Axioms for Constant Function Market Makers

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Introduction

- A constant function market maker (CFMM) is a simple mechanism to trade financial assets.
- Originally introduced for prediction markets
- Most popular for trading cryptocurrencies

Motivation of DEXes

- A decentralized alternative to an order-book exchange
  - In the spirit of blockchains: less trust, faster, happening on-chain
  - simple design because computation on-chain is expensive
  - No censorship, no KYC, no need for intermediation
- Around 10% of crypto trading:
  - An average of \$3.4B trading volume a day in 2022
  - \$45B total value locked

Informal Description

- Liquidity is pooled
- Liquidity providers (LPs) provide tokens of type A and type B in a fixed ratio to the pool
- Traders swap tokens with the pool
- The market is made such that (ignoring trading fees) some function of inventories is kept constant
- To compensate LPs for the risk they take on, a trading fee is levied

Informal Description

- Example: Constant Product Market Maker (CPMM): Invariant is the product of inventories
- Two tokens A and B with current inventories $I = (I_A, I_B)$.
- A trader can swap x type A tokens for y type B tokens such that

$$(I_A + x)(I_B - y) = I_A I_B.$$

Contribution

- Systematic approach to constructing invariants
- Axiomatic approach: distill desiderata for useful trading functions
- Characterize a simple class of trading functions by a combination of intuitive axioms
- Characterize the CPMM as "trader optimal" within that class
- Bonus: Get an axiomatic characterization of LMSR MMs from prediction markets almost for free

Model

- A finite set of asset types $\mathcal{A}$
- A **trading function** is a strictly increasing, continuous function $f : \mathbb{R}_+^{\mathcal{A}} \to \mathbb{R}$ (other authors also require quasi-concavity)
- $f(I)$ is the value of the inventory I measured in a numéraire
- A **trade** $r \leq I$ changes the inventory from I to $I - r$.
- To execute a trade, a trader needs to compensate the MM with $f(I) - f(I - r)$ units of numéraire

Trading and the numéraire

- In a DEX there are w.l.o.g. two types of trades:

  Swapping: Let $A, B \in \mathcal{A}$. If inventory prior to trading is $I \in \mathbb{R}_+^{\mathcal{A}}$, then $x > 0$ type A tokens are exchanged for $y > 0$ type B tokens such that

  $$f(I) = f(I_A + x, I_B - y, I_{-\{A,B\}})$$

  Adding/Removing Liquidity: Let $\lambda \leq 1$. A trader can trade λI in exchange for $f(I) - f((1-\lambda)I)$ shares of the pooled liquidity

- What is the numéraire in a DEX?
- The numéraire is a tokenized version of a share of the pooled liquidity

Remarks

- We ignore trading fees.
- AMMs can be defined in more general terms.
- Restriction to CFMMs is wlog under "path independence" (terms of trade are a function of current inventory only): Abernathy et al. 2013
- In prediction markets it is more customary to work with cost functions or scoring function. Equivalence between
  - trading and cost functions is straightforward
  - scoring and cost functions needs translation invariance (see, however, Frongillo et al. 2023)

Examples

- Variants of the CFMM:
  - More than two asset types are pooled and a (weighted) geometric mean is maintained
    $$f(I) = I_A^{\alpha_A} I_B^{\alpha_B} I_C^{\alpha_C} \cdots$$
  - "Curve" used for swapping stablecoins implicitly defined by
    $$\alpha n^n \sum_{A \in \mathcal{A}} I_A + f(I) = \alpha n^n f(I) + \frac{\alpha f(I)^{n+1}}{n^n \prod_{A \in \mathcal{A}} I_A}$$
  - "Concentrated" liquidity, "Uniswap V3"
    $$(I + \alpha)(J + \beta)$$
    where $\alpha, \beta > 0$ are constants

Model (cont.)

