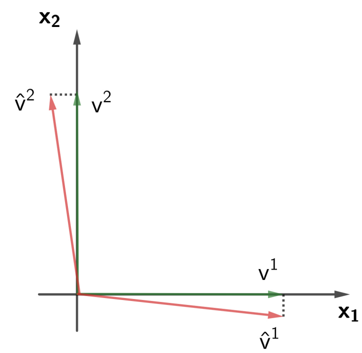


Figure 7: Example for equation (8)

Clearly, K is a closed convex cone in $\mathbb{R}^k$ such that $int\, K \cup \{0\}$ contains an orthant of $\mathbb{R}^k$. Define G by

$$G = \left\{ x \in \mathbb{R}^N \mid (x_{i_1}, \ldots, x_{i_k}) \in K \right\}.$$

By construction, we have $G \subseteq T_X(x^*)$. For the final step of the proof, we will show that $G \subseteq F_X(x^*)$.

Denote by I the identified type $\{i_1, \ldots, i_k\}$. If $I = F$, then $G \subseteq F_X(x^*)$ by the construction of G because in this case K is a cone in $\mathbb{R}^N$ and $K \subseteq F_X(x^*)$. Otherwise, let V be the subspace generated by $\{\mathbf{1}_{i'}\}_{i' \in F \setminus I}$. Define the slice S of $F_X(x^*)$ as

$$S = F_X(x^*) \cap V.$$

Note that S contains all directions $y \in F_X(x^*)$ such that $y_i = 0$ for all $i \in I$. Clearly, S is a convex cone.

We will show that $S = V$. Assume, to the contrary, that this does not hold. Any convex cone that is not the whole space must be contained in a closed half-space. Applying this to V, there exists $v \in V$, $v \neq 0$, such that $\langle v, y \rangle \leq 0$ for all $y \in S$. Since $v \neq 0$, the set I' of indices $i' \in F$ for which $v_{i'} \neq 0$ is not empty. Then, we can

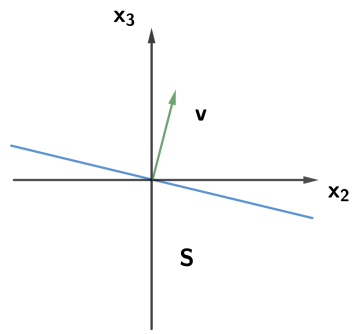


Figure 8: Example in the 23-plane. Possible types are 1 and 123

show (Figure 8 contains an example) that type $I \cup I'$ is possible with

$$e(i) = \begin{cases} e_I(i) & \text{for } i \in I \\ 1 & \text{for } i \in I' \text{ such that } v_i > 0 \\ -1 & \text{for } i \in I' \text{ such that } v_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

Indeed, if $x \in \mathbb{R}^N$, $e(i)x_i \geq e(i)x_i^*$ for all $i \in I \cup I'$, and $e(j)x_j > e(j)x_j^*$ for some $j \in I \cup I'$, then, for type I to be possible, it must be $x \notin X$ or $e(i)x_i = e(i)x_i^*$ for all $i \in I$. In the second case, we have $x - x^* \in V$ and $j \in I'$. Hence,

$$\langle v, x - x^* \rangle = \sum_{v_i > 0} v_i(x_i - x_i^*) + \sum_{v_i < 0} v_i(x_i - x_i^*) > 0$$

by the construction of e. This implies $x - x^* \notin S$ and $x \notin X$, from which it follows that $I \cup I'$ is a possible type. Since the possibility of both I and $I \cup I'$ contradicts full identification, we have $S = V$.

Therefore, $F_X(x^*)$ contains V, from which $G \subseteq F_X(x^*)$ follows by convexity. This concludes the proof of the theorem.

