Sequential Sampling Beyond Decisions?
A Normative Model of Decision Confidence

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Decision confidence

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  - Subjective assessment of one's own decision quality

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  - It affects future decisions
  - We communicate it to others
  - It helps to adapt strategies

Car choice example: Audi vs. BMW?

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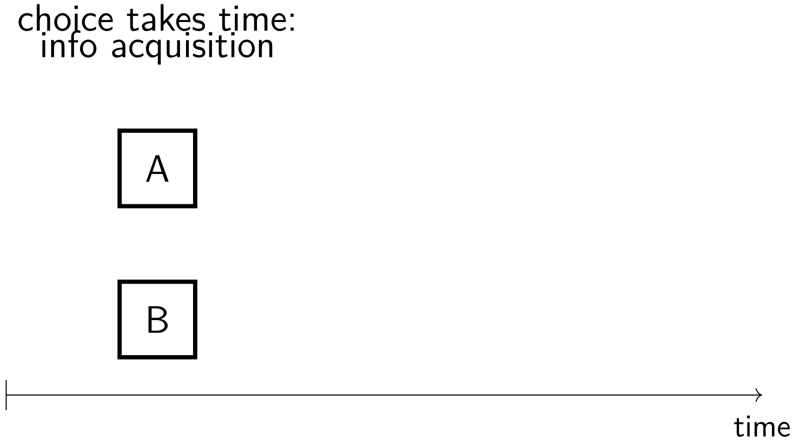

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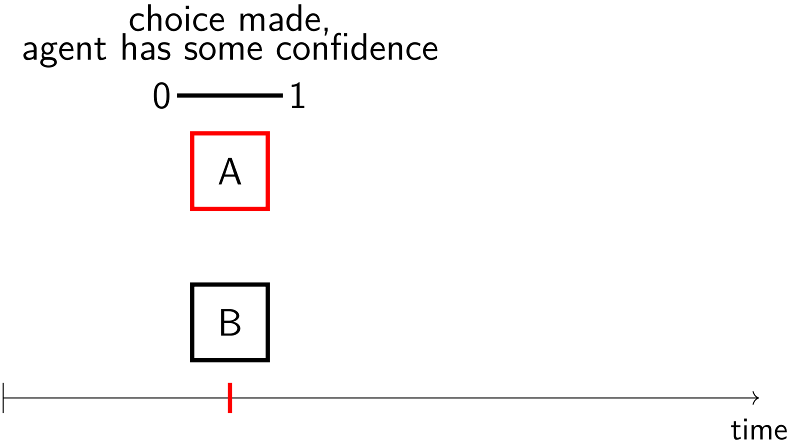

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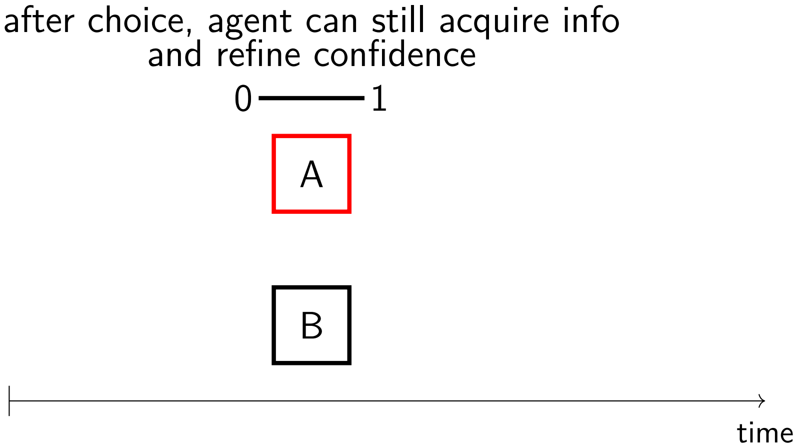

