Comparative Rationality*

Mauricio Ribeiro†

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Abstract

This paper introduces a natural criterion to compare the incompatibility of choices with preference maximization. Whereas previous approaches use indices to assess this incompatibility, the criterion I propose leads to an incomplete rationality ordering over choice correspondences that has intuitive implications to several models of boundedly rational choice. Despite its incompleteness, no index of incompatibility (that I am aware of) fully agrees with it. I characterize when an index would do so using a notion of predictive mistakes of the preference maximization model. I then propose a method to build indices of incompatibility that agree with the criterion and use it to define two new indices of incompatibility, one of them based on the Houtman-Maks index. As an empirical application, I compare how my approach differs from existing approaches in assessing the rationality of choices in an earlier choice elicitation experiment.

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†School of Economics, University of Bristol. E-mail: mauricio.ribeiro@bristol.ac.uk

1 Introduction

In choice theory, choices are rational if they can be explained by the maximization of a complete and transitive (preference) relation.[1] However, choices are often not "rational" in this sense, and for several reasons.

People can maximize preferences, but these preferences might be incomplete as in Eliaz and Ok (2006), or intransitive as in Bordes (1976). Even if people maximize a complete and transitive preference relation, they can make mistakes (cf. Nielsen and Rehbeck (2020)) or face menu-dependent constraints that are unobserved by the analyst (cf. Masatlioglu et al. (2012)). Finally, people might use a different choice procedure (cf. Manzini and Mariotti (2007)). In each of these cases, their choices can be incompatible with rationality.

Deviations from rationality, however, admit of gradation. One's choices may deviate from rationality on some isolated occasion (which may perhaps be considered a "mistake") whereas another's choices can be wildly incompatible with it. Therefore, we need methods that can measure the "extent" to which choices deviate from rationality.

These methods may prove to be useful for applied Economics. First, they can help to identify in what choice domains rationality provides a "sufficiently" good description of people's choices (cf. Brocas et al. (2019)). For instance, a person might behave "more rationally" in the case of simple choice alternatives (e.g., beverages or clothes), but not in the case of lotteries or state-contingent claims. This knowledge can help researchers not only in choosing the appropriate model for their purposes but also in directing their efforts to develop more descriptively accurate models of choice for domains where rationality fails to be an "approximation" of behavior. Second, methods to compare the rationality of choices allow one to run comparative statics on rationality using individual characteristics of interest such as level of education, intelligence, income, and gender (cf. Choi et al. (2014)). Finally, comparing the rationality of people's choices in a given domain at different points in time can be used to investigate whether people "learn" to be rational (cf. Harbaugh et al. (2001) and Brocas et al. (2019)).

Economists have proposed several ways to measure how incompatible observed choices are with preference maximization. The standard approach relies on indices of incompatibility, which assign to a person's choices a number that measures their degree of incompatibility with rationality. These numbers are then used to make comparative

[1] For this reason, I use preference maximization and rationality interchangeably throughout.

judgments of rationality: if the number an index assigns to Alice's choices is smaller than the one it assigns to Bob's, then the index deems Alice's choices as more rational than Bob's.

Importantly, indices of incompatibility compare the rationality of choices *indirectly*, through the numbers they assign to choices. But what if two indices disagree when comparing the rationality of Alice's and Bob's choices? How can we know if one index gets it right while the other gets it wrong — or if the comparison is difficult — without directly comparing the rationality of their choices?

Moreover, this index-oriented approach implicitly assumes that we can compare the rationality of the choices of any two people. Although completeness is a desirable feature of a ranking, imposing it from the get-go ignores that choices can fail to be rational in many different ways and, hence, that we might be more confident about some comparisons than about others. By directly comparing the rationality of choices, we might learn that some comparisons are "easy" whereas others are "hard." Indices of incompatibility should then agree with the "easy" comparisons while being free to disagree in the "difficult" ones.

In this spirit, I propose a stringent criterion to directly compare the rationality of choices. To state it, suppose we elicit Alice and Bob's choices from a given collection of menus. The criterion says that Alice's choices are at least as rational as Bob's if, for every sub-collection of menus in this collection, whenever Bob's choices are compatible with preference maximization in the sub-collection, so are Alice's.

This criterion induces an *incomplete* ordering on the space of choices defined over the same collection of menus, which I call the *rationality ordering*.[2] The rationality ordering cannot compare pairs that violate rationality in different sub-collections of menus. For instance, if a person exhibits a cycle on menus with two alternatives, while another exhibits a violation on menus with more than two alternatives, which of these violations should be deemed more "serious?" To answer this question, we need either to make further assumptions about types of violations or use information about the specific choice domain. The criterion then defers the judgment to the analyst.[3]

[2]Incomplete rankings play an important role in other well-known comparison problems. The Lorenz ordering over income distributions, the first-order stochastic dominance ordering over lotteries, and the Pareto ordering over allocations are examples of partial orderings that provide a minimal basis for comparisons in their respective domains. The rationality ordering I propose aims to provide such a basis for comparisons of rationality.

[3]If we have more information about the choice domain, we can extend the rationality ordering I propose without (necessarily) completing it. For instance, if the alternatives are sufficiently similar to each other in the relevant dimensions, e.g., they cost the same, then we might impose that $\succsim_{\text{rat}}$ be invariant to the relabeling of alternatives. I discuss three possible ways of extending my rationality

In Section 3, I provide an example in which it is intuitively clear how to make rationality comparisons. The criterion is duly consistent with this intuition, pointing to its applicability. Interestingly, as shown in Section 5.2, only one type of index (that I am aware of) can do so, namely, indices that count violations of the preference maximization model. But how we count these violations matters, and, by extending the example, I show that existing methods of counting violations cannot account for the extended ranking of rationality, while the criterion can (Examples 5 and 6).

In Section 4, I show that the rationality ordering can capture the extent of the departure from rationality in several models of boundedly rational choice. In Section 4.1, I begin with models that drop only one of the following four assumptions of the preference maximization model: (i) the preference relation is transitive; (ii) it is complete (iii) an element is chosen from a menu if, and only if, it is a maximum of the preference; and (iv) the same preference is maximized in all the menus we observe. I show that, in these models, the severity of the violation of the assumption decreases as choices become more rational according to the rationality ordering. To illustrate, suppose Alice's and Bob's choices are compatible with the maximization of an intransitive preference. Then, if Alice's choices are at least as rational as Bob's, and Alice's choices violate transitivity in a triple of alternatives, then so does Bob's.

In Section 4.2, I show that the same holds for other well-known models of boundedly rational choice. For each model considered, as choices compatible with the model become more rational according to the rationality ordering, we can find a representation of choices by the model that becomes "closer" to preference maximization, where the meaning of "closer" is model-specific. I end section 4.2 by discussing the framework Bernheim and Rangel (2009) propose for conducting welfare analysis when choices are not compatible with preference maximization. I show that the welfare inferences that one can make using this framework become coarser as choices become less rational according to the rationality ordering.

A model is, at least partially, judged by its ability to predict new observations given past observations. In Section 5.1, I show that we can interpret the rationality ordering as comparing the predictive errors an analyst would make if she used the preference maximization model to predict choices from some menus given the choices from other menus. To illustrate, suppose you tell me you would choose a from $\{a, b\}$, b from $\{b, c\}$, and a from $\{a, c\}$. If you are a preference maximizer, I can infer that you prefer a to b and c, and b to c. I would then predict that you would choose a

ordering in Appendix B.

from the menu $\{a, b, c\}$. If you instead choose b, this prediction fails. I build on this idea to prove that we can equivalently state the criterion I propose as follows: Alice's choices are least as rational as Bob's if, for every sub-collection of menus $\mathcal{B}$, whenever the preference maximization model incorrectly predicts Alice's choice in menu A given her revealed preferences in $\mathcal{B}$, it also incorrectly predicts Bob's choice in menu A given his revealed preferences in $\mathcal{B}$.

