Persistent Winners and Reserve Prices in Repeated Auctions*

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Abstract

A seller running repeated auctions with bidders who have constant valuations over time can exploit the information obtained in past auctions to set reserve prices in future ones. We consider an environment where bidders are myopic, losers are replaced by new bidders, and past winners leave with an exogenous probability. Our model reflects the main characteristics of the market for online display advertising, where publishers use real-time first- or second-price auctions to sell impressions to advertisers. The optimal reserve price in infinitely repeated auctions is either equal to the value of the last winner, or lower than it when the last winner's value is sufficiently high. In this second case, the optimal reserve price is *decreasing* in the last winner's value in a first-price auction, while it is independent of it in a second-price auction, and typically lower than in a first-price auction. The second-price auction yields a higher seller's revenue than the first-price auction when the persistence of past winners is sufficiently low, because in the second-price auction a past winner who is outbid acts as a reserve price. We also describe typical paths of reserve prices and characterize the stationary distribution of winners' values. The probability of trade may be non-monotonic in the persistence of past winners.

1 Introduction

Repeated auctions for identical objects are often won by the same bidders. For example, in auctions for online display advertising, an advertiser often acquires multiple impressions to the same user across different websites.[1] This is not surprising because many buyers want to acquire multiple objects with similar characteristics, and because a bidder who outbids all competitors in an auction

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[1] Many of us are probably familiar with the experience of repeatedly observing an advertisement for a product we are supposedly interested in, across different websites that we visit while surfing the web.

is likely to be able to do it again in a later auction for a similar object. Winning multiple objects on sale in repeated auctions, however, requires bidders to bid repeatedly and, by doing so, to provide information about their valuations to sellers. Sellers can then exploit the information conveyed by past bids, for example by choosing the reserve price in an auction based on the outcomes of past auctions for similar objects.[2]

In this paper we analyze an environment with a single seller who runs an infinite sequence of auctions,[3] each for a single identical object and with a fixed number of bidders. Bidders are drawn from a pool of ex-ante identical potential buyers, have a constant private valuation for the objects on sale and a maximum number of objects that they are willing to acquire. The seller observes all bids in past auctions and chooses a reserve price in each auction contingent on this information. In this environment, we compare the two most common types of sealed-bid auctions: the first-price auction (FPA) and the second-price auction (SPA).

The main application of our analysis is auctions for online display advertising, a large and complex market with the following principal actors.[4] Final sellers are publishers that own digital ad spaces on a website and sell impressions — where an impression is defined as the event that a particular user sees an advertisement shown on the publisher's website. Hence, an impression occurs any time a user opens a website and an advertisement is visible. Publishers sell multiple impressions (with identical or very similar characteristics) available at different moments in time through an intermediary, called the Ad Exchange. The Ad Exchange is an online platform that runs real-time auctions (in the instant between a click on a website and a display ad appearing): it sets reserve prices,[5] collects bids from potential advertisers, and implements allocations and payments according to the auction rules. Both first-price and second-price sealed-bid auctions have historically been used in this market.[6] Bidders are advertisers who wish to buy impressions — i.e., to show their online advertisement to a particular user on a specific publisher's website. Typically, large advertisers bid through another intermediary, called a Demand Side Platform (or DSP). The role of the DSP is twofold: (i) it allows advertisers to manage large and complex advertising campaigns, that typically require to spend a fixed budget to purchase a variety of impressions with certain characteristics over a specific time interval; and (ii) to implement automatic, real-time, bidding across a large number of auctions. There is also a large number of small, and often unsophisticated, advertisers

[2]On reserve prices in dynamic auctions see Amin *et al.* (2013), Amin *et al.* (2014), Caillaud and Mezzetti (2004), Deng *et al.* (2021), and Kanoria and Nazerzadeh (2021). For a survey of the literature on behavior-based pricing strategies, where the seller adjust prices based on buyers' past behavior, see Fudenberg and Villas-Boas (2007).

[3]The literature on sequential auctions include Ashenfelter (1989), Beggs and Graddy (1997), Bernhardt and Scoones (1994), Bikhchandani (1988), Gale and Stegeman (2001), Ginsburgh (1998), Hu and Zou (2015), McAfee and Vincent (1993, 1997), Mezzetti (2011), Milgrom and Weber (2000), and Weber (1983).

[4]The US market for online display advertising was estimate to be worth $143 billion in 2022. US website publishers alone sell more than 13 billion online advertisements every day. For a more detailed description of the market see McAfee (2011) and Muthukrishnan (2009).

[5]Ad Exchanges typically have programs designed to increase publisher's revenue by setting appropriate reserve prices — see, e.g., https://support.google.com/admanager/answer/11385824?hl=en. In a large-scale field experiment, Ostrovsky and Schwarz (2023) show the relevance of reserve prices in sponsored search auctions conducted by Yahoo! to sell advertisements. Although sponsored search auctions use a different selling mechanism (the generalized SPA) and simultaneously allocate multiple slots, the evidence is suggestive that reserve price adjustments can substantially increase revenues in practical settings that are close to the one that we analyze.

[6]Because of time constraints, it would be infeasible to run dynamic auctions, like the ascending auction, in this context. Moreover, each auction sells a single impression, since each impression is a unique event available at a different point in time.

that bid for impressions using much simpler and less customizable buying tools, called Advertiser Ad Networks. The vast majority of transactions in this market (around 90%) use some type of automatic technology.

Our model is a stylized version of the market for online display advertising that only includes two types of agents: a single seller and many ex-ante identical bidders. Our seller can be interpreted as an Ad Exchange that runs repeated auctions for identical objects — e.g., impressions on a given website to different users with identical characteristics (who visit the website at different moments) or, alternatively, multiple impressions to exactly the same user who visits different websites (at different moments). Bidders can be interpreted as different Demand Side Platforms and Advertiser Ad Networks that are willing to purchase multiple objects sold in the repeated auctions — e.g., a DSP bidding on behalf of an advertiser that wants to repeatedly show its ad to a specific user,[7] or to acquire multiple impressions with certain characteristics on different websites.

We assume that the seller uses either a first-price or a second-price sealed-bid mechanism for all auctions and, in each auction, she sets a reserve price and observes all bids.[8] Each bidder has the same valuation for all the objects on sale, but has a stochastic capacity that represents the maximum number of objects that he is willing to acquire. This can be interpreted as either a true capacity constraint (arising from, e.g., a fixed budget) or, in repeated auctions for impressions to the same user, a maximum number of times that an advertiser wants to shows its ad to that user (because, e.g., showing the ad too many times can become ineffective). The capacity constraint implies that the winner of an auction may not participate in the subsequent ones; in this case, we assume that he is replaced by a new bidder. Losers instead always leave the auctions and are replaced by new bidders. In Section 2, we provide a microfoundation of this participation behavior, based on (*i*) the presence of many sellers and (*ii*) seller-specific and time-specific match values.

We make the simplifying assumption that bidders are myopic: specifically, in each auction all bidders choose the optimal bid for a static auction with a fixed number of symmetric bidders, given the reserve price chosen by the seller. This assumption allows us to take bidding behavior in each auction as given and focus on the seller's strategy across auctions and on the dynamic implications of the seller's choices for revenue, efficiency, and the paths of observed reserve price and auction winners. As we are going to discuss in Section 2, the assumption implies that bidders neglect both that their bids in an auction reveal information that the seller may use in future auctions, and that the reserve price chosen by the seller in an auction may reflect information about a competitor's value.[9]

Besides, we believe that the assumption of myopic bidding is a reasonable approximation of the actual behavior of many bidders in auctions for display advertising, because of presence of a large

[7]For example, an advertiser may be the seller of a product that a user has searched for (and possibly inserted in a shopping cart or list), but has not purchased yet. The advertiser may want to repeatedly display an ad for that product to that particular user, to induce him to complete the purchase.

[8]Consistent with the actual functioning of the market for online display advertising, we assume that the auction format is fixed in all auctions, and that the only instrument available to the seller is the reserve price, that the seller chooses in every auctions. We also focus on the two most commonly-used static auction mechanisms.

[9]Pagnozzi and Sartori (in preparation) analyze the implications of bidders' sophistication in an environment similar to the one of this paper: two sequential second-price sealed-bid auctions. In this setting, bidders shade their bid away from the dominant strategy equilibrium, strategically concealing their valuation in the first auction to prevent price discrimination in the second one. See also Caillaud and Mezzetti (2004).

number of small bidders and because of the use of automatic bidding algorithms in those auctions. These are software programs that are often trained and learn in computer-simulated static and independent auctions, where variations in reserve prices predominantly reflect sellers' heterogeneity — e.g., different sellers' valuations or outside options — or information about a common valuation of the object on sale, and therefore have no effect on the distributions of competitors' bids or affect these distributions symmetrically.

What is the optimal path of reserve prices for the seller and how is it affected by the possibility of having repeated winners? How are total revenue and efficiency effected by the auction format, the primitives of our model, and the observable characteristics of this market (like the winner's tenure)?

In order to address these questions and isolate the key forces at play, we first focus on a static auction, where the seller receives exogenous information about the valuation of one of the bidders — which we call the incumbent — who participates in the auction with some fixed probability. We show that, in choosing the reserve price, the seller faces a trade off between exploiting the possible presence of the incumbent or targeting new bidders.[10] In both the FPA and the SPA, the optimal reserve price has the following features: if the incumbent's value is low, the reserve price *excludes the incumbent* — i.e., it is higher than his value; if the incumbent's value is intermediate, it *tracks the incumbent* — i.e., it is equal to his value; if the incumbent's value is high, it *tails the incumbent* — i.e., it is lower than his value.

The intuition for this pattern of the optimal reserve price is the following. Tracking the incumbent guarantees that the seller extracts the whole bidders' surplus in case the incumbent participates in the auction and wins, but may result in a reserve price that is too far from the optimal one for new bidders. Hence, tracking is optimal when the incumbent's value is neither too high, nor too low. When the incumbent's value is low, the seller sets a reserve price that the incumbent is not willing to pay, because the surplus that the seller can extract from the incumbent is limited and hence the seller prefers to only target new bidders.

Finally, when the incumbent's value is high tracking the incumbent requires the seller to set a reserve price that is sub-optimally high in case the incumbent does not participate, because it implies a high probability of no trade with new bidders. In this case, the seller prefers to set a reserve price lower than the incumbent's value that fails to maximize her revenue when the incumbent participates, but guarantees a higher revenue when the incumbent does not participate (compared to tracking).

Comparing the optimal reserve price in first- and second-price auctions we show the following two main results. First, when the seller tails the incumbent, she always chooses a lower reserve price in the SPA than in the FPA. The reason is that the bid submitted by the incumbent is a perfect substitute for the seller's reserve price when the incumbent loses in the SPA, but not in the FPA. Hence, the seller has a lower incentive to raise the reserve price in the SPA. Second, the tailing reserve price does not depend on the incumbent's value in the SPA, but it is *decreasing* in the incumbent's value in the FPA. The intuition for this surprising result is that, in the FPA, the

[10] There is a vast literature on optimal reserve prices in static auctions that considers risk averse bidders (Hu *et al.*, 2010), correlated types (Levin and Smith, 1996), interdependent values (Quint, 2017; Hu *et al.*, 2019), endogenous entry (McAfee, 1993; Levin and Smith, 1994; Peters and Severinov, 1997), level-k bidders (Crawford *et al.*, 2009), taste projection (Gagnon-Bartsch *et al.*, 2021), and loss-averse bidders (Balzer and Rosato, 2023). Focusing on internet markets, Bajari and Hortacsu (2004) describe eBay auctions and Ostrovsky and Schwarz (2023) analyze optimal reserve prices in sponsored search auctions.

marginal benefit of increasing the reserve price is decreasing in the incumbent's value, because bids of high-value bidders are less responsive to the reserve price than bids of low-value bidders;[11] while the cost of increasing the reserve price (which reflects the risk that no new bidder bids above the reserve price) is independent of the incumbent's value, because this cost materializes only if the incumbent does not participate. In the SPA, by contrast, both the cost and the benefit of increasing the reserve price are independent of the incumbent's value, because bids are independent of the reserve price.

The revenue ranking between FPA and SPA depends on the incumbent's value. When the incumbent's value is low, the seller's revenue is higher in the FPA than in the SPA. The reason is that, when the seller either excludes or tracks the incumbent, the expected payment of myopic bidders is higher in the FPA. In this case, as we are going to show, myopic bidders overestimate competition in the auction (because they neglect that the incumbent never bids more than the reserve price), which induces them to bid more aggressively in the FPA but not in the SPA. This effect of myopic bidding tends to yield a higher revenue in the FPA. When the incumbent's value is high, by contrast, the seller's revenue is higher in the SPA than in the FPA. The reason is that the bid by a high-value incumbent is a substitute for the seller's reserve price in the SPA. Therefore, when the seller tails a high-value incumbent, in the SPA the incumbent's bid increases the expected seller's revenue if the incumbent stays but loses, while the seller's reserve price (which in the SPA can be set closer to the optimal reserve price for new bidders, as a result of the presence of the high incumbent bid) allows her to also obtain a high revenue when the incumbent leaves.

The analysis of the static model assumes that the seller receives exogenous information about an incumbent. Of course, in our dynamic environment the seller's information about an incumbent arises endogenously, because it represents the valuation of a previous winner. Moreover, the probability of the incumbent participating to the static auction reflects a winner's capacity in the repeated auctions. This implies that, in addition to the static effects that we have discussed, the reserve price chosen by the seller in a period also has a dynamic effect, because it affects the information observed by the seller. More precisely, increasing the reserve price in repeated auctions reduces the seller's information, because it reduces the set of possible incumbent's valuations that the seller may observe.

Despite this additional dynamic effect, we show that in both the FPA and the SPA the optimal reserve price in repeated auctions has the same qualitative features as in the static environment, as long as the incumbent's value is not too low: the seller tracks the incumbent if he has an intermediate value, while she tails the incumbent if he has a high value. Because of the additional dynamic cost of increasing the reserve price, however, the reserve price in case of tailing is always lower than in the static environment. Moreover, the seller never excludes the incumbent because exclusion requires an excessively low reserve price in a period that allows a low-value incumbents to win, even if the information on this low-value incumbent obtained by the seller is not used to set the reserve price in the subsequent period, which is never optimal.

In the static environment, revenue is higher in the FPA than in the SPA for a large set of the

[11] Formally, the derivative of the FPA bidding function with respect to the reserve price is strictly positive (for any type higher than the reserve price), but it is decreasing the bidder's type.

incumbent's values; but revenue is higher in the SPA precisely when the incumbent has a relatively high valuation. In repeated auctions, incumbents are winners of past auctions and, therefore, they are more likely to have high rather than low valuations. As a consequence, the seller may obtain a higher revenue in the SPA than in the FPA, especially when the expected bidders' capacity is not too high. The reason is that, when capacity is high, the seller always tracks the incumbent and, in this case, the seller's revenue is always higher in the FPA. Moreover, the seller's revenue is increasing in bidders' capacity in both the FPA and SPA: the higher is the expected number of objects that bidders are willing to acquire, the higher is the distribution of bidders' values in an auction, because high-value winners of past auctions are more likely to participate again in future ones. This increases the seller's revenue in both auction formats.

We also describe typical paths of reserve prices and winners' values in repeated auctions and we characterize the stationary distribution of winners' values. This distribution allows us to analyze trade — i.e., the probability that an auction results in a sale to one of the bidders — and how trade varies with bidders' capacity. The stationary distribution of winners' values is increasing in bidders' capacity (in the FOSD sense). It may be expected that a higher capacity always increases trade too, because it has the direct effect of making it less likely that a past winner leaves the repeated auctions. This is not necessarily true, however, because bidders' capacity also affects the reserve prices chosen by the seller and, in particular, a lower capacity tends to reduce reserve prices. In fact, we show that trade may not be monotonic in bidders' capacity, and may actually decrease in capacity when capacity is high. This happens precisely when the reserve price effect overcomes the direct effect of an increase in capacity.

Moreover, when there is no tailing, trade is higher in the FPA than in the SPA. The reason is that the seller is willing to track lower-value incumbents in the FPA, and hence sets a lower minimum reserve price in the FPA than in the SPA (and this is the only difference in the reserve price policies adopted in the two auction formats). This has a positive effect on total efficiency in the FPA.

Finally, we analyze the information that may be inferred in repeated auctions by only observing the tenure of the current winner — i.e., the number of consecutive periods in which a bidder has won. A higher tenure implies a higher expected valuation of the winner and, in the SPA, a higher expected revenue for the seller and a higher probability of future trade. In the FPA, by contrast, a higher tenure may reduce the expected seller's revenue. The reason is that, since the reserve price is decreasing in the incumbent's value in the FPA, a higher expected valuation of the winner may actually reduce the bids of the highest bidders (through the reduction of the reserve price).

The rest of the paper is organized as follows. Section 2 describes the model. In Section 3, we begin by considering reserve prices (Section 3.1) and the seller's revenue (Section 3.2) in a static environment with exogenous information. Section 4 contains the main analysis and results: Section 4.1 characterizes the optimal reserve prices in the FPA and SPA, Section 4.2 describes typical paths of reserve prices and winners' values in repeated auctions, and Section 4.3 compares the seller's revenue in the two auction formats. In Section 5, we characterize the stationary distribution of winners' values, and use it to analyze the long-run probability of trade in Section 5.1. Finally, Section 6 studies how the tenure of the current auction winner affects the probability of trade and the seller's expected revenue. Section 7 concludes. All proofs are in the Appendix.

2 Model

We consider a discrete-time model of infinitely repeated auctions with a single (female) seller who maximizes her total expected profits, discounted at rate β. In every period $t \in \{0, 1, \dots\}$, the seller runs an auction to allocate an identical object to a fixed number n of bidders. At the beginning of each period t, the seller chooses a public reserve price R_t, conditional on the history of bids in all previous periods. We consider both first-price sealed-bid auctions (FPA) and second-price sealed-bid auctions (SPA).

Bidders are drawn from an infinite pool of ex-ante symmetric potential buyers. Each bidder i has a constant valuation θ_i for the object, distributed according to $F(\theta_i)$ on $[0, 1]$, with increasing virtual value $\psi(\theta_i) := \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}$. The seller's value is normalized to 0.

Bidders' behavior is characterized by the following two assumptions.

Assumption 1. *(Participation) After losing an auction, a bidder always leaves — i.e., he does not participate to any subsequent auction. A bidder who wins an auction — henceforth, the incumbent — leaves with probability η and participates to the next auction with probability $(1-\eta)$. Bidders who leave are replaced by new bidders.*

The winner of a previous auction may participate in future auctions, but losers do not (e.g., because they realize that they are facing a higher-value competitor).[12] Assumption 1 implies that, in every period, the auction has either n new bidders (which happens if there was no winner in the previous period or if the winner of the previous period left) or $(n-1)$ new bidders in addition to the winner of the previous period. We will interpret $(1-\eta)$ as a measure of the persistence of winning bidders and we will analyze how our results change as this persistence varies.

Hence, since the seller always observes the winning bid,[13] in any period all previous histories can be summarized by one of the following two types of states, that reflect the information of the seller.

1. State $\emptyset$: there are n new bidders (and no incumbent);
2. State θ: with probability η there are n new bidders; with probability $(1-\eta)$ there are $(n-1)$ new bidders and one bidder with value θ.

State $\emptyset$ is the initial state and also arises whenever no bidder bids above the reserve price in the previous-period auction, while state θ arises whenever in the previous period the auction was won by a bidder with value θ.

Assumption 2. *(Myopic Bidding) In every period t, all bidders bid as in a static auction with n symmetric bidders and reserve price R_t.*

Assumption 2 greatly simplifies bidders' behavior: regardless of the history and of the current reserve price, in each auction bidders behave as in a static auction with $n-1$ symmetric competitors, and this is common knowledge. Therefore, bidders perfectly reveal their valuations to the seller when they bid in an auction. The assumption implies that bidders neglect both that (i) their bids reveal

[12]Shortly, we are going to provide a microfoundation for this assumption. The alternative assumption that losers continue to participate in the repeated auctions does not affect our qualitative results on the optimal reserve prices.

[13]Under our Assumption 2, bidders adopt monotone bidding strategies.

information to that may be used in future periods to set reserve prices (*forward myopia*) and that (ii) the current reserve price may be informative about the valuation of a competitor (*backward myopia*).

We make two observations about Assumption 2, which may be judged very restrictive for bidders' behaviour. First, notice that backward myopia is irrelevant in the SPA, because bidders have a dominant strategy, which is therefore independent of competitors' values.[14]

Second, Assumption 2 is satisfied in an environment where bidders are myopic only in the very first period in which they participate in the repeated auctions. This requires a much weaker and arguably more realistic form of unsophistication for bidders, namely *one-shot myopia*. The reason is the following. After winning an auction, an incumbent reveals all his private information to the seller, so that forward myopia is irrelevant after the first period. Moreover, after winning an auction, in each subsequent period the incumbent best responds to the strategies of his competitors. This is obvious for the SPA, while for the FPA notice that, after winning, the incumbent plays an optimal strategy for an auction in which he competes with $n-1$ new bidders, which is indeed the case. In other words, backward myopia implies that bidders in the FPA neglect the information about a competitor's value — i.e., the incumbent — that is conveyed by the reserve price; but this can have no effect for the incumbent himself, because he already knows his own value.

2.1 Microfoundation of Bidders' Behavior

We are now going to provide a microfoundation for bidders' valuations and participation decision in the repeated auctions, which is based on the presence of many alternative auctions, as well as auction- and time-specific match values for bidders.

There are many sellers who simultaneously run auctions for similar objects. For example, each seller repeatedly sells impressions for the same unique user, but there are multiple users (and hence sellers) with very similar characteristics. Bidders randomly choose to enter one of the auctions.[15] Upon entering the auction of seller s at time t, a bidder i learns his potential match value $\theta_i(s,t)$ for winning that auction. Potential match values are both seller-specific and time-specific and are uncorrelated across sellers and periods: in our application to online advertising, they represent an advertiser's valuation for a particular impression available at time t, which depends on the exact moment at which the corresponding user may view the ad.

A bidder who wins the auction obtains the match value $\theta_i(s,t)$, and this also represents the value of winning another auction by seller s in any subsequent period, after winning it at time t. For online advertising, this can be interpreted as the per-impression value (which is constant for simplicity) obtained by repeatedly showing an ad to a particular user, after showing it for the first time. Winners pay an arbitrarily small transaction cost for switching to a different seller (which arises, for example, for giving up a realized match value and drawing a new one). The match value of a winner drops to 0 with fixed probability η in every period. For online advertising, this can be interpreted as the probability that the targeted user buys the product sold by the advertiser, or as

[14] This also implies that, in the SPA with $\beta = 0$, bidders are perfectly rational.