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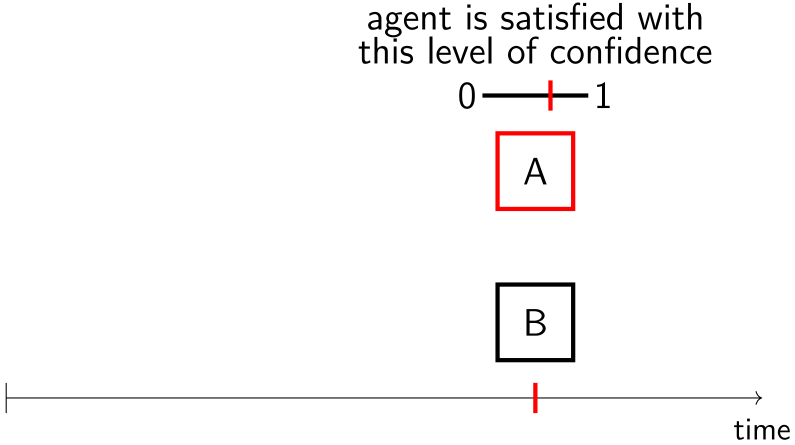

Structure of modeled data

Trial	Difficulty ($u_A - u_B$)	RT (hours)	Decision (A or B)	RT2 (hours)	Confidence ($\in [0, 1]$)
1	5	3	A	0.1	0.6
2	-25	0.5	B	1	0.8
⋮	⋮	⋮	⋮	⋮	⋮

Experimental paradigm

Evidence of post-decisional info acquisition

- Post-decisional display of options affects confidence resolution (Moran et al., 2015)

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- Post-decisional display of options affects confidence resolution (Moran et al., 2015)
- Spontaneous error recognition and changes of mind (Charles and Yeung, 2019)
- Different neural processes involved in decisions and confidence judgements (Hilgenstock et al., 2014)

Research question and contribution

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- **Cognitive science** – confidence models: Pleskac and Busemeyer (2010) (2DSD), Moran et al. (2015) (CCB)

$\Rightarrow$ post-decisional info acquisition not derived from first principles

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- Goal: provide statistical foundation for post-decisional mechanism in confidence formation

- **Cognitive science** – confidence models: Pleskac and Busemeyer (2010) (2DSD), Moran et al. (2015) (CCB)

$\Rightarrow$ post-decisional info acquisition not derived from first principles

- **Statistics** – sequential sampling models: Wald (1947), Chernoff (1961), Fudenberg et al. (2018)

$\Rightarrow$ not modeling decision confidence

Main takeaways

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  - **Not always**

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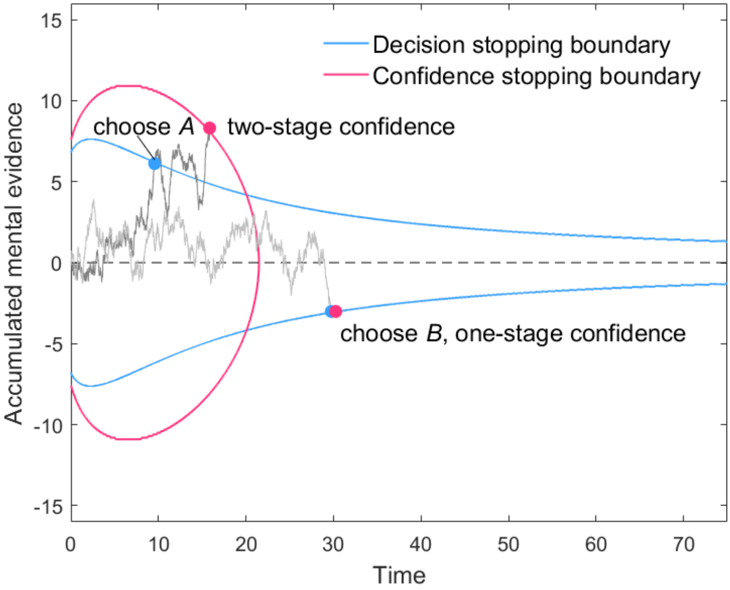
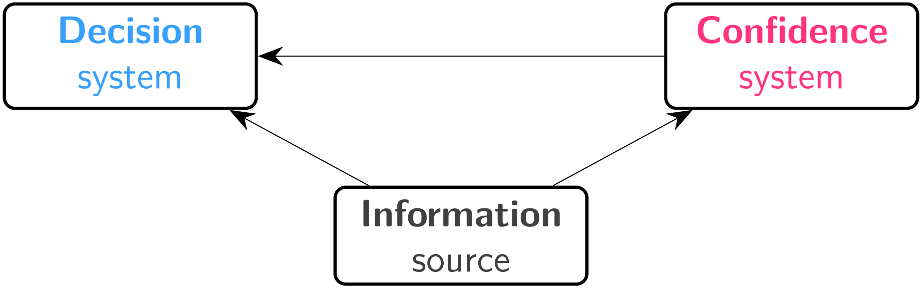
  - Some **scope** only for relatively **fast decisions**