This new interpretation provides an intuitive characterization of when an index of incompatibility never disagrees with the rationality ordering. An index will do so if (and only if) it increases with the size of the sets of predictive errors based on the different sub-collections of menus. I use this characterization to understand why existing indices of incompatibility disagree with the rationality ordering, even though two of these indices satisfy a weaker type of agreement (one of which is the well-known Houtman-Maks index (Houtman and Maks, 1985)). I then describe a method to build indices of incompatibility that agree with the ordering from indices that satisfy this weaker type of agreement. The method gives rise to two new indices of incompatibility: the Probability index (Example 7) and the Average Houtman-Maks index (Example 8).

In Section 6, I use the data from a choice elicitation experiment run by Bouacida (2021) to compare how different measures of incompatibility assess the rationality of choices in the data. Despite being incomplete, the rationality ordering I propose can make non-trivial comparisons between the elicited choice correspondences. Existing indices disagree with at least 20% of these comparisons, which has implications for the assessed rationality of choices in the experiment. Moreover, the Probability and the Average Houtman-Maks indices are more discerning than existing indices in comparing the rationality of choices. The Average Houtman-Maks index, for instance, needs 64 levels of rationality to classify the rationality of choices in the data, while the Houtman-Maks index only needs 6.

To understand why the choices of subjects are not rational, I study when they violate well-known properties of rational choice[4] and how frequent these violations are. I also identify in which sub-collections of menus the choices of subjects more often violate rationality. By doing so, I identify a frequent violation of rational choice that, to the best of my knowledge, has not received much attention in the choice theory literature.

In Section 7, I review the relevant literature. The proofs of the results in the paper

[4]See Table 5 for a list of the properties considered.

are in Appendix A.

2 Notation and Definitions

Let X be a nonempty set. We interpret X as the set of all (mutually exclusive) alternatives and keep it arbitrarily fixed throughout. For any nonempty collection $\mathcal{A}$ of nonempty subsets of X, we refer to an element $A \in \mathcal{A}$ as a *menu* and to the ordered pair $(X, \mathcal{A})$ as a *choice space*. For each $i \in \mathbb{N}$, we denote the collection of all subsets of X with i alternatives by $\mathcal{M}_i(X)$. When $i = 2$, we say that $\mathcal{M}_2(X)$ is the collection of *pairwise menus* of X.

Given any binary relation R on X, we denote by $R^>$ its *strict part*, i.e., $xR^>y$ if, and only if, xRy and not yRx, and by $R^=$ its *symmetric part*, i.e., $xR^=y$ if, and only if, xRy and yRx. We denote the set of R-*incomparable* pairs by

$$\mathrm{Inc}(R) := \left\{(x, y) \in X^2 : \text{ Neither } xRy \text{ nor } yRx\right\}.$$

We say that a binary relation R on X is *acyclic* if we cannot find elements $x_1, \ldots, x_n \in X$ such that $x_1R^>x_2R^> \ldots R^>x_nR^>x_1$. We say that a binary relation R on X is *Suzumura-consistent* if we cannot find elements $x_1, \ldots, x_n \in X$ such that $x_1Rx_2R \ldots Rx_nR^>x_1$.

For any binary relation $\succsim$ on X, $A \subseteq X$ and $x \in X$, $x \succsim A$ means that $x \succsim y$, for every $y \in A$, while $x \succ A$ means that $x \succ y$, for every $y \in A$. We say that x is a $\succsim$-*maximum* of A if $x \in A$ and $x \succsim A$. We denote the set of all $\succsim$-maxima of A by $\max(A, \succsim)$. We say that x is a $\succsim$-*maximal* of A if there is no $y \in A$ such that $y \succ x$. We denote the set of all $\succsim$-maximal elements of A by $\mathrm{MAX}(A, \succsim)$.

We say that a binary relation $\succsim$ on X is a *preorder* on X whenever it is reflexive and transitive. A *partial order* on X is an antisymmetric preorder and a *total order* on X is a complete partial order. A *preference relation* is a complete preorder. Finally, a binary relation $\trianglerighteq$ on X *extends* a binary relation $\succsim$ if $\succsim \subseteq \trianglerighteq$ and $\succ \subseteq \triangleright$.

A *choice correspondence* on the choice space $(X, \mathcal{A})$ is a correspondence $c : \mathcal{A} \rightrightarrows X$ such that $\emptyset \neq c(A) \subseteq A$ for all $A \in \mathcal{A}$. If $|c(A)| = 1$ for all $A \in \mathcal{A}$, we say that c is a *choice function*. Given a sub-collection $\mathcal{B} \subseteq \mathcal{A}$, we denote the restriction of c to $\mathcal{B}$ by $c|_{\mathcal{B}}$. We denote the set of all choice correspondences on $(X, \mathcal{A})$ by $\mathcal{C}(X, \mathcal{A})$.

Definition 1. Given a choice correspondence $c \in \mathcal{C}(X, \mathcal{A})$ and a nonempty $\mathcal{B} \subseteq \mathcal{A}$, we say that c **is rationalized by** $\succsim$ **on** $\mathcal{B}$ if $\succsim$ is a complete preorder on $\cup\mathcal{B}$ and

$c(A) = \max(A, \succsim)$ for all $A \in \mathcal{B}$. If there exists such a preorder, we say that c is **rationalizable** on $\mathcal{B}$.

Given a choice correspondence $c \in \mathcal{C}(X, \mathcal{A})$ and a nonempty sub-collection $\mathcal{B}$ of $\mathcal{A}$, we say that x is *revealed preferred to* y in $\mathcal{B}$ if there is a set $B \in \mathcal{B}$ such that $x \in c(B)$ and $y \in B$. We then write $xR_{\mathcal{B}}(c)y$. We say that x is *revealed strictly preferred to* y in $\mathcal{B}$ if there is a set $B \in \mathcal{B}$ such that $x \in c(B)$ and $y \in B \backslash c(B)$. We then write $xP_{\mathcal{B}}(c)y$. When $\mathcal{B} = \mathcal{A}$, we simply write $R(c)$ and $P(c)$ for $R_{\mathcal{A}}(c)$ and $P_{\mathcal{A}}(c)$. When $\mathcal{B} = \mathcal{M}_2(X)$, we write B_c for $R_{\mathcal{M}_2(X)}(c)$, and say that B_c is the relation revealed in pairwise menus. Therefore, xB_cy if, and only if, $x \in c(\{x, y\})$.

We say that c *reveals a cycle* on $\mathcal{B}$ if we can find $x_1, \dots, x_n \in \cup\mathcal{B}$ such that

$$x_1 R_{\mathcal{B}}(c) \cdots R_{\mathcal{B}}(c) x_n R_{\mathcal{B}}(c) x_1,$$

with $x_i P_{\mathcal{B}}(c) x_j$, for some distinct $i, j \in \{1, \dots, n\}$. We then say that $(x_1, \dots, x_n)$ is a *revealed cycle of* c on $\mathcal{B}$. If, moreover, $x_1, \dots, x_n$ are distinct, we say that $(x_1, \dots, x_n)$ is a *proper revealed cycle of* c on $\mathcal{B}$. We note without proof the following well-known result due to Richter (1966).