[15] An alternative and equivalent microfoundation has only one auction and exactly n bidders; with losers drawing new values in every period and the winner keeping his value with a fixed probability in every period.

a function of the optimal number of times that an ad should be shown to the same user in order to induce him to buy.[16]

In this environment, bidders who lose an auction in period t become new bidders in period $t+1$ because they draw a new potential match value.[17] The winner of seller s's auction, by contrast, keeps his realized match values and continues bidding in the auctions run by seller s, until either the match value drops to 0 or he loses an auction, in which cases he also becomes a new bidder.

This microfoundation basically implies that a generic auction in any period t becomes a repeated auction only if there is a winner, and that sellers effectively run repeated auctions only for a finite but stochastic number of periods — i.e., until the winner's value drops to 0 or he is outbid by a new bidder.

Myopic bidding arises if bidders maximize their static profit in every period (hence assigning weight 0 to future profits) and, in addition, attribute all variations in reserve prices across auctions to differences in sellers' valuation or outside option. For example, in auctions for online display advertising, reserve prices depend on (an are often equal to) the price that the seller would obtain through a direct sale of the impression — i.e., by allocating the impression to guaranteed contracts that publishers directly negotiate with advertisers to supply a given number of generic impressions with some predefined characteristics. A loose interpretation of myopic bidding is that, when a bidder participates in a repeated auction, he only maximizes the profit obtained by winning an impression to a specific user for the first time, and disregards all future payoffs that he may obtain by acquiring multiple impressions to the same user.[18]

2.2 Preliminaries

In contrast to the SPA, in the FPA bidders' misperception about the environment (number of effective bidders and valuation of the incumbent) affects their bids both in level and in their sensitivity to the reserve price. This implies that (i) revenue equivalence between FPA and SPA does not hold and (ii) in the FPA the seller can adjust the reserve price to exploit the response of myopic bidders.

In particular, by Assumption 2, in the FPA with reserve price price R, a bidder with value θ bids:

$$b\left(\theta, R, n\right) := \mathbb{I}\left[\theta > R\right] \mathbb{E}\left[\max\left\{y, R\right\}\right],$$

where y is the highest among $(n-1)$ values drawn form the distribution F. This is the seller's revenue in the FPA when a bidder with value θ wins the auction. Notice that, by the Revenue Equivalence Theorem, $b(\theta, R, n)$ is also the expected payment of a bidder with value θ conditional on winning any standard auction with n total bidders (and hence $(n-1)$ competitors) and reserve price R.

Moreover, since it is a dominant strategy for all bidders to bid their value in the SPA, when there

[16]An alternative interpretation is that bidders have a stochastic capacity constraint that is distributed according to a geometric distribution with parameter η, so that an auction winner hits the constraint in every period with probability η.

[17]Of course, this can also be interpreted as losers leaving the auction of seller s and entering a new auction in period $t+1$.

[18]For example, when bidding for an impression in an auction, bidders may be unaware that with some probability they will want to purchase multiple impressions to the same user.

is an incumbent with value θ the expected payment of a new bidder with value θ' can be written as $b(\theta', \max\{\theta, R\}, n-1)$. The reason is that the bid equal to θ by the incumbent is equivalent to a reserve price, so that the expected payment by a new bidder is the same as in a standard auction with only $(n-2)$ real competitors and a reserve price equal to $\max\{\theta, R\}$. This also implies that $b(\theta', \max\{\theta, R\}, n-1)$ is the seller's expected revenue in the SPA when a new bidder with value θ' outbids the incumbent and wins the auction.

In our analysis, we are going to use the function $b(\theta, R, n)$ to characterize the seller's expected revenue in all possible states, in both the FPA and the SPA. The following facts will be useful.

Fact 1. $b(\theta, R, n)$ *is increasing in* θ*,* R *and* n*.*

Fact 1 implies that, when there is an incumbent and the seller sets a reserve price equal to his value, myopic new bidders bid "more aggressively" — in the sense that they have a higher expected payment — in the FPA than in the SPA. The reason is the following. In the SPA, the expected payment of a new bidder with value θ conditional on winning is $b(\theta, R, n-1)$. In the FPA, by contrast, when there is an incumbent new bidders bid as in an auction with n symmetric bidders, even if in reality there are only $(n-1)$ new bidders, as a consequence of myopic bidding. Therefore, the expected payment of a new bidder with value θ conditional on winning in the FPA is $b(\theta, R, n)$, which is higher than in the SPA. As we are going to show, this effect of myopic bidding tends to increase revenue in the FPA.

Fact 2. $\frac{\partial^2}{\partial R \partial n} b(\theta, R, n) < 0$*,* $\frac{\partial^2}{\partial R \partial \theta} b(\theta, R, n) < 0$*, and* $\frac{\partial^2}{\partial \theta \partial n} b(\theta, R, n) > 0$*.*

Fact 2 implies that in the FPA bidders with higher values are less sensitive to changes in the reserve price: increasing the reserve price induces all bidders (with values higher than the reserve price) to bid more aggressively, but less so the higher the value of the bidder.

3 Static Auction

In this section, we start our analysis by considering a static environment with exogenous information about one of the n bidders: before an auction, the seller learns that a bidder with value θ (the incumbent) is going to participate in the auction with probability $(1-\eta)$. We first characterize the optimal reserve price for the seller in Section 3.1, and then compare the seller's revenue in the FPA and SPA in Section 3.2.

Notice preliminary that a seller who has no information — which we refer to as information $\emptyset$ — faces a familiar problem. By the Revenue Equivalence Theorem, we can express her revenue in both the FPA and the SPA — as well as in any efficient auction — as a function of the reserve price R by integrating the expected payment of the bidder with the highest valuation:

$$\pi_n(R) \coloneqq \int_R^\infty b(\theta, R, n) \mathrm{d}F^n(\theta). \tag{3.1}$$

Maximizing this revenue yields the standard monopoly price condition (Myerson, 1981; Riley and Samuelson, 1981).

Fact 3. *Independently of n, η and of the auction format, the optimal reserve price in state $\emptyset$ is $R_\emptyset = r^M$, where r^M solves $\psi(r^M) = 0$.*

In the next section we characterize the optimal reserve price in the FPA and SPA given any possible value of θ, and then compare the expected seller's revenue in the two auction formats.

3.1 Static Reserve Prices with an Incumbent

When the seller has information θ, she knows that with probability $(1-\eta)$ a bidder with valuation θ will participate to the auction. The seller then chooses the reserve price R to maximize a weighted sum of her expected revenues in auction format $i = F, S$ when the incumbent participates and when he does not — i.e., she maximizes

$$\pi^i(\theta, R) \coloneqq \eta \pi_n(R) + (1-\eta)\, \pi^i_{n-1,\theta}(R), \tag{3.2}$$

where $\pi^i_{n-1,\theta}(R)$ represents the seller's revenue in the iPA, $i = F, S$, with reserve price R and an incumbent with value θ (as well as $n-1$ symmetric bidders).[19] The seller faces a trade off between setting the optimal reserve price for the standard symmetric bidders and setting a reserve price aimed at extracting surplus from the incumbent with value θ if he participates. The next proposition characterizes the optimal solution to this trade off.

Proposition 4. *The optimal reserve price in auction format $i = F, S$ with information θ is*

$$R^i(\theta) = \begin{cases} \underline{R}^i & \text{if } 0 \le \theta < \underline{\theta}^i \quad \textit{Exclusion} \\ \theta & \text{if } \underline{\theta}^i \le \theta < \overline{\theta}^i \quad \textit{Tracking} \\ \overline{R}^i(\theta) & \text{if } \theta \ge \overline{\theta}^i \quad \textit{Tailing} \end{cases}$$

where $\underline{R}^i > \underline{\theta}^i$, $\underline{R}^S = r^M > \underline{R}^F$, and

- *$\overline{R}^S(\theta) = \overline{\theta}^S$ solves*

$$\psi(\overline{\theta}^S) = \frac{1-\eta}{n\eta f(\overline{\theta}^S)}, \tag{3.3}$$

- *$\overline{R}^F(\theta)$ solves*

$$\psi(\overline{R}^F(\theta)) = \frac{1-\eta}{n\eta f(\overline{R}^F(\theta))} - \frac{(1-\eta)(n-1)\log(F(\theta))}{n\eta f(\overline{R}^F(\theta))} \tag{3.4}$$

  and is strictly decreasing in θ and $\overline{R}^F(\overline{\theta}^F) = \overline{\theta}^F$.

Figure 3.1 displays typical reserve prices in the FPA and SPA.[20] In both auction formats, if θ is relatively small, the seller chooses to exclude the incumbent from the auction by setting a reserve price higher than his value. We define this policy *exclusion.* For intermediate values of θ (between $\underline{\theta}^i$ and $\overline{\theta}^i$), the seller sets a reserve price exactly equal to the incumbent's value in both the FPA and SPA. This policy, that we define *tracking*, allows the seller to capture the whole surplus generated by the incumbent, when he participates in the auction.

[19] See the Appendix for a formal definition.

[20] All figures assume that θ is uniformly distributed on $[0, 1]$.

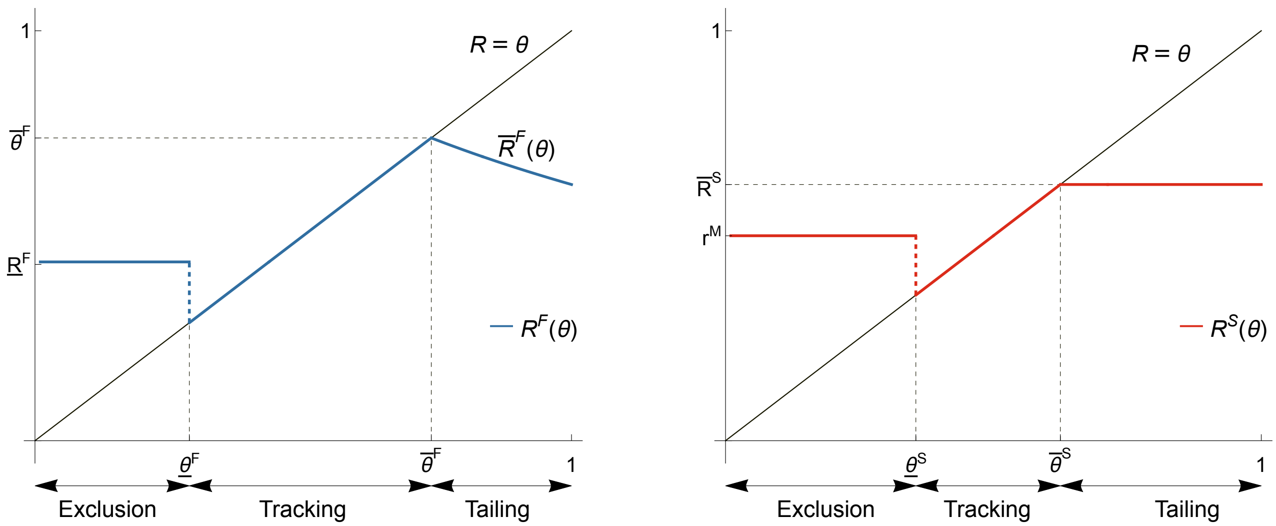

Figure 3.1: Optimal static reserve prices in FPA (left) and SPA (right) as a function of the seller's information, for $\theta \sim \mathcal{U}[0,1]$, $\eta = 0.5$, and $n = 4$.

Finally, for high values of θ, the seller sets a reserve price lower than the incumbent's value and higher than r^M.[21] This policy that we define *tailing*, however, requires the seller to choose a reserve price that is qualitatively different across the two auction formats. The tailing reserve price is constant in the SPA: the seller sets the same reserve price for all high types that she does not want to track. By contrast, in the FPA the reserve price is decreasing in θ: the higher the value of the incumbent, the lower the reserve price set by the seller. Finally, this reserve price is higher in the FPA than in the SPA, with the difference being decreasing in θ, and it is equal in the two auction formats when $\theta = 1$.[22]

The optimal reserve price is shaped by the following effects. Setting R above θ reduces the seller's revenue discretely by

$$(1-\eta)\, F(\theta)^{n-1}\, \theta \tag{3.5}$$

i.e., by the valuation of the incumbent times the probability that he wins (because he participates and no other bidder exceeds his valuation). By contrast, if the incumbent does not participate, the seller's revenue is maximized at r^M. Hence, tracking an incumbent pushes the seller away from the static optimum: when the incumbent θ is higher than r^M, the reserve price is too high for the standard bidders and the seller bears an excessive risk of no trade; when $\theta < r^M$, the reserve price is too low for standard bidders and the seller loses revenue from infra-marginal bidders.[23] Moreover, the cost of tracking increases as the incumbent's value is further away from r^M, therefore *tracking* is optimal for intermediate values of θ, while the optimal policy is different for high and low values of θ.

[21] In the equations characterizing the reserve price in the tailing region (equations (3.3) and (3.4)) there is a positive wedge relative to the monopoly price condition.

[22] Formally, the difference between conditions (3.3) and (3.4) is proportional to $-\log F(\theta)$, which is positive, decreasing in θ, and converging to zero as $F(\theta) \to 1$. Hence, $\overline{R}^F(1) = \overline{R}^S(1) = \overline{\theta}^S$.

[23] Notice that the loss for $\theta < r^M$ is present even if the incumbent participates, while the no trading loss for $\theta > r^M$ materializes only if the incumbent does not participate. As a consequence, when $\eta = 0$, we have perfect tracking for high θ but not for low θ (see Proposition 5).

Exclusion is optimal if θ is low enough. The discrete revenue loss that arises when setting R above θ, expressed by (3.5), becomes negligible for θ low, so that the benefit of setting a reserve price that is optimal for the standard bidders eventually dominates. In this case, the seller sets the same reserve price regardless of the value of the incumbent that she wants to exclude, and this reserve price maximizes a convex combination of her revenues with n and $(n-1)$ standard bidders.

Finally, *tailing* is optimal if θ is very high. In this case, tracking the incumbent is too costly: if the incumbent does not participate, the reserve price is excessively higher than r^M. The seller still takes into account the possible presence of an incumbent, but only follows him at a distance by setting a reserve price lower than θ. In the SPA, $\overline{R}^S$ is independent of θ because, conditional on tailing, (i) the cost of marginally increasing the reserve price is independent of θ, as this cost arises only when the incumbent does not participate (and no other bidder meets the reserve price) and (ii) the benefit of marginally increasing the reserve price is also independent of θ, as it is equal to the probability that all standard bidders are lower than the reserve price, so that the reserve binds when the incumbent participates.

By contrast, in the FPA $\overline{R}^F(\theta)$ is decreasing in θ because, conditional on tailing, (i) the cost of marginally increasing the reserve price is independent of θ, as in the SPA, but (ii) the benefit does depend on θ, as it represents the effect of a higher reserve price on the winning bid (either by the incumbent, or by the standard bidder with a higher value) when the incumbent participates. By Fact 2 bids of higher-value bidders are less sensitive to the reserve price, so that the marginal benefit of increasing the reserve price in the FPA is decreasing in θ.

The fact that $\overline{R}^F(1) = \overline{R}^S(1)$, combined with the fact that the tailing reserve is decreasing in the FPA (and constant in the SPA) allows us to conclude that $\overline{\theta}^S < \overline{\theta}^F$: the FPA tracks some incumbents which the SPA tails. The next proposition shows that the FPA does more tracking overall — i.e., the FPA also tracks some incumbents which the SPA excludes — and performs some comparative statics.

Proposition 5. *The following results hold:*

- *The tracking region is larger in the FPA than in the SPA:* $\underline{\theta}^F < \underline{\theta}^S < \overline{\theta}^S < \overline{\theta}^F$.
- *In both FPA and SPA, the set of incumbent's values that the seller tracks is decreasing in η and n:* $\frac{\partial \underline{\theta}^i}{\partial \eta}, \frac{\partial \underline{\theta}^i}{\partial n} > 0$ and $\frac{\partial \overline{\theta}^i}{\partial \eta}, \frac{\partial \overline{\theta}^i}{\partial n} < 0$.
- $\exists \overline{\eta} : \overline{\theta}^i = 1, \quad \forall \eta < \overline{\eta}$.
- $\lim_{\eta \to 1} R^i(\theta) = r^M$ *and* $\lim_{n \to \infty} R^S(\theta) = r^M$, *while* $\lim_{n \to \infty} \underline{\theta}^F < r^M < \lim_{n \to \infty} \overline{\theta}^F \forall \eta < 1$.

Figure 3.2 shows the optimal reserve price in the FPA and SPA for different values of η, which is our measure of winners' persistence. In both auction formats, for sufficiently low values of η, the seller never tails an incumbent and tracks all high-value incumbents.[24] In this case, the risk that the incumbent does not participate in the auction is small, so that the seller always chooses the reserve price that fully expropriates high incumbents. Proposition 5 implies that $\overline{\theta}^S = 1$ if and only if $\overline{\theta}^F = 1$ — i.e., whenever the FPA tracks the highest incumbent's type, so does the SPA, and vice versa. Moreover, the tracking region is wider in the FPA than in the SPA.

[24] The threshold $\overline{\eta}$ is independent of the auction format and solves $\frac{1-\overline{\eta}}{\overline{\eta}} = n\psi(1) f(1)$.

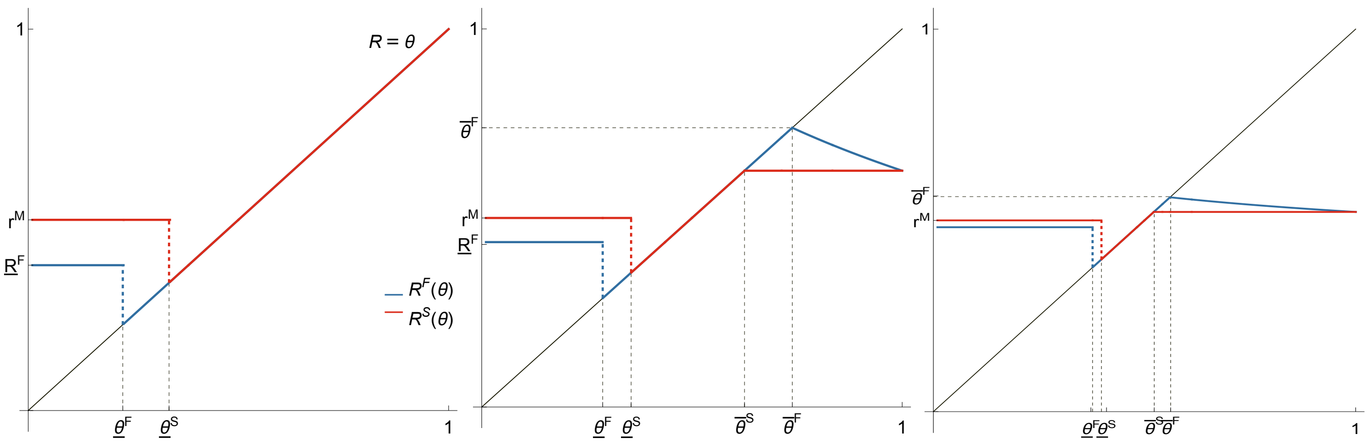

Figure 3.2: Effect of η on the static reserve prices in FPA (blue) and SPA (red) for $\theta \sim \mathcal{U}[0,1]$, and $n = 2$.

Increasing either η or n reduces the seller's incentives to track the incumbent, because he is more likely to lose or not participate in the auction. Intuitively, the higher is η, the riskier it is for the seller to set a reserve price that departs from the optimal reserve price for standard bidders. As $\eta \to 1$, the incumbent never participates and the seller disregards the information and sets the reserve price equal to r^M in both auction formats. As $n \to \infty$, the probability of the incumbent winning goes to zero, and the seller always sets the reserve price equal to r^M in the SPA. In the FPA, instead, even when $n \to \infty$ the reserve price depends on the value of the incumbent and there exists a region of incumbent's types that the seller tails.[25]

The differences among auction formats hinge on the fact that, conditional on the incumbent participating, the seller's reserve price plays a stronger role in the FPA than in the SPA. In fact, in the FPA, the reserve price affects the bids of standard bidders and hence the seller's revenue when the incumbent loses.[26] By contrast, in the SPA the seller's reserve price is irrelevant when the incumbent loses, since the incumbent's bid (which is equal to his value and hence higher than the seller's reserve price with tailing) has the same role as a seller's chosen reserve price.[27] Intuitively, in the FPA the effect of the reserve price hinges on the reserve price being observable by bidders, while in the SPA any bid is equivalent to a reserve price, from the seller's point of view. Hence, the seller has a stronger incentive to increases the reserve price in the FPA than in the SPA, whenever the reserve price tails the incumbent.

3.2 Seller's Revenue in a Static Auction

When she chooses the optimal reserve price, the seller's expected revenue in the static auction format iPA, $i = F, S$, is

$$\Pi^i(\theta) = \max_R \pi^i(\theta, R).$$

[25]In this case, the tailing reserve price $R_{lim}(\theta)$ solves $\psi(R_{lim}(\theta)) = -\frac{(1-\eta)\log(F(\theta))}{\eta f(R_{lim}(\theta))}$.

[26]Notice that an increase in either θ or n reduces this effect. The reason is that, if either the incumbent's value or the number of new bidders increase, so does the value of a wining entrant, and the reserve price has a weaker effect on higher-value bidders.

[27]Of course, in a standard auction without incumbents the role of a reserve price in the two formats is the same.

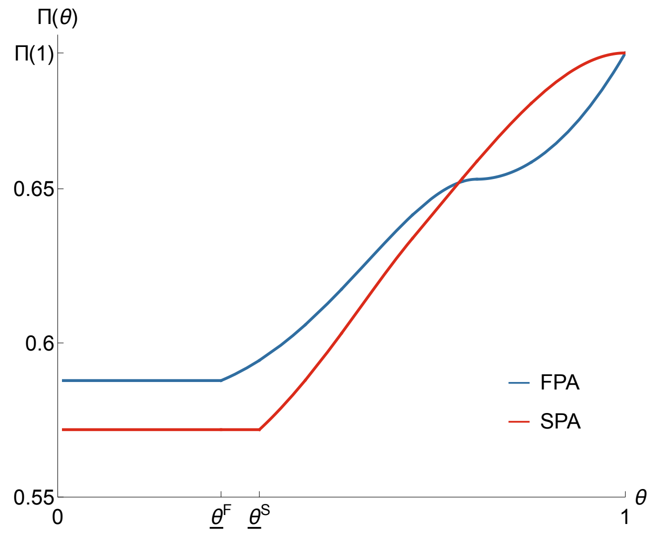

Figure 3.3: Seller's revenues in FPA and SPA as a function of the incumbent's value, with $\theta \sim \mathcal{U}[0,1]$, $\eta = 0.5$, and $n = 4$.