Decision stopping boundary
Confidence stopping boundary
choose A
two-stage confidence
choose B, one-stage confidence
Accumulated mental evidence
Time
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60
70

Model structure



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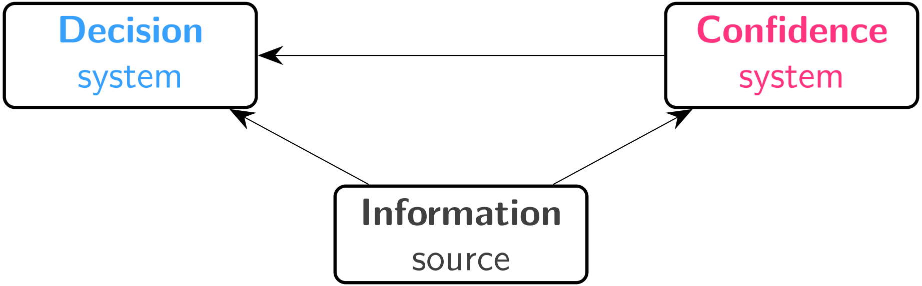
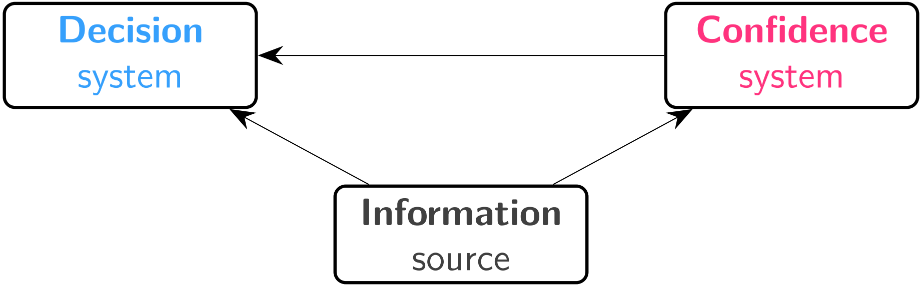


- **DS**'s goal: choose optimal option

Model structure



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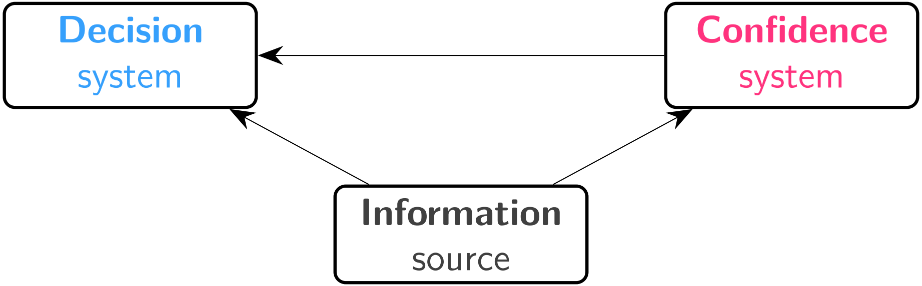

- **DS**'s goal: choose optimal option
- **CS**'s goal: provide feedback in form of confidence
- **Information**: costly and noisy mental evidence about true difference between A and B

Model

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- Prior $N(X_0, 2\sigma_0^2)$

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- **Decision problem**:
  - decision rule: given info, max. expected utility
  - info acquisition stopping rule:

max E[payoff – cost of time]

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$$d_t = \text{sgn}\left(\text{E}\left[\theta|\mathcal{F}_t^Z\right]\right)$$

$$\max_{\tau} \text{E}\left[|\theta|\mathbb{1}\{d_\tau = \text{sgn}(\theta)\} - c\tau\right]$$