- If no liquidity is removed/added we can consider a fixed curve/surface

$$f(I) = const.$$

- Notion of **equivalence**: invariance under monotone transformations
- If f is differentiable, we can define **marginal exchange rates**:

$$p_{A,B}(I) = \frac{\frac{\partial f}{\partial I_A}}{\frac{\partial f}{\partial I_B}} = -\frac{dI_B}{dI_A}.$$

Independence

Independence: For each subset of token types $\mathcal{B} \subseteq \mathcal{A}$ and inventories I, J

$$f(I_{\mathcal{B}}, J_{-\mathcal{B}}) = f(J) \Leftrightarrow f(I) = f(J_{\mathcal{B}}, I_{-\mathcal{B}})$$

- For differentiable f, exchange rates only depend on the inventory of exchanged assets

$$p_{A,B}(I_{A,B}, I_{-A,B}) = p_{A,B}(I_{A,B}, J_{-A,B}),$$

for each $A, B \in \mathcal{A}$ and $I, J \in \mathbb{R}_+^{\mathcal{A}}$

Independence

- Geometric means are independent,

$$p_{A,B}(I) = \frac{I_B}{I_A}$$

- Curve rules are not independent,

$$p_{A,B}(I) = \frac{81 + \frac{f(I)^4}{I_A^2 I_B I_C}}{81 + \frac{f(I)^4}{I_A I_B^2 I_C}}$$

Further Axioms

Scale invariance: If liquidity is added/removed proportionally, then the market is made in the same way, formally, for $\lambda > 0$ and $I, J \in \mathbb{R}_+^{\mathcal{A}}$ we have $f(I) = f(J) \Rightarrow f(\lambda I) = f(\lambda J)$

- Scale invariance allows to make liquidity positions fungible.
- A stronger version is:

Homogeneity: For $\lambda > 0$ and $I \in \mathbb{R}_+^{\mathcal{A}}$ we have $f(\lambda I) = \lambda f(I)$

- Each scale invariant trading function is equivalent to a homogeneous one
- All previous examples except for Uniswap V3 satisfy homogeneity

Axioms (cont.)

Sufficient funds: The AMM should be able to make a market at each inventory level and for arbitrarily large trade sizes: For $I, J \in \mathbb{R}_+^{\mathcal{A}}$ with $f(I) = f(J)$ we have

$$I_A > 0 \text{ for each } A \in \mathcal{A} \Rightarrow J_A > 0 \text{ for each } A \in \mathcal{A}$$

- Geometrically: liquidity curves do not intersect the axes
- Product rule, curve rule and geometric averages satisfy this.

Constant Inventory Elasticity

- Geometric averages satisfy proportionality between exchange rates and inventory ratios
- it is a natural idea to generalize this to:

$$p_{A,B}(I) = \frac{I_B^m}{I_A^m}$$

for a constant $m \geq 0$

- This leads to the class of constant inventory elasticity MMs (CEMM):

$$f(I) = \begin{cases} \left(\sum_{A \in \mathcal{A}} I_A^{1-m}\right)^{1/(1-m)}, & \text{for } m \neq 1, \\ \prod_{A \in \mathcal{A}} I_A, & \text{for } m = 1, \end{cases}$$

where $1/m$ is the inventory elasticity of the exchange rate

Constant Inventory Elasticity (cont.)

- The class can be generalized to asymmetric AMMs by introducing different coefficients
- They are sufficiently funded for $m \geq 1$
- By construction, CEMMs are scale invariant and independent

Main Result

Theorem

A trading function for $|\mathcal{A}| > 2$ assets is independent, scale invariant, and sufficiently funded if and only if it is equivalent to a CEMM with $m \geq 1$.

Proof sketch:

- Independence gives an additive representation through monotone transformation by a classical result of Debreu (1960)
- Continuity, scale-invariance, and separability pin down functional form

Discussion

- The result is not true for the two-dimensional case (we introduce another axiom later)
- Need continuous liquidity curves but no differentiability assumption
- If sufficient funds is dropped, then we still get constant inventory elasticity, but m is not restricted
- A natural alternative axiom quasi-concavity gives $m \geq 0$.
- If symmetry is added we obtain CEMMs *with equal coefficients*

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- A natural alternative axiom quasi-concavity gives $m \geq 0$.
- If symmetry is added we obtain CEMMs *with equal coefficients*
- If SI is replaced by Homogeneity we get charaterization of functions instead of equivalence classes under monotonic transformation.