We now analyze how $\Pi^i(\theta)$ changes as a function of the value of the incumbent and across auction formats. First, notice that if the seller has no information, then by the Revenue Equivalence Theorem the seller's revenue is independent of the auction format. By contrast, if the seller has information about an incumbent with value θ,

Proposition 6. *The revenue ranking between auction formats depends on the value of the incumbent. If $\theta \leq \overline{R}^S$, then $\Pi^F(\theta) > \Pi^S(\theta)$. If $\overline{R}^S < 1$, then there exists a cutoff $\tilde{\theta} > \bar{R}^S$ such that: $\Pi^S(\theta) > \Pi^F(\theta)$ if and only if $\theta > \tilde{\theta}$. Revenue equivalence holds only if $\theta = 1$.*

Figure 3.3 shows the seller's revenue in the FPA and SPA as a function of θ. Revenue is higher in the FPA than in the SPA when θ is sufficiently low — i.e., when both auctions either exclude or track the incumbent. When both auctions tail the incumbent, revenue is higher in the SPA than in the FPA if and only if θ is sufficiently high. Notice that at $\theta = 1$ the reserve price is the same in the two auction formats by Proposition 4 and revenue equivalence holds. The reason is that if the incumbent participates in the auction, then he wins for sure; while if the incumbent does not participate, then there is a standard auction with n symmetric bidders.[28]

The intuition for the revenue ranking in Proposition 6 is the following. When the incumbent does not participate in the auction, the FPA and SPA yield the same revenue as a function of the reserve price. Moreover, when the incumbent participates and wins, the FPA and SPA also yield the same revenue as a function of the reserve price — i.e., $b(\theta, R, n)$ — because the incumbent bids rationally in both auction formats. Hence, the revenue ranking hinges on the differences between the auction formats when the incumbent participates and loses, that we now discuss.

First notice that, when $\theta = 1$, since the incumbent always wins when he participates, revenue

[28] To see why revenue is higher in the SPA for θ sufficiently high, consider a marginal reduction in the incumbent's value starting from $\theta = 1$, which reduces the seller's revenue in both auction formats. But the reduction in θ has a first-order effect on revenue in the FPA, because the incumbent pays his bid (which depends on θ) when the incumbent wins (which happens with a high probability). The effect in the SPA, by contrast, is second-order because the incumbent's payment is independent of θ.

is the same function of the reserve price in both auction formats. Therefore, the seller chooses the same optimal reserve price and the FPA and SPA are revenue equivalent.

When the incumbent loses the auction to a bidder with value θ', the expected seller's revenue is $b(\theta', R, n)$ in the FPA — i.e., the bid submitted by the winner — and $b(\theta', \max\{\theta, R\}, n-1)$ in the SPA — i.e., the expected payment of the winner in an auction with $(n-1)$ new bidders and the incumbent bidding θ. There are two differences between these expressions. First, the last argument of the function $b(\cdot)$ is higher in the FPA because of the effect of myopic bidding. This tends to increase revenue in the FPA by Fact 1. Second, the second argument is higher in the SPA, and strictly so with tailing — a *reserve price effect*. This reflects the fact that the losing bid by the incumbent has exactly the same effect as a reserve price equal to θ in the SPA, but not in the FPA where reserve price needs to be announced to bidders in order to affect their bids. This reserve price effect tends to increase revenue in the SPA.

When there is tracking in both auction formats, the reserve price effect vanishes because $R = \theta$. Therefore, the only difference in revenues across the two auction formats is driven by the effect of myopic bidding, which makes standard bidders pay more conditional on winning in the FPA.[29] Intuitively, if the $n-1$ standard bidders have a value lower than the incumbent, then the seller's revenue in both auctions is equal to the reserve price (when the incumbent stays). If the incumbent loses to an entrant with value θ', however, then the winner's expected payment in the SPA is $b(\theta', \theta, n-1)$, which is lower than $b(\theta', \theta, n)$ — the winner's expected payment in the FPA. The intuition is that, by Assumption 1, standard bidders fail to realize that a reserve price equal to the value of a competitor means that there are only $(n-1)$ real competitors in the auction, and hence they overestimate competition. This has no effect in the SPA, while it leads bidders to bid more aggressively in the FPA.

With tailing, by contrast, the reserve price effect adds a counteracting force: when the incumbent loses to an entrant with value θ', the seller's revenue is $b(\theta', \theta, n-1)$ in the SPA (independently of the reserve price that she sets) and $b(\theta', \overline{R}^F(\theta), n)$ in the FPA, because in the SPA the incumbent's bid equal to θ acts as a reserve price which is higher than $\overline{R}^F(\theta)$. Since this reserve price effect is increasing in θ, overall the seller's revenue is higher in the SPA if and only if the incumbent's value is sufficiently high.

Although revenue in the static environment is higher in the FPA for a larger set of incumbent's values than in the SPA, as displayed in Figure 3.3, in repeated auctions an incumbent's value is relatively more likely to be high than low, since incumbents are winners of previous auctions. And the static revenue is higher in the SPA precisely when the incumbent's value is high. So it is not surprising that, in the dynamic setting, the seller may obtain a higher revenue in the SPA than in the FPA, as we going to show in Section 4.

[29] Moreover, revenue in the FPA is also higher when there is exclusion in the SPA. First, when both the FPA and SPA exclude the incumbent, the two auction formats yield the revenue of a static auction with $n-1$ standard bidders. Since these bidders bid in the FPA as if they had other $(n-1)$ competitors, the seller's revenue is strictly higher than in the SPA for any given reserve price. Therefore, when the seller optimally chooses the reserve price, the seller's revenue is still strictly higher in the FPA than in the SPA. Second, with exclusion in the SPA and tracking in the FPA, the FPA would do better than the SPA if it also excluded the incumbent, which is not optimal in the FPA, so it must yield a higher revenue with the optimal reserve price.

4 Repeated Auctions

In this section, we consider our dynamic model, where the seller's information about the incumbent arises endogenously, since the incumbent in an auction is the winner of a previous auction. In Section 4.1, we start by analyzing the optimal reserve prices chosen by the seller in the FPA and SPA, and then study the characteristics of the minimum reserve price (Section 4.1.1). Section 4.2 describes typical paths of reserve prices and winners' values in repeated auction and Section 4.3 analyzes and compares the seller's revenue in the FPA and SPA.

In every period, the seller maximizes the sum of current and discounted future profits in the repeated auctions, by choosing a reserve price as a function of the whole history of bids. By assumption 1, the seller's problem can be described as a dynamic control problem, in which the state is either the valuation of the incumbent who won the auction in the previous period, or state $\emptyset$ if there was no bid above the reserve price in the previous period. We are going to show that, in contrast to the static environment, by choosing a reserve price the seller affects not only her revenue in the current period, but also the transition dynamics between the possible states (i.e., no trade or an incumbent's value) and therefore her future payoff.

4.1 Optimal Reserve Prices

The value function of the dynamic control problem of the seller in a generic state θ has the following recursive representation:[30]

$$V(\theta) = \max_R \,[\text{static revenue}] + \beta \cdot \mathbb{E}_{\theta,R}\left[V\left(\theta'\right)\right],$$

where the expectation operator $\mathbb{E}_{\theta,R}[\cdot]$ indicates that the transition between the current state θ and the future state θ' depends on the reserve price R.

In Theorem 7 we will prove that, on path, the seller never excludes an incumbent — i.e., she never chooses a reserve price that is higher than the value of a bidder that may win an auction.[31] Anticipating this result, we can write the value functions of the seller's problem in the iPA, $i = F, S$, as:

$$\begin{aligned} V^i(\theta) = \max_{R\leq\theta} \pi^i(\theta, R) + \beta \Bigg[\eta \left(F(R)^n V^i_\emptyset + \int_R^1 V^i\left(\theta'\right) \mathrm{d}F\left(\theta'\right)^n\right) + \\ + (1-\eta)\left(F(\theta)^{n-1} V^i(\theta) + \int_\theta^1 V^i\left(\theta'\right) \mathrm{d}F\left(\theta'\right)^{n-1}\right)\Bigg], \end{aligned} \tag{4.1}$$

and

$$V^i_\emptyset = \max_R \pi_n(R) + \beta\left[F(R)^n V^i_\emptyset + \int_R^1 V^i\left(\theta'\right) \mathrm{d}F\left(\theta'\right)^n\right]. \tag{4.2}$$

[30] Slightly abusing notation, the expression includes $\emptyset$ as a possible state.

[31] Intuitively, this is because in order to sell today to a type that the seller wants to exclude tomorrow, she has to set a reserve price that is suboptimally low: this reduces today's expected profits and does not provide any benefit tomorrow. The Appendix describes the general expressions for the value functions that take into account the possibility of exclusion.

In state $\emptyset$, the expected continuation value reflects the fact that the next-period state is going to be: (i) $\emptyset$ if no bidder has a value above R or (ii) the value of the auction winner θ'. By contrast, when there is an incumbent with value θ, the next-period state is going to be: (i) $\emptyset$ if the incumbent leaves and no new bidder has a value above R; (ii) the incumbent's value θ if he stays and wins again; (iii) the value of a new bidder θ' if he outbids the incumbent (when he stays) and all other new bidders.

Notice that the expression for the static revenue in auction format i, given by (3.2), represents the only difference between the value functions in the FPA and SPA. The reason is that, given any initial state and reserve price, the transition between states is identical in both auction formats, since the seller always learns the value of the highest bidder if this is higher than the reserve price and this is the only thing that affects the transition dynamic.[32]

The expressions for the value functions show that the reserve price affects the transition dynamics only when there is no incumbent (i.e., either in state θ when the incumbent leaves or in state $\emptyset$), because in this case a new bidder has to bid above the reserve price to become the incumbent.[33] Therefore, a marginal increase in the reserve price R shifts the continuation value from $V(R)$ to $V_\emptyset$ in case there is no incumbent and the highest new bidder's value is exactly equal to R. This dynamic effect is independent of the incumbent's value and represents either a benefit or a cost for the seller, depending on whether $V(R)$ is higher or lower than $V_\emptyset$.

Our main result characterizes the dynamically optimal reserve price in the two auction formats.[34]

Theorem 7. *Consider auction format $i = F, S$. In state $\emptyset$, the optimal reserve price $R^i_\emptyset$ solves*

$$\psi(R^i_\emptyset) = -\beta\left(V^i\left(R^i_\emptyset\right) - V^i_\emptyset\right). \tag{4.3}$$

The optimal reserve price in state θ is

$$R^i(\theta) = \begin{cases} \theta & \text{if } R^i_\emptyset \leq \theta < \overline{\theta}^i \\ \overline{R}^i(\theta) & \text{if } \theta \geq \overline{\theta}^i \end{cases}$$

where $\overline{R}^i(\overline{\theta}^i) = \overline{\theta}^i$ and

– $\overline{R}^S(\theta) = \overline{\theta}^S$ *such that*

$$\psi(\overline{\theta}^S) = \frac{1-\eta}{n\eta f(\overline{\theta}^S)} - \beta(V^S(\overline{\theta}^S) - V^S_\emptyset), \tag{4.4}$$

– $\overline{R}^F(\theta)$ *is strictly decreasing in θ and solves*

$$\psi(\overline{R}^F(\theta)) = \frac{(1-\eta)(1-(n-1)\log(F(\theta)))}{n\eta f(\overline{R}^F(\theta))} - \beta(V^F(\overline{R}^F(\theta)) - V^F_\emptyset). \tag{4.5}$$

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[32]Clearly, this does not imply that the two auction formats will have the same dynamics, because optimal reserve prices (as a function of the incumbent's value) will generically be different in the FPA and SPA.

[33]Since there is no exclusion, the reserve has no dynamic role when the incumbent stays, since his bid is never lower than the reserve price and, hence, the reserve price never binds for new bidders.

[34]Abusing notation, we use the same notation for the optimal reserve prices in a static auction and in the repeated auctions. This reflects the fact that, as we are going to show, these two types of reserve price share the same qualitative features.

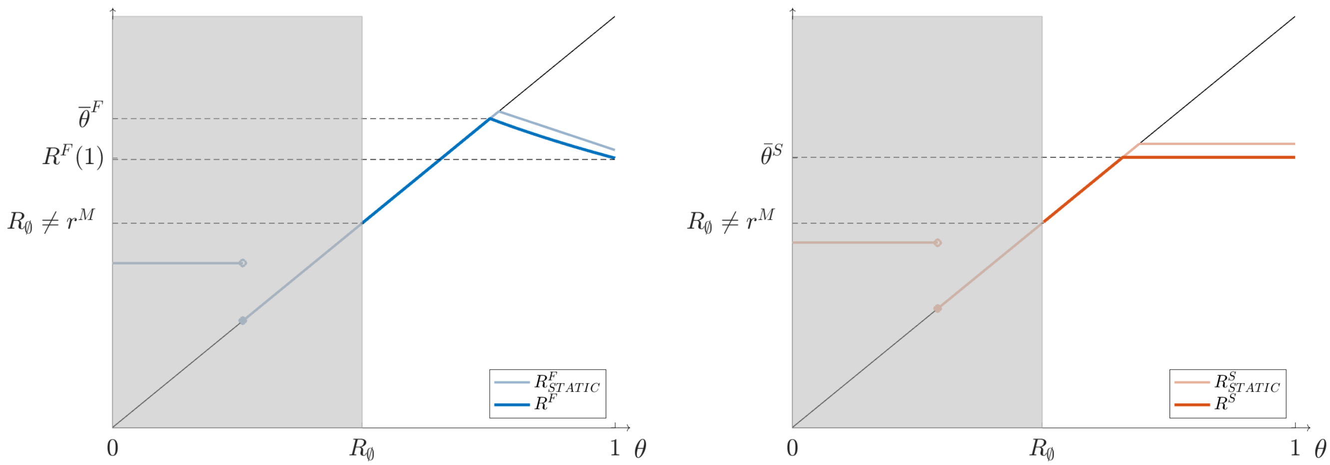

Figure 4.1: Optimal dynamic (bright color) and static (shaded color) reserve prices in FPA (left) and SPA (right) as a function of the state θ, with $\theta \sim \mathcal{U}[0,1]$, $\beta = 0.5$, $\eta = 0.5$, and $n = 2$.

Figure 4.1 shows the optimal reserve prices in the FPA and SPA, and compares it with the ones in the static environment. The optimal reserve prices characterized by Theorem 7 have the same qualitative characteristics as in the static setting, with three notable differences. First, the reserve price is always weakly greater than the initial reserve price $R^i_\emptyset$ that the seller chooses in state $\emptyset$, which implies that the seller never observes an incumbent with a value lower than $R^i_\emptyset$. Second, the seller never excludes an incumbent. Third, in the tailing region, the reserve price is lower than in the static setting because of dynamic concerns.

The intuitive reason why there is no exclusion on path is that, if the seller wanted to exclude an incumbent in a period, then in the previous period she had to set a reserve price that did not exclude him, which cannot be optimal. Formally, in the proof of Theorem 7 we show that the seller only wants to exclude types that are lower than the minimum reserve price $R^i_\emptyset$, and hence that can never win an auction and become incumbents.

In the tailing region — i.e., when $\theta \geq \overline{\theta}^i$ — the conditions in Theorem 7 for the optimal reserve prices $\overline{R}^i(\theta)$ in the FPA and SPA include the dynamic wedge $\beta(V^i(\overline{R}^i(\theta)) - V^i_\emptyset)$, $i = F, S$, in addition to the static wedges of Proposition 4. This wedge reflects the effect of increasing the reserve price on the transition dynamics, that we have discussed above. Since this dynamic wedge does not depend on θ, as in the static setting the tailing reserve price is decreasing in θ in the FPA and independent of θ in the SPA. Moreover, we show that when tailing the dynamic wedge is always positive (since $V^i(\overline{R}^i(\theta)) > V^i_\emptyset$) and hence that the reserve price is strictly lower than in the static setting.

The dynamic wedge vanishes if either $\beta = 0$ or $\eta = 1$. When $\beta = 0$, the repeated auctions are equivalent to a sequence of static auctions where the seller maximizes her profit given the information of the previous auction (as analyzed in Section 3).[35] When $\eta = 1$, the incumbent always leaves and we have a sequence of static auctions with symmetric bidders. In this case the seller to chooses a reserve price equal to r^M, regardless of the state, yielding a constant value function equal to $\frac{\pi_n(r^M)}{1-\beta}$.[36]

[35] In fact, and one can think of the dynamic model as offering a microfoundation for the presence of the incumbent in a static auction.

[36] In the limit $n \to \infty$, the seller can sell to a bidder with value 1 in every period, so that the value function is

Of course, exactly as in the static setting, when $\eta = 0$ the seller always tracks the incumbent.

The same dynamic wedge also determines the optimal reserve price $R^i_\emptyset$ that the seller chooses when there is no incumbent (see condition (4.3)). In this case, however, $V^i\left(R^i_\emptyset\right) - V^i_\emptyset$ might be either positive or negative, as we discuss in Section 4.1.1.

Notice that Theorem 7 allows to significantly simplify the analysis of our dynamic model.[37] In the SPA, the whole dynamics of the repeated auctions is completely pinned down by the two extreme values of the tracking region. Hence studying these two values (and their comparative statics) allows us to fully characterize all the effects of changes in the model primitives. For the FPA, by contrast, since the tailing reserve price is not constant, a complete analysis requires the whole schedule of reserve prices in $[\overline{\theta}^F, 1]$. For many qualitative results, however, since the tailing reserve price is strictly decreasing for those incumbent's values, we only need the three thresholds $R^F_\emptyset$ and $\overline{\theta}^F$ and $\overline{R}^F(1)$.

4.1.1 Minimum Reserve Price

Since the seller never chooses a reserve price higher than the incumbent's value, as previously discussed, $R^i_\emptyset$ represents the lowest possible bidder that the seller excludes from the repeated auctions. This induces the seller to distort the initial reserve price away from its static benchmark r^M. Recalling condition (4.3),

$$\psi(R^i_\emptyset) = -\beta\left(V^i\left(R^i_\emptyset\right) - V^i_\emptyset\right),$$

the seller sets $R^i_\emptyset < r^M$ if and only if $V^i\left(R^i_\emptyset\right) > V^i_\emptyset$: the seller reduces the reserve price relative to r^M if and only if she prefers an auction with the lowest possible incumbent rather than one with no incumbent at all.

As discussed in the previous section, the dynamic distortion on the right-hand-side of condition (4.3) vanishes when either $\beta = 0$ or $\eta = 1$ and, hence, $R^i_\emptyset = r^M$. The following proposition shows that, for specific values of the model's parameters, there are conditions on primitives that allow to sign the dynamic wedge $V^i\left(R^i_\emptyset\right) - V^i_\emptyset$ and determine the direction of the distortion of $R^i_\emptyset$.

Proposition 8. *Local to $\beta = 0$ or $\eta = 1$, $R^i_\emptyset > r^M$ if and only if*

$$\pi^i_{n-1,r^M}\left(r^M\right) - \pi_n\left(r^M\right) < 0. \tag{4.6}$$

When $\eta = 0$, $R^i_\emptyset > r^M$ if and only if

$$\pi^i_{n-1,r^M}\left(r^M\right) - \pi_n\left(r^M\right) < \beta\int_{r^M}^1 \frac{\mathrm{d}}{\mathrm{d}\theta'}\pi^i_{n-1,\theta'}\left(\theta'\right)\frac{F\left(\theta'\right)^{n-1}\left(1-F\left(\theta'\right)\right)}{\left(1-\beta F\left(\theta'\right)^{n-1}\right)}\mathrm{d}\theta'. \tag{4.7}$$

Condition (4.6) shows that, local to $\beta = 0$ or $\eta = 1$, the difference between $R^i_\emptyset$ and r^M is proportional to the difference between $\pi^i_{n-1,r^M}\left(r^M\right)$ and $\pi_n\left(r^M\right)$ — i.e., it only depends on static

also constant (and equal to $\frac{1}{1-\beta}$). But while both $\eta = 1$ and $n \to \infty$ yield the same reserve price r^M when there is no incumbent, they induce different reserve prices when there is an incumbent. The reason is that the static wedge vanishes in the SPA but not in the FPA (where the reserve price depends on the incumbent's value even when $n \to \infty$ by Proposition 5).

[37] The theorem also has direct implications of economic relevance that we detail in Section 4.2.

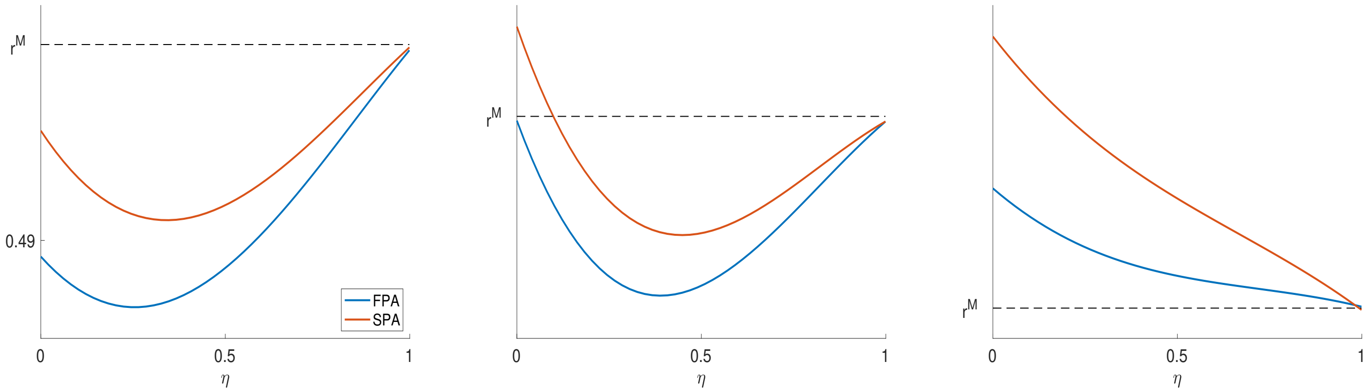

Figure 4.2: Initial reserve price $R_\emptyset$ as a function of η in FPA and SPA with $\theta \sim \mathcal{U}[0,1]$, $\beta \approx 0.6$ and, from the left, $n = 2$, $n = 3$ and $n = 6$.

revenues. Therefore, $R^i_\emptyset < r^M$ if and only if the seller prefers to insure against no trade (i.e., to have a bidder with value r^M and $n-1$ new bidders, obtaining r^M even when all $n-1$ new bidders are below r^M), rather than having one additional bidder (which increase her profit from $\pi^i_{n-1}(r^M)$ to $\pi_n(r^M)$). We show in the Appendix that this comparison depends on the number of bidders n: when $\eta = 1$, there exists $\bar{n}^i$ such that $R^i_\emptyset > r^M$ if and only if $n > \bar{n}^i$.[38] This is displayed in Figure 4.2, that shows the reserve price $R^i_\emptyset$ in the FPA and SPA for different values on n.

At $\eta = 0$, instead, the sign of the difference between $R_\emptyset$ and r^M is determined by condition (4.7), which depends on an additional term, on top of the difference between $\pi^i_{n-1}(r^M)$ and $\pi_n(r^M)$. Since this term is always positive,[39] if condition (4.6) is satisfied, then condition (4.7) is satisfied too. Therefore, $R^i_\emptyset > r^M$ local to $\eta = 1$ implies $R^i_\emptyset > r^M$ local to $\eta = 0$.[40] When condition (4.7) is satisfied but condition (4.6) is not, $R_\emptyset > r^M$ at $\eta \approx 0$ but $R_\emptyset < r^M$ at $\eta \approx 1$, implying that $R_\emptyset$ is non-monotone in η. The central panel of Figure 4.2 shows an example of this.