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- **Confidence problem**:
  - confidence: posterior probability of being correct
  - info acquisition stopping rule:

min E[loss from imprecise confidence + cost of time]

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- **Confidence problem**:

$$conf_t = \Pr\left(d_\tau = \text{sgn}(\theta)|\mathcal{F}_t^Z\right)\ \ t \geq \tau$$

$$\min_{\tau_c \text{ s.t. } \tau_c \geq \tau} \mathsf{E}\left[\left(conf_{\tau_c} - \mathbb{1}\{d_\tau = \text{sgn}(\theta)\}\right)^2 + \bar{c}(\tau_c - \tau)\right]$$

Main result

Theorem 1

$\exists T > 0 \ s.t. \ P(\tau^* > T \ \& \ \tau_C^* = \tau^*) = P(\tau^* > T) > 0.$

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$\exists T > 0$ *s.t.* $P(\tau^* > T \ \& \ \tau_C^* = \tau^*) = P(\tau^* > T) > 0.$

$\Rightarrow$ There are decisions with **no post-decisional** info acquisition

$\Rightarrow$ **Scope for post-decisional** stage only for relatively **fast decisions**

Boundedness of (unconstrained) **confidence** stopping

Lemma 1

Optimal stopping time in the unconstrained confidence stopping problem is bounded almost surely by $T_c := \max\{\frac{1}{2\pi\bar{c}} - \frac{\alpha^2}{\sigma_0^2}, 0\}$.

Boundedness of (unconstrained) **confidence** stopping

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Proof idea: Bound = rightmost point of region beyond which cost of sampling dominates expected gains from confidence refinement.

More details

Unboundedness of **decision** stopping

Lemma 2 (Fudenberg et al., 2018)

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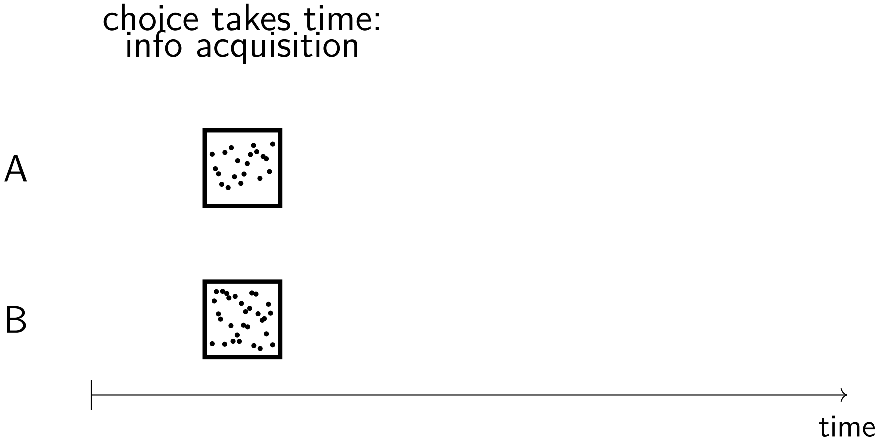
- Conjecture based on numerical solution ⏩ Numerics and conjecture
- Empirical evaluation ⏩ Empirical patterns
- Simpler version based on Wald model [paper]
- Deterministic stopping times [paper]

Extras

2-alternative choice-followed-by-confidence paradigm

Back to data

2-alternative choice-followed-by-confidence paradigm



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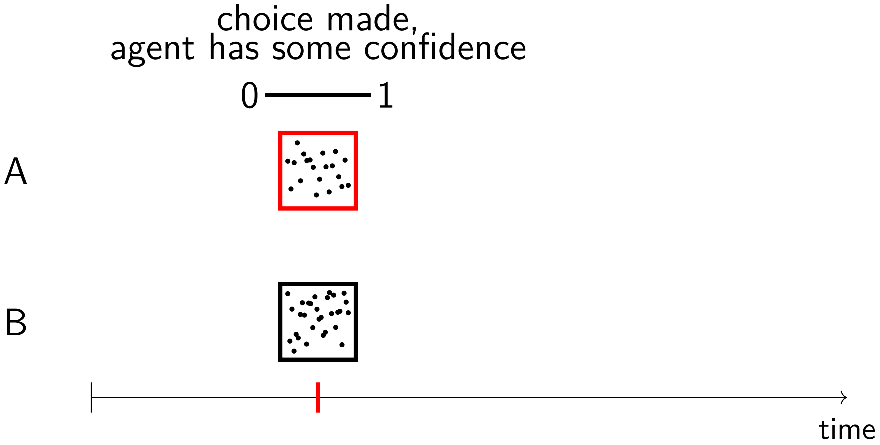

