Equity In Allocating Identical Objects Through Reserve Categories

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A very simple problem

- Allocating identical objects to a set of agents such that
  - each agent receives at most one object,
  - each object is assigned to at most one agent.

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- Example: vaccines, ICU's or other medical units to patients

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- Priority mechanisms

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- Priority mechanisms
  - widely used
  - widely criticized: because they could leave certain groups of patients with no or very little access

How to provide a wider access?

- Reserve systems:
  - units are divided into reserve categories, e.g. disadvantaged community member, essential worker, death likely within 1 year,
  - a certain number of units is reserved for each category and
  - each category has its own priority ordering of patients.

The solution would be straightforward unless...

- patients belong to multiple categories in general, e.g. a patient could be an essential worker from a disadvantaged community.
- An example:
  - i: an essential worker (c_1) from a disadvantaged community (c_2)
  - j: an essential worker
  - k: a disadvantaged community member
  - there are two units in total and one unit is reserved for each category

π_{c_1}	π_{c_2}
$\{i\}$	$\{i\}$
$\{j\}$	$\{k\}$

Alternative solution: randomization

- The Department of Health, Pennsylvania
  - A weighted lottery mechanism for the allocation of medications to treat COVID-19
  - “all patients who meet clinical eligibility criteria should have a chance to receive treatment" (“Pandemic Guidelines for the Interim Pennsylvania Crisis Standards of Care")

How does randomization work in Pennsylvania?

- p: general community changes (total available units divided by the estimated number of eligible patients)

Group	Chances to receive treatment
Disadv. community member (c_1)	1.25 x p
Essential worker (c_2)	1.25 x p
Death likely within 1 year (c_3)	0.5 x p
c_1 and c_2	1.5 x p
c_1 and c_3	0.75 x p
c_2 and c_3	0.75 x p

Our approach

- A general class of random allocation mechanisms under reserves to satisfy:
  - Efficiency and respecting priorities within any "wider access framework" (i.e. weak ordering of priorities)
- Specific mechanisms within this class to satisfy:
  - Fairness-egalitarianism

Our model

- $\mathcal{I}$: a set of **agents**
- $\mathcal{C}$: a set of **reserve categories**
- for each $c \in \mathcal{C}$:
    - q_c identical units are reserved, and
    - there is a **weak priority order** π_c over $\mathcal{I}$

Random allocation

Definition

Given a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c \in \mathcal{C}}, (q_c)_{c \in \mathcal{C}})$, a **random allocation** is a stochastic $|\mathcal{I}| \times |\mathcal{C}|$ matrix Z where for each i and c, z_{ic} is the probability with which agent i is assigned one unit from category c such that

i. for each $i \in \mathcal{I}$, $\sum_{c \in \mathcal{C}} z_{ic} \leq 1$,

ii. for each $c \in \mathcal{C}$, $\sum_{i \in \mathcal{I}} z_{ic} \leq q_c$.

Two main properties

Definition

non-wastefulness: if a unit remains (partially) unassigned, then each agent receives a unit w/p 1

Definition

respecting priorities: if an agent receives a unit from a category with positive probability, then each strictly higher priority agent is assigned a unit w/p 1 (not necessarily from that category)

Priority-Based Sequential Welfare Improvement (PBSWI)

- The simplest possible idea:
  1. each agent has zero utility initially
  2. for each category, start with the agents at the top (of the priority ordering)
  3. **update the utility profile**: (weakly) increase agents' utility *feasibly* by some amount
  4. if an agent reaches utility one, then continue with the next agent in the priority ordering (*respecting priorities*)
  5. until all units are allocated (*non-wastefulness*)

Priority-Based Sequential Welfare Improvement (PBSWI)

- But this idea is not easy to execute.
- Challenge: how to update the utility profile *feasibly* and *comprehensively*?

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- But this idea is not easy to execute.
- Challenge: how to update the utility profile *feasibly* and *comprehensively*?
- First, comprehensiveness:

$$\begin{array}{cc} \underline{\pi_{c_1}} & \underline{\pi_{c_2}} \\ \{i\} & \{i,j\} \\ \{k\} & \{k\} \end{array}$$

- $\Gamma_c(v)$: the set of **claimants** for category c under v

Priority-Based Sequential Welfare Improvement (PBSWI)

- A clear exception:

$$\begin{array}{cc} \pi_{c_1} & \pi_{c_2} \\ \hline \{i\} & \{j\} \\ \{k\} & \{k\} \end{array}$$

- For the reservation profile $v = (v_i, v_j, v_k) = (1, 1, 0)$, all agents are claimants for all categories.
- But, any random allocation such that a unit is (probabilistically) assigned to k does not respect priorities.