Proposition 1 (**Richter's Theorem**). *Given a choice correspondence $c \in \mathcal{C}(X, \mathcal{A})$ and a nonempty $\mathcal{B} \subseteq \mathcal{A}$, c is rationalizable on $\mathcal{B}$ iff there is no (proper) revealed cycle of c on $\mathcal{B}$.*

3 The Rationality Ordering

Suppose you are in a bar with five friends, and you are curious about the rationality of their choices. The bar offers three beers: a, b, and c. So, you ask your friends what they would choose from the menus in the first row of Table 1, where you also record their answers. From menu $\{a, b, c\}$, for example, Friend 1 says he would choose c, and Friend 2 says she would choose either a or c.

You start by analyzing their choices from pairwise menus. You find that the choices of Friends 1 to 4 are rationalizable in these menus, whereas the choices of Friend 5 are not. You then reason that if Friends 1 to 4 maximize preferences, you can use their revealed preferences in pairwise menus to predict their choices from the menu $\{a, b, c\}$.

Your predictions turn out to be right for Friends 1 and 2, but wrong for Friends 3 and 4. You conclude that Friends 1 and 2 are rational whereas Friends 3 to 5 are not. But you wonder whether you can say more about the rationality of their choices.

Table 1: The Leading Example

$\mathcal{A}$	$\{a,b,c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$
Friend 1	c	a	c	c
Friend 2	a,c	a	c	a,c
Friend 3	a	a	c	c
Friend 4	a	b	c	c
Friend 5	a	a	b	c

By comparing the choices of Friends 3 and 4, you notice that they only choose differently from the menu $\{a,b\}$, where Friend 3 chooses a, and Friend 4 chooses b. Both friends share a violation of rationality in the menus $\{a,b,c\}$ and $\{a,c\}$, and this is the only violation in the choices of Friend 3. Given, however, that Friend 4 chooses b from $\{a,b\}$, her choices also violate rationality in the menus $\{a,b,c\}$ and $\{a,b\}$ whereas the choices of Friend 3 do not. You conclude that Friend 3 is more rational than Friend 4.

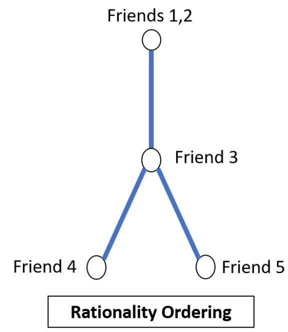
By comparing the choices of Friends 3 and 5, you notice that they only choose differently from the menu $\{b,c\}$, where Friend 3 chooses c, and Friend 5 chooses b. Again, both friends share a violation of rationality in the menus $\{a,b,c\}$ and $\{a,c\}$. Given, however, that Friend 5 chooses b from $\{b,c\}$, his choices violate rationality when restricted to pairwise menus whereas the choices of Friend 3 do not. You conclude that Friend 3 is also more rational than Friend 5.

You cannot, however, decide who is more rational, Friend 4 or Friend 5. On the one hand, the choices of Friend 4 violate rationality in the menus $\{a,b,c\}$ and $\{a,b\}$ whereas the choices of Friend 5 do not. On the other hand, the choices of Friend 5 violate rationality in pairwise menus whereas the choices of Friend 4 do not. Which of these two violations is a more "serious" violation of rationality? Undecided, you suspend judgment.

Therefore, Friends 1 and 2 are the most rational; Friend 3 is less rational than Friends 1 and 2, but more rational than Friends 4 and 5; and Friends 4 and 5 are hard to compare. This example suggests that, whenever we need to compare the rationality of choices, we should comparatively check for violations of rationality. If one friend violates rationality whenever the other does, then the latter should be declared to be at least as rational as the former. The next definition formalizes this intuition.

Definition 2. For any $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$, we say that c_1 is **at least as rational as** c_2, and write $c_1 \succsim_{\text{rat}} c_2$, if, for every $\mathcal{B} \subseteq \mathcal{A}$, whenever c_1 is *not* rationalizable on $\mathcal{B}$, then

Figure 1: Ranking of friends by the rationality of choices



c_2 is *not* rationalizable on $\mathcal{B}$. We call $\succsim_{\text{rat}}$ the **rationality ordering**.[5]

Clearly, $\succsim_{\text{rat}}$ is a preorder on $\mathcal{C}(X, \mathcal{A})$, which will be incomplete in all interesting applications. Going back to our example, it is easily checked that $\succsim_{\text{rat}}$ delivers the proposed ranking of the rationality of the choices of Friends 1 to 5, namely

$$\text{Friend 1} \sim_{\text{rat}} \text{Friend 2} \succ_{\text{rat}} \text{Friend 3} \succ_{\text{rat}} \{\text{Friend 4}, \text{Friend 5}\},$$

whereas Friends 4 and 5 are not compared by $\succsim_{\text{rat}}$. Figure 1 displays the Hasse Diagram of the partial order[6] induced by $\succsim_{\text{rat}}$ once we identify friends whose choices are equally rational.

The definition of $\succsim_{\text{rat}}$ requires that we compare the rationality of choices in all possible sub-collections of menus. But, in fact, we only need to consider either of two types of sub-collections.

Definition 3. Given a choice space $(X, \mathcal{A})$ and $c \in \mathcal{C}(X, \mathcal{A})$, we say that $\mathcal{B} \subseteq \mathcal{A}$ is a **minimal incompatible collection** of c, if c cannot be rationalized in $\mathcal{B}$ but it is rationalizable in any $\mathcal{B}' \subset \mathcal{B}$. The set of *minimal incompatible collections* of c is denoted by $\text{MIC}(c)$. Dually, we say that $\mathcal{B} \subseteq \mathcal{A}$ is a **maximal compatible collection** of c if c is rationalizable in $\mathcal{B}$ but c cannot be rationalized in any $\mathcal{B}' \supset \mathcal{B}$. The set of *maximal compatible collections of* c is denoted by $\text{MCC}(c)$.

Minimal incompatible collections identify the "essential" violations of rationality in choices, and, hence, they can help us to determine why rationality fails (see Section 6).

[5]Strictly speaking, I should write $\succsim_{\text{rat}}^{(X,\mathcal{A})}$ to emphasize the dependence of the ordering on the underlying choice space $(X, \mathcal{A})$, but I refrain to do so to simplify notation.

[6]A Hasse Diagram of a partial order $\succcurlyeq$ on a finite set X is a graph whose nodes are the elements of x and whenever $x, y \in X$ are such that $x \succ y$ and there is no $z \in X \backslash \{x, y\}$ such that $x \succ z \succ y$, we draw downward edge linking x to y. Hence, maximal elements are represented in the top of the diagram, whereas minimal elements are represented in the bottom.

maximal compatible collections are the largest collections in which a person chooses "as if" maximizing a preference relation, and, hence, they can guide our search for possible candidates for revealed preferences. The next proposition shows that knowing either of these sub-collections is sufficient to compare the rationality of choices through $\succsim_{\text{rat}}$.