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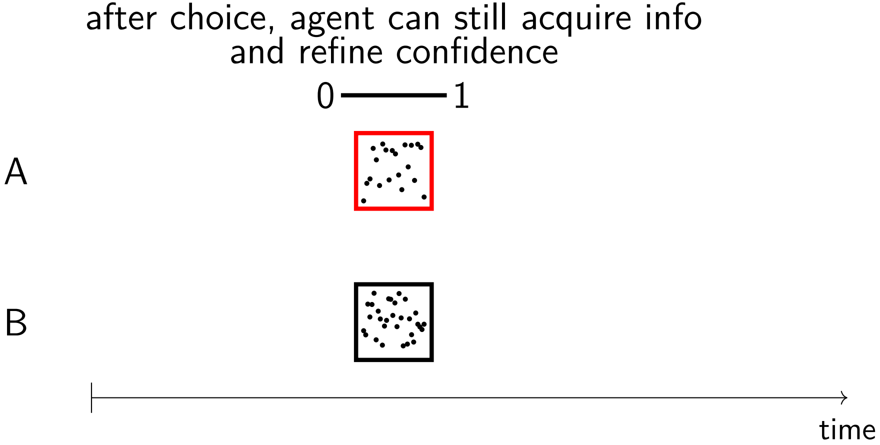

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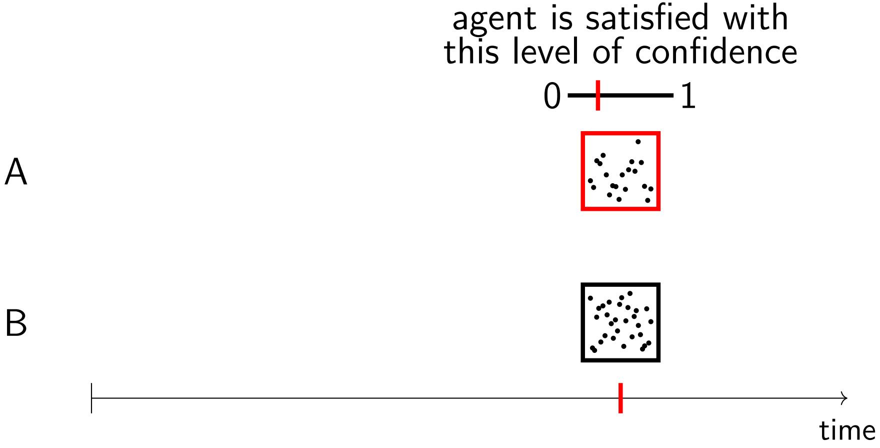

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Beliefs

Lemma 3

Posterior beliefs after observing $\{Z_s\}_{s \leq t}$ *are*

$$N(X_t, \sigma_t^2)$$

$$X_t = \frac{\sigma_0^{-2} X_0 + \alpha^{-2} Z_t}{\sigma_0^{-2} + \alpha^{-2} t}$$

$$\sigma_t^2 = \frac{2}{\sigma_0^{-2} + \alpha^{-2} t}.$$

Confidence objective

$$conf_t = \Phi\left(\frac{X_t}{\sigma_t}\right)\mathbb{1}\{X_\tau \geq 0\} + \Phi\left(-\frac{X_t}{\sigma_t}\right)\mathbb{1}\{X_\tau < 0\}$$

