Inequality measurement for bounded variables

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Motivation

- Inequality: a hotly debated topic and of policy interest
  - Piketty (2015), Bourguignon (2017), Atkinson (2018), Milanovic (2018)
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- Many non-pecuniary indicators are bounded
  - i.e., take values from a closed finite interval with fixed limits (a lower bound and an upper bound)
  - We refer to them as bounded variables (Lambert and Zheng 2011)

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- Present an illustration showing how a different picture can emerge in practice

Income growth, sectoral shift and change in inequality

- Inequality ranking of the following distributions due to income growth

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- Temkin (1986) and Bosmans (2007): 1, 2, 3; Fields (1998): 1, 2, 4

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- **Maximality principle**: For a bounded variable with a given lower bound and a given upper bound, whenever we pick any two (non-trivial) MIDs, the corresponding levels of inequality must coincide
  - *Two judges who accepted bribes in all of their cases might be equally corrupt, even if one tried fewer cases* (Temkin 1986)

Notation

- Distribution of achievements: $\mathbf{x} = (x_1, ..., x_n),\ x_i \in [0, U]\ \forall i = 1, \ldots, n$
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  - Bipolar distribution: $(\underbrace{0, \ldots, 0}_{n-n'}, \underbrace{U, \ldots, U}_{n'})$ for $n' < n$
  - Almost bipolar distribution: $(\underbrace{0, \ldots, 0}_{n-n'}, \varepsilon, \underbrace{U, \ldots, U}_{n'})$; $\varepsilon = [n\mu(\mathbf{x}) - n'U]$

Properties

- Fundamental properties:
  - **Anonymity**: $I(\mathbf{y}) = I(\mathbf{x})$ whenever $\mathbf{y}$ is obtained from $\mathbf{x}$ through permutation
  - **Transfer principle**: $I(\mathbf{y}) > I(\mathbf{x})$ when $\mathbf{y}$ is obtained from $\mathbf{x}$ by a regressive transfer (poor to rich); $I(\mathbf{y}) < I(\mathbf{x})$ when $\mathbf{y}$ is obtained from $\mathbf{x}$ by a progressive transfer (rich to poor)

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- Other properties:
  - **Equality principle**: Inequality is minimal and equal to zero when all units feature exactly the same achievement value, i.e. $x_1 = x_2 = \cdots = x_n$
  - **Population principle** (for variable-population comparisons): $I(\mathbf{y}) = I(\mathbf{x})$, whenever $\mathbf{y}$ is obtained from $\mathbf{x}$ by a *replication*

The class of inequality measures (fixed population)

Theorem 1

For any $\mathbf{x} \in \mathcal{X}_n$, an inequality index I satisfies anonymity, the transfer principle, the equality principle and the maximality principle if and only if

$$I(\mathbf{x}) = \begin{cases} M\left[\dfrac{f(\mathbf{x}) - f(\bar{\mathbf{x}})}{f(\hat{\mathbf{x}}) - f(\bar{\mathbf{x}})}\right] & \text{if } \mathbf{x} \in \mathcal{X}_n \setminus \{\mathbf{0}, \mathbf{U}\} \\ 0 & \text{if } \mathbf{x} \in \{\mathbf{0}, \mathbf{U}\} \end{cases}, \tag{1}$$

where $0 < M < +\infty$ is a proportionality constant, $\bar{\mathbf{x}}$ is the egalitarian distribution with the same mean as $\mathbf{x}$, $\hat{\mathbf{x}}$ is an MID (bipolar/almost-bipolar) for $\mathbf{x}$, $f : \mathcal{X}_n \to \mathbb{R}_{++}$ is a symmetric and strictly S-convex function, and $\mathbf{0}$ and $\mathbf{U}$ are the two extreme egalitarian distributions.

The class of inequality measures (variable population)

Theorem 2

For any $\mathbf{x} \in \mathcal{X}_n$, an inequality index I satisfies anonymity, the transfer principle, the equality principle, the restricted maximality principle and *the population principle* if and only if

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Example: Normalised Gini (G^*)

$$G^*(\mathbf{x}) = \begin{cases} \dfrac{G_a(\mathbf{x})U}{\mu(\mathbf{x})(U-\mu(\mathbf{x}))} & \text{if } \mathbf{x} \in \mathcal{X}_n \setminus \{\mathbf{0}, \mathbf{U}\} \text{ and } \hat{\mathbf{x}} \text{ bipolar} \\ \dfrac{G_a(\mathbf{x})n^2}{(n-n'-1)(\varepsilon+n'U)+n'(U-\varepsilon)} & \text{if } \mathbf{x} \in \mathcal{X}_n \setminus \{\mathbf{0}, \mathbf{U}\} \text{ and } \hat{\mathbf{x}} \text{ almost-b} \\ 0 & \text{if } \mathbf{x} \in \{\mathbf{0}, \mathbf{U}\} \end{cases}$$