Proposition 2. *For any $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$, the following statements are equivalent:*

(i) $c_1 \succsim_{rat} c_2$ *;*

(ii) *$MIC(c_1)$ is coarser than $MIC(c_2)$, that is, for every $\mathcal{B} \in MIC(c_1)$, there is a $\mathcal{B}' \in MIC(c_2)$ such that $\mathcal{B}' \subseteq \mathcal{B}$;*

(iii) *$MCC(c_2)$ is finer than $MCC(c_1)$, that is, for every $\mathcal{B} \in MCC(c_2)$, there is a $\mathcal{B}' \in MCC(c_1)$ such that $\mathcal{B} \subseteq \mathcal{B}'$.*

Remark 1. Since the number of sub-collections of $\mathcal{A}$ grows exponentially with $|\mathcal{A}|$, comparing the rationality of choices in every sub-collection of menus becomes infeasible for moderately large values of $|\mathcal{A}|$. But when X is finite and $|X| < |\mathcal{A}|$, Proposition 2 suggests a different strategy to compute $\succsim_{\text{rat}}$.

The key insight is that a minimal incompatible collection of $c \in \mathcal{C}(X, \mathcal{A})$ can have at most $|X|$ menus. In fact, any collection where c violates rationality must give rise to at least one *proper* revealed cycle, and proper revealed cycles are revealed by, at most, $|X|$ menus. If a minimal incompatible collection had more than $|X|$ menus, we could eliminate one menu while keeping the proper revealed cycle, contradicting the minimality of the collection.

Therefore, by Proposition 2, to compare the rationality of choices we only need to check for rationality in sub-collections of, at most, $|X|$ menus. The number of such collections is

$$\sum_{i=2}^{|X|} \binom{|\mathcal{A}|}{i},$$

which grows polynomially with $|\mathcal{A}|$ for a fixed $|X|$. The smaller the cardinality of X, the fewer the sub-collections of $\mathcal{A}$ we need to check for the rationality of choices.[7] In the empirical application in Section 6, this strategy allows us to reduce the number of collections we need to check for the rationality of choices by roughly 73%.

[7] To calculate the minimal incompatible collections of a choice correspondence, we can use a backtracking with pruning algorithm.

4 $\succsim_{\text{rat}}$ and Boundedly Rational Choice

Economists have proposed models of choice that subsume the preference maximization model as a particular case but can account for some of the violations of rationality that have been identified. In this section, I show that for some of these models, as choices become more rational according to $\succsim_{\text{rat}}$, we can find representations of choices in the model that become "closer" to preference maximization, where the meaning of closer is model-specific.

I first consider models that drop one assumption of the preference-maximization model while retaining the others, making it easier to define when their representation becomes "closer" to preference maximization. I then study other models of boundedly rational choice that subsume the preference-maximization model but either introduce new primitives in their representations of choices (e.g., constraints) or capture more general choice procedures (e.g., sequential choice). Finally, I study the implications of $\succsim_{\text{rat}}$ to the framework Bernheim and Rangel (2009) propose for conducting welfare analysis when choices are not rational.

Since the goal here is to show that the rationality ordering has interesting conceptual implications, I sometimes impose further structure on the collection of menus $\mathcal{A}$, e.g., that $\mathcal{A}$ contains all pairwise menus.

4.1 The Four Pillars of Preference Maximization

The preference maximization model postulates that people have preferences that are: (i) *menu-independent*; (ii) *transitive*; (iii) *complete;* and (iv) *maximized from every menu*. Suppose that we drop one of these assumptions but keep the other three.

If we drop menu-independence, what is the minimum number of complete and transitive relations we need to account for choices? If we drop transitivity, how intransitive is the rationalizing relation? If we drop completeness, how incomplete is the rationalizing relation? And if people fail to maximize preferences, how often do they do so?

4.1.1 Dropping Menu-Independence

Kalai et al. (2002) study a model in which the decision-maker maximizes different preferences from different menus, which they call *Rationalization by Multiple Rationales* (RMR). Formally, $c \in \mathcal{C}(X, \mathcal{A})$ is *RMR-representable* if there exists a collection $\mathcal{R}$ of preference relations such that, for every $A \in \mathcal{A}$, there exists a preference $\succsim \in \mathcal{R}$

such that

$$c(A) = \max(A, \succsim).$$

We then say that $\mathcal{R}$ is an *RMR-representation* of c. In this model, "irrationality" is captured by the minimum cardinality of an RMR-representation of c. As the next result shows, $\succsim_{\text{rat}}$ is duly consistent with this intuition.

Proposition 3. *For every $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$, if $c_1 \succsim_{rat} c_2$, then the minimum cardinality of a RMR-representation of c_1 is at at least as small as the minimum cardinality of a RMR-representation of c_2.*

4.1.2 Dropping Transitivity

Choices are often incompatible with the maximization of a *transitive* binary relation. Dropping transitivity, however, requires that we define what maximizing an intransitive relation means because $\max(A, \succsim)$ can be empty when $\succsim$ is not transitive.[8]

Given a choice space $(X, \mathcal{A})$ and a complete binary relation R on X, a *R-maximizer* Γ_R on (X, A) is a correspondence such that

(i) For every menu $A \in \mathcal{A}$, $\emptyset \neq \Gamma_R(A) \subseteq A$;

(ii) For every menu $A \in \mathcal{A}$, if $\max(A, R) \neq \emptyset$, then $\Gamma_R(A) = \max(A, R)$.

For instance, if R is complete and quasi-transitive, then $\Gamma(\cdot, R) := \max(\cdot, R)$ is a R-maximizer. If R is complete, then most well-known tournament solutions are R-maximizers (e.g. the top-cycle set).

Given a choice correspondence $c \in \mathcal{C}(X, \mathcal{A})$, we say that c is *rationalizable by a complete relation* if we can find a complete binary relation R and a R-maximizer Γ_R such that, for every $A \in \mathcal{A}$,

$$c(A) = \Gamma_R(A).$$

When choices are rationalized by a complete relation, we can measure departures from rationality by how intransitive the rationalizing relation is. We thus need a way to compare the transitivity of binary relations.

Definition 4. Given a binary relation R on X and $\{x, y, z\} \subseteq X$, we say that R *satisfies transitivity* in $\{x, y, z\}$ if R is transitive when restricted to $\{x, y, z\}$. A binary relation R_1 is *at least as transitive* as a binary relation R_2 if, for every triple $\{x, y, z\} \subseteq X$, whenever R_2 satisfies transitivity in $\{x, y, z\}$, then so does R_1.

[8] The literature on tournaments deals with this issue. For an overview of it, see Laslier (1997).

Proposition 4 shows that $\succsim_{\text{rat}}$ is duly consistent with the intuition that rationality in this model is captured by how intransitive the rationalizing relation is.

Proposition 4. *Assume that $\mathcal{M}_2(X) \subseteq \mathcal{A}$, and let $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ be rationalizable by the complete relations R_1 and R_2 with maximizers Γ_{R_1} and Γ_{R_2}. If $c_1 \succsim_{rat} c_2$, then R_1 is at least as transitive as R_2.*

4.1.3 Dropping Completeness

Choices can also fail to be compatible with the maximization of a *complete* relation (cf. Aumann (1962), Bossert et al. (2005) and Eliaz and Ok (2006)). When the rationalizing relation $\succsim$ is incomplete, we need to redefine what maximization means because $\max(A, \succsim)$ can be empty for some menus $A \in \mathcal{A}$. I follow the literature on incomplete preferences and replace the notion of $\succsim$-maximum with the one of $\succsim$-maximal alternatives.