Aside: Prediction Markets

- In prediction market instead of multiplicative one usually considers additive scaling:

  Translation Invariance: for each $I, J \in \mathbb{R}_+^{\mathcal{A}}$ and $\alpha \in \mathbb{R}$ we have

  $$f(I) = f(J) \Rightarrow f(I + \alpha \mathbb{1}) = f(J + \alpha \mathbb{1}).$$

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$$f(I) = f(J) \Rightarrow f(I + \alpha \mathbb{1}) = f(J + \alpha \mathbb{1}).$$

- Replacing SI with TI in the previous characterization gives us **Logarithmic Market Scoring Rule** (and Constant Sum) MMs

$$f_{LMSR}(I) = -b \log \left(\sum_{A \in \mathcal{A}} e^{(c_A - I_A)/b} \right)$$

Aside: Prediction Markets

- Replacing SI with TI in the characterization gives us LMSR (and Constant Sum) MMs.

$$f_{LMSR}(I) = -b \log \left(\sum_{A \in \mathcal{A}} e^{(c_A - I_A)/b} \right)$$

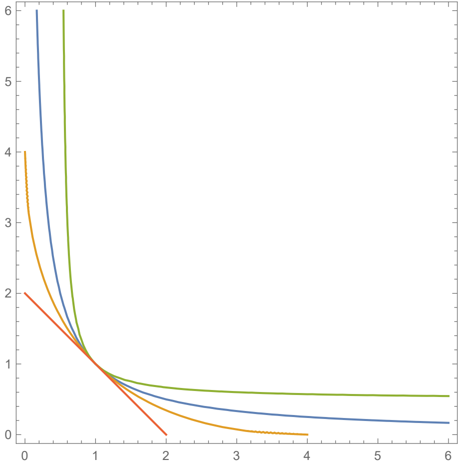
- Intuition: Change of variables. There is a bijection between SI and TI trading functions defined by

$$g(I) := \log(f(\{e^{I_A}\}_{A \in \mathcal{A}})),$$

- Independence is preserved under the above bijection.
- Transformation maps CEMM with parameter m to LMSR with parameter $b = 1 - m$.
- $\Rightarrow$ Axiomatic Foundation for LMSR, See the newest WP

CEMM Class

- We can rank CEMMs by convexity



Optimality

- This can be formalized in different ways, e.g. the following:
  - Call f **more favorable** to traders than g if starting from a balanced inventory the trader obtains more through trading, formally for each I and for each $A, B \in \mathcal{A}$ with $I_A = I_B$, $x > 0$ and $y, y' > 0$ such that $f(I_A + x, I_B - y, I_{-A,B}) = f(I)$ and $g(I_A + x, I_B - y', I_{-A,B}) = g(I)$ we have $y \geq y'$.

Theorem

Let $|\mathcal{A}| > 2$. The constant product rule $f_{product}$ is trader optimal within the class of scale invariant, independent, symmetric and sufficiently funded AMMs.

The two dimensional case

- The previous characterization is not true for the case of two assets.
- We need separability, i.e. liquidity curves of the form

$$\phi_A(l_A) + \phi_B(l_B) = const,$$

  for increasing one-dimensional functions ϕ_A, ϕ_B which is no longer implied by the other axioms
- We can impose "Liquidity Additivity" to get separability
- We can also characterize a much larger class without separability
- $\rightarrow$ See the WP for details

Conclusion

- Have introduced an axiomatic approach for AMMs for DeFi
- Combination of homogeneity and separability gives meaningful restrictions on viable trading functions
- Characterize the CPMM as "trader optimal" within that class
- Replacing homogeneity by translation invariance gives most popular prediction market trading functions
- Separability is arguably not always desirable: what can we impose instead?

Thank you! Questions?