Theorem 4

Sufficiency. Obviously, all vectors $a \in ri\, S$ share the same set of indices i with non-zero entries a_i. Let $I \subseteq F$ be the set of all such indices. By Lemma 1, no type $J \subset I$ is possible at x^*. To see this observe that otherwise there would exist a type $K \subseteq J$ which is linear at x^*, contradicting that all vectors $a \in ri\, S$ share the same set of indices i with non-zero entries a_i. Now, consider all remaining types $J \subseteq F$, $J \neq I$. For such types, there exists $j \in J$ such that $j \notin I$. By $x^* \in ri\, Y$, where Y is a face of X parallel to $|F \setminus I|$ coordinate axes with $\dim Y \geq |F \setminus I|$, there exist $y^+, y^- \in X$ such that $y^+(j) > x^*(j) > y^-(j)$ and $y^+(i) = x^*(i) = y^-(i)$, for all $i \neq j$. It follows that no such type J is possible at x^*. Hence, type I is fully identified at x^*. A similar argument can be made for the non-zero vectors of a 1-dimensional subspace.

Necessity. In view of Lemma 4 in the previous proof, we only show the necessity of the second part of (i). Let I denote the type $\{i_1, \ldots, i_k\}$ identified at x^*. If $I = F$, then the second part of (i) is trivially satisfied. To see this, observe that then $k = |I| = |F| = N$. In this case, $x^* \in \text{ri}\, Y$, for Y being a face of X with $\dim Y \geq N - N = 0$ always holds, because the relative interior of any point is the point itself. Analogously, the condition on Y being parallel to $N - N = 0$ coordinate axes also holds in a trivial way.

Next, let V be the subspace generated by $\{\mathbf{1}_{i'}\}_{i' \in F \setminus I}$ and Y be the face of X with $x^* \in Y$. Define the slice S of Y as

$$S = Y \cap V$$

Note that S contains all directions $y \in Y$ such that $y_i = 0$, for all $i \in I$. Clearly, S is a convex set.

We will show that $\mathbf{1}_{i'} \in V$ implies that both $(x^* + \varepsilon \mathbf{1}_{i'}) \in S$ and $(x^* - \varepsilon \mathbf{1}_{i'}) \in S$, for some $\varepsilon > 0$. Now, assume to the contrary, that this does not hold. Note that any convex set S for which this does not hold must be contained in a closed half-space of V that contains x^*. As such, there exists $v \in V$, $v \neq 0$, such that $\langle v, y \rangle \leq 0$, for all $y \in S$. Since $v \neq 0$, the set I' of indices $i' \in F$ for which $v_{i'} \neq 0$ is not empty. Then, we can show that type $I \cup I'$ is possible using essentially the same argument used in the proof of Theorem 3. It follows that there exists $\varepsilon > 0$ such that for all $\mathbf{1}_{i'} \in V$ it holds that $(x^* + \varepsilon \mathbf{1}_{i'}) \in S$ and $(x^* - \varepsilon \mathbf{1}_{i'}) \in S$.

Hence, $x^* \in ri\, Y$, for some face Y of X parallel to $|F \setminus I| = N - k$ coordinate axes with $\dim Y \geq |F \setminus I| = N - k$. This concludes the proof.

Lemma 2

One direction is a special case of Lemma 1. For the other direction, recall that by a standard result (Arrow *et al* [4], Theorem 1) for a convex polytope $S \subset \mathbb{R}^N$ the set of points $x \in S$ such that $y \geq x \,\&\, y \neq x \implies y \notin S$ coincides with the set of points in S that maximise a linear function $\langle a, . \rangle$ where a has strictly positive components. A straightforward extension of this result is that, for an evaluation function e such that $e(i) \neq 0$ for all i, the set of e-admissible points coincides with the set of points in S that maximise a linear function $\langle a, . \rangle$ where a satisfies $sign(a_i) = e_i$ (the result in Arrow *et al* [4] corresponds to the case $e(i) = 1$ for all i). If (e, I) is possible at $x^* \in X$, then the projection x' of x^*on $\mathbb{R}^I$ has the property that $e(i)\, y_i \geq e(i)\, x'_i \,\forall i \in I \,\&\, y \neq x' \implies y \notin X'$, where X' denotes the projection of X on $\mathbb{R}^I$. What is more, $e(i) \neq 0$ for all $i \in I$. Since X' is also a convex polytope, by the previous result (applied to the subspace), there exists $a' \in \mathbb{R}^I$ with non-zero components such that $\langle a', x' \rangle \geq \langle a', y' \rangle$ for all $y' \in X'$. Therefore $\left\langle a^I, x \right\rangle \geq \left\langle a^I, y \right\rangle$ for all $y \in X$ where $a^I_i = a'_i$ for $i \in I$. Thus, I is a linear type at x.