- Note: G_a is the absolute Gini coefficient (same G^* for relative Gini)

Illustration

Consistency requirement

- Most bounded indicators are expressed as attainments or short-falls
  - Mortality rate vs. survival rate; literacy rate vs. illiteracy rate

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Consistency requirement

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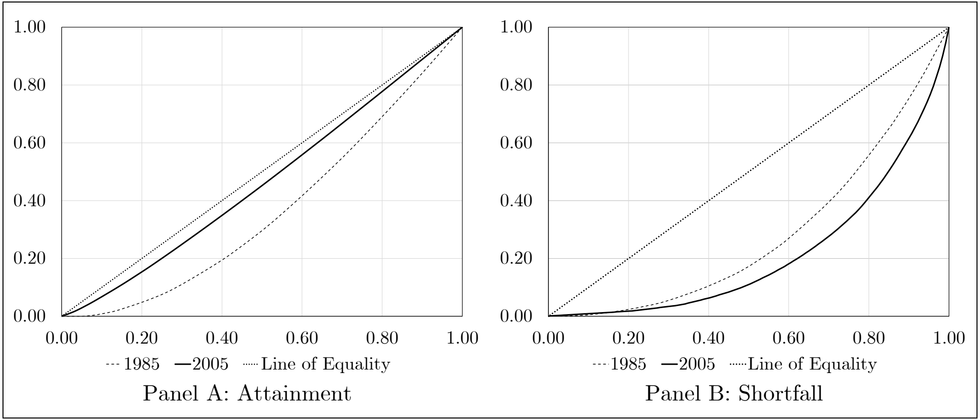
  - Mortality rate vs. survival rate; literacy rate vs. illiteracy rate
- No a priori reason to prefer one representation (attainment or shortfall) over the other, but ...
- How to ensure that inequality assessments guarantee consistent comparisons when switched between attainments and shortfalls?
  - Micklewrite and Stewart (1999); Erreygers (2009); Lambert and Zheng (2011); Lasso de la Vega and Aristondo (2012); Bosmans (2016)

Inconsistent Lorenz comparisons (relative)

Cross-country BCG immunisation rates



Consistency properties and proposed solutions

- Properties
  - **Perfect complementarity**: No change in inequality index when switched between attainments and shortfalls (Erreygers, 2009)
  - **Consistency**: No change in inequality ordering when switched between attainments and shortfalls (Lambert and Zheng, 2011; Bosmans, 2016)

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- Solutions
  - Absolute indices (Erreygers, 2009; Lambert and Zheng, 2011; Chakravarty et al., 2015; Seth and Alkire, 2017)
  - Equally weighted general means of any inequality index on attainments and the same on shortfalls (Lasso de la Vega and Aristondo 2012)
  - A pair of inequality indices: I^A for attainments and $I^S = \phi(I^A)$ for the shortfalls, where ϕ is strictly increasing (Bosmans 2016)

Strongly consistent normalised measures

Proposition

An inequality index is strongly consistent iff it satisfies perfect complementarity.

Theorem 3

The inequality indices I characterised in theorems 1 and 2 are also strongly consistent if and only if

$$f(\mathbf{x}^S) = p(\mathbf{x}) f(\mathbf{x}) + q(\mathbf{x}), \tag{2}$$

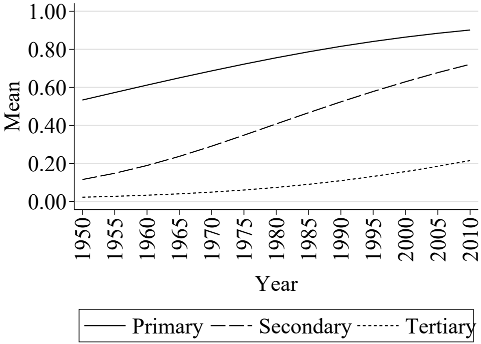
where $p(\mathbf{x}) = \dfrac{f(\hat{\mathbf{x}}^S) - f(\bar{\mathbf{x}}^S)}{f(\hat{\mathbf{x}}) - f(\bar{\mathbf{x}})}$, $q(\mathbf{x}) = \dfrac{f(\hat{\mathbf{x}}) f(\bar{\mathbf{x}}^S) - f(\hat{\mathbf{x}}^S) f(\bar{\mathbf{x}})}{f(\hat{\mathbf{x}}) - f(\bar{\mathbf{x}})}$ and $\mathbf{x}^S = \mathbf{U} - \mathbf{x}$.