Confidence objective

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$$\begin{aligned}
&\mathrm{E}[(conf_{\tau_c} - \mathbb{1}\{d_\tau = \mathrm{sgn}(\theta)\})^2] \\
&\quad = \mathrm{E}[\mathrm{E}[(conf_{\tau_c} - \mathbb{1}\{d_\tau = \mathrm{sgn}(\theta)\})^2 | \mathcal{F}_{\tau_c}]] \\
&\quad = \mathrm{E}[\mathrm{var}(\mathbb{1}\{d_\tau = \mathrm{sgn}(\theta)\} | \mathcal{F}_{\tau_c})] \\
&\quad = \mathrm{E}[conf_{\tau_c}(1 - conf_{\tau_c})] \\
&\quad = \mathrm{E}\left[\Phi\left(\frac{X_{\tau_c}}{\sigma_{\tau_c}}\right)\Phi\left(-\frac{X_{\tau_c}}{\sigma_{\tau_c}}\right)\right]
\end{aligned}$$

Auxiliary unconstrained confidence problem

$$\min_{\tau_c} \mathrm{E}\left[\Phi\left(\frac{X_{\tau_c}}{\sigma_{\tau_c}}\right)\Phi\left(-\frac{X_{\tau_c}}{\sigma_{\tau_c}}\right)+\bar{c}\tau_c\right]$$

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Loss function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$

$$f(t,z) := \Phi\left(\frac{\sigma_0^{-2}X_0 + \alpha^{-2}z}{\sqrt{2(\sigma_0^{-2} + \alpha^{-2}t)}}\right)\Phi\left(-\frac{\sigma_0^{-2}X_0 + \alpha^{-2}z}{\sqrt{2(\sigma_0^{-2} + \alpha^{-2}t)}}\right) + \bar{c}t$$

Confidence subcontinuation region

Lemma 4

Let $\mathring{C}_C$ denote

$$\left\{(t,z)\in[0,\infty)\times\mathbb{R}\colon \frac{\alpha^{-2}}{\sigma_0^{-2}+\alpha^{-2}t}\varphi^2\left(\frac{\sigma_0^{-2}X_0+\alpha^{-2}z}{\sqrt{2(\sigma_0^{-2}+\alpha^{-2}t)}}\right)>\bar{c}\right\}.$$

It is not optimal to stop sampling for confidence in $\mathring{C}_C$.

Idea of proof for Lemma 4

- Innovation representation of evidence process $\{Z_s\}$ $\Rightarrow$ process $\{Y_s^{(t,z)}\}_{s\geq 0}$, $Y_s^{(t,z)} := (t+s, Z_s^z)'$ for $s \geq 0$,

$$\mathrm{d}Y_s^{(t,z)} = \begin{pmatrix} 1 \\ \frac{\sigma_0^{-2}X_0+\alpha^{-2}Z_s^z}{\sigma_0^{-2}+\alpha^{-2}(t+s)} \end{pmatrix} \mathrm{d}s + \begin{pmatrix} 0 \\ \alpha\sqrt{2} \end{pmatrix} \mathrm{d}\bar{B}_s, \ s \geq 0, \ Y_0^{(t,z)} = (t,z)$$

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- Examine generator A of process Y acting on loss function f (Itô) because

$$\mathrm{E}\left[f(Y_{\tau'}^{(t,z)})\right] = f(t,z) + \mathrm{E}\left[\int_0^{\tau'} Af(t+s, Z_s^z)ds\right]$$

Idea of proof for Lemma 4 (cont.)

- Region of obvious expected decrease of loss function

$$\mathring{C}_C = \{(t, z) \in [0, \infty) \times \mathbb{R} \colon Af(t, z) < 0\}$$

Boundedness of (unconstrained) confidence stopping

Lemma 5

Optimal stopping time in the unconstrained confidence stopping problem is bounded almost surely by $T_c := \max\{\frac{1}{2\pi\bar{c}} - \frac{\alpha^2}{\sigma_0^2}, 0\}$.

Boundedness of (unconstrained) confidence stopping

Lemma 5

Optimal stopping time in the unconstrained confidence stopping problem is bounded almost surely by $T_c := \max\{\frac{1}{2\pi\bar{c}} - \frac{\alpha^2}{\sigma_0^2}, 0\}$.

Proof idea: Bound = rightmost point of $\mathring{C}_C$ beyond which cost of sampling dominates any expected gains from learning ($\mathcal{A}f > 0$).