Theorem 5

Suppose that type I is possible at x^*. By Lemma 2, this means that there exists an $a^I \in \mathbb{R}^N$ such that x^* is an optimal solution to the linear program

$$\max_{Bx \leq c} \left\langle a^I, x \right\rangle$$

By the Strong Duality Theorem (Bertsimas and Tsitsiklis [6], p. 148), the dual problem

$$\min_{B^T y = a^I,\, y \geq 0} \langle c, y \rangle$$

also has an optimal solution y^*. Since y^* is feasible, we have $y^* \geq 0$ and $a^I = B^T y^*$. By Complementary Slackness (Bertsimas and Tsitsiklis [6], p. 151),

$$y^*_j \left(B^{(j)} x^* - c_j \right) = 0, \text{ for all } j = 1, \ldots, m$$

where $B^{(j)}$ is the j-row of B, $j = 1, \ldots, m$. Therefore, $y^*_j = 0$ for the constraints that are not active at x^*. This implies that $a^I = \bar{B}^T(x^*)\, \bar{y}^*$ for $\bar{y}^*$ obtained from y^* by filtering out non-active constraints.

Conversely, suppose that $a^I = \bar{B}^T(x^*)\, y$ for some $y \geq 0$ and $a^I \in \mathbb{R}^N$. Let $y^*_j = y_j$ if the constraint j is active at x^* and $y^*_j = 0$ otherwise. Since $y^* \geq 0$ and

$a^I = B^T(x^*)\, y^*$, y^* is a feasible solution of the dual problem. By Complementary Slackness, x^* is an optimal solution of the primal problem $\max_{Bx \leq c} \langle a^I, x \rangle$, which implies that type I is linear, and therefore possible, at x^*.

Theorem 6

In this proof for clarity we will omit to denote the dependence of $\bar{B}$ on x^*. Suppose that $\left(a^I, I\right)$ is possible at x^*. Let e^j be a vector in $\mathbb{R}^N$ such that $e_i^j = -1$ for $i = j$ and $e_i^j = 0$ otherwise, and let $d^j = \bar{B}^{-1} e^j$. Then for $x(\lambda) = x^* + \lambda d^j$, $\lambda > 0$, we have

$$\bar{B}x = \bar{B}x^* + \lambda \bar{B} d^j = \bar{c} + \lambda e^j \leq \bar{c}.$$

Thus, the N constraints that are active at x^*, $\bar{B}x^* = \bar{c}$, will remain satisfied at $x(\lambda)$ for any $\lambda > 0$. Hence, there exists $\lambda_0 > 0$ such that $x(\lambda_0) \in X$. Since type I is possible at x^*, we have $\langle a^I, x(\lambda_0) \rangle \leq \langle a^I, x^* \rangle$ which implies $\langle a^I, d^j \rangle \leq 0$. Thus,

$$\left\langle a^I, -d^j \right\rangle = \sum_{i=1}^{N} a_i \bar{B}_{ij}^{-1} \geq 0,$$

which implies condition (2).

Conversely, suppose that $\left(a^I, I\right)$ is not possible at x^*. Then there exists an adjacent vertex $x^j \in X$ such that $\langle a^I, x^j \rangle > \langle a^I, x^* \rangle$. Note that $\bar{B}x^j = \bar{c}'$, where $\bar{c}_i' < \bar{c}_i$ for $i = j$ and $\bar{c}_i' = \bar{c}_i$ otherwise. Let $d^j = \frac{1}{\lambda}\left(x^j - x^*\right)$ for $\lambda = \bar{c}_i - \bar{c}_i' > 0$. Then $x^j = x^* + \lambda d^j$ implies $\langle a^I, d^j \rangle > 0$. On the other hand,

$$\bar{B} d^j = \frac{1}{\lambda}\left(\bar{B}x^j - \bar{B}x^*\right) = \frac{1}{\lambda}\left(\bar{c}' - \bar{c}\right) = e^j,$$

which implies $d^j = \bar{B}^{-1} e^j$. Hence,

$$\left\langle a^I, -d^j \right\rangle = \sum_{i=1}^{N} a_i \bar{B}_{ij}^{-1} < 0,$$

which implies that condition (2) is violated.