Priority-Based Sequential Welfare Improvement (PBSWI)

- $C(i, v)$: the set of reserve categories, for which agent i is a *claimant* under the reservation profile v.
- $C(I, v) = \bigcup_{i \in I} C(i, v)$.

Definition

Given a reservation profile $v = (v_i)_{i \in \mathcal{I}}$, agents in I have **exclusive rights** over the set of reserve categories $C(I, v)$ if $\sum_{i \in I} v_i = \sum_{c \in C(I,v)} q_c$.

Theorem

(The Supply-Demand Theorem (Gale, 1957)) Let $v = (v_i)_{i \in \mathcal{I}}$ *be a reservation profile. There is a random allocation* Z *such that (i) for each* $i \in \mathcal{I}$, $u_z(i) \geq v_i$*, and (ii)* $z_{ic} > 0$ *implies* $i \in \Gamma_c(v)$*, if and only if, for each subset* I *of agents*

$$\sum_{i \in I} v_i \leq \sum_{c \in C(I,v)} q_c. \tag{1}$$

Priority-Based Sequential Welfare Improvement (PBSWI)

Step 0. Let the reservation profile be $v^0 = 0$.

For each $n \geq 1$ and the reservation profile v^{n-1}, the following steps are executed.

Step n.1 For each set of agents I with exclusive rights over $C(I, v^{n-1})$,

i. for each $i \in I$, let $v_i^n = v_i^{n-1}$, and
ii. mark each reserve category in the set $C(I, v^{n-1})$ as unavailable.

Let A_n denote the set of available reserve categories.

Step n.2 If $A_n = \emptyset$, then let $Z^\star$ with $u_{Z^\star} = v^{n-1}$ be the outcome. Otherwise, proceed to Step n.3.

Step n.3 (Welfare improvement) Select a feasible reservation profile $v^n \neq v^{n-1}$ such that for each i, $v_i^n = v_i^{n-1} + \lambda_i^n$ where $\lambda_i^n \in [0, 1]$, and for each $i \notin \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, $\lambda_i^n = 0$.

Egalitarianism

Definition

A random allocation $Z \in \mathcal{Z}^a(R)$ is **egalitarian** if it is Lorenz dominant in the set $\mathcal{Z}^a(R)$.

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Theorem

No rule is egalitarian.

Egalitarianism

- More interestingly...

π_{c_1}	π_{c_2}
$\{i, j\}$	$\{i, j\}$
$\{i_1, i_2\}$	$\{j_1, j_2\}$
$\{k, l\}$	$\{k, l\}$

Procedural fairness

Definition

A random allocation is procedurally fair if it obtained as a sequence of feasible reservation profiles such that, at each step, the selected reservation profile Lorenz dominates any other feasible reservation profile that can be selected at that step.

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The Priority-Based Rawlsian (PBR) rule

- Rawlsian principle: Maximizing the minimum welfare.
- At each step: utilities of the most disadvantaged agents are increased continuously as long as the constraints embedded through *claimants* and *reservation profile* are not binding.

The Priority-Based Rawlsian (PBR) rule

Step n.3 (Welfare improvement selection rule of the *PBR*)
The agents with the minimum reservation value are *selected* among agents, who are *claimants* for at least one *available* category. Their reservation values are increased equally up to the minimum of the following two, while other agents' reservation values do not change:

- The reservation value of a *non-selected* agent, who is a *claimant* for at least one *available* category.
- The level at which a subset of *claimants* for at least one *available* category has *exclusive rights* over the categories for which they are*claimant*.

Main theorem

Theorem

A rule is procedurally fair if and only if φ *is welfare-equivalent to the PBR.*

Proof

- At each step: construct a particular network to find out the bottleneck set with the maximum increase of the least advantaged agents
- By an extension of the Max-flow Min-cut Theorem (Ford, Fulkerson, 1956)

Theorem

Let (V, A, l, k) be a supply-demand network such that there exists a flow f. Then, the maximum value of a flow is equal to the minimum value of

$$k(\delta^{out}(V')) - l(\delta^{in}(V'))$$

taken over $V' \subseteq V$ with $s \in V'$ and $t \notin V'$.