Figure 4.2 also provides insight on how the minimum reserve price varies across auction formats. In the Appendix we show that the distance between $R^i_\emptyset$ and r^M at $\eta = 0$ and $\eta = 1$ depends on the magnitude of the functions in conditions (4.6) and (4.7). The next proposition exploits this to formalize the comparison between the initial reserve prices in the FPA and SPA at the extreme values of η.

Notice that, by Fact 15, the revenues conditional on tracking are higher in FPA than in SPA. Since a type $R_\emptyset$ is always tracked, $\pi^F_{n-1,r^M}\left(r^M\right) - \pi_n\left(r^M\right) > \pi^S_{n-1,r^M}\left(r^M\right) - \pi_n\left(r^M\right)$.[41] Therefore,

Proposition 9. *Local to $\eta = 0$ and $\eta = 1$, $R^F_\emptyset < R^S_\emptyset$.*

At $\eta = 1$, by conditions 7.17 and 7.21 the comparison between $R^F_\emptyset$ and $R^S_\emptyset$ is proportional to the difference in $\pi^i_{n-1}(r^M)$ across auction formats. Since by Fact 15 $\pi^F_{n-1}(r^M) > \pi^S_{n-1}(r^M)$, the initial reserve price is lower in the FPA than in the SPA. The intuition is that at $\eta = 1$ the reserve price is

[38] Notice that when $\eta \to 1$ also the static wedges vanish and the seller sets $R = r^M$ in all periods, regardless of the incumbent, since the repeated auctions collapse to a sequence of static ones.

[39] This is because $\frac{\mathrm{d}}{\mathrm{d}\theta'}\pi^i_{n-1,\theta'}\left(\theta'\right) > 0$.

[40] The converse is not true because, when condition (4.7) is satisfied, $\pi^i_{n-1,r^M}\left(r^M\right) - \pi_n\left(r^M\right)$ can be either positive or negative.

[41] $\pi_n\left(r^M\right)$ is the same in the FPA and SPA by revenue equivalence.

the same in both auction formats, but slightly reducing η has a higher effect on FPA than on SPA, therefore $R_{\emptyset}^{F} < R_{\emptyset}^{S}$.

From conditions (7.22) and (7.21), at $\eta = 1$ and at $\eta = 0$ the value of $R_{\emptyset}^{i} - r^{M}$ is proportional to $\pi_{n-1,r^M}^{i}\left(r^{M}\right) - \pi_{n}\left(r^{M}\right)$, which is larger in FPA than in SPA, therefore $R_{\emptyset}^{F} < R_{\emptyset}^{S}$. At $\eta = 1$ the intuition is that the reserve price is the same in both auction formats, but slightly reducing η has a higher effect on FPA than on SPA.

Consistently with the static setting, the fact that the initial reserve price in the FPA is lower than in the SPA reflects the fact that the seller is more willing to track an incumbent in the FPA than in the SPA. Moreover, the initial reserve price has relevant implications for the probability of trade, especially when the seller never tails an incumbent (e.g., when $\eta = 0$). In this case, Proposition 9 immediately implies that the FPA is more efficient than the SPA, as we discuss in Section 5.1.

4.2 Paths of Reserve Prices and Auction Winners

Theorem 7 allows us to describe the dynamics of reserve prices and incumbents that can be observed in repeated auctions, when the seller sets the optimal reserve prices. Figure 4.3 displays an example of typical paths of reserve prices and winners' values in the FPA and SPA. Notice that the paths restart with the initial reserve price $R_{\emptyset}$ and no incumbent after any period in which there was no trade. The paths are often increasing after a period of trade, but sometimes decreasing because of the optimal reserve price schedule in repeated auctions.

In each period, there is a lower bound on the possible value of a winner, which is either the value of the previous winner (if he stays), or the reserve price (if the previous winner leaves). In the tracking region, these two bounds coincide and, when there is trade, the value of the auction winner can only increase. In the tailing region, instead, the value of the winner can decrease if the incumbent leaves, because the reserve price is lower than the incumbent's value.

The dynamics of the winner's values gives only a partial view of the dynamics of the reserve prices, as the two move in the same direction only with tracking. In the tailing region, the reserve price is unresponsive to the value of the winner in the SPA, and moves *against* it in the FPA. For this reason, in the SPA all downward movements in the winner's value have no consequence on the reserve price, which either increases until it reaches $\bar{\theta}_S$ and then remains constant, or reverts back to $R_{\emptyset}$ after a period of no trade. In FPA, by contrast, the reserve price is strictly increasing only up to $\tilde{R}(1)$, while it can both increase and decrease in the interval $[R^{F}(1), \overline{\theta}_{F}]$. This depends on whether a previous winner does not win again because he faces a new bidder with a higher value (in which case the reserve price decreases) or because he leaves the auction (in which case the reserve price can both increase or decrease, depending on the whether the value of the new winner is higher or lower than the value of the previous one).

Finally, notice that although the two auction formats have the same transition dynamic for a given reserve price, they will generically induce different realizations of paths of winners' values and reserve prices, because the seller chooses different reserve prices in the two formats. For example, consider the simple case where $\beta = 0$ (so that the dynamic distortions discussed in Section 4.1 vanish). In this case, for a given incumbent, the reserve price is higher in FPA than in SPA (and strictly so when there is tailing in the SPA). Therefore, for any incumbent's value, the FPA results in lower trade and efficiency than the SPA. Of course, total efficiency depends also on the stationary distribution of incumbents, which in turn depends on the reserve prices chosen by the seller and therefore differs in the two auction formats. We analyze this stationary distribution and its effects on trade in Section 5.

4.3 Seller's Revenue: FPA vs. SPA

In this section, we compare the seller's revenue in the FPA and SPA. Recall from the analysis of the static environment (Proposition 5) that the revenue ranking of the two auction formats depends on the incumbent's value: the FPA benefits from myopic bidding with low-value incumbents, while the SPA exploits high-value incumbents as implicit reserve prices. These two effects also determine the revenue ranking among auction formats in the dynamic setting, together with the stationary distribution of the incumbents' values.

Our dynamic setting provides a natural microfoundation for the existence of incumbent bidders in repeated auction, and affects the optimal reserve prices chosen by the seller. Since incumbents are winners of previous auctions, they tend have relatively high valuations,[42] which skews the distribution or realized incumbents' valuations towards regions where the SPA dominates in the static setting. When values are uniformly distributed, this results in a higher seller's revenue in the SPA than in the FPA for a wide range of our model's parameters (see Figure 4.4).[43]

Moreover, because η affects the expected valuation of an incumbent and also the seller's reserve prices, it is not surprising that the revenue ranking between FPA and SPA depends on η. In particular, the higher is η, the higher is seller's incentive to tail the incumbent by setting a reserve price lower than his valuation. In this case, the SPA allows the seller to choose a reserve price closer to the optimal one for new bidders than in the FPA (as we have discussed in Section 3), which tends to increase revenue in the SPA. By contrast, a sufficiently low η induces the seller to always track the incumbent, which favors the FPA because of myopic bidding (as we have discussed in Section 3).

The next proposition formalizes this intuition and compares the seller's revenue in the FPA and SPA when either η is very small, or very large.

Proposition 10. *There exists a neighborhood $\mathcal{N}_0$ of $\eta = 0$ such that $V_0^F(\eta) > V_0^S(\eta)$, $\forall \eta \in \mathcal{N}_0$. There exists a neighborhood $\mathcal{N}_1$ of $\eta = 1$ such that $V_0^S(\eta) > V_0^F(\eta)$, $\forall \eta \in \mathcal{N}_1$.*

It is not surprising that the expected revenue is higher in the FPA than in the SPA when $\eta = 0$, because in this case both auction formats always track the incumbent and they only differ in the initial reserve price $R_\emptyset^i$. But recall that myopia induces new bidders to overpay in the FPA when the reserve price is equal to the value of the incumbent (see the discussion following Proposition 6). Hence, if the seller chooses an initial reserve price equal to $R_\emptyset^S$ in the FPA, she obtains exactly the

[42]Higher than the unconditional distribution of a new bidder's value.

[43]Figure 4.4 also shows that the seller's revenue in both auction formats is decreasing in η, because an increase in η reduces bidders' capacity and hence worsens the distribution of incumbents' values. More precisely, for a given reserve price policy, a higher η results in a distribution of bidders' values that FOSD the distribution with a lower η, in every period, and therefore a higher expected revenue. The expected revenue is even higher with a higher η when the seller chooses the optimal reserve price policy.

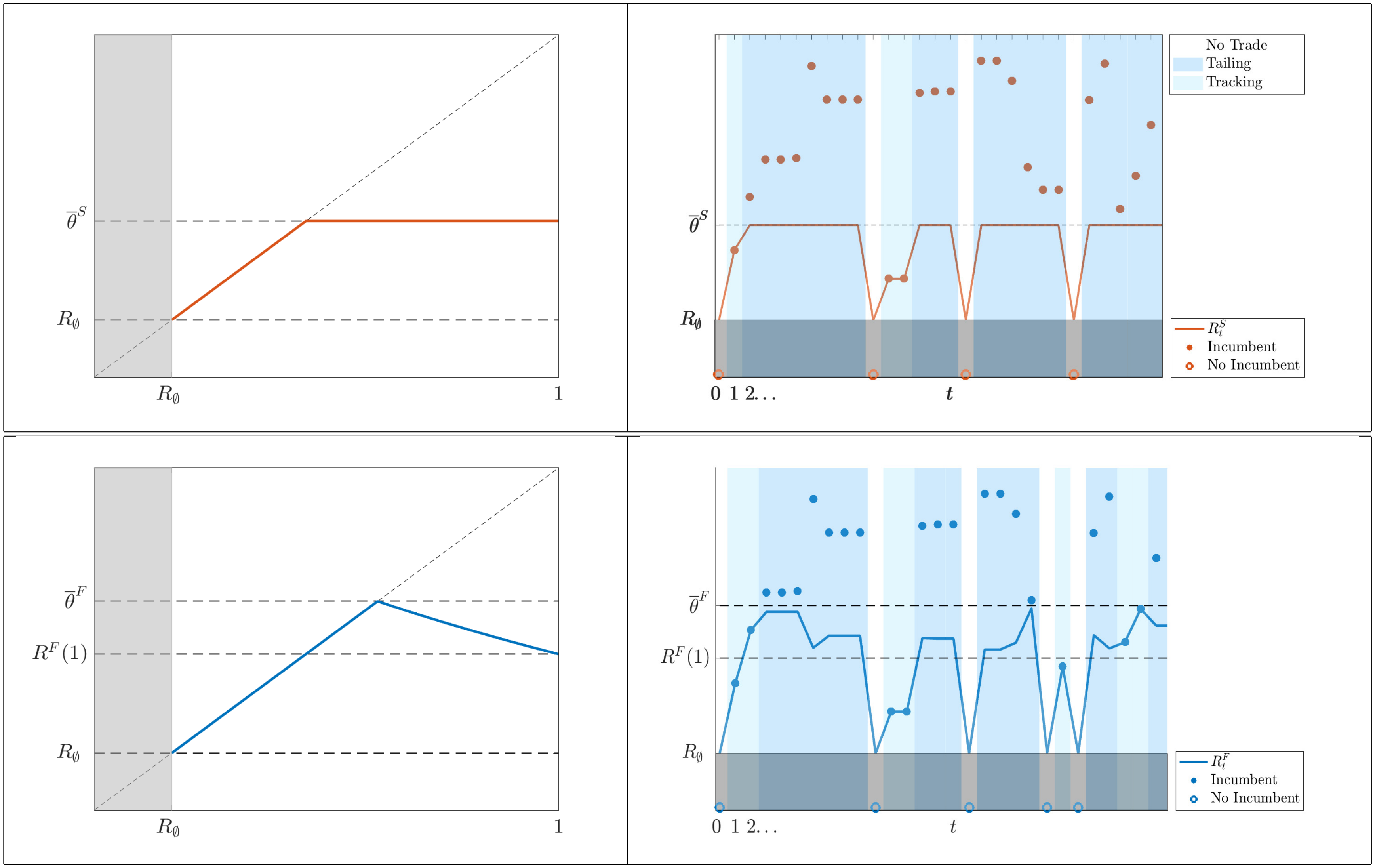

Figure 4.3: Reserve price schedules (left) and typical paths of incumbent's values (dots) and reserve prices (continuous line) in the repeated auctions (right), for SPA (top) and FPA (bottom). In the paths plots, the white areas highlight periods with no incumbent, the dark shaded areas periods with tracking, and the light shaded areas period with tailing.

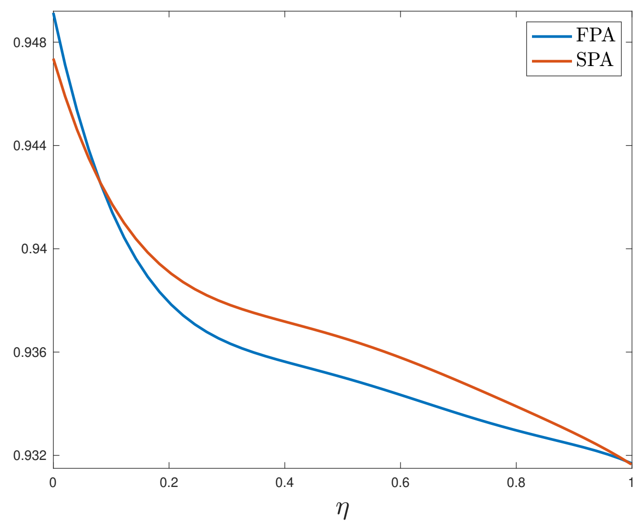

Figure 4.4: Seller's value in FPA and SPA as a function of η, with $\theta \sim \mathcal{U}[0,1]$, $\beta = 0.15$ and $n = 10$.

same path of winners as in the SPA and a higher revenue. Choosing the optimal initial reserve price $R_{\emptyset}^{F}$ can only further increase revenue in the FPA compared to the SPA.[44]

At $\eta = 1$, by contrast, myopic bidding has no effect because the incumbent always leaves and the seller chooses the same reserve price in the FPA and the SPA, so that the two auction formats are revenue equivalent. The revenue ranking when η is close to 1 depends on a mechanical difference in the two auction formats. Local to $\eta = 1$, since changes in the reserve prices and the bidders' distribution is second order, we can interpret the effect of a marginal reduction in η as the effect of replacing an arbitrary bidder in a static auction with an optimal reserve price with a "special" bidder.

If this special bidder has a value which is drawn from the unconditional distribution $F(\cdot)$, then by the Revenue Equivalence Theorem the seller would obtain the same expected revenue in the FPA and SPA. Conditional on the *actual value* of the special bidder, however, we show that the seller's revenue is higher in the FPA if this value is low and higher in the SPA if this value is high.[45] Now, in our environment of repeated auctions, the special bidder introduced in the static auction is actually an incumbent and, hence, his value is drawn form a distribution that FOSD F — the incumbent's value is the maximum of n draws from the unconditional distribution F, truncated above the reserve price. This shifts weight towards higher realizations of the special bidder's values, which results in a higher revenue in the SPA than in the FPA.

The previous argument hinges on the fact that the two auctions have the same reserve price and the same distribution of incumbents. Of course, changes in η also affect the reserve prices in the FPA and SPA, which also influence the seller's revenue when the incumbent leaves. As we have discussed in Section 3.2, the fact that the seller typically sets a lower reserve price in the SPA (closer

[44]Notice that this argument holds for any η that is sufficiently low to induce the seller to always track the incumbent in both auction formats (which is the case, for example, when $\beta = 0$ and $\eta < \overline{\eta}$, where $\overline{\eta}$ is defined in the proof of Proposition 5).

[45]This property is further explored in our companion paper Carannante *et al.* (in preparation).

to r^M) favors this auction format in this case. This additional force explains the fact that revenue in the SPA can often be higher than in the FPA — as is apparent from inspection of Figure 4.4 — despite the effect of myopic bidding that may induce to believe that the FPA should be preferred by the seller. Therefore, our quantitative exercise suggests that the effect of myopic bidding is small relative to the role that incumbents play as implicit reserve prices in the SPA, which allows the seller to tailor the reserve price to the event that the incumbent leaves.

5 Stationary Distribution of Winners' Values and Trade

In this section, we study the stationary distribution of incumbent's values in the repeated auctions, and use it to analyze the probably of trade. All results refer to both the FPA and SPA, unless otherwise stated.[46]

Let $G \in \Delta(\{\emptyset \cup [R_\emptyset, 1]\})$ be stationary distribution of incumbent's values. This distribution has a point mass $G_\emptyset \in [0,1]$ at $\emptyset$ and is absolutely continuous on $[R_\emptyset, 1]$. Hence, letting $G(\theta)$ be the absolutely continuous part with density g such that,

$$G_\emptyset + \int_{R_\emptyset}^{1} \mathrm{d}G\left(\theta'\right) = 1 \tag{5.1}$$

with $G\left(R_\emptyset\right) = 0$, $G\left(1\right) = 1 - G_\emptyset$.

Given the possible transition dynamics between states, the stationary distribution solves the following system of equations

$$\begin{aligned}
G_\emptyset &= G_\emptyset F\left(R_\emptyset\right)^n + \int_{R_\emptyset}^{1} \eta F^n\left(R\left(\theta\right)\right)\mathrm{d}G\left(\theta\right), \\
\mathrm{d}G\left(\theta\right) &= \mathrm{d}G\left(\theta\right)\left(1-\eta\right)F\left(\theta\right)^{n-1} + G_\emptyset \mathrm{d}F^n\left(\theta\right) + \int_{\{\theta' : R(\theta') \le \theta \le \theta'\}} \eta \mathrm{d}F^n\left(\theta\right)\mathrm{d}G\left(\theta'\right) \\
&\quad + \left[\eta \mathrm{d}F^n\left(\theta\right) + \left(1-\eta\right)\mathrm{d}F^{n-1}\left(\theta\right)\right]G\left(\theta\right).
\end{aligned}$$

Notice that state $\emptyset$ can be reached either (i) starting from state $\emptyset$ itself, when no bidder bids above $R_\emptyset$, or (ii) starting from a state θ, when the incumbent leaves and no new bidder bids above $R\left(\theta\right)$. Similarly, a state $\theta \ge R_\emptyset$ can be reached either (i) starting from θ itself, when the incumbent θ stays and wins again,[47] or (ii) starting from states $\emptyset$ or θ', when the highest bidders has value θ.[48]

If the seller tracks an incumbent θ', then θ' represents a lower bound for a new incumbent's value: in this case it is not possible to transition directly — i.e. without going through state $\emptyset$ — to a state with a lower incumbent. By contrast, if the seller tails an incumbent θ', since the reserve price is strictly lower than θ', it is possible to transition directly to a state with a lower incumbent (but this requires the old incumbent θ' to leave). The set of all states that allow a transition to a

[46] Throughout this section, to simplify notation, we suppress the superscript i that refers to the auction format.

[47] This event has probability $\left(1-\eta\right)F\left(\theta\right)^{n-1}$.

[48] In the last case, the auction has a first-time winner. Notice that, if state θ is reached starting form a state θ', then θ must be higher either of the old incumbent θ' (in case θ' stays) or of $R(\theta')$ (in case θ' leaves).

state with a lower-value incumbent is

$$\left\{\theta' : R\left(\theta'\right) \leq \theta \leq \theta'\right\} = \begin{cases} \emptyset & \text{if } \theta < R^i(1) \\ [\tilde{R}^{-1}(\theta), 1] & \text{if } \theta \in [R^i(1), \overline{\theta}^i] \\ [\theta, 1] & \text{if } \theta > \overline{\theta}^i \end{cases} \tag{5.2}$$

The case $\theta \in [R^i(1), \overline{\theta}^i]$ in (5.2) is only relevant in the FPA,[49] where such a state can be reached either starting from *low* incumbents (with values below θ) or from *high* incumbents (with values above $\tilde{R}^{-1}(\theta)$), but not from *intermediate* incumbents (with values $\in (\theta, \tilde{R}^{-1}(\theta))$) that induce a reserve price above θ.

When incumbents are always tracked — i.e., when $\overline{\theta}^i = 1$ — the set defined by (5.2) is empty (for all θ) and the expression for $\mathrm{d}G(\theta)$ becomes

$$\mathrm{d}G(\theta) = \frac{G_\emptyset \mathrm{d}F^n(\theta) + \left[\eta \mathrm{d}F^n(\theta) + (1-\eta)\,\mathrm{d}F^{n-1}(\theta)\right] G(\theta)}{\left(1 - (1-\eta) F^{n-1}(\theta)\right)}, \tag{5.3}$$

which can be solved on $[R_\emptyset, 1]$ using the boundary condition $G(R_\emptyset) = 0$. Equation (5.1) implies $G(1) = 1 - G_\emptyset$, giving the final condition that allows to pin down $G_\emptyset$.[50]

When transitions to a state with a lower-value incumbent are possible — i.e., when the set defined by (5.2) is not empty — the solution of the differential equations is obtained connecting at most three solutions (on the three intervals on which the functions are defined), and the continuity of G gives the additional boundary conditions which jointly determine $G_\emptyset$ (and, from there, the whole stationary distribution).

Figure 5.1 shows the stationary distributions of winners' values in the repeated auctions for different values of η. Notice that, as η decreases, the distribution assign more weight to higher rather than lower values: $G(\cdot)$ with a low η FOSD $G(\cdot)$ with a high η. In the next section, we are going to use this stationary distribution to analyze trade — i.e., the probability that in an auction the object is sold to one of the bidders.

5.1 Trade

By definition, trade T is the long-run probability that an object is sold in one of the repeated auctions. Since in all auctions the object is sold in all states except state $\emptyset$, this is equal to one minus the stationary distribution of state $\emptyset$ — i.e.,[51]

$$T = 1 - G_\emptyset.$$

The next proposition analyzes how long run trade depends on the persistence of bidders. It may be expected that a higher persistence always increases trade, because it makes it less likely that a

[49] In the SPA, $R^S(1) = \overline{\theta}^S$.

[50] Notice indeed that $G(1)$ obtained from solving forward (5.3) is increasing in $G_\emptyset$, as $G_\emptyset$ increases the forcing term.

[51] In the market for online display advertising, impressions that are not sold through the auction are often allocated to guaranteed contracts of the publisher. In this case, $G_\emptyset$ reflects the percentage of impressions allocated to guaranteed contracts.