Formally, we say that $c \in \mathcal{C}(X, \mathcal{A})$ is *rationalizable by an incomplete preference* if there exists a (possibly) incomplete preorder $\succsim$ on X such that, for every $A \in \mathcal{A}$,

$$c(A) = \text{MAX}(A, \succsim).$$

Intuitively, departures from rationality in this model are related to the cardinality of the set of incomparable elements of the most complete preorder that rationalizes choices. But for this statement to make sense, we need to be sure that there is such a preorder. The next lemma takes care of that.[9]

Lemma 1. *Assume that $\mathcal{M}_2(X) \subseteq \mathcal{A}$. If $c \in \mathcal{C}(X, \mathcal{A})$ is rationalizable by an incomplete preference relation, then there is a unique preorder $\succsim^\star$ that rationalizes c and such that, for any other preorder $\succsim$ that rationalizes c, we have $\succsim \subseteq \succsim^\star$. Moreover, for every $x, y \in X$, $(x, y) \in Inc(\succsim^\star)$ if, and only if, there exists a $z \in X$ such that either $|c(\{x, z\})| = 2$ and $|c(\{y, z\})| = 1$, or $|c(\{x, z\})| = 1$ and $|c(\{y, z\})| = 2$.*

Lemma 1 implies that, provided we take the most complete preorder consistent with choices, we can identify incomparabilities through choices. In fact, if $\succsim^\star$ is the most complete preorder that rationalizes $c \in \mathcal{C}(X, \mathcal{A})$, whenever $(x, y) \in \text{Inc}(\succsim^\star)$ there is a $z \in X$ such that either $|c(\{x, z\})| = 2$ and $|c(\{y, z\})| = 1$, or $|c(\{x, z\})| = 1$ and $|c(\{y, z\})| = 2$. In either case, c violates rationality in the sub-collection $\{\{x, y\}, \{y, z\}, \{x, z\}\}$.

[9] Lemma 1 was first proved in Ribeiro and Riella (2017).

In fact, a converse of this result is true: given $c \in \mathcal{C}(X, \mathcal{A})$, if c is rationalizable by an incomplete preference and violates rationality in the collection $\{\{x, y\}, \{y, z\}, \{x, z\}\}$, then c chooses two alternatives from two of the menus in the collection and one alternative from the remaining one, which implies that alternatives that are both chosen from the same menu in the sub-collection must be incomparable. Whenever this holds, we say that c *reveals incomparability* in $\{x, y, z\}$, and say that the menu $A \in \{\{x, y\}, \{y, z\}, \{x, z\}\}$ with $|c(A)| = 1$ is a *c-revealer.*

Therefore, if $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ are rationalizable by incomplete preferences and $c_1 \succsim_{\text{rat}} c_2$, then, for every $\{x, y, z\} \subseteq X$, if c_1 reveals incomparability on $\{x, y, z\}$, then c_2 also reveals incomparability on $\{x, y, z\}$. This suggests that the most complete relation that rationalizes c_1 is more complete than the most complete relation that rationalizes c_2. The next results shows that this holds under some conditions.[10]

Proposition 5. *Assume that $\mathcal{M}_2(X) \subseteq \mathcal{A}$ and that $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ are rationalizable by incomplete preferences with $c_1 \succsim_{rat} c_2$, and let $\succsim_1$ and $\succsim_2$ be the most complete preorders that rationalize c_1 and c_2.*

(i) *For every $x, y \in X$, if $|c_2(\{x, y\})| = 2$ whenever $|c_1(\{x, y\})| = 2$, then $Inc\,(\succsim_1) \subseteq Inc\,(\succsim_2)$;*

(ii) *For every $A \in \mathcal{M}_2(X)$, if A is a c_2-revealer whenever it is a c_1-revealer, then $Inc\,(\succsim_1) \subseteq Inc\,(\succsim_2)$.*

(iii) *If X is finite and the number of c_1-revealers is at least as large as the number of c_2-revealers, then $|Inc\,(\succsim_1)\,| \leqslant |Inc\,(\succsim_2)\,|$.*

4.1.4 Dropping the Maximization Assumption

Although a person can have complete and transitive preferences, she might fail to maximize these preferences, especially from menus with more than two alternatives. In this section, I propose a model where choices from pairwise menus reveal a complete and transitive preference that might not be maximized in menus with more than two alternatives. Departures from rationality in this model are then captured by the collection of menus where preferences are not maximized.

Formally, assume that $\mathcal{M}_2(X) \subseteq \mathcal{A}$ and recall that B_c denotes the relation revealed from pairwise menus. Given $c \in \mathcal{C}(X, \mathcal{A})$, we say that c is *rational in pairwise menus* if c is rationalizable on $\mathcal{M}_2(X)$. Moreover, for every $A \in \mathcal{A} \backslash \mathcal{M}_2(X)$, we say that

[10] An example that shows why we need these conditions is available under request.

c *fails to maximize preferences in* A if $c(A) \neq \max(A, B_c)$. The next result follows directly from these definitions.

Proposition 6. *Assume that* $\mathcal{M}_2(X) \subseteq \mathcal{A}$ *and that* $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ *are rational in pairwise menus. If* $c_1 \succsim_{rat} c_2$*, then, for every* $A \in \mathcal{A}$*, if* c_1 *fails to maximize preferences in* A*, then so does* c_2*.*

In Proposition 6, all failures of maximization are equally important. Going back to the leading example, this means that Friends 3 and 4 are equally rational in terms of maximization failures. But if preferences are revealed in pairwise menus, there is a sense in which Friend 3 makes a "more serious" mistake than Friend 4, because Friend 3 chooses her second-best alternative from the menu $\{a, b, c\}$, whereas Friend 4 chooses her third-best alternative. We have seen that $\succsim_{\text{rat}}$ indeed ranks Friend 3 as more rational than Friend 4. The next example generalizes this intuition.

Example 1. Let $X = \{x_1, \ldots, x_n\}$ and $\mathcal{A} = \mathcal{M}_2(X) \cup \{X\}$. For each $k \geqslant 1$, let $\succsim_k$ be the total order in which x_1 is ranked in the k-th position of the ranking, i.e., $x_1 \succ_1 x_2 \succ_1 \cdots \succ_1 x_n$, $x_2 \succ_2 x_1 \succ_2 x_3 \cdots \succ_2 x_n$, and so on. Define

$$c_k(A) := \begin{cases} \max(A, \succsim_k) & , \text{ if } A = \{x, y\} \\ \{x_1\} & , \text{ if } A = X \end{cases}.$$

We now show that $c_k \succ_{\text{rat}} c_{k+1}$ for every $k = 1, \ldots, n$. Let $\mathcal{B} \in \text{MIC}(c_k)$, i.e., let $\mathcal{B}$ be a minimal incompatible collection of c_k (see Definition 3). Then, there exists $i \in \{1, \ldots, k\}$ such that $\mathcal{B} = \{X, \{x_1, x_i\}\}$. But then $c_k(\{x_1, x_i\}) = \{x_i\}$ and, hence, $c_{k+1}(\{x_1, x_i\}) = \{x_i\}$. Thus, $\mathcal{B} \in \text{MIC}(c_{k+1})$. Moreover, $\{X, \{x_1, x_{k+1}\}\} \in \text{MIC}(c_{k+1})$, but $\{X, \{x_1, x_{k+1}\}\} \notin \text{MIC}(c_k)$. By Proposition 2, $c_k \succ_{\text{rat}} c_{k+1}$, as required.