Corollary 5

If column i has a positive and a negative entry, then same is true for the vector $D(x^*)\,a^{\{i\}}$. Hence, type $\{i\}$ is not possible by condition (2). Conversely, suppose that every column of $D(x^*)$ has only entries of one sign or zero. Then, by choosing appropriately the sign of a_i^I for each column i, we can satisfy condition (2) for any type I, which means that all types are possible.

Corollary 6

Obviously, we can always find y with positive coordinates such that

$$(D^{-1}(x^*)\,y)_i \neq 0$$

for all $i = 1, \ldots, N$. Hence, we have $D(x^*)\,a^F = y > 0$ for $a^F = D^{-1}(x^*)\,y$, so that the full type F is possible.

Proposition 2

Obviously, a type that is possible at x^* is linear in this case as well. Then $\left(a^I, I\right)$ is possible if and only if for any $x \in X$, we have $\left\langle a^I, x^* \right\rangle \geq \left\langle a^I, x \right\rangle$, which is equivalent to $\left\langle a^I, x^* - x \right\rangle \geq 0$, from which condition (3) follows.

Theorem 7

"only if": Suppose there is an orthant K such that $\bigcup_t F\left(x^t\right) \subseteq K$. We can show using the construction in the first part of the proof of Theorem 2 that for any type I there exists e such that (e, I) is possible at all t. Hence, the type is not identified.

"if": Suppose there is no orthant K such that $\bigcup_t F\left(x^t\right) \subseteq K$. Then, there exist $v, w \in \bigcup_t F\left(x^t\right)$ such that $v_i > 0$ and $w_i < 0$ for some $i \in \{1, \ldots, N\}$. Clearly, $v \in F\left(x^t\right)$ and $w \in F(x^s)$ for some $t, s \in T$ (possibly with $t = s$). Hence, $(e, \{i\})$ is possible at both $\left(X_t, x^t\right)$ and (X_s, x^s) only if $e(i) = -1$ (follows from $v_i > 0$) and $e(i) = 1$ (follows from $w_i < 0$), a contradiction. Therefore, type $\{i\}$ is not possible at $\mathcal{O}$ and the type is partially identified.

Theorem 8

For sufficiency, the same argument of the proof of Theorem 3 can be used here with obvious adaptations, replacing (X, x^*) with a suitable element of the collection $\mathcal{O}$.

For the necessity proof, note that the intersection of the normal cones, $\bigcap_t N_{X_t}(x^t)$, must satisfy the first condition of Lemma 4: otherwise, multiple linear types would be possible at $\mathcal{O}$. Using this fact, we see that the proof of Lemma 5 remains valid with $T_X(x^*)$ replaced by $conv\left(\bigcup_t T_{X_t}(x^t)\right)$. Indeed, the only non-trivial step (needed for the third paragraph of the proof of Lemma 5) is to show that

$$\left(\bigcap_t N_{X_t}(x^t)\right)^0 = conv\left(\bigcup_t T_{X_t}(x^t)\right), \tag{9}$$

To prove (9), note that, by the Polar Cone Theorem (Bertsekas [5], p. 100), we have

$$\left(C^0\right)^0 = cl\,(conv\,(C))$$

for any nonempty cone C. By applying this equation to $C = \bigcup_t T_{X_t}(x^t)$, we get

$$\left(\left(\bigcup_t T_{X_t}(x^t)\right)^0\right)^0 = cl\left(conv\left(\bigcup_t T_{X_t}(x^t)\right)\right). \tag{10}$$

Clearly, the convex hull of a finite union of closed convex cones is closed, so we can omit the closure operator. Using the fact that the normal cone is the polar cone of the tangent cone, it is straightforward to check that

$$\left(\bigcup_t T_{X_t}(x^t)\right)^0 = \bigcap_t N_{X_t}(x^t).$$

By combining this with (10), we get the desired result (9). Therefore, the first condition of Lemma 5 holds with $T_X(x^*)$ replaced by $conv\left(\bigcup_t T_{X_t}(x^t)\right)$. For the rest of the proof, we can use the same argument as in the final part of the proof of Theorem 3, replacing (X, x^*) with a suitable element of the collection $\mathcal{O}$.

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