Equity In Allocating Identical Objects Through Reserve Categories

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A very simple problem

- Allocating identical objects to a set of agents such that
  - each agent receives at most one object,
  - each object is assigned to at most one agent.
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- Example: vaccines, ICU’s or other medical units to patients
A very simple solution

- Priority mechanisms
A very simple solution

- Priority mechanisms
  - widely used
  - widely criticized: because they could leave certain groups of patients with no or very little access
How to provide a wider access?

- Reserve systems:
  - units are divided into reserve categories, e.g. disadvantaged community member, essential worker, death likely within 1 year,
  - a certain number of units is reserved for each category and
  - each category has its own priority ordering of patients.
The solution would be straightforward unless...

- patients belong to multiple categories in general, e.g. a patient could be an essential worker from a disadvantaged community.

An example:
- \(i\): an essential worker (\(c_1\)) from a disadvantaged community (\(c_2\))
- \(j\): an essential worker
- \(k\): a disadvantaged community member
- there are two units in total and one unit is reserved for each category

\[
\begin{array}{c|c}
\pi_{c_1} & \pi_{c_2} \\
\{i\} & \{i\} \\
\{j\} & \{k\}
\end{array}
\]
Alternative solution: randomization

- The Department of Health, Pennsylvania
  - A weighted lottery mechanism for the allocation of medications to treat COVID-19
  - "all patients who meet clinical eligibility criteria should have a chance to receive treatment" ("Pandemic Guidelines for the Interim Pennsylvania Crisis Standards of Care")
How does randomization work in Pennsylvania?

- $p$: general community changes (total available units divided by the estimated number of eligible patients)

<table>
<thead>
<tr>
<th>Group</th>
<th>Chances to receive treatment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disadv. community member ($c_1$)</td>
<td>$1.25 \times p$</td>
</tr>
<tr>
<td>Essential worker ($c_2$)</td>
<td>$1.25 \times p$</td>
</tr>
<tr>
<td>Death likely within 1 year ($c_3$)</td>
<td>$0.5 \times p$</td>
</tr>
<tr>
<td>$c_1$ and $c_2$</td>
<td>$1.5 \times p$</td>
</tr>
<tr>
<td>$c_1$ and $c_3$</td>
<td>$0.75 \times p$</td>
</tr>
<tr>
<td>$c_2$ and $c_3$</td>
<td>$0.75 \times p$</td>
</tr>
</tbody>
</table>
Our approach

- A general class of random allocation mechanisms under reserves to satisfy:
  - Efficiency and respecting priorities within any “wider access framework" (i.e. weak ordering of priorities)
- Specific mechanisms within this class to satisfy:
  - Fairness-egalitarianism
Our model

- $\mathcal{I}$: a set of agents
- $\mathcal{C}$: a set of reserve categories
- for each $c \in \mathcal{C}$:
  - $q_c$ identical units are reserved, and
  - there is a weak priority order $\pi_c$ over $\mathcal{I}$
Definition

Given a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c \in \mathcal{C}}, (q_c)_{c \in \mathcal{C}})$, a random allocation is a stochastic $|\mathcal{I}| \times |\mathcal{C}|$ matrix $Z$ where for each $i$ and $c$, $z_{ic}$ is the probability with which agent $i$ is assigned one unit from category $c$ such that

i. for each $i \in \mathcal{I}$, $\sum_{c \in \mathcal{C}} z_{ic} \leq 1$,

ii. for each $c \in \mathcal{C}$, $\sum_{i \in \mathcal{I}} z_{ic} \leq q_c$. 
Two main properties

**Definition**

**non-wastefulness**: if a unit remains (partially) unassigned, then each agent receives a unit w/p 1

**Definition**

**respecting priorities**: if an agent receives a unit from a category with positive probability, then each strictly higher priority agent is assigned a unit w/p 1 (not necessarily from that category)
Priority-Based Sequential Welfare Improvement (PBSWI)

The simplest possible idea:

1. each agent has zero utility initially
2. for each category, start with the agents at the top (of the priority ordering)
3. **update the utility profile**: (weakly) increase agents’ utility *feasibly* by some amount
4. if an agent reaches utility one, then continue with the next agent in the priority ordering (*respecting priorities*)
5. until all units are allocated (*non-wastefulness*)
Priority-Based Sequential Welfare Improvement (PBSWI)

- But this idea is not easy to execute.
- Challenge: how to update the utility profile feasibly and comprehensively?
But this idea is not easy to execute.

Challenge: how to update the utility profile \textit{feasibly} and \textit{comprehensively}?

First, comprehensiveness:

\[
\begin{align*}
\pi_{c_1} & \{i\} \\
\pi_{c_2} & \{i, j\} \\
\{k\} & \{k\}
\end{align*}
\]

\(\Gamma_c(v)\): the set of \textit{claimants} for category \(c\) under \(v\)
A clear exception:

\[
\begin{array}{c c}
\pi_{c_1} & \pi_{c_2} \\
\{i\} & \{j\} \\
\{k\} & \{k\}
\end{array}
\]

For the reservation profile \( v = (v_i, v_j, v_k) = (1, 1, 0) \), all agents are claimants for all categories.