Back to results

Unboundedness of decision stopping

Lemma 6 (Fudenberg et al., 2018)

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Proof idea: At indifference ($Z_t = 0$), benefit of waiting for small enough ε is of order $\sqrt{\varepsilon}$, while cost is of order ε.

Main analytical result

Theorem 2

Let $T_c := \max\{\frac{1}{2\pi\bar{c}} - \frac{\alpha^2}{\sigma_0^2}, 0\}$.

Then $P(\tau^* > T_c \;\&\; \tau^*_C = \tau^*) = P(\tau^* > T_c) > 0$.

Figure: Numerical solution ($\bar{c} = 0.007, c = 0.02, \alpha = 2, \sigma_0 = 1.8$)

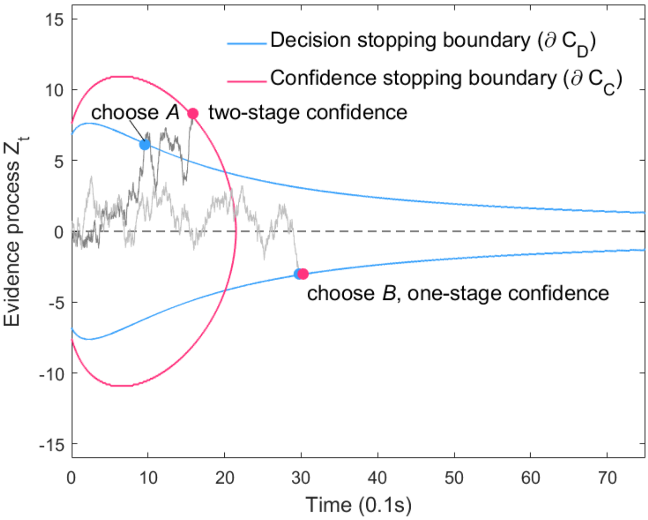
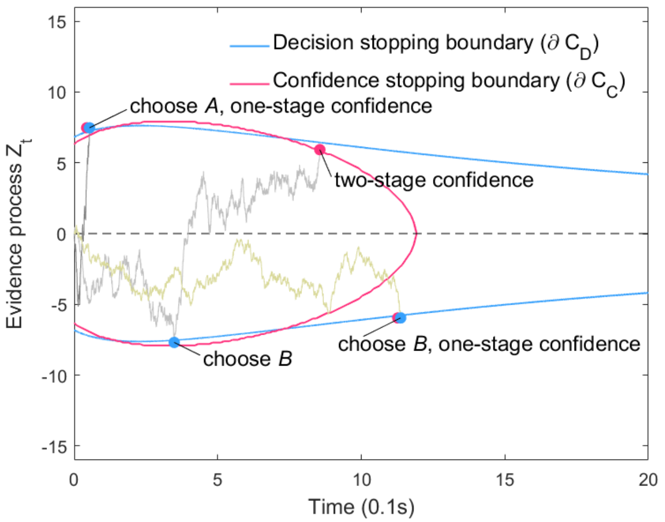


Figure: Numerical solution ($\bar{c} = \mathbf{0.012}, c = 0.02, \alpha = 2, \sigma_0 = 1.8$)



Conjecture

$\exists$ cases s.t. **very fast decisions** (as well as slow decisions) lead to **one-stage confidence** and moderately fast decisions lead to two-stage confidence

Back to conclusion

Empirical evaluation: patterns of Moran et al. (2015)

1. Speed-accuracy trade-off: higher error rate under time pressure
2. Slow errors
3. Negative correlation of confidence and difficulty
4. Negative correlation of decision time and confidence
5. Lower confidence under time pressure
6. Positive confidence resolution: lower confidence in errors
7. Increased confidence resolution under time pressure
8. Positive correlation of RT2 and difficulty
9. Lower RT2 in correct choices
10. Negative correlation of RT2 and confidence
11. Positive correlation of RT2 and RT

Empirics (cont.)

12. Decreased confidence resolution for difficult decisions
13. Higher RT2 for correct choices and lower RT2 for errors under higher difficulty

Back to conclusion