Remark 2. Example 1 suggests that $\succsim_{\text{rat}}$ reacts to the ordinal intensity of the failure in maximizing behavior, but this is only partially true. In fact, let $X = \{x_1, \ldots, x_n\}$ and $\mathcal{A} = \mathcal{M}_2(X) \cup \{X\}$. Let $\succsim$ be the total order $x_1 \succ x_2 \succ \cdots \succ x_n$. Define, for each $A \in \mathcal{A}$,

$$c_k(A) := \begin{cases} \max(A, \succsim) & , A = \{x, y\} \\ x_k & , A = X \end{cases}.$$

Then, although $c_1 \succ_{\text{rat}} c_k$ for all $k \in \{2, \ldots, n\}$, it is easily checked that c_i and c_j are $\succsim_{\text{rat}}$-incomparable for any distinct $i, j \in \{2, \ldots, n\}$.[11]

[11]Nevertheless, if we extend $\succsim_{\text{rat}}$ so that it becomes invariant to the relabeling of alternatives (see

4.2 Other Models of Boundedly Rational Choice

I now consider other well-known models of boundedly rational choice. In each of these models, I show that we can find a representation of choices that becomes "closer" to preference-maximization as choices become more rational according to $\succsim_{\text{rat}}$.

4.2.1 Rational Shortlist Method

Manzini and Mariotti (2007) propose a two stage procedure in which a decision-maker has two criteria in mind. Each criterion is represented by an asymmetric relation $\succ_i$, $i = 1, 2$. In the first stage, she eliminates alternatives that are dominated according to $\succ_1$. In the second stage, she eliminates alternatives that are dominated according to $\succ_2$, arriving at her final choices. They call this procedure a *Rational Shortlist Method* (RSM).

Formally, $c \in \mathcal{C}(X, \mathcal{A})$ is **RSM-representable** if there exists a pair of asymmetric relations $(\succ_c^1, \succ_c^2)$ on X such that, for all $A \in \mathcal{A}$,

$$c(A) = \text{MAX}(\text{MAX}(A, \succ_c^1), \succ_c^2).$$

We then say that $(\succ_c^1, \succ_c^2)$ is a *RSM-representation* of c.

When c is a *choice function* and we observe choices from all finite menus, it is without loss of generality to assume that $\succ_1$ is acyclic (but not $\succ_2$) in the definition of RSM-representability. If, in addition, c is rationalizable on $\mathcal{A}$, we can take $\succ_1$ to be the standard revealed relation, $\succ_2$ to be the empty relation, and conclude that $c(A) = \max(A, \succ_1)$. Therefore, when choices are rational, the decision-maker simply maximizes her (first-stage) preferences.

In this spirit, the next result establishes that if $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ are RSM-representable choice functions and if $c_1 \succsim_{\text{rat}} c_2$, then we can find RSM-representations of c_1 and c_2 such that the first stage relation of c_1 is more complete than the one of c_2, both being acyclic, and the second stage relation of c_1 is less complete than the one of c_2. Thus, as choices become more rational according to $\succsim_{\text{rat}}$, their representation in the model become closer to the choices of a (one-stage) preference maximizer.

Proposition 7. *Assume that all finite menus of X are in $\mathcal{A}$, and let $c_1, c_2 \in \mathcal{C}(X, A)$ be RSM-representable choice functions. If $c_1 \succsim_{rat} c_2$, then we can find RSM-representations $(\succ_{c_1}^1, \succ_{c_1}^2)$ and $(\succ_{c_2}^1, \succ_{c_2}^2)$ of c_1 and c_2 such that:*

Appendix B), then the extended ordering will deem c_k as strictly more rational than c_{k+1} for every $k \in \{1, \ldots, n-1\}$.

(i) $\succ^1_{c_1}$ *and* $\succ^1_{c_2}$ *are acyclic;*

(ii) *There is an injection* $\psi_1 : \succ^1_{c_2} \to \succ^1_{c_1}$*, and, moreover,* $Inc(\succ^1_{c_1}) \subseteq Inc(\succ^1_{c_2})$*;*

(iii) *There is an injection* $\psi_2 : \succ^2_{c_1} \to \succ^2_{c_2}$*, and, moreover,* $Inc(\succ^2_{c_2}) \subseteq Inc(\succ^2_{c_1})$*.*

If, in addition, $B_{c_1} = B_{c_2}$*, then* $\succ^1_{c_2} \subseteq \succ^1_{c_1}$*,* $\succ^1_{c_1}$ *is at least as transitive as* $\succ^1_{c_2}$*, and* $\succ^2_{c_1} \subseteq \succ^2_{c_2}$*.*

4.2.2 Categorize Then Choose

Manzini and Mariotti (2012) study a choice procedure in which a decision-maker ranks both categories of alternatives and alternatives. When choosing from a menu, she first eliminates the alternatives in the menu whose categories are dominated in her ranking of categories. She then uses her ranking over alternatives to select among the alternatives that survive the first stage. They call this procedure *Categorize Then Choose* (CTC).

Formally, given an asymmetric binary relation $\succ_s$ on $2^X \setminus \{\emptyset\}$, define

$$A_{\succ_s} \coloneqq \{x \in A : \ \forall B, B' \subseteq A, \ B' \succ_s B \implies x \notin B\}.$$

We say that $c \in \mathcal{C}(X, \mathcal{A})$ is *CTC-representable* if there exists an asymmetric relation $\succ^c_s$ on $2^X \setminus \{\emptyset\}$ and a complete and asymmetric relation $\succ_c$ on X such that, for all $A \in \mathcal{A}$,

$$c(A) = \max(A_{\succ^c_s}, \succ_c).$$

We then say that $(\succ^c_s, \succ_c)$ is a *CTC-representation* of c.

A decision-maker that categorizes and then chooses departs from preference maximization both because of the categorization stage and because her second-stage preference is not transitive. Therefore, a natural way to capture rationality in this model is to require that, as choices become more rational, the categorization stage becomes less relevant, while the relation used in the second stage becomes more transitive. The next result shows that if $c \in \mathcal{C}(X, \mathcal{A})$ is CTC-representable, we can find a CTC-representation of c such that $\succsim_{\text{rat}}$ is consistent with this intuition.

Proposition 8. *Assume that* $\mathcal{A}$ *contains all finite menus of* X*. Let* $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ *be CTC-representable choice functions. If* $c_1 \succsim_{rat} c_2$*, then we can find CTC-representations* $(\succ^{c_1}_s, \succ_{c_1})$ *and* $(\succ^{c_2}_s, \succ_{c_2})$ *such that:*

(i) *There is an injection* ψ *from* $\succ^{c_1}_s$ *to* $\succ^{c_2}_s$*, i.e.,* $\succ^{c_1}_s$ *is more incomplete than* $\succ^{c_2}_s$*;*

(ii) $\succ_{c_1}$ *is at least as transitive as* $\succ_{c_2}$.

4.2.3 Order Rationalization Theory

Cherepanov et al. (2013) study a model of choice in which the decision-maker has a collection of binary relations they call *rationales*, in addition to a (complete and transitive) preference relation. In the first stage, the decision-maker eliminates all alternatives that she cannot rationalize, i.e., which are not a maximum according to some rationale. In the second stage, she maximizes her preference relation over the alternatives that survive the first stage. They call this model *Order Rationalization* (OR).