But, any random allocation such that a unit is (probabilistically) assigned to \( k \) does not respect priorities.
Priority-Based Sequential Welfare Improvement (PBSWI)

- $C(i, v)$: the set of reserve categories, for which agent $i$ is a claimant under the reservation profile $v$.
- $C(I, v) = \bigcup_{i \in I} C(i, v)$.

Definition

Given a reservation profile $v = (v_i)_{i \in I}$, agents in $I$ have exclusive rights over the set of reserve categories $C(I, v)$ if $\sum_{i \in I} v_i = \sum_{c \in C(I, v)} q_c$. 
Priority-Based Sequential Welfare Improvement (PBSWI)

Theorem

(The Supply-Demand Theorem (Gale, 1957)) Let $v = (v_i)_{i \in \mathcal{I}}$ be a reservation profile. There is a random allocation $Z$ such that (i) for each $i \in \mathcal{I}$, $u_Z(i) \geq v_i$, and (ii) $z_{ic} > 0$ implies $i \in \Gamma_c(v)$, if and only if, for each subset $I$ of agents

$$\sum_{i \in I} v_i \leq \sum_{c \in C(I,v)} q_c. \quad (1)$$
Priority-Based Sequential Welfare Improvement (PBSWI)

Step 0. Let the reservation profile be \( v^0 = 0 \).

For each \( n \geq 1 \) and the reservation profile \( v^{n-1} \), the following steps are executed.

Step n.1 For each set of agents \( I \) with exclusive rights over \( C(I, v^{n-1}) \),

i. for each \( i \in I \), let \( v^i_n = v^i_{n-1} \), and

ii. mark each reserve category in the set \( C(I, v^{n-1}) \) as unavailable.

Let \( A_n \) denote the set of available reserve categories.

Step n.2 If \( A_n = \emptyset \), then let \( Z^* \) with \( u_{Z^*} = v^{n-1} \) be the outcome. Otherwise, proceed to Step n.3.

Step n.3 (Welfare improvement) Select a feasible reservation profile \( v^n \neq v^{n-1} \) such that for each \( i \),

\[ v^i_n = v^i_{n-1} + \lambda^i_n \]

where \( \lambda^i_n \in [0, 1] \), and for each \( i \not\in \bigcup_{c \in A_n} \Gamma_c(v^{n-1}) \), \( \lambda^i_n = 0 \).
Egalitarianism

Definition
A random allocation \( Z \in \mathcal{Z}^a(R) \) is \textit{egalitarian} if it is Lorenz dominant in the set \( \mathcal{Z}^a(R) \).
Egalitarianism

Definition
A random allocation \( Z \in \mathcal{Z}^a(R) \) is egalitarian if it is Lorenz dominant in the set \( \mathcal{Z}^a(R) \).

Theorem
No rule is egalitarian.
Egalitarianism

- More interestingly...

\[ \pi_{c_1} \]
\{i, j\} \quad \{i, j\}
\{i_1, i_2\} \quad \{j_1, j_2\}
\{k, l\} \quad \{k, l\}
Procedural fairness

Definition

A random allocation is procedurally fair if it obtained as a sequence of feasible reservation profiles such that, at each step, the selected reservation profile Lorenz dominates any other feasible reservation profile that can be selected at that step.
Procedural fairness

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The Priority-Based Rawlsian (PBR) rule

- Rawlsian principle: Maximizing the minimum welfare.
- At each step: utilities of the most disadvantaged agents are increased continuously as long as the constraints embedded through claimants and reservation profile are not binding.
The Priority-Based Rawlsian (PBR) rule

Step n.3 (Welfare improvement selection rule of the PBR)
The agents with the minimum reservation value are selected among agents, who are claimants for at least one available category. Their reservation values are increased equally up to the minimum of the following two, while other agents’ reservation values do not change:

- The reservation value of a non-selected agent, who is a claimant for at least one available category.
- The level at which a subset of claimants for at least one available category has exclusive rights over the categories for which they are claimant.
Main theorem

Theorem

A rule is procedurally fair if and only if $\varphi$ is welfare-equivalent to the PBR.
Proof

- At each step: construct a particular network to find out the bottleneck set with the maximum increase of the least advantaged agents
- By an extension of the Max-flow Min-cut Theorem (Ford, Fulkerson, 1956)

**Theorem**

Let \((V, A, I, k)\) be a supply-demand network such that there exists a flow \(f\). Then, the maximum value of a flow is equal to the minimum value of

\[
k(\delta^{\text{out}}(V')) - I(\delta^{\text{in}}(V'))
\]

taken over \(V' \subseteq V\) with \(s \in V'\) and \(t \notin V'\).