Evolution of cross-country inequality in education

- We study the evolution of cross-country inequality in three education indicators, 1950-2010
  - Share of total adult population with at least some primary education
  - Share of total adult population with at least some secondary education
  - Share of total adult population with at least some tertiary education
  - Source of the education data: `http://www.barrolee.com/`
- In order to study inequality, we use:
  - The absolute Gini index, the relative Gini index and the normalised Gini index
- We treat each country as an observation (133 countries)

Evolution of cross-country inequality in education

Mean and the absolute Gini index, $U = 1$ Max



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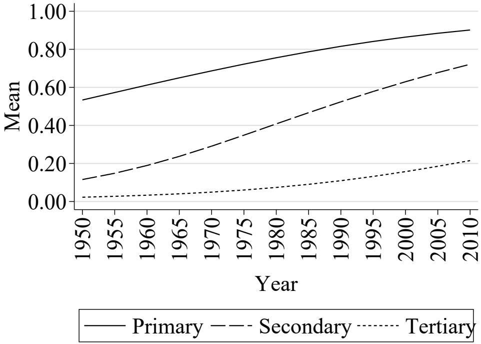

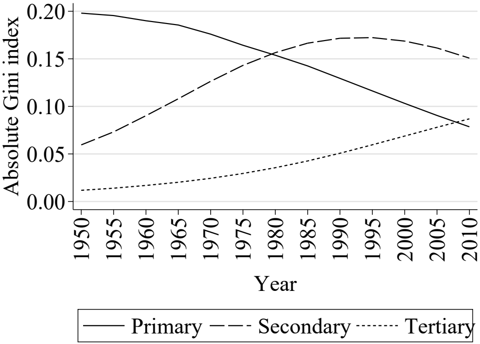

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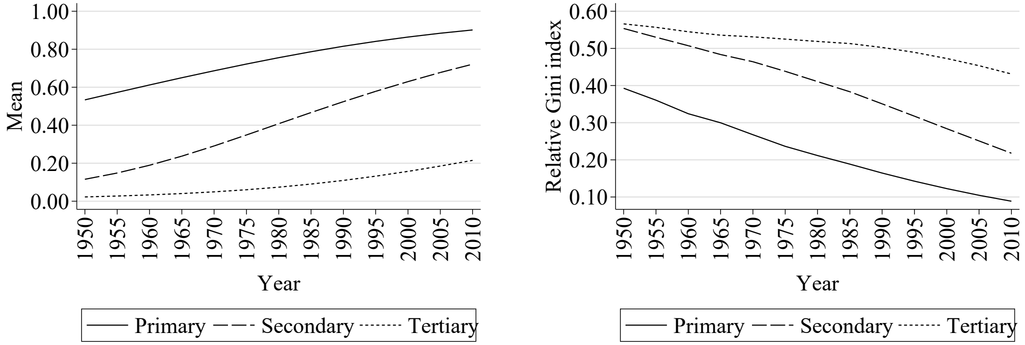
Mean and the relative Gini index, $U = 1$



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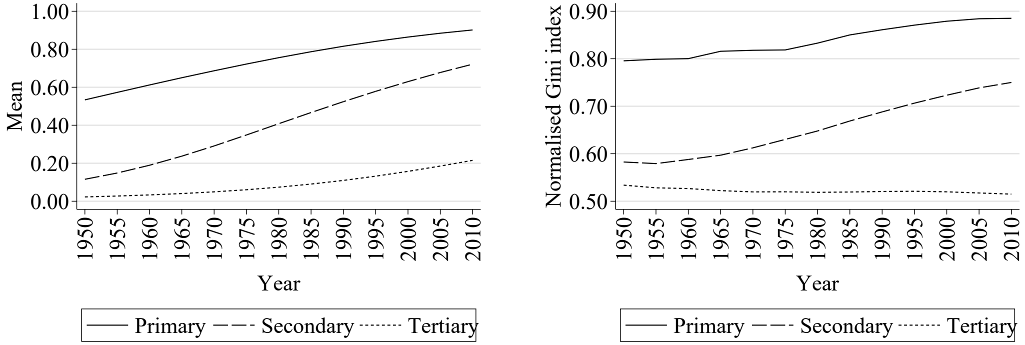
Mean and the normalised Gini index, $U = 1$



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- Future research:
  1. Address robustness of inequality comparisons to changes in upper bound
  2. Identify related partial orderings (and related stochastic dominance conditions)

Thank you

Questions and comments are welcome!

Mean and maximum inequality

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