Cherepanov et al. (2013) call the constraint induced by the rationales in the first stage a *psychological constraint* and show that a constraint is generated by a set of rationales if, and only if, it satisfies property α, i.e., for every $A, B \in \mathcal{A}$ if $A \subseteq B$, then $\varphi(B) \cap A \subseteq \varphi(A)$. Therefore, a psychological constraint φ is a choice correspondence on $(X, \mathcal{A})$ that satisfies property α.

Formally, we say that $c \in \mathcal{C}(X, \mathcal{A})$ is *OR-representable* if there is a psychological constraint $\varphi_c \in \mathcal{C}(X, \mathcal{A})$ and a complete and transitive binary relation $\succ_c$ such that, for every $A \in \mathcal{A}$,

$$c(A) = \max(\varphi_c(A), \succ_c).$$

We then say that $(\varphi_c, \succ_c)$ is a *OR-representation* of c.

In this model, rationality should be related to the size of the largest constraint φ that OR-represents c. The next result shows that, provided that c_1 and c_2 share choose the same from pairwise menus, $\succsim_{\text{rat}}$ delivers this intuition.

Proposition 9. *Assume that X is finite and assume that $\mathcal{A} = 2^X \backslash \{\emptyset\}$. Let $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ be OR-representable choice functions with $B_{c_1} = B_{c_2}$, where B_{c_1} is a total order. If $c_1 \succsim_{rat} c_2$, then the largest constraints φ_{c_1} and φ_{c_2} consistent with the Order Rationalization of c_1 and c_2 satisfy $\varphi_2(A) \backslash \{c_2(A)\} \subseteq \varphi_1(A) \backslash \{c_1(A)\}$.*

4.2.4 Revealed (P)Reference Theory

Ok et al. (2015) propose a model in which the existence of a *reference* alternative in some menus distorts preference maximization. More specifically, the existence of a reference in a menu eliminates from consideration alternatives in the menu that are worst than the reference according to some criterion that is relevant to the decision-maker. They call this model the *Reference-Dependent Choice* (RDC) model.

Formally, we say that c is *RDC-representable* if there is a complete and transitive preference relation $\succsim_c$, a nonempty set $\mathcal{U}_c$ of real maps on X, a function $\mathbf{r}_c : \mathcal{A} \to X \cup \{\Diamond\}$ with $\mathbf{r}_c(A) \in A \backslash c(A)$ whenever $\mathbf{r}_c(A) \neq \Diamond$, such that for every $A \in \mathcal{A}$,

1. if $\mathbf{r}_c(A) = \Diamond$, then $c(A) = \max(A, \succsim_c)$;

2. if $\mathbf{r}_c(A) \neq \Diamond$, then $c(A) = \max(A \cap \mathcal{U}_c^{\uparrow}(\mathbf{r}_c(A)), \succsim_c)$, where

$$\mathcal{U}_c^{\uparrow}(\mathbf{r}_c(A)) \coloneqq \{x \in X : u(x) \geqslant u(\mathbf{r}_c(A)),\ \text{for all } u \in \mathcal{U}_c\};$$

3. for any $A' \subseteq A$ such that $\mathbf{r}_c(A) \in A'$ and $c(A) \cap A' \neq \emptyset$, we have $\mathbf{r}_c(A') \neq \Diamond$ and

$$c(A') = \max(A' \cap \mathcal{U}_c^{\uparrow}(\mathbf{r}_c(A)), \succsim_c).$$

We then say that $(\succsim_c, \mathcal{U}_c, \mathbf{r}_c)$ is a *RDC-representation* of choices. Intuitively, $\mathbf{r}_c$ maps menus to their references, and, whenever $\mathbf{r}_c(A) = \Diamond$ for some $A \in \mathcal{A}$, no reference influences the choices from the menu A. $\mathcal{U}_c$ represents the set of criteria that are relevant for the decision-maker in the menus where a reference exists.

In this model, departures from rationality are related to the collection of menus where references influence choice. The next proposition shows that $\succsim_{\text{rat}}$ is consistent with this intuition.

Proposition 10. *Assume all non-empty finite subsets of X are in $\mathcal{A}$. Let $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ be RDC-representable with $c_1 \succsim_{rat} c_2$. Then, we can find RDC-representations of $(\succsim_{c_1}, \mathcal{U}_{c_1}, \boldsymbol{r}_{c_1})$ and $(\succsim_{c_2}, \mathcal{U}_{c_2}, \boldsymbol{r}_{c_2})$ of c_1 and c_2 such that:*

(i) $\succsim_{c_i} = B_{c_i}$*;*

(ii) *For every $A \in \mathcal{A}$, $\boldsymbol{r}_{c_i}(A) = \Diamond$ if, and only if, $c_i(A) = \max(A, \succsim_{c_i})$;*

(iii) *For every $A \in \mathcal{A}$, if $\boldsymbol{r}_{c_1}(A) \neq \Diamond$, then $\boldsymbol{r}_{c_2}(A) \neq \Diamond$.*

Remark 3. The RDC-representation used in Proposition 10 only attributes references to menus where the decision-maker does not maximize the preferences revealed from pairwise menus. Therefore, it only departs from rational choice when strictly necessary. In particular, if c is rationalizable, the representation never assigns a reference to a menu.

4.2.5 Preference Structures

Nishimura and Ok (2020) propose a model of choice in which a decision-maker has a pair of binary relations $(\succsim, R)$ on X. $\succsim$ is an incomplete preorder that encodes the rankings she deems uncontroversial and R is a complete binary relation that represents her revealed preferences. The two relations are intuitively linked. First, R extends $\succsim$. Second, R is transitive with respect to $\succsim$, i.e., for all $x, y, z \in X$, if either $xRy \succsim z$ or $x \succsim yRz$, then xRz. When these conditions hold, Nishimura and Ok (2020) say that $(\succsim, R)$ is a *preference structure*.

Given a preference structure $(\succsim, R)$, the decision-maker chooses by first eliminating dominated elements according to $\succsim$ and then "maximizing" R over the remaining elements, where the maximization of R in a set $A \subseteq X$ is understood in the sense of selecting all the alternatives in A that are in the top-cycle of R, denoted by $\bigcirc(A, R)$.[12] Nishimura and Ok (2020) call this procedure a *Choice by a Preference Structure* (CPS).

Formally, $c \in \mathcal{C}(X, \mathcal{A})$ is *CPS-representable* if there is a preorder $\succsim_c$ and a complete binary relation R_c such that $(\succsim_c, R_c)$ is a preference structure and such that, for every $A \in \mathcal{A}$,

$$c(A) = \bigcirc(\mathrm{MAX}(A, \succsim_c), R_c).$$

We then say that $(\succsim_c, R_c)$ *CPS-represents* c.

We can think of $\succsim_c$ as the decision-maker's (welfare) preferences over alternatives and of R_c as her revealed preference. Consistent with this interpretation, if $c \in \mathcal{C}(X, \mathcal{A})$ is CPS-representable and we observe choices from all pairwise menus, we have that $R_c = B_c$ for any pair $(\succsim_c, B_c)$ that CPS-represents c.

Rationality in the CPS model is thus related to the completeness of $\succsim_c$ and to the transitivity of R_c. The next proposition shows that $\succsim_{\text{rat}}$ is consistent with this intuition under some conditions.