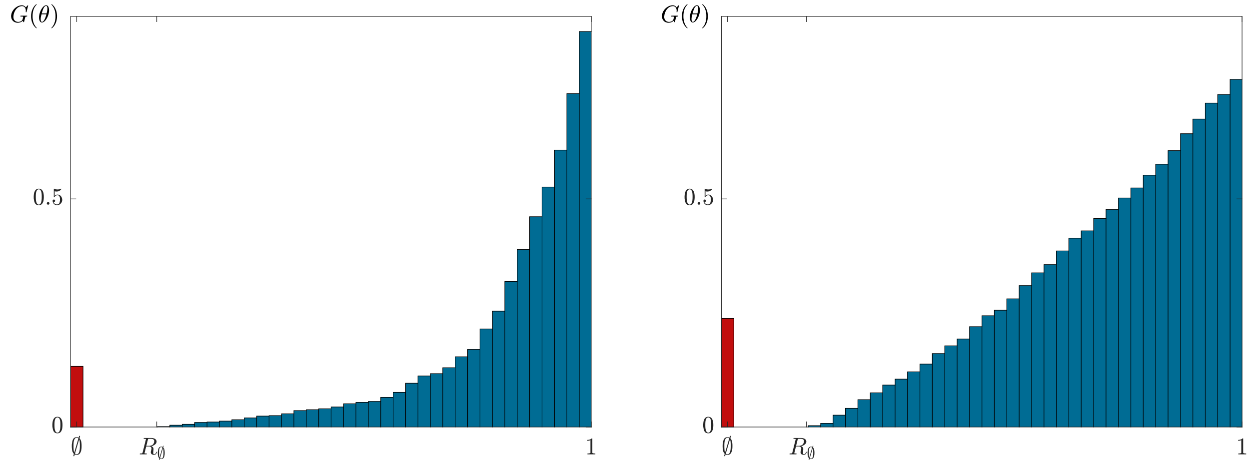

Figure 5.1: Stationary distributions over incumbent's types with η low (left) and high (right).

past winner leaves the repeated auctions and trade always occurs when a past winner stays (because the seller always chooses a reserve price that is lower than his value). This is not necessarily true, however, because the persistence of bidders also affects the reserve prices chosen by the seller and, in particular, a lower persistence weakly reduces the reserve prices in every state θ.[52]

Proposition 11. *T is continuous in η and satisfies $T(0) = 1$, $T(1) = 1 - F\left(r^M\right)^n$. T can be non-monotonic in η.*

Notice that, when $\eta = 0$ — i.e., when the incumbent always stays in the auction — there is always trade, i.e. $G_\emptyset = 0$; while when $\eta = 1$, the probability of no trade is the same as in a static auction, i.e. $G_\emptyset = 1 - F^n\left(r^M\right)$. Figure 5.2 plots $G_\emptyset$ as a function of η and shows that trade may be increasing in η when η is low, but decreasing in η when η is high.

In general, η affects $G_\emptyset$ through three different channels. First, there is a *direct effect* that reflects the change in $G_\emptyset$ implied by the incumbent leaving more often. This effect is equal to

$$\int_{R_\emptyset}^{1} F^n\left(R\left(\theta\right)\right) g\left(\theta\right) \mathrm{d}\theta,$$

which is always positive. Second, there is a *reserve price effect* that reflects the change in $G_\emptyset$ implied by the fact that the seller adjusts the reserve price when η changes. This effect is equal to

$$G_\emptyset \frac{\mathrm{d}R_\emptyset}{\mathrm{d}\eta} \mathrm{d}F^n\left(R_\emptyset\right) + \eta \int_{R_\emptyset}^{1} \frac{\mathrm{d}R\left(\theta\right)}{\mathrm{d}\eta} \mathrm{d}F^n\left(R\left(\theta\right)\right) g\left(\theta\right) \mathrm{d}\theta.$$

While the effect of η on $R_\emptyset$ is ambiguous (see Section 4.1.1), $\frac{\mathrm{d}R(\theta)}{\mathrm{d}\eta} \leq 0$ with a strict inequality in the tailing region. The latter effect dominates the former, as shown in Figure 5.2. This reflects the fact that an increase in η induces the seller to reduce the reserve price when she does not track the incumbent, which has a positive effect on trade that counteracts the direct effect.

[52]Moreover, the bidders' persistence also directly affect the stationary distribution of winners' values.

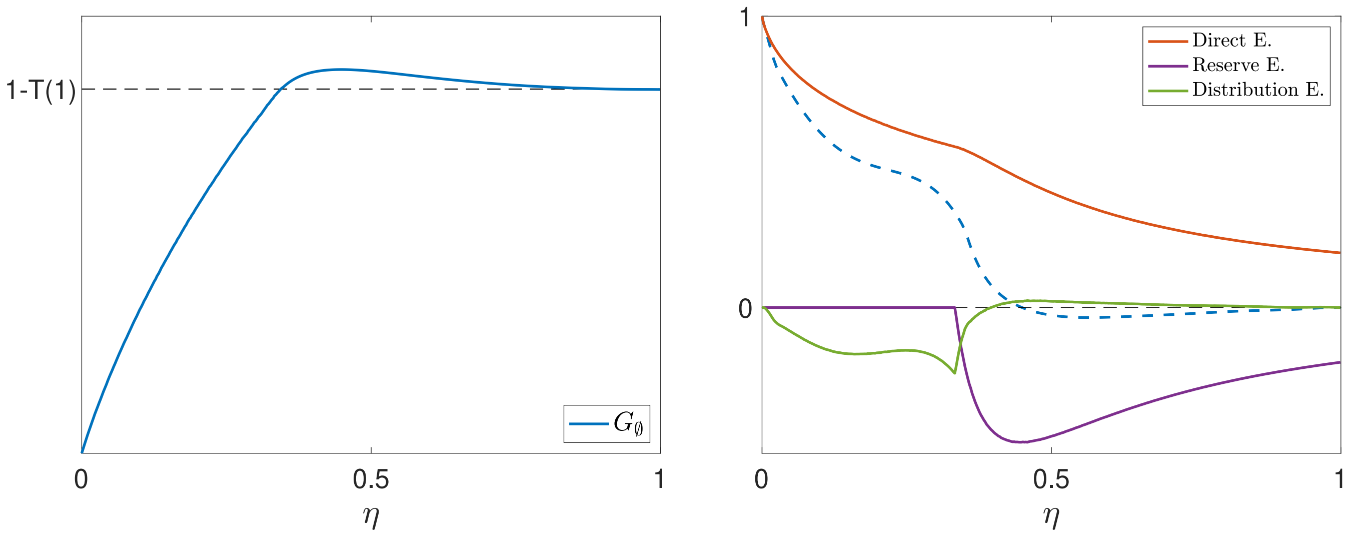

Figure 5.2: Left: $G_\emptyset$ as a function of η. Right: Decomposition of direct effect, reserve price effect and distribution effect. Both graphs refer to a SPA with $\beta = 0$ and $n = 2$.

Third, there is a *distribution effect* that reflects the change in $G_\emptyset$ resulting from a change in the stationary distribution of the incumbent's values: the long run frequency of each incumbent's value is directly affect by η. This effect is equal to

$$\frac{\mathrm{d}G_\emptyset}{\mathrm{d}\eta} F^n\left(R_\emptyset\right) - \eta \frac{\mathrm{d}R_\emptyset}{\mathrm{d}\eta} g\left(R_\emptyset\right) F^n\left(R_\emptyset\right) + \eta \int_{R_\emptyset}^{1} F^n\left(R\left(\theta\right)\right) \frac{\mathrm{d}}{\mathrm{d}\eta} g\left(\theta\right) \mathrm{d}\theta.$$

The total effect of η on $G_\emptyset$ is obtained summing these three terms,

$$\frac{\mathrm{d}G_\emptyset}{\mathrm{d}\eta} = \text{Direct Effect} + \text{Reserve Price Effect} + \text{Distribution Effect}.$$

Figure 5.2 shows the three different effects of η on trade — i.e., $1 - G_\emptyset$.[53] Notice that the distribution effect is relatively small, the direct effect is always positive, while the reserve price effect is zero when the seller always tracks an incumbent (i.e., for η low) and strictly negative when there is tailing of high-value incumbents.

This decomposition of the effects of changes in η on $G_\emptyset$ provides an interpretation of the transition dynamic associated with a permanent change in η.[54] The direct effect reflects the short-term effect of an increase in the bidder's persistence on trade, when neither the reserve prices nor the stationary distribution of winners' value change. This effect always reduces trade. The additional reserve price effect reflects a medium-term response, when the seller can react by adjusting the reserve prices in response to changes in bidders' persistence. This effects tends to be negative and may overcome the direct effect, resulting in an overall increase of trade. Finally, in the long term, the stationary distribution of winners' values also changes as reflected by the distribution effect.

[53] In the Appendix we derive reserve price and distribution effect for the SPA with $\beta = 0$ and $n = 2$, used in the plot.

[54] A permanent reduction in η may reflect a structural change arising, for example, because a reduction in users' attention require advertisers to show their ad for a higher number of times before inducing a purchase.

Long run trade allows us to compare the FPA and SPA in terms of efficiency.

Proposition 12. *The FPA is more efficient than the SPA when* $\overline{\theta}^F = \overline{\theta}^S = 1$ *— i.e., for* η *small enough.*

The condition $\overline{\theta}^i = 1$ implies that the seller always tracks the incumbent. Hence, long-run trade only depends on the initial reserve price, that determines the lowest possible value of a bidder who may win an auction. The proposition follows from the fact that $R_{\emptyset}^{FPA} < R_{\emptyset}^{SPA}$ (by Proposition 9). The intuition is that, because of myopic bidding, the seller is more willing to track low-value incumbents in the FPA than in the SPA, and this has a positive effect on the total efficiency of the repeated auctions.

Recall that, by Proposition 10, when η is small also the seller's revenue is higher in the FPA. Therefore, in this case the FPA dominates the SPA both in terms of revenue and total efficiency, despite the fact the assumption of myopic bidding has stronger effects when η is low because there is full tracking, and only in the FPA, which may lead one to expect lower efficiency in the FPA.

6 Winners' Tenure

What information can be inferred in repeated auction by an analyst who only observes some limited characteristics of the environment? In this section, we analyze information that can be inferred by only observing the tenure of the current winner — i.e., the number of consecutive periods in which a bidder has won the repeated auctions — and not, for example, the auction prices or the reserve prices chosen by the seller (let alone the incumbent's value that is a sufficient statistic for all the auction moments).[55]

Let $\tau \in \{1, 2, \dots\}$ be the number of consecutive periods in which the current winner has participated in the repeated auctions: when $\tau = i$, the current winner won for the last i periods. Let further $g(\theta, \tau)$ be the probability that the current winner has value θ and tenure τ.[56] The probability that the current winner with value θ will win for one additional period is $(1-\eta)F^{n-1}(\theta)$ — i.e., the probability that the incumbent stays in the auction and the $(n-1)$ new bidders have a lower value than him. This probability is independent of the winner's tenure. Therefore, recursive substitution yields

$$g(\theta, \tau) = \left[(1-\eta)F^{n-1}(\theta)\right]^{\tau-1} g(\theta, 1). \tag{6.1}$$

Let $g_\tau(\theta) \equiv \frac{g(\theta,\tau)}{\int_{R_\emptyset}^1 g(\theta,\tau)d\theta}$ be the conditional distribution of the values of winners with tenure τ.

Lemma 13. *The conditional distributions* $g_\tau(\theta)$ *are increasing in tenure (in FOSD sense) and converge in probability to* 1.

[55]In our application to display advertisement, this is equivalent to inferring information such as the likelihood of trade or the expected seller's revenue, given the number of periods in which an identical advertisement has been displayed.

[56]Notice that so far, with a with slight abuse of notation, we have denoted $g(\theta)$ as the marginal distribution of $g(\tau, \theta)$. Of potential independent interest is the marginal distribution on τ, namely the distribution of the number of consecutive auctions with the same winner (or the number of periods in which a winner manages to win again, until either he is outbid or he reaches his capacity).

The length of tenure is an indicator of the expected valuation of an incumbent, which is the main driver of all relevant statistics of the repeated auctions. In fact, a winner with tenure τ: (i) did not leave the auction for $\tau - 1$ periods and (ii) outbid a total of $(n-1)\tau$ new bidders. While the first event is independent of the bidder's valuation (and therefore does not affect $g_\tau(\theta)$), the probability of the second event is strictly increasing in the winner's value. Hence, longer tenures reflect higher winner's valuations which, in the limit $\tau \to \infty$, converge to the upper bound of the support of valuations.[57]

Lemma 13 has immediate and economically relevant implications because the valuation of the auction winner is sufficient to determine all the relevant statistics of the repeated auctions, and tenure is a signal of this valuation.

Proposition 14. *In the SPA, a higher tenure of the winner in period t implies a higher expected seller's revenue in period t but a lower probability of trade in period $t+1$. In the FPA, an increase in the tenure of the auction winner in period t has a non-monotone effect on the probability of trade in period $t+1$ if there is tailing, and eventually reduces the expected seller's revenue in period t if and only if*

$$\frac{d}{d\theta} b\left(\theta, R^F(\theta)\right)\Big|_{\theta=1} < 0.$$

Figure 6.1 shows how the winner's expected payment varies with his tenure in the two auction formats. In the SPA, the expected payment of a winner with value θ is

$$e^S(\theta) = \mathbb{E}\left[\theta' | \theta' \text{ is the highest of } R^S(\theta) \text{ and } n-1 \text{ bidders below } \theta\right].$$

This is an increasing function of θ, since $R^S(\theta)$ is (weakly) increasing in θ and so is the expected value of the second-highest bidder. Hence, the expected payment of a generic winner with tenure τ in the SPA,

$$e^S(\tau) = \int_{R_\emptyset}^1 e^S(\theta)\, g_\tau(\theta)\, \mathrm{d}\theta,$$

is increasing in τ by Lemma 13.

In the FPA, by contrast, the payment of a winner with value θ is

$$b\left(\theta, R^F(\theta), n\right),$$

which may be decreasing in θ local to $\theta = 1$, when the reserve price $R^F(\theta)$ is decreasing in θ. In this case, at $\theta \approx 1$ the winner's payment in the FPA is decreasing in his tenure too, by Lemma 13. The intuition is that, in this case, a higher tenure implies a higher winner's value but also a lower seller's reserve price, which induces all bidders to bid relatively less aggressively and reduces the seller's revenue.

Trade can be defined as

$$T(\theta) = (1-\eta) + \eta\left(1 - F^n(R(\theta))\right).$$

[57]Moreover, the survival function — i.e., the probability that a winner with tenure τ will win for one more period — is also increasing in τ and converges to $1-\eta$.

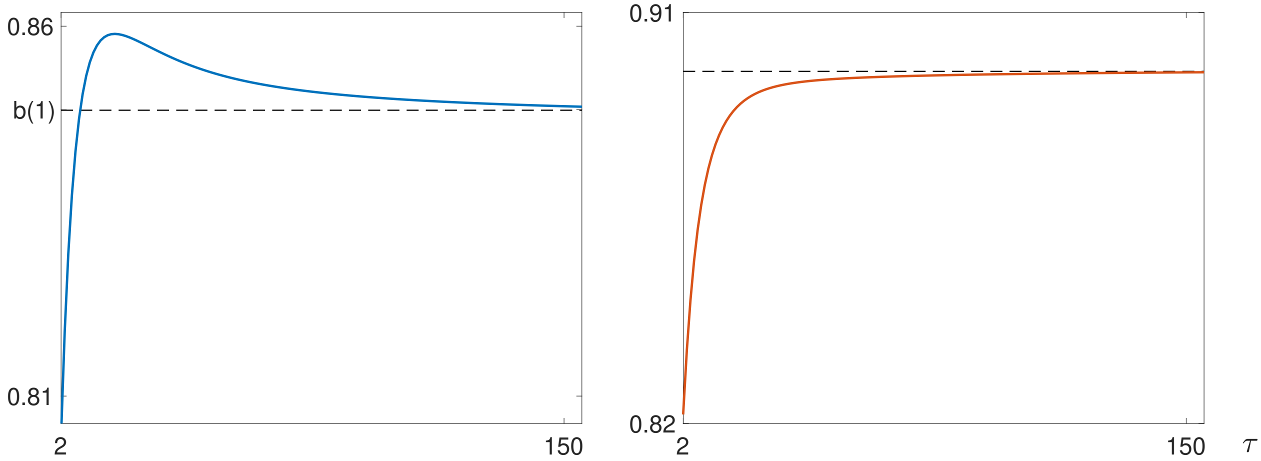

Figure 6.1: Winner's expected payment in FPA (left) and SPA (right) as a function of his tenure τ.

This is a decreasing function of the reserve price. Since in the SPA the reserve price is weakly increasing in the value of the incumbent, the probability of trade in the next period, $\int_{R_\emptyset}^{1} T(\theta)\, g_\tau(\theta)\, \mathrm{d}\theta$, is increasing in the tenure of the winner in the current period. By contrast, in the FPA trade is a non-monotone function of winner's tenure, because the reserve price may be decreasing in the value of the incumbent. Specifically, trade in the FPA is always increasing in tenure when there is tailing.

7 Conclusions

We have analyzed a stylized model of the market for online display advertising, focusing on how the seller adjusts reserve prices using the information conveyed by past bids. In our model, the seller auctions identical objects for infinitely many periods — reflecting the high frequency of unique transactions in this market — and bidders are myopic — reflecting the large number of small and unsophisticated advertisers that populate the market. After a successful auction, the seller distorts the reserve price to extract surplus from the winning bidder, who has revealed his valuation and might remain as an incumbent in future auctions. This surplus extraction is risky, however, as it requires raising the reserve price above the static optimal price, which maximizes the seller's revenue in the event the incumbent leaves the auction. If the probability of this event is high enough, the seller adopts a *tailing* strategy, leaving some surplus to incumbents with high values.

The two auction formats used to sell online display advertisements — the FPA and the SPA — differ sharply in the optimal tailing strategy. This reflects both differences in how each auction format exploits bidders' myopia and a mechanical difference in how the presence of an incumbent interacts with the reserve price in determining the seller's revenue. In the SPA, the incumbent substitutes for the reserve price, allowing the seller to set a lower reserve price than in the FPA. Moreover, the reserve price in the FPA depends negatively on the incumbent's value, implying that the winners' tenure has counter-intuitive predictions for the seller's revenue and the probability of re-trade. The optimal tailing strategies also determine the minimum reserve price (which is the one following a period without trade) and, overall, drive the differences in the dynamics, trade and revenue across the two auction formats.

Surprisingly, given that myopia induces static overbidding only in the FPA, the SPA yields higher dynamic value to the seller, unless incumbents' persistence is extremely high. This is because the SPA better exploits the possible presence of a bidder with a relatively high value — i.e. one drawn from the stationary distribution of incumbents. We also characterize the long-run probability of trade and show that incumbents' persistence affect this probability both directly (since trade can fail only if the incumbent leaves) and indirectly through the response of the seller's reserve price policy. Because of these contrasting effects, trade may be non-monotonic in incumbents' persistence.

The results we derived and the very tractability of our dynamic setting largely rely on our assumptions about the stochastic process for bidders' preferences and on myopic bidding. As we detailed in our microfoundation, we believe that both assumptions are reasonable descriptions of a high frequency market where relatively unsophisticated buyers aim to run user-specific advertisement campaigns that require showing the same ad for a fixed — but unknown to the seller — number of times. Recall that myopia is restrictive only in the first period bidders participate to the auction, since incumbents possess no residual private information and best respond to their (myopic) competitors. Therefore, bidders' behavior in our model is consistent with any learning dynamics whose starting point is the static auction — say because the algorithm used by bidders to choose their initial strategy is trained in simulated auctions where different reserve prices simply reflect different sellers' valuations or outside options.[58]

Relaxing our assumption of myopic bidding is a natural extension of our analysis. We believe there are two avenues for addressing the tractability challenges presented by this extension. The first is to focus on an empirical investigation of how automatic bidding systems learn and respond to seller's discrimination through reserve prices. The recent literature at the intersection of market design and machine learning provide tools to explore this question. The second avenue requires to simplify the model to obtain a tight characterization of the equilibrium even in a context where bidders internalize sellers' discrimination. This fits in a more standard strand of the literature related to the ratchet effect and durable goods markets. For example, Bonatti and Cisternas (2020) recently analyzed a dynamic setting with (myopic) sellers and a single sophisticated buyer who reduces the quantity demanded to hide his valuation and lower discrimination. Relatedly, allowing for competition among multiple buyers, Caillaud and Mezzetti (2004) and Pagnozzi and Sartori (2023) study the optimal reserve price in English auctions where bidders are sophisticated, but restricting to a simpler (two-period) environment than the one analyzed in this paper.

[58]Notice that backward and forward myopia in the first period originate from the same misspecified model (i.e., the static auction); if bidders do not realize that their bids will be used to price discriminate them in the future, there is no reason why they should think that the *current* reserve is used to discriminate — hence informative of — a potential competitor, and vice versa.

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Appendix

Proof of Facts 1, 2 and 3

Consider the function

$$\begin{aligned} b(\theta, R, n) &= \mathbb{E}\left[\max\{y, R\}\, | y \text{ is max of } n-1 \text{ below } \theta\right] \\ &= \frac{\int_0^R R \mathrm{d}F^{n-1}(y) + \int_R^\theta y \mathrm{d}F^{n-1}(y)}{F(\theta)^{n-1}}. \end{aligned}$$

Partial and cross differentiation yield

$$\frac{\partial}{\partial R} b(\theta, R, n) = \frac{R\mathrm{d}F^{n-1}(R) - R\mathrm{d}F^{n-1}(R) + \int_0^R \mathrm{d}F^{n-1}(y)}{F(\theta)^{n-1}} = \left(\frac{F(R)}{F(\theta)}\right)^{n-1} > 0 \tag{7.1}$$

$$\frac{\partial}{\partial \theta} b(\theta, R, n) = \frac{(n-1)F^{n-2}(\theta) f(\theta) \int_R^\theta F^{n-1}(y)\mathrm{d}y}{F(\theta)^{2(n-1)}} = \frac{(n-1) f(\theta) \int_R^\theta F^{n-1}(y)\mathrm{d}y}{F(\theta)^n} > 0 \tag{7.2}$$

$$\frac{\partial^2}{\partial R \partial \theta} b(\theta, R, n) = -(n-1)\, f(\theta) \frac{F(R)^{n-1}}{F(\theta)^n} < 0 \tag{7.3}$$

$$\frac{\partial^2}{\partial R \partial n} b(\theta, R, n) = \left(\frac{F(R)}{F(\theta)}\right)^{n-1} \log\left(\frac{F(R)}{F(\theta)}\right) < 0 \tag{7.4}$$

where the last inequality follows from $\frac{F(R)}{F(\theta)} < 1$. The fact that $b(\theta, R, n) > b(\theta, R, n-1)$ follows from the facts that $\max\{y, R\}$ is increasing in y and that the maximum of $n-1$ identical draws stochastically dominates the maximum of $n-2$ identical draws.

Finally, direct differentiation of the objective function

$$\pi_n(R) = \int_R^1 b(\theta, R, n)\, \mathrm{d}F^n(\theta)$$

gives the critical condition

$$-b(R, R, n)\, \mathrm{d}F^n(R) + \int_R^1 \frac{\partial}{\partial R} b(\theta, R, n)\, \mathrm{d}F^n(\theta) = 0.$$

Since $b(x, x, n) = x$, substituting (7.1) and integrating we obtain

$$\begin{aligned} -R\mathrm{d}F^n(R) + \int_R^1 \left(\frac{F(R)}{F(\theta)}\right)^{n-1} \mathrm{d}F^n(\theta) &= -R\mathrm{d}F^n(R) + \int_R^1 nF(R)^{n-1} f(\theta)\, \mathrm{d}\theta \\ &= -R\mathrm{d}F^n(R) + nF(R)^{n-1}(1 - F(R)) \\ &= nF(R)^{n-1}(1 - F(R) - Rf(R)) = -\mathrm{d}F^n(R)\, \psi(R), \end{aligned}$$

which is satisfied if and only if $\psi(R) = 0$. Notice moreover that we obtain the equivalent rewriting

$$\pi_n(R) = \pi_n(1) - \int_R^1 \frac{\mathrm{d}}{\mathrm{d}x} \pi_n(x)\, \mathrm{d}x = \int_R^1 \psi(x)\, \mathrm{d}F^n(x).$$

Proof of Proposition 4

In either auction format and for all θ, the function $\pi^i(\theta, R)$ has a different specification when $R \leq \theta$ and when $R > \theta$, with a downward discontinuity at $R = \theta$. Formally, [59]

$$\pi^F(\theta, R) = \eta\pi_n(R) + (1-\eta) \cdot \begin{cases} F(\theta)^{n-1} b(\theta, R, n) + \int_\theta^1 b(x, R, n)\, \mathrm{d}F(x)^{n-1} & \text{if } R \leq \theta \\ \int_R^1 b(x, R, n)\, \mathrm{d}F(x)^{n-1} & \text{if } R > \theta \end{cases} \tag{7.5}$$

$$\pi^S(\theta, R) = \eta\pi_n(R) + (1-\eta) \cdot \begin{cases} F(\theta)^{n-1} b(\theta, R, n) + \int_\theta^1 b(x, \theta, n-1)\, \mathrm{d}F(x)^{n-1} & \text{if } R \leq \theta \\ \int_R^1 b(x, R, n-1)\, \mathrm{d}F(x)^{n-1} & \text{if } R > \theta \end{cases} \tag{7.6}$$

Throughout, let $\pi^{+,i}(\theta, R)$ and $\pi^{-,i}(R)$ be the revenue specification corresponding to the upper and lower branch respectively (notice that $\pi^{-,i}$ does not depend on the incumbent's value because it presumes that the incumbent is excluded), and $\pi^{=,i}(\theta) = \pi^i(\theta, \theta)$ be the tracking revenue. Moreover, let $\Pi^{-,i} = \max_R \pi^{-,i}(R)$ and $R^{-,i}$ be the maximizer. Finally, let $R^{i,+}(\theta)$ be the maximizer of $\pi^{+,i}(\theta, R)$. Because $\pi^{+,i}(\theta, R) \geq \pi^{-,i}(R)$, whenever feasible $R^{+,i}(\theta)$ also maximizes $\pi^i(\theta, R)$, that is $R^{i,+}(\theta) \leq \theta \Rightarrow R^{i,+}(\theta) = \arg\max \pi^i(\theta, R)$. Otherwise, to solve the seller's problem we need to compare $\Pi^{-,i}$ and $\pi^{=,i}(\theta)$.

We begin by characterizing the tailing reserve price $R^{i,+}(\theta)$ and the region where it is feasible. In the SPA, $R^{S,+}(\theta)$ is obtained by setting to zero the following function:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}R}\pi^{+,S}(\theta, R) &= \frac{\mathrm{d}}{\mathrm{d}R}\left[\eta\pi_n(R) + (1-\eta)\left(F(\theta)^{n-1} b(\theta, R, n) + \int_\theta^1 b(x, \theta, n-1)\, \mathrm{d}F(x)^{n-1}\right)\right] \\ &= \eta\frac{\mathrm{d}}{\mathrm{d}R}\pi_n(R) + (1-\eta) F(\theta)^{n-1} \frac{\mathrm{d}}{\mathrm{d}R} b(\theta, R, n) \\ &= \eta\frac{\mathrm{d}}{\mathrm{d}R}\pi_n(R) + (1-\eta) F(R)^{n-1} \\ &= F(R)^{n-1}\left(\eta n (1 - F(R) - Rf(R)) + (1-\eta)\right) \\ &= F(R)^{n-1}\left((1-\eta) - \eta n f(R)\psi(R)\right). \end{aligned} \tag{7.7}$$

This yields the condition $\psi\left(R^{S,+}(\theta)\right) = \frac{1-\eta}{\eta n f(R^{S,+}(\theta))}$, which is independent of θ and feasible if and only if $\theta \geq R^{S,+}$.

In the FPA, we preliminary show that $R^{F,+}(\theta)$ is decreasing in θ by proving that the objective $\pi^{+,F}(\theta, R)$ is submodular and appealing to the Topkis theorem. We have

$$\begin{aligned} \frac{\partial^2}{\partial R \partial \theta}\pi^{+,F}(\theta, R) &= (1-\eta)\left(\frac{\partial^2 b(\theta, R)}{\partial\theta\partial R} \cdot F(\theta)^{n-1} + \frac{\partial b(\theta, R)}{\partial R} \cdot \mathrm{d}F(\theta)^{n-1} - \frac{\partial b(\theta, R)}{\partial R} \cdot \mathrm{d}F(\theta)^{n-1}\right) \\ &\propto \frac{\partial^2 b(\theta, R)}{\partial\theta\partial R}, \end{aligned}$$

which is negative per Fact 2. Analytically, $R^{F,+}(\theta)$ is obtained by setting the following derivative

[59] This writing is less then trivial, probably we need to explain why it is true (especially for the SPA) and what are the differences across auction formats.

equal to zero:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}R}\pi^{+,F}(\theta,R) &= \frac{\mathrm{d}}{\mathrm{d}R}\left[\eta\pi_n(R)+(1-\eta)\left(F(\theta)^{n-1}b(\theta,R,n)+\int_\theta^1 b(x,R,n)\,\mathrm{d}F(x)^{n-1}\right)\right]\\
&= F(R)^{n-1}\left((1-\eta)-\eta n f(R)\psi(R)\right)+(1-\eta)\int_\theta^1 \frac{\mathrm{d}}{\mathrm{d}R}b(x,R,n)\,\mathrm{d}F(x)^{n-1}\\
&= F(R)^{n-1}\left((1-\eta)-\eta n f(R)\psi(R)\right)+(1-\eta)\int_\theta^1 \left(\frac{F(R)}{F(x)}\right)^{n-1}\mathrm{d}F(x)^{n-1}\\
&= F(R)^{n-1}\left((1-\eta)-\eta n f(R)\psi(R)+(1-\eta)\int_\theta^1 \frac{\mathrm{d}F(x)^{n-1}}{F(x)^{n-1}}\right)\\
&= F(R)^{n-1}\left((1-\eta)-\eta n f(R)\psi(R)+(1-\eta)\int_\theta^1 (n-1)\,\mathrm{d}\log(F(x))\right)\\
&= F(R)^{n-1}\left((1-\eta)-\eta n f(R)\psi(R)-(1-\eta)(n-1)\log(F(\theta))\right). \qquad (7.8)
\end{aligned}
$$

Rearranging yields the expression in the statement.

The feasibility condition $R^{F,+}(\theta)\leq\theta$ is satisfied as long as $\theta\geq\overline{\theta}^F$ such that

$$
(1-\eta)-\eta n f\left(\overline{\theta}^F\right)\psi\left(\overline{\theta}^F\right)-(1-\eta)(n-1)\log\left(F\left(\overline{\theta}^F\right)\right).
$$

By concavity, whenever $R^{F,+}(\theta)$ is not feasible then the profit function conditional on no exclusion is decreasing in the relevant domain $R\geq\theta$, so the seller compares the tracking revenue $\pi^{=,i}$ with the exclusion revenue Π^{-i}.

Notice the following

Fact 15. *The tracking profit satisfy* $\pi^{=,S}(\theta)<\pi^{=,F}(\theta)$ *but* $\frac{d}{d\theta}\pi^{=,S}(\theta)>\frac{d}{d\theta}\pi^{=,F}(\theta)$.

Proof. (Of Fact 15) Direct computation gives

$$
\pi^{=,F}(\theta)-\pi^{=,S}(\theta)=\int_\theta^1 b(x,\theta,n)\,\mathrm{d}F(x)^{n-1}-\int_\theta^1 b(x,\theta,n-1)\,\mathrm{d}F(x)^{n-1}>0
$$

by Fact 1. Moreover,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\pi^{=,F}(\theta)-\frac{\mathrm{d}}{\mathrm{d}\theta}\pi^{=,S}(\theta) &= -b(\theta,\theta,n)+b(\theta,\theta,n-1)+\int_\theta^1\frac{\partial}{\partial R}b(x,\theta,n)\,\mathrm{d}F(x)^{n-1}\\
&\quad-\int_\theta^1\frac{\partial}{\partial R}b(x,\theta,n-1)\,\mathrm{d}F(x)^{n-1}\\
&=\int_\theta^1\underbrace{\frac{\partial}{\partial R}b(x,\theta,n)-\frac{\partial}{\partial R}b(x,\theta,n-1)}_{\leq 0}\mathrm{d}F(x)^{n-1}\leq 0,
\end{aligned}
$$

where the central inequality is Fact 2. ∎

We now proceed to characterize $R^{-,i}$. For the SPA, $\pi^{-,s}(R)=\eta\pi_n(R)+(1-\eta)\pi_{n-1}(R)$ is a convex combination of functions which — per Fact 3 — are both maximized at r^M. It then follows that $R^{-,s}=r^M$ and $\Pi^{-,s}=\eta\pi_n\left(r^M\right)+(1-\eta)\pi_{n-1}\left(r^M\right)$.

Now, $\underline{\theta}^i$ solves $\Pi^{-,i} = \pi^{=,i}\left(\underline{\theta}^i\right)$. For the SPA, we get

$$\begin{aligned}\eta\pi_n\left(r^M\right) + (1-\eta)\,\pi_{n-1}\left(r^M\right) &= \eta\pi_n\left(\underline{\theta}^s\right) + (1-\eta)\left[F\left(\underline{\theta}^s\right)^{n-1}\underline{\theta}^s + \int_{\theta}^{1} b\left(x,\theta,n-1\right)\mathrm{d}F\left(x\right)^{n-1}\right] \\ &= \eta\pi_n\left(\underline{\theta}^s\right) + (1-\eta)\,\pi_{n-1}\left(\underline{\theta}^s\right) + (1-\eta)\,F\left(\underline{\theta}^s\right)^{n-1}\underline{\theta}^s,\end{aligned}$$

which is rearranged to

$$\eta\left[\pi_n\left(r^M\right) - \pi_n\left(\underline{\theta}^s\right)\right] + (1-\eta)\left[\pi_{n-1}\left(r^M\right) - \pi_{n-1}\left(\underline{\theta}^s\right)\right] = (1-\eta)\,F\left(\underline{\theta}^s\right)^{n-1}\underline{\theta}^s. \tag{7.9}$$

This condition has a clear economic interpretation: since — per Fact 3 — $\pi_n\left(r^M\right) \geq \pi_n\left(x\right)$ for every n and x, the seller sets a suboptimally low reserve in exchange for the insurance to trade because, in the event that none of the $(n-1)$ standard bidders bids above the reserve price $\underline{\theta}^s$, the (surviving) tracked incumbent guarantees revenue $\underline{\theta}^s$.

Notice that the LHS of (7.9) is decreasing (as a function of $\underline{\theta}^S$) because $\pi_n\left(x\right)$ and $\pi_{n-1}\left(x\right)$ are both increasing in $\left[0, r^M\right]$, while the RHS is increasing. Moreover, the LHS is positive when $\underline{\theta}^S = 0$ and equal to 0 when $\underline{\theta}^S = r^M$, while the RHS is equal to 0 when $\underline{\theta}^S = 0$ and positive when $\underline{\theta}^S = r^M$. Hence, they cross in a unique point in $\left[0, r^M\right]$, implying that $\underline{\theta}^S < r^M$.

Similar arguments and computations also hold in the FPA and yield the optimal reserve price $R^{-,F} < r^M$ and the exclusion threshold $\underline{\theta}^F$. For the former notice that $\underline{R}^F$ solves

$$\max_{R}\eta\int_{R}^{1}\psi(x)\mathrm{d}F^n(x) + (1-\eta)\int_{R}^{1} b(x,R,n)\mathrm{d}F^{n-1}\left(x\right)$$

and the FOC gives

$$\begin{aligned}&-\eta\psi(R)\mathrm{d}F^n(R) + (1-\eta)\left[-R\mathrm{d}F^{n-1}\left(R\right) - \int_{R}^{1}\frac{\mathrm{d}b(x,R,n)}{\mathrm{d}R}\mathrm{d}F^{n-1}\left(x\right)\right] > \\ &-\eta\psi(R)\mathrm{d}F^n(R) + (1-\eta)\left[-R\mathrm{d}F^{n-1}\left(R\right) - \int_{R}^{1}\frac{\mathrm{d}b(x,R,n-1)}{\mathrm{d}R}\mathrm{d}F^{n-1}\left(x\right)\right] = \\ &-\psi(R)\left(\eta\mathrm{d}F^n(R) + (1-\eta)\mathrm{d}F^{n-1}\left(R\right)\right),\end{aligned}$$

where the inequality uses (7.4). This means that the FOC at r^M is positive, hence $\underline{R}^F < r^M$. The same manipulations as for the SPA yield that $\underline{\theta}^F$ solves

$$\eta\left[\pi_n\left(\underline{R}^F\right) - \pi_n\left(\underline{\theta}^F\right)\right] + (1-\eta)\left[\pi_{n-1}\left(\underline{R}^F\right) - \pi_{n-1}\left(\underline{\theta}^F\right)\right] = (1-\eta)\,F\left(\underline{\theta}^F\right)^{n-1}\underline{\theta}^F. \tag{7.10}$$

Using the same argument as for the SPA — i.e., LHS decreasing to 0 in $\left[0, \underline{R}^F\right]$ and RHS increasing from 0 in $\left[0, \underline{R}^F\right]$ — yields that there is a unique solution to (7.10) in $\left[0, \underline{R}^F\right]$.

Proof Proposition 5

Tracking across Auction Formats We start by showing that $\underline{\theta}^F < \underline{\theta}^S < r^M < \overline{\theta}^S < \overline{\theta}^F$. If $\overline{\theta}^S < 1$, then $\overline{\theta}^S = R^F\left(1\right) < \overline{\theta}^F$ because the tailing reserve is strictly decreasing in the FPA. The fact that $\overline{\theta}^S > r^M$ follows from $\psi\left(\overline{\theta}^S\right) = \frac{1-\eta}{\eta n f\left(\overline{\theta}^S\right)} > 0 = \psi\left(r^M\right)$. The fact that $\underline{\theta}^S < r^M$ was established in Proposition 4.

Finally, notice from equation (7.9) and (7.10) that $\underline{\theta}^i$ solves $g^i(x) = h(x)$, where $h(x) = (1-\eta) F(x)^{n-1} x$, which is increasing and $g^i(x) = \eta\left[\pi_n\left(\underline{R}^i\right) - \pi_n(x)\right] + (1-\eta)\left[\pi_{n-1}\left(\underline{R}^i\right) - \pi_{n-1}(x)\right]$ which is decreasing and such that

$$g^S(x) - g^F(x) = \eta\left[\pi_n\left(\underline{R}^F\right) - \pi_n\left(r^M\right)\right] + (1-\eta)\left[\pi_{n-1}\left(\underline{R}^F\right) - \pi_{n-1}\left(r^M\right)\right],$$

which is negative by Fact 3. The claim then follows.

Comparative Statics of Tracking Recall that in the SPA $\overline{\theta}^S$ solves

$$g(x) = \frac{1-\eta}{n\eta},$$

where we assume that $g(x) = \psi(x) f(x)$ is increasing in $[0,1]$. Then

$$\frac{\mathrm{d}\overline{\theta}^S}{\mathrm{d}\eta} \propto \frac{\mathrm{d}}{\mathrm{d}\eta}\frac{1-\eta}{n\eta} < 0$$

and

$$\frac{\mathrm{d}\overline{\theta}^S}{\mathrm{d}n} \propto \frac{\mathrm{d}}{\mathrm{d}n}\frac{1-\eta}{n\eta} < 0.$$

For the FPA, $\overline{\theta}^F$ solves

$$g(x) = \frac{(1-\eta)}{\eta n}\left[1 - (n-1)\log(F(x))\right],$$

where the RHS is positive but decreasing in x. To show the result, it is then sufficient to notice the RHS shifts down (as a function of x) because

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{(1-\eta)}{\eta n}\left[1 - (n-1)\log(F(x))\right] \propto \frac{\mathrm{d}}{\mathrm{d}\eta}\frac{(1-\eta)}{\eta n} < 0.$$

Full Tracking The threshold $\overline{\theta}^i$ solves $g(x) = z^i(x)$, where g is (strictly) increasing and z^i is weakly decreasing (constant if $i = S$). A necessary and sufficient condition for there to be a (unique) solution is that $g(1) = f(1) > z^i(1) = \frac{1-\eta}{n\eta}$, which is equivalent to

$$\eta > \overline{\eta} = \frac{1}{f(1)n + 1}.$$

The threshold $\overline{\eta}$ is non-trivial if $f(1) > 0$. The idea is that there must be positive density around 1, otherwise it is suboptimal to track at the upper bound.

Limit Results Since $\lim_{\eta\to 1}\frac{1-\eta}{n\eta} = 0$ then $g\left(\overline{\theta}^i\right) \to 0$ for both $i \in \{F, S\}$. The fact that $\overline{\theta}^i \to r^M$ then follows from continuity of g. Likewise, as $\lim_{n\to\infty}\frac{1-\eta}{n\eta} = 0$, we also get $\overline{\theta}^S \to r^M$ as $n \to \infty$.

On the contrary, $\lim_{n\to\infty} z^F(x) = -\frac{1-\eta}{\eta}\log(F(x))$ and, from continuity of both functions, we get that $\overline{\theta}^F \to \overline{\theta}^{F,\infty}$ where $\overline{\theta}^{F,\infty} > r^M$ is such that

$$\psi\left(\overline{\theta}^{F,\infty}\right) f\left(\overline{\theta}^{F,\infty}\right) = -\frac{1-\eta}{\eta}\log\left(F\left(\overline{\theta}^{F,\infty}\right)\right).$$

For all θ, $R^F(\theta) \to R^{F,\infty}(\theta)$ that solves

$$\psi\left(R^{F,\infty}(\theta)\right) f\left(R^{F,\infty}(\theta)\right) = -\frac{(1-\eta)}{\eta}\log\left(F(R^{F,\infty}(\theta))\right).$$

Proof Proposition 6

We first show that if, $\theta \le \overline{\theta}^S$, then $\Pi^S \le \Pi^F$. Since $\underline{\theta}^F \le \underline{\theta}^S \le \overline{\theta}^S$ by Proposition 5, so distinguish three cases. First, if $\theta \le \underline{\theta}^F$, then $\pi^{-,F}(R) - \pi^{-,S}(R) = \int_R^1 b(x,R,n) - b(x,R,n-1)\,\mathrm{d}F^{n-1}(x) > 0$ for all R so that $\Pi^{-,F} = \max_R \pi^{-,F}(R) > \max_R \pi^{-,S}(R) = \Pi^{-,S}$. Second, if $\theta \in \left(\underline{\theta}^F, \underline{\theta}^S\right)$, then the FPA tracks but the SPA excludes and $\Pi^F(\theta) = \pi^{=,F}(\theta) > \Pi^{-,F} > \Pi^{-,S} = \Pi^S(\theta)$. Third, if $\theta \in (\underline{\theta}^S, \overline{\theta}^S)$, then both auction formats track and — per Fact 15 — $\Pi^F(\theta) = \pi^{=,F}(\theta) > \pi^{=,S}(\theta) = \Pi^S(\theta)$.

Next we show that if there is tailing, then the SPA dominates in a neighborhood of 1. Suppose $\overline{\theta}^S = \overline{R}^F(1) < 1$, that is both auction formats tail a positive measure of incumbents. If $\theta \in [\overline{\theta}^F, 1]$, it holds $\Pi^i(\theta) = \pi^{+,i}\left(\theta, R^{+,i}(\theta)\right)$ for $i \in \{S, F\}$. By direct inspection, notice that $\pi^{+,F}(1,R) = \pi^{+,S}(1,R)$ for all R — since all differences in revenues arise when the incumbent stays but loses — and at $\theta = 1$ revenues are the same function of the reserve, giving the same maxim and and maximum in both auction formats. Using the envelope theorem, for $\theta \in (\overline{\theta}^F, 1)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\Pi^F(\theta) &= (1-\eta)\left[-b(\theta, R^{+,F}(\theta), n)\mathrm{d}F^{n-1}(\theta) + b(\theta, R^{+,F}(\theta), n)\mathrm{d}F^{n-1}(\theta) + \frac{\partial}{\partial\theta}b(\theta, R^{+,F}(\theta), n)F^{n-1}(\theta)\right] \\ &= (1-\eta)\frac{\partial}{\partial\theta}b(\theta, R^{+,F}(\theta), n)F^{n-1}(\theta),\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\Pi^S(\theta) &= (1-\eta)\left[\theta\mathrm{d}F^{n-1}(\theta) - b(\theta,\theta,n-1)\mathrm{d}F^{n-1}(\theta) + \int_\theta^1 \frac{\partial}{\partial R}b(x,\theta,n-1)\mathrm{d}F^{n-1}(x)\right] \\ &= (1-\eta)\int_\theta^1 \frac{\partial}{\partial R}b(x,\theta,n-1)\mathrm{d}F^{n-1}(x) \\ &= (1-\eta)\int_\theta^1 \left(\frac{F(\theta)}{F(x)}\right)^{n-2}\mathrm{d}F^{n-1}(x) \\ &= (1-\eta)(n-1)F(\theta)^{n-2}(1-F(\theta)).\end{aligned}$$

It follows that $\lim_{\theta\to1}\frac{\mathrm{d}}{\mathrm{d}\theta}\Pi^S(\theta) = 0 < \lim_{\theta\to1}\frac{\mathrm{d}}{\mathrm{d}\theta}\Pi^F(\theta)$. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Pi^F(\theta) > \frac{\mathrm{d}}{\mathrm{d}\theta}\Pi^S(\theta) \quad\Leftrightarrow\quad \frac{\partial}{\partial\theta}b(\theta, R^{+,F}(\theta), n)F(\theta) > (n-1)(1-F(\theta))$$

$$\Leftrightarrow \quad f(\theta)\int_{R^{+,F}(\theta)}^{\theta} F^{n-1}(y)\mathrm{d}y > (1-F(\theta))\,F(\theta)^{n-1}$$

$$\Leftrightarrow \quad \int_{R^{+,F}(\theta)}^{\theta}\left(\frac{F(y)}{F(\theta)}\right)^{n-1}\mathrm{d}y > \frac{(1-F(\theta))}{f(\theta)}, \tag{7.11}$$

which is satisfied (at least) in a left neighborhood of 1 (because when $\theta = 1$, $R^{+,F}(1) < 1$ and hence the LHS is positive, and the RHS is equal to 0).

For all points θ in such neighborhood, it then holds

$$\begin{aligned}\Pi^{S}(\theta) - \Pi^{F}(\theta) &= \Pi^{S}(1) - \int_{\theta}^{1}\frac{\mathrm{d}}{\mathrm{d}\theta'}\Pi^{S}(\theta') - (\Pi^{F}(1) - \frac{\mathrm{d}}{\mathrm{d}\theta'}\Pi^{F}(\theta')\,\mathrm{d}\theta')\\ &= \int_{\theta}^{1}\frac{\mathrm{d}}{\mathrm{d}\theta'}\Pi^{F}(\theta') - \frac{\mathrm{d}}{\mathrm{d}\theta'}\Pi^{S}(\theta')\,\mathrm{d}\theta' > 0,\end{aligned}$$

where the second equality uses the fact that, by the Revenue Equivalence Theorem, $\Pi^{S}(1) = \Pi^{F}(1)$.

Proof of Theorem 7

We can write the seller's problem in recursive form where the state is the winner of the previous period winner — or incumbent. Accounting for the possibility that there is no such winner (i.e., that no bidder bid above the reserve price in the previous period auction), the state space is $\emptyset \cup [0,1]$. The initial value is

$$V_{\emptyset}^{i} = \max_{R}\pi_{n}(R) + \beta\left[F(R)^{n}V_{\emptyset}^{i} + \int_{R}^{1}V^{i}(\theta')\,\mathrm{d}F(\theta')^{n}\right].$$

The value of starting the dynamic auctions with incumbent θ is

$$V^{i}(\theta) = \max\left\{V^{-,i}, V^{+,i}(\theta)\right\},$$

where

$$\begin{aligned}V^{-,i} = \max_{R}\pi^{-,S}(R) + \beta\Big\{\eta\Big(F(R)^{n}V_{\emptyset}^{i} + \int_{R}^{1}V^{i}(y)\,\mathrm{d}F(y)^{n}\Big) +\\ (1-\eta)\Big(F(R)^{n-1}V_{\emptyset}^{i} + \int_{R}^{1}V^{i}(y)\,\mathrm{d}F(y)^{n-1}\Big)\Big\}\end{aligned} \tag{7.12}$$

is the value conditional on exclusion[60] and

$$\begin{aligned}V^{+,i}(\theta) = \max_{R\le\theta}\pi^{+,i}(\theta,R) + \beta\Big\{\eta\Big(F(R)^{n}V_{\emptyset}^{i} + \int_{R}^{1}V^{i}(y)\,\mathrm{d}F(y)^{n}\Big) +\\ (1-\eta)\Big(F(\theta)^{n-1}V^{i}(\theta) + \int_{\theta}^{1}V^{i}(y)\,\mathrm{d}F(y)^{n-1}\Big)\Big\}.\end{aligned} \tag{7.13}$$

Lemma 16. *$V^{i}(\theta)$ is increasing in θ.*

[60] Notice it is immaterial to restrict to cases in which the reserve price actually excludes ($R < \theta$) because in that case $V^{-,i} < V^{+,i}(\theta)$.

Proof. If $\theta > \theta'$, then $\pi^i(\theta, R) > \pi^i(\theta', R)$ and $F(\theta_{t+1}|\theta, R) \trianglerighteq^{\text{FOSD}} F(\theta_{t+1}|\theta', R)$ for all R.[61] The result then follows from Theorem 9.7 in SLP. □

The rest of the proof is organized as follows. First, Lemmas 17 and 18 derive conditions (4.3), (4.4), (4.5) by direct differentiation of the value function.[62] Second, Lemmas 19 and 20 establish that, on path, the seller does not optimally exclude any incumbent and, hence, that $V^i(\theta) = V^{+,i}(\theta)$. (Recall that in the exposition we have implicitly assumed that this is the case — see (4.1).)

Lemma 17. *The initial reserve price $R^i_\emptyset$ solves $\psi\left(R^i_\emptyset\right) = -\beta\left(V^i\left(R^i_\emptyset\right) - V^i_\emptyset\right)$.*

Proof. Recall that $R^i_\emptyset$ solves

$$\max_R \pi_n(R) + \beta\left[F(R)^n V^i_\emptyset + \int_R^1 V^i(\theta')\,\mathrm{d}F(\theta')^n\right].$$

The FOC of this problem is

$$-\mathrm{d}F(R)^n\,\psi(R) + \beta\mathrm{d}F(R)^n\left(V^i_\emptyset - V^i(R)\right) = 0,$$

which is rearranged to the desired expression. □

Lemma 18. *If $R^i(\theta) < \theta$, then $R^i(\theta)$ satisfies conditions (4.4)-(4.5) in the text. $R^S(\theta)$ is flat and $R^F(\theta)$ is decreasing in θ.*

Proof. Differentiating the relevant branch of the value (7.13), we get that $R^i(\theta)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}R}\pi^{+,i}(\theta, R) + \beta\eta\mathrm{d}F^n(R)\left(V^i_\emptyset - V^i(R)\right) = 0. \tag{7.14}$$

Substituting the static marginal revenues (7.7)-(7.8) and rearranging yields conditions (4.4)-(4.5). Notice the dynamic wedge $\beta\eta\mathrm{d}F^n(R)\left(V^i_\emptyset - V^i(R)\right)$ is independent of θ. Since $\frac{\mathrm{d}}{\mathrm{d}R}\pi^{+,S}(\theta, R)$ is also independent of θ, the R that solves (7.14) is also independent of θ. Since

$$\frac{\partial^2}{\partial R\partial\theta}\eta\left(F(R)^n V^i_\emptyset + \int_R^1 V^i(y)\,\mathrm{d}F(y)^n\right) + (1-\eta)\left(F(\theta)^{n-1}V^i(\theta) + \int_\theta^1 V^i(y)\,\mathrm{d}F(y)^{n-1}\right) = 0,$$

submodularity of $\pi^{+,F}$ implies submodularity of the dynamic objective. This implies that, as in the static case, $R^F(\theta)$ is also decreasing in the tailing region. □

Lemma 19. *Let R^i_{Ex} be the optimal reserve price conditional on excluding an incumbent. It holds that $R^S_{Ex} = R^S_\emptyset$ and $R^F_{Ex} < R^F_\emptyset$*

[61] If the incumbent leaves, the dynamics depends only on the reserve; if he stays then, let y be the max of $n-1$ standard bidders. If $y < \theta'$ then $\theta_{t+1}|R,\theta = \max\{R,\theta\} > \max\{R,\theta'\} = \theta_{t+1}|R,\theta'$. If $\theta' < y < \theta$ then $\theta_{t+1}|R,\theta = \max\{R,\theta\} > \max\{R,y\} = \theta_{t+1}|R,\theta'$. If $y > \theta$ then $\theta_{t+1}|R,\theta = \max\{R,y\} = \max\{R,y\} = \theta_{t+1}|R,\theta'$.

[62] All the expressions for the static marginal are from the proof of Proposition 4.

Proof. Recall that R^i_{Ex} solves

$$\max_R \pi^{-,i}(R) + \beta \left\{ \eta \left(F(R)^n V^i_\emptyset + \int_R^1 V^i(y)\, \mathrm{d}F(y)^n \right) + \right.$$
$$\left. (1-\eta) \left(F(R)^{n-1} V^i_\emptyset + \int_R^1 V^i(y)\, \mathrm{d}F(y)^{n-1} \right) \right\}.$$

For the SPA, the FOC of this problem is

$$\frac{\mathrm{d}}{\mathrm{d}R}\pi^{-,S} + \beta \left(V^S_\emptyset - V^S(R) \right) \left[\mathrm{d}F(R)^n \eta + (1-\eta)\, \mathrm{d}F(R)^{n-1} \right] = 0 \tag{7.15}$$

$$-\psi(R) \left[\eta \mathrm{d}F(R)^n + (1-\eta)\, \mathrm{d}F(R)^{n-1} \right] + \beta \left(V^S_\emptyset - V^S(R) \right) \left[\mathrm{d}F(R)^n \eta + (1-\eta)\, \mathrm{d}F(R)^{n-1} \right] = 0.$$

Since the term $\eta \mathrm{d}F(R)^n + (1-\eta)\, \mathrm{d}F(R)^{n-1}$ factors out, we obtain the same condition as in Lemma 17. This implies that $R^S_{Ex} = R^S_\emptyset$.

For the FPA, the FOC is

$$\frac{\mathrm{d}}{\mathrm{d}R}\pi^{-,S}(R) + \zeta(R) + \beta \left(V^F_\emptyset - V^F(R) \right) \left[\mathrm{d}F(R)^n \eta + (1-\eta)\, \mathrm{d}F(R)^{n-1} \right] = 0, \tag{7.16}$$

where

$$\zeta(R) := \frac{\mathrm{d}}{\mathrm{d}R}\pi^{-,F} - \frac{\mathrm{d}}{\mathrm{d}R}\pi^{-,S} = (1-\eta) \frac{\mathrm{d}}{\mathrm{d}R} \left[\int_R^1 b(x,R,n)\, \mathrm{d}F(x)^{n-1} - \int_R^1 b(x,R,n-1)\, \mathrm{d}F(x)^{n-1} \right]$$
$$= (1-\eta) \int_R^1 \left[\underbrace{\frac{\mathrm{d}}{\mathrm{d}R} b(x,R,n) - \frac{\mathrm{d}}{\mathrm{d}R} b(x,R,n-1)}_{<0 \text{ by } (7.4)} \right] \mathrm{d}F(x)^{n-1} < 0.$$

By Lemma 17, $R^F_\emptyset$ solves $\frac{\mathrm{d}}{\mathrm{d}R}\pi^{-,S}(R) + \beta \left(V^F_\emptyset - V^F(R) \right) \left[\mathrm{d}F(R)^n \eta + (1-\eta)\, \mathrm{d}F(R)^{n-1} \right] = 0$. Combining this with $\zeta < 0$, we obtain that the FOC for exclusion evaluated at $R^F_\emptyset$ is negative, which means that the optimum R^F_{Ex} must be lower than $R^F_\emptyset$. □

Lemma 20. *If R is a tailing reserve price, then (i) $R^i > R^i_\emptyset$ and (ii) $V^i(R^i) > V^i_\emptyset$.*

Proof. For the SPA, we have a unique tailing reserve $\overline{R}^S$ that solves

$$\frac{\mathrm{d}}{\mathrm{d}R}\pi^{+,S}(\theta, R) + \beta \eta \mathrm{d}F^n(R) \left(V^i_\emptyset - V^i(R) \right) = 0.$$

Substituting (7.7) and rearranging yields

$$F(R)^{n-1} \left((1-\eta) - \eta n f(R) \psi(R) \right) + \beta \eta \mathrm{d}F^n(R) \left(V^i_\emptyset - V^i(R) \right) = 0$$

$$F(R)^{n-1} (1-\eta) + \eta \mathrm{d}F^n(R) \left[\beta \left(V^i_\emptyset - V^i(R) \right) - \psi(R) \right] = 0.$$

By Lemma 17, the second addendum is zero at $R^S_\emptyset$ which implies that $\overline{R}^S$ must be higher than $R^S_\emptyset$.

For the FPA, since $R^F(\theta)$ is decreasing (Lemma 18), it is sufficient to show that $R^F(1) > R^F_\emptyset$.

By (4.5) and (7.8), $R^F(1)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}R}\pi^{+,F}(\theta,R)+\beta\eta\mathrm{d}F^n(R)\left(V^i_\emptyset-V^i(R)\right)=0.$$

Since $\pi^{+,F}(1,R)=\pi^{+,S}(1,R)$, the argument developed above for the SPA also establishes that $R^F(1)>R^F_\emptyset$.

Finally it can also be shown that $V(R)>V_\emptyset$. □

This concludes the proof of Theorem 7.

Proof of Proposition 8

We first show that, local to $\beta=0$ or $\eta=1$, $R^i_\emptyset>r^M\Leftrightarrow\pi^i_{n-1,r^M}\left(r^M\right)<\pi_n\left(r^M\right)\Leftrightarrow n$ is large. Since $R^i_\emptyset=r^M$ when $\beta=0$ or $\eta=1$ (by Lemma 17), this is equivalent to proving that the derivatives of $R^i_\emptyset$ w.r.t. β local to $\beta=0$ and w.r.t. η local to $\eta=1$, evaluated at r^M, are positive when condition (4.6) is met.

Differentiating w.r.t. β the characterizing equation — Lemma 17 — for $R^i_\emptyset$ we get

$$\left[\psi'\left(R^i_\emptyset\right)+\beta\frac{\mathrm{d}}{\mathrm{d}\theta}V^i\left(R^i_\emptyset\right)\right]\frac{\mathrm{d}}{\mathrm{d}\beta}R^i_\emptyset=-\left(V^i\left(R^i_\emptyset\right)-V^i_\emptyset\right)-\beta\frac{\partial}{\partial\beta}\left(V^i\left(R^i_\emptyset\right)-V^i_\emptyset\right).$$

Lemma 16 and $\psi'>0$ yield that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}R^i_\emptyset\propto-\left(V^i\left(R^i_\emptyset\right)-V^i_\emptyset\right)-\beta\frac{\partial}{\partial\beta}\left(V^i\left(R^i_\emptyset\right)-V^i_\emptyset\right).$$

Recall that at $\beta=0$, $R^i_\emptyset=r^M$ and the seller's values are equal to the static profits. Therefore,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\beta}R^i_\emptyset\right|_{\beta=0}\propto\pi^i_{n-1,r^M}\left(r^M\right)-\pi_n\left(r^M\right),\tag{7.17}$$

which implies that $\left.\frac{\mathrm{d}}{\mathrm{d}\beta}R^i_\emptyset\right|_{\beta=0}>0$ if and only if $\pi^i_{n-1,r^M}\left(r^M\right)-\pi_n\left(r^M\right)<0$.[63]

Differentiating the expression in Lemma 17 w.r.t. η instead yields

$$\left[\psi'\left(R^i_\emptyset\right)+\beta\frac{\mathrm{d}}{\mathrm{d}\theta}V^i\left(R^i_\emptyset\right)\right]\frac{\mathrm{d}}{\mathrm{d}\eta}R^i_\emptyset=-\beta\frac{\partial}{\partial\eta}\left(V^i\left(R^i_\emptyset\right)-V^i_\emptyset\right).$$

By the same argument used above for the derivative w.r.t. β,

$$\frac{\mathrm{d}}{\mathrm{d}\eta}R^i_\emptyset\propto-\frac{\partial}{\partial\eta}\left(V^i\left(R^i_\emptyset\right)-V^i_\emptyset\right).\tag{7.18}$$

[63] Still need to justify the claim (in the text) that this condition is met for n small enough, namely that $\pi^i_{n-1,r^M}\left(r^M\right)-\pi_n\left(r^M\right)$ is decreasing in n.

By the envelope theorem,

$$\begin{aligned}\frac{\partial}{\partial\eta}V^i(R^i_\emptyset) &= \pi_n\left(R^i_\emptyset\right) - \pi^i_{n-1,R^i_\emptyset}\left(R^i_\emptyset\right) + \beta\left(F\left(R^i_\emptyset\right)^n V^i_\emptyset + \int_{R^i_\emptyset}^1 V^i\left(\theta'\right)\mathrm{d}F\left(\theta'\right)^n\right)\\ &\quad - \beta\left(F\left(R^i_\emptyset\right)^{n-1} V^i\left(R^i_\emptyset\right) + \int_{R^i_\emptyset}^1 V^i\left(\theta'\right)\mathrm{d}F\left(\theta'\right)^{n-1}\right)\\ &\quad + \beta\eta\left(F\left(R^i_\emptyset\right)^n \frac{\partial}{\partial\eta}V^i_\emptyset + \int_{R^i_\emptyset}^1 \frac{\partial}{\partial\eta}V^i\left(y\right)\mathrm{d}F\left(y\right)^n\right)\\ &\quad + \beta\left(1-\eta\right)\left(F\left(R^i_\emptyset\right)^{n-1} \frac{\partial}{\partial\eta}V^i_\emptyset + \int_{R^i_\emptyset}^1 \frac{\partial}{\partial\eta}V^i\left(y\right)\mathrm{d}F\left(y\right)^{n-1}\right). \qquad (7.19)\end{aligned}$$

At $\eta = 1$, $V^i_\emptyset = V^i(\theta) = \frac{\pi_n\left(r^M\right)}{1-\beta}$ for all θ and $R^i_\emptyset = r^M$. So the previous expression simplifies to

$$\begin{aligned}\frac{\partial}{\partial\eta}V^i(R^i_\emptyset) &= \frac{\partial}{\partial\eta}\pi^{i,=}\left(R^i_\emptyset\right) + \beta\frac{\pi_n\left(r^M\right)}{1-\beta}\left(F\left(R^i_\emptyset\right)^n + \int_{R^i_\emptyset}^1 \mathrm{d}F\left(\theta'\right)^n - F\left(R^i_\emptyset\right)^{n-1} - \int_{R^i_\emptyset}^1 \mathrm{d}F\left(\theta'\right)^{n-1}\right)\\ &\quad + \beta\left(F\left(R^i_\emptyset\right)^n \frac{\partial}{\partial\eta}V^i_\emptyset + \int_{R^i_\emptyset}^1 \frac{\partial}{\partial\eta}V^i\left(y\right)\mathrm{d}F\left(y\right)^n\right).\end{aligned}$$

Moreover,

$$\frac{\partial}{\partial\eta}V^i_\emptyset = \beta\left(F\left(R^i_\emptyset\right)^n \frac{\partial}{\partial\eta}V^i_\emptyset + \int_{R^i_\emptyset}^1 \frac{\partial}{\partial\eta}V^i\left(y\right)\mathrm{d}F\left(y\right)^n\right). \qquad (7.20)$$

Therefore, at $\eta = 1$,

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\eta}R^i_\emptyset\right|_{\eta=1} &\propto -\left[\pi_n\left(r^M\right) - \pi^i_{n-1,r^M}\left(r^M\right)\right]\\ &\quad + \beta\frac{\pi_n\left(r^M\right)}{1-\beta}\left(F\left(r^M\right)^{n-1} + 1 - F\left(r^M\right)^{n-1} - F\left(r^M\right)^n - 1 + F\left(r^M\right)^n\right)\\ &\propto \pi^i_{n-1,r^M}\left(r^M\right) - \pi_n\left(r^M\right), \qquad (7.21)\end{aligned}$$

which establishes the claim.

In addition, at $\eta = 0$, $\frac{\mathrm{d}}{\mathrm{d}\eta}R^i_\emptyset \propto \pi^i_{n-1,R^i_\emptyset}\left(R^i_\emptyset\right) - \pi_n\left(R^i_\emptyset\right)$, indeed from (7.18) $\frac{\mathrm{d}}{\mathrm{d}\eta}R^i_\emptyset \propto -\frac{\partial}{\partial\eta}\left(V^i\left(R^i_\emptyset\right) - V^i_\emptyset\right)$.

At $\eta = 0$,

$$\begin{aligned}\frac{\partial}{\partial\eta}\left(V^i\left(R^i_\emptyset\right) - V^i_\emptyset\right)\Big|_{\eta=0} &\propto \pi_n\left(R^i_\emptyset\right) - \pi^i_{n-1,R^i_\emptyset}\left(R^i_\emptyset\right) + \beta\left[\left(F\left(R^i_\emptyset\right)^n V^i_\emptyset + \int_{R^i_\emptyset}^1 V^i\left(\theta'\right)\mathrm{d}F\left(\theta'\right)^n\right)\right.\\ &\quad\left. - \left(F\left(R^i_\emptyset\right)^{n-1} V^i\left(R^i_\emptyset\right) + \int_{R^i_\emptyset}^1 V^i\left(\theta'\right)\mathrm{d}F\left(\theta'\right)^{n-1}\right)\right]\\ &= \pi_n\left(R^i_\emptyset\right) - \pi^i_{n-1,R^i_\emptyset}\left(R^i_\emptyset\right) + \beta\left[\left(F\left(R^i_\emptyset\right)^n V^i + V^i\int_{R^i_\emptyset}^1 \mathrm{d}F\left(\theta'\right)^n\right)\right.\\ &\quad\left. - \left(F\left(R^i_\emptyset\right)^{n-1} V^i + V^i\int_{R^i_\emptyset}^1 \mathrm{d}F\left(\theta'\right)^{n-1}\right)\right]\\ &= \pi_n\left(R^i_\emptyset\right) - \pi^i_{n-1,R^i_\emptyset}\left(R^i_\emptyset\right), \qquad (7.22)\end{aligned}$$

where the second equality uses that at $\eta = 0$ $V^i_\emptyset = V^i(\theta) = V^i$.

It can also be shown that, at $\eta = 0$, $R^i_\emptyset > r^M$ if and only if

$$\pi^i_{n-1,r^M}\left(r^M\right) - \pi_n\left(r^M\right) < \beta\int_{r^M}^1 \frac{\mathrm{d}}{\mathrm{d}\theta'}\pi^{i,=}\left(\theta'\right)\frac{F\left(\theta'\right)^{n-1}\left(1 - F\left(\theta'\right)\right)}{\left(1 - \beta F\left(\theta'\right)^{n-1}\right)}\mathrm{d}\theta'.$$

Proof of Proposition 9

Expressions 7.17 and 7.21 also imply that the distance between $R^i_\emptyset$ and r^M is proportional to $\pi^i_{n-1,r^M}\left(r^M\right) - \pi_n\left(r^M\right)$ at $\eta = 1$. By Fact 15, this difference is larger in the FPA than in the SPA. Therefore, $R^F_\emptyset < R^S_\emptyset$ local to $\eta = 1$ (because, at $\eta = 1$, the two reserve prices are equal but the one in the FPA is more sensitive to change in η than the one in the SPA).

A similar argument also hold at $\eta = 0$, using that, by Fact 15, the marginal tracking revenue $\frac{\mathrm{d}}{\mathrm{d}\theta'}\pi^{i,=}(\theta)$ is lower in the FPA.

Proof of Proposition 10

We prove the results in the limit cases $\eta = 0$ and $\eta = 1$. Since the value functions are continuous in η in our environment (see Theorem 4.6 in Stokey, Lucas and Prescott, 1989), the results also hold in a neighborhood of these points.

We first prove that whenever there is full tracking in both auction formats, i.e. $\overline{\theta}^S = \overline{\theta}^F = 1$, then $V^F_\emptyset > V^S_\emptyset$. Fix η that satisfies this property and let $\hat{R}^S_\eta$ be the SPA reserve price *policy* associated to η, which prescribes a minimum reserve price and full tracking afterwards and that achieves the value $v^S\left(\hat{R}^S_\eta\right) = V^S_\emptyset$. If the seller replicated $\hat{R}^S_\eta$ in the FPA, he would *i*) obtain revenue equivalence in state $\emptyset$ and *ii*) induce a pointwise identical dynamic for the incumbent's value. For each realization of the incumbent, Fact 15 implies that the FPA has (strictly) higher expected static revenue. Therefore, if the seller follows strategy $\hat{R}^S_\eta$ in the FPA, she obtains dynamic utility $v^F\left(\hat{R}^S_\eta\right)$ that is higher than the dynamic utility of the SPA. Of course, the value function of the FPA is higher than the one of the SPA when the seller chooses the optimal reserve prices in the FPA — i.e., $V^F_\emptyset \geq v^F\left(\hat{R}^S_\eta\right) > v^S\left(\hat{R}^S_\eta\right) = V^S_\emptyset$.

At $\eta = 1$ the revenue is the same in both auction formats. We show that revenue in the SPA is higher than in the FPA at $\eta \approx 1$ by showing that $\frac{\partial}{\partial \eta}\left(V_\emptyset^S - V_\emptyset^F\right)|_{\eta=1} < 0$ and, hence, that slightly reducing η reduces revenue in both auction formats, but less so in the SPA. The proof proceeds as follows: we first show that the dynamic comparison between FPA and SPA can be reduced to a comparison of the static revenues in the two auction formats with the optimal reserve price and one "special bidder"; then we show that this static comparison favors the SPA.

Recall that $V_\emptyset^i = V^i(\theta) = \frac{\pi_n\left(r^M\right)}{1-\beta}$ and $R_\emptyset^i = R^i(\theta) = r^M$ for every θ and i. Using these facts and (7.20) we get

$$\frac{\partial}{\partial \eta}\left(V_\emptyset^S - V_\emptyset^F\right) = \beta\left(F\left(r^M\right)^n \frac{\partial}{\partial \eta}\left(V_\emptyset^S - V_\emptyset^F\right) + \int_{r^M}^1 \frac{\partial}{\partial \eta}\left(V^S(y) - V^F(y)\right) \mathrm{d}F(y)^n\right)$$

$$\frac{\partial}{\partial \eta}\left(V_\emptyset^S - V_\emptyset^F\right)\left(1 - \beta F\left(r^M\right)^n\right) = \beta \int_{r^M}^1 \frac{\partial}{\partial \eta}\left(V^S(y) - V^F(y)\right) \mathrm{d}F(y)^n. \tag{7.23}$$

By the envelope theorem (which holds because all incumbents are tailed), for $i = F, S$,

$$\frac{\partial}{\partial \eta} V^i(\theta)\Big|_{\eta=1} = \frac{\partial}{\partial \eta}\pi^{+,i}\left(\theta, r^M\right) + \beta \frac{\pi_n\left(r^M\right)}{1-\beta}\left(\underbrace{F\left(r^M\right)^n + \int_{r^M}^1 \mathrm{d}F\left(\theta'\right)^n - F\left(r^M\right)^{n-1} - \int_{r^M}^1 \mathrm{d}F\left(\theta'\right)^{n-1}}_{=0 \text{ for both } i \in \{S,F\}}\right)$$
$$+ \beta\left(F\left(r^M\right)^n \frac{\partial}{\partial \eta} V_\emptyset^i + \int_{r^M}^1 \frac{\partial}{\partial \eta} V^i(y) \mathrm{d}F(y)^n\right). \tag{7.24}$$

Therefore,

$$\frac{\partial}{\partial \eta}\left(V^S(y) - V^F(y)\right) = \frac{\partial}{\partial \eta}\pi^{+,S}\left(y, r^M\right) - \frac{\partial}{\partial \eta}\pi^{+,F}\left(y, r^M\right)$$
$$+ \beta F\left(r^M\right)^n \frac{\partial}{\partial \eta}\left(V_\emptyset^S - V_\emptyset^F\right) + \beta \int_{r^M}^1 \frac{\partial}{\partial \eta}\left(V^S(x) - V^F(x)\right) \mathrm{d}F(x)^n.$$

Integrating,

$$\int_{r^M}^1 \frac{\partial}{\partial \eta}\left(V^S(y) - V^F(y)\right) \mathrm{d}F(y)^n = \int_{r^M}^1 \left[\frac{\partial}{\partial \eta}\pi^{+,S}\left(y, r^M\right) - \frac{\partial}{\partial \eta}\pi^{+,F}\left(y, r^M\right)\right] \mathrm{d}F(y)^n$$
$$+ \beta\left(1 - F\left(r^M\right)^n\right)\left[F\left(r^M\right)^n \frac{\partial}{\partial \eta}\left(V_\emptyset^S - V_\emptyset^F\right)\right]$$
$$+ \beta\left(1 - F\left(r^M\right)^n\right)\left[\int_{r^M}^1 \frac{\partial}{\partial \eta}\left(V^S(x) - V^F(x)\right) \mathrm{d}F(x)^n\right]$$

$$\Leftrightarrow \quad \int \frac{\partial}{\partial \eta}\left(V^S(y) - V^F(y)\right) \mathrm{d}F(y)^n =$$
$$= \frac{\int_{r^M}^1 \left[\frac{\partial}{\partial \eta}\pi^{+,S}\left(y, r^M\right) - \frac{\partial}{\partial \eta}\pi^{+,F}\left(y, r^M\right)\right] \mathrm{d}F(y)^n + \beta\left(1 - F\left(r^M\right)^n\right) F\left(r^M\right)^n \frac{\partial}{\partial \eta}\left(V_\emptyset^S - V_\emptyset^F\right)}{\left(1 - \beta\left(1 - F\left(r^M\right)^n\right)\right)}.$$

Substituting in (7.23) yields

$$\frac{\partial}{\partial\eta}\left(V_{\emptyset}^{S}-V_{\emptyset}^{F}\right)\left(1-\beta F\left(r^{M}\right)^{n}\right)=$$

$$=\frac{\int_{r^M}^{1}\left[\frac{\partial}{\partial\eta}\pi^{+,S}\left(y,r^{M}\right)-\frac{\partial}{\partial\eta}\pi^{+,F}\left(y,r^{M}\right)\right]\mathrm{d}F\left(y\right)^{n}+\beta\left(1-F\left(r^{M}\right)^{n}\right)F\left(r^{M}\right)^{n}\frac{\partial}{\partial\eta}\left(V_{\emptyset}^{S}-V_{\emptyset}^{F}\right)}{\left(1-\beta\left(1-F\left(r^{M}\right)^{n}\right)\right)}$$

$$\Leftrightarrow\quad\frac{\partial}{\partial\eta}\left(V_{\emptyset}^{S}-V_{\emptyset}^{F}\right)\left(1-\beta F\left(r^{M}\right)^{n}-\frac{\beta\left(1-F\left(r^{M}\right)^{n}\right)F\left(r^{M}\right)^{n}}{\left(1-\beta\left(1-F\left(r^{M}\right)^{n}\right)\right)}\right)=$$

$$=\frac{\int_{r^M}^{1}\left[\frac{\partial}{\partial\eta}\pi^{+,S}\left(y,r^{M}\right)-\frac{\partial}{\partial\eta}\pi^{+,F}\left(y,r^{M}\right)\right]\mathrm{d}F\left(y\right)^{n}}{\left(1-\beta\left(1-F\left(r^{M}\right)^{n}\right)\right)}.$$

Hence,[64]

$$\frac{\partial}{\partial\eta}\left(V_{\emptyset}^{S}-V_{\emptyset}^{F}\right)\propto\int_{r^M}^{1}\left[\frac{\partial}{\partial\eta}\pi^{+,S}\left(y,r^{M}\right)-\frac{\partial}{\partial\eta}\pi^{+,F}\left(y,r^{M}\right)\right]\mathrm{d}F\left(y\right)^{n}.$$

Notice that

$$\frac{\partial}{\partial\eta}\pi^{+,i}\left(y,r^{M}\right)=\pi_{n}\left(r^{M}\right)-\pi_{n-1,y}^{i}\left(r^{M}\right)$$

so that

$$\begin{aligned}\frac{\partial}{\partial\eta}\pi^{+,S}\left(y,r^{M}\right)-\frac{\partial}{\partial\eta}\pi^{+,F}\left(y,r^{M}\right)&=\pi_{n-1,y}^{F}\left(r^{M}\right)-\pi_{n-1,y}^{S}\left(r^{M}\right)\\&=\int_{y}^{1}\left(b\left(x,r^{M},n\right)-b\left(x,y,n-1\right)\right)\mathrm{d}F\left(x\right)^{n-1}.\end{aligned}$$

Therefore, $\frac{\partial}{\partial\eta}\left(V_{\emptyset}^{S}-V_{\emptyset}^{F}\right)<0$ holds if and only if

$$\int_{r^M}^{1}\left[\int_{y}^{1}\left(b\left(x,r^{M},n\right)-b\left(x,y,n-1\right)\right)\mathrm{d}F\left(x\right)^{n-1}\right]\mathrm{d}F\left(y\right)^{n}<0. \tag{7.25}$$

Claim. For all r,n

$$\int_{r}^{1}\left[\int_{y}^{1}\left(b\left(x,r,n\right)-b\left(x,y,n-1\right)\right)\mathrm{d}F\left(x\right)^{n-1}\right]\mathrm{d}F\left(y\right)<0.$$

Proof. By revenue equivalence, for all r,n it holds $\mathbb{E}\left[\pi_{n-1,y}^{F}\left(r\right)\right]=\mathbb{E}\left[\pi_{n-1,y}^{S}\left(r\right)\right]$ where the expectation is taken under the unconditional distribution F of the bidder y (see Carannante *et al.*, in

[64]We use that

$$\left(1-\beta F\left(r^{M}\right)^{n}-\frac{\beta\left(1-F\left(r^{M}\right)^{n}\right)F\left(r^{M}\right)^{n}}{\left(1-\beta\left(1-F\left(r^{M}\right)^{n}\right)\right)}\right)=\frac{\left(1-\beta\right)\overbrace{\left(1-\beta\left(1-F\left(r^{M}\right)^{n}\right)F\left(r^{M}\right)^{n}\right)}^{>0}}{\underbrace{1-\beta\left(1-F\left(r^{M}\right)^{n}\right)}_{>0}}>0.$$

.

preparation). Writing the expectation explicitly, using expressions (7.5)-(7.6), we obtain

$$\begin{aligned}
0 &= \int_0^1 \pi_{n-1,y}^F(r) - \pi_{n-1,y}^S(r)\,\mathrm{d}F(y) \\
&= \int_0^r \pi_{n-1,y}^F(r) - \pi_{n-1,y}^S(r)\,\mathrm{d}F(y) + \int_r^1 \pi_{n-1,y}^F(r) - \pi_{n-1,y}^S(r)\,\mathrm{d}F(y) \\
&= F(r)\left[\int_r^1 \left[b(x,r,n) - b(x,r,n-1)\right]\mathrm{d}F(x)^{n-1}\right] \\
&\quad + \int_r^1 \left(\int_y^1 \left[b(x,r,n) - b(x,y,n-1)\right]\mathrm{d}F(x)^{n-1}\right)\mathrm{d}F(y),
\end{aligned}$$

and therefore

$$\int_r^1 \left(\int_y^1 \left[b(x,r,n) - b(x,y,n-1)\right]\mathrm{d}F(x)^{n-1}\right)\mathrm{d}F(y) = -F(r)\left[\int_r^1 \left[b(x,r,n) - b(x,r,n-1)\right]\mathrm{d}F(x)^{n-1}\right].$$

The claim follows since the right-hand-side is negative because $b(x,r,n) > b(x,r,n-1)$ by Fact (1). □

The condition in Claim 7 differs from the sufficient condition (7.25) as the latter is evaluated at $r = r^M$ and computed under the distribution of the maximum $\mathrm{d}F(y)^n$, rather than of a generic bidder $\mathrm{d}F(y)$.

Claim. Let

$$\varsigma(y) = \int_y^1 \left(b\left(x, r^M, n\right) - b(x,y,n-1)\right)\mathrm{d}F(x)^{n-1}.$$

It holds that: (i) $\varsigma(1) = 0$, (ii) $\varsigma\left(r^M\right) > 0$, and (iii) $\varsigma'(y) < 0 \Leftrightarrow y < y^\star$, where $y^\star$ is such that $b\left(y^\star, r^M, n\right) = \psi(y^\star)$.

Proof. $\varsigma(1) = 0$ is obvious; $\varsigma\left(r^M\right) > 0$ follows from Fact (1).

Recall that, for all r, n,

$$\int_r^1 b(x,r,m)\,\mathrm{d}F^m(x) = \int_r^1 \psi(x)\,\mathrm{d}F^m(x).$$

Hence, letting $r = y$ and $m = n-1$ in the previous expression, we can rewrite

$$\varsigma(y) = \int_y^1 \left(b\left(x, r^M, n\right) - \psi(x)\right)\mathrm{d}F(x)^{n-1}$$

and

$$\varsigma'(y) = \mathrm{d}F(y)^{n-1}\left[\psi(y) - b\left(y, r^M, n\right)\right].$$

The result follows from monotonicity of $\psi(y) - b\left(y, r^M, n\right)$. □

Notice that the claim implies that $\varsigma(y)$ is negative if and only if y is large. The sufficient condition (7.25) requires that the expectation of $\varsigma(y)$ with respect to the highest draw above r^M — which stochastically dominates a generic draw above r^M — is negative, given that by Claim 7 this expectation with respect to a generic draw above r^M is already negative.

Claim. A sufficient condition for (7.25) is

$$\int_{r^M}^{y^\star} \left|\varsigma'(y)\right| \mathrm{d}y > \int_{y^\star}^{1} \left(1 + F(y) - F(y^\star)\right) \varsigma'(y)\, \mathrm{d}y.$$

Proof. Condition (7.25) is

$$\int_{r^M}^{1} \varsigma(y)\, \mathrm{d}F(y)^n < 0.$$

Write

$$\int_{r^M}^{1} \varsigma(y)\, \mathrm{d}F(y)^n = \int_{r^M}^{y^\star} \varsigma(y)\, \mathrm{d}F(y)^n + \int_{y^\star}^{1} \varsigma(y)\, \mathrm{d}F(y)^n$$

By Claim 7, the second term is negative and

$$\int_{r^M}^{y^\star} \varsigma(y)\, \mathrm{d}F(y)^n < \int_{r^M}^{y^\star} \varsigma(y)\, \mathrm{d}F(y)$$

because ς is monotonically decreasing in $\left[r^M, y^\star\right]$ and $\mathrm{d}F(y)^n \trianglerighteq^{\text{FOSD}} \mathrm{d}F(y)$. Hence, Condition (7.25) is implied by $\int_{r^M}^{y^\star} \varsigma(y)\, \mathrm{d}F(y) < 0$.

We have

$$\int_{r^M}^{y^\star} \varsigma(y)\, \mathrm{d}F(y) = \int_{r^M}^{1} \varsigma(y)\, \mathrm{d}F(y) - \int_{y^\star}^{1} \varsigma(y)\, \mathrm{d}F(y),$$

and

$$\int_{r^M}^{1} \varsigma(y)\, \mathrm{d}F(y) = \int_{r^M}^{1} \left[b\left(x, r^M, n\right) - \psi(x)\right] \mathrm{d}F(x)^{n-1}$$

by the Revenue Equivalence Theorem. Substituting, we get

$$\int_{r^M}^{y^\star} \varsigma(y)\, \mathrm{d}F(y) = \int_{r^M}^{1} \left[b\left(y, r^M, n\right) - \psi(y)\right] \mathrm{d}F(y)^{n-1} - \int_{y^\star}^{1} \left(F(y) - F(y^\star)\right) \left[\psi(y) - b\left(y, r^M, n\right)\right] \mathrm{d}F^{n-1}(y),$$

which can be rearranged to the condition in the Claim. □

Summing up, Claim 7 provides a sufficient condition for the dynamic revenues in the SPA exceeding those in the FPA in a neighborhood of $\eta = 1$.

Proof of Propositions 11 and 12 and $\frac{\mathbf{d}G_\emptyset}{\mathbf{d}\eta}$ in SPA

The stationary distribution is continuous in η as a direct consequence of Theorem 12.13 in Stokey, Lucas and Prescott (1989). Recall that

$$G_\emptyset = G_\emptyset F(R_\emptyset)^n + \int_{R_\emptyset}^{1} \eta F^n(R(\theta))\, \mathrm{d}G(\theta). \tag{7.26}$$

If $\eta = 0$ we have

$$G_\emptyset = G_\emptyset F(R_\emptyset)^n$$

that can hold only for $G_\emptyset = 0$. At $\eta = 1$ we get

$$\begin{aligned} G_\emptyset &= G_\emptyset F\left(r^M\right)^n + \int_{R_\emptyset}^1 F^n\left(r^M\right) \mathrm{d}G(\theta) \\ &= G_\emptyset F\left(r^M\right)^n + F^n\left(r^M\right)(1 - G_\emptyset) \\ &= F^n\left(r^M\right). \end{aligned}$$

That $G_\emptyset$ can be non-monotonic is proved by the example in the text. To obtain the decomposition of the effect of η, total differentiation of (7.26) (where $R_\emptyset, R(\theta)$ and $G(\theta)$ all depend on η) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\eta} G_\emptyset = \frac{\mathrm{d}}{\mathrm{d}\eta} G_\emptyset F(R_\emptyset)^n + G_\emptyset \frac{\mathrm{d}}{\mathrm{d}\eta} R_\emptyset \mathrm{d}F(R_\emptyset)^n - \eta \frac{\mathrm{d}}{\mathrm{d}\eta} R_\emptyset F^n(R_\emptyset) \mathrm{d}G(R_\emptyset) + \int_{R_\emptyset}^1 F^n(R(\theta)) \mathrm{d}G(\theta) + \\ + \eta \left[\int_{R_\emptyset}^1 \frac{\mathrm{d}}{\mathrm{d}\eta} R(\theta) \mathrm{d}F(R(\theta))^n \mathrm{d}G(\theta) + \int_{R_\emptyset}^1 F(R(\theta))^n \mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}\eta} G(\theta)\right) \right]. \end{aligned}$$

This expression contains the three effects described in the main text (direct, reserve price and distribution).

When both the SPA and the FPA induce full tracking, the stationary distribution is completely pinned down by $R_\emptyset$. By Proposition 9 the result follows because, in this case, $\frac{\partial G_\emptyset}{\partial R_\emptyset} < 0$: conditionally on always tracking all bidders above $R_\emptyset$, reducing the initial reserve price $R_\emptyset$ increases efficiency.

Example When $\theta \sim \mathcal{U}(0,1)$, $\beta = 0$ and $n = 2$, the reserve price effect in the SPA is

$$\begin{aligned} G_\emptyset \frac{\mathrm{d}R_\emptyset}{\mathrm{d}\eta} \mathrm{d}F^n(R_\emptyset) + \eta \int_{R_\emptyset}^{\bar{\theta}^S} \frac{\mathrm{d}R(\theta)}{\mathrm{d}\eta} \mathrm{d}F^n(R(\theta))\, g(\theta)\, \mathrm{d}\theta + \eta \int_{\bar{\theta}^S}^1 \frac{\mathrm{d}R(\theta)}{\mathrm{d}\eta} \mathrm{d}F^n(R(\theta))\, g(\theta)\, \mathrm{d}\theta \\ = G_\emptyset \frac{\mathrm{d}R_\emptyset}{\mathrm{d}\eta} \mathrm{d}F^n(R_\emptyset) + \eta \frac{\mathrm{d}\bar{\theta}^S}{\mathrm{d}\eta} \mathrm{d}F^n\left(\bar{\theta}^S\right)\left(1 - G_\emptyset - G(\bar{\theta}^S)\right) \\ = -\frac{1}{2\eta n} \mathrm{d}\left(\bar{\theta}^S\right)^n \left(1 - G_\emptyset - G(\bar{\theta}^S)\right) \end{aligned}$$

where the second equality follows from the fact that $\frac{\mathrm{d}R(\theta)}{\mathrm{d}\eta} = 0$ in the tracking region. The third equality follows from the fact that when $\beta = 0$, $\frac{\mathrm{d}R_\emptyset}{\mathrm{d}\eta} = 0$ and that with uniform distribution, $\bar{\theta}^S = \frac{1+\eta(n-1)}{2n\eta}$ so $\frac{\mathrm{d}\bar{\theta}^S}{\mathrm{d}\eta} = -\frac{1}{2\eta^2 n}$. With $n = 2$, the reserve price effect is

$$-\frac{1}{2\eta} \bar{\theta}^S \left(1 - G_\emptyset - G(\bar{\theta}^S)\right)$$

The distribution effect is

$$\begin{aligned} \frac{\mathrm{d}G_\emptyset}{\mathrm{d}\eta} F^n(R_\emptyset) - \eta \frac{\mathrm{d}R_\emptyset}{\mathrm{d}\eta} g(R_\emptyset) F^n(R_\emptyset) + \eta \int_{R_\emptyset}^1 F^n(R(\theta)) \frac{\mathrm{d}}{\mathrm{d}\eta} g(\theta)\, \mathrm{d}\theta \\ = \frac{1}{4} \frac{\mathrm{d}G_\emptyset}{\mathrm{d}\eta} + \eta \int_{\frac{1}{2}}^1 R(\theta)^2 \frac{\mathrm{d}}{\mathrm{d}\eta} g(\theta)\, \mathrm{d}\theta, \end{aligned}$$

where the second equality uses $\frac{\mathrm{d}R_\emptyset}{\mathrm{d}\eta} = 0$ and $R_\emptyset = \frac{1}{2}$ at $\beta = 0$.

Proof of Lemma 13 and Proposition 14

We prove that the distribution conditional on tenure satisfies the MLR property which implies FOSD.

By 6.1 we have that

$$\frac{g_{\tau}(\theta)}{g_{\tau-1}(\theta)} = \frac{\frac{g(\theta,\tau)}{g(\tau)}}{\frac{g(\theta,\tau-1)}{g(\tau-1)}} = \frac{\frac{(1-\eta)F^{n-1}(\theta)g(\theta,\tau-1)}{g(\tau)}}{\frac{g(\theta,\tau-1)}{g(\tau-1)}} = \frac{g(\tau-1)}{g(\tau)}(1-\eta)F^{n-1}(\theta),$$

where $g(\tau) = \int_{R_{\emptyset}}^{1} g(\theta,\tau)\mathrm{d}\theta$ and hence the MLR property follows from $F^{n-1}(\theta)$ being increasing in θ. Moreover,

$$\frac{g_{\tau}(\theta)}{g_{\tau}(\theta')} = \left(\frac{F^{n-1}(\theta)}{F^{n-1}(\theta')}\right)^{\tau-1} \frac{g(\theta,1)}{g(\theta',1)},$$

from which we get that $\theta < \theta' \Rightarrow \lim_{\tau\to\infty} \frac{g_{\tau}(\theta)}{g_{\tau}(\theta')} = 0$.