Proposition 11. *Assume that $\mathcal{M}_2(X) \subseteq \mathcal{A}$, and let $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ with $c_1 \succsim_{rat} c_2$ be CPS-representable. If $\succsim_{c_1}$ and $\succsim_{c_2}$ are the most complete relations such that $(\succsim_{c_1}, B_{c_1})$ and $(\succsim_{c_2}, B_{c_2})$ are CPS-representations of c_1 and c_2, then:*

(i) *B_{c_1} is at least as transitive as B_{c_2};*[13]

[12] For every binary relation R on X and any subset $A \subseteq X$, $\bigcirc(A, R)$ is the smallest $B \subseteq A$ such that $xR^{>}y$ for every $x \in B$ and $y \in A \setminus B$.

[13] See Definition 4.

(ii) *If, in addition,* $|c_1(\{x,y\})| = |c_2(\{x,y\})|$*, for every* $x, y \in X$*, then* $Inc(\succsim_1) \subseteq Inc(\succsim_2)$*;*

(iii) *If, in addition,* $c_1(\{x,y\}) = c_2(\{x,y\})$ *for all* $x, y \in X$*, then*

$$\succ_{c_2} \subseteq \succ_{c_1} \text{ and } \sim_{c_2} \subseteq \sim_{c_1} .$$

4.2.6 The Unambiguously Preferred Relation

When choices are rational, we can infer preferences from choices and then conduct welfare analysis based on these preferences. But how to conduct welfare analysis when choices are not rational? Green and Hojman (2007) and Bernheim and Rangel (2009) tackle this question and, through different routes, they get to the same conclusion: we should use a stronger criterion to conclude that one alternative is revealed preferred to another.

Formally, given a choice correspondence $c \in \mathcal{C}(X, \mathcal{A})$ and $x, y \in X$, we say that x is *unambiguously preferred to* y, and write $xP_c^\star y$, if for every $A \in \mathcal{A}$ with $x \in A$, $y \notin c(A)$. When all finite menus are in $\mathcal{A}$, Bernheim and Rangel (2009) show that this relation is the most complete acyclic relation that satisfies the property that, for every $A \in \mathcal{A}$, $c(A) \subseteq \text{MAX}(A, P_c^\star)$.

I follow Green and Hojman (2007) and Bernheim and Rangel (2009) in the definition of $P_c^\star$ but propose a different version of the $R_c^\star$ relation in Bernheim and Rangel (2009).[14] Given $c \in \mathcal{C}(X, \mathcal{A})$, we say that x is *unambiguously indifferent to* y, and write $xI_c^\star y$, if, for all $A \in \mathcal{A}$ with $x, y \in A$, we have that $x \in c(A)$ if, and only if, $y \in c(A)$. Define $R_c^\star := P_c^\star \cup I_c^\star$. Analogously to $P_c^\star$, when all finite menus are in $\mathcal{A}$, $R_c^\star$ is the most complete Suzumura-consistent relation that satisfies the property that, for every $A \in \mathcal{A}$, $c(A) \subseteq \text{MAX}(A, R_c^\star)$.

Intuitively, $xR_c^\star y$ when there are no inconsistencies in the revealed rankings of x and y from different menus. One can thus interpret that the decision-maker has a well-defined ranking between these alternatives. Moreover, when we observe choices from pairwise menus, this ranking is revealed from the choice in the menu $\{x, y\}$.

Bernheim (2016) states that

[14] Bernheim and Rangel (2009) define $xR_c^\star y$ if, and only, it is not the case that $yP_c^\star x$. $R_c^\star$ is then the largest relation that has $P^\star$ as its strict part. However, when making weak welfare comparisons, Bernheim and Rangel (2009) use the relation R', which they define as $xR'y$ if, for every menu $A \in \mathcal{A}$ with $x, y \in A$, $x \in c(A)$ whenever $y \in c(A)$. Therefore, the symmetric part of R' is what I call the unambiguously indifferent relation.

> In settings where choice inconsistencies are pervasive, $P^\star_c$ *may not be very discerning*. Whether the resulting ambiguity undermines our ability to draw useful welfare conclusions depends on the context ... That said, *a lack of discernment will certainly prove problematic* in some instances. (Italics added)

The next proposition shows that $\succsim_{\text{rat}}$ can be used to formalize the statement that the unambiguous welfare relation "may not be very discerning".

Proposition 12. *Assume that all finite subsets of X are in $\mathcal{A}$. Let $c_1, c_2 \in \mathcal{C}(X,\mathcal{A})$ and assume that $c_1 \succsim_{rat} c_2$. Then, if $xR^\star_{c_2}y$, then either $xR^\star_{c_1}y$ or $yR^\star_{c_1}x$, and, hence, $Inc(R^\star_{c_1}) \subseteq Inc(R^\star_{c_2})$. If, in addition, $c_1(\{x,y\}) = c_2(\{x,y\})$ for all $x, y \in X$, then $xP^\star_{c_2}y$ implies $xP^\star_{c_1}y$ and $xI^\star_{c_2}y$ implies $xI^\star_{c_1}y$.*

Remark 4. The incompleteness of $R^\star_c$ is problematic for individual welfare analysis. The same holds *a fortiori* for the welfare analysis of groups because the incompleteness of the welfare relations of the members of the group will lead to the incompleteness of the corresponding Pareto relations of the group. Therefore, the set of Pareto optimal allocations will likely increase as the choices of the members of the group become more and more incompatible with preference maximization (according to $\succsim_{\text{rat}}$). This implies that the Pareto criterion proposed by Bernheim and Rangel (2009) will not be "very discerning," which might be "problematic in some instances."

5 $\succsim_{\textbf{rat}}$ and Indices of Incompatibility

The standard approach to comparisons of rationality relies on indices of incompatibility. I now define what an index of incompatibility is and when it "agrees" with $\succsim_{\text{rat}}$.

Definition 5. Given a choice space $(X,\mathcal{A})$, an **index of incompatibility in** $(X,\mathcal{A})$ is a function $I : \mathcal{C}(X,\mathcal{A}) \to \mathbb{R}_+$ such that $I(c) = 0$ if, and only if, c is rationalizable on $\mathcal{A}$. We say that an index of incompatibility I is:

(i) **weakly consistent with** $\succsim_{\text{rat}}$ if $c_1 \succsim_{\text{rat}} c_2$ implies $I(c_1) \leqslant I(c_2)$;

(ii) **consistent with** $\succsim_{\text{rat}}$ if, in addition, $c_1 \succ_{\text{rat}} c_2$ implies $I(c_1) < I(c_2)$.

In this section, I propose an intuitive characterization of (weak) consistency with $\succsim_{\text{rat}}$. The characterization sheds light on why the existing indices of incompatibility (that I am aware of) disagree with $\succsim_{\text{rat}}$, which I discuss in the context of my leading example (Table 1).

As mentioned in the introduction, the only indices of incompatibility in the literature that can deliver the ranking in the leading example are those that count violations of rationality. If, however, we introduce one more friend whose choices are intuitively less rational than those of the other five friends in the example, the methods of counting violations (that I am aware of) do not deliver this ranking whereas $\succsim_{\text{rat}}$ does (see Examples 5 and 6).

Although no existing index is consistent with $\succsim_{\text{rat}}$, some are weakly consistent with $\succsim_{\text{rat}}$, including the well-known Houtman-Maks index (Example 2). I then propose a method to adjust weakly consistent indices to get to consistent ones and use it to propose two new indices of incompatibility.

5.1 Predictive Errors and Consistency with $\succsim_{\text{rat}}$

I motivated $\succsim_{\text{rat}}$ by arguing that an analyst who has to judge the rationality of choices based only on the choices cannot disregard the only type of evidence she has access to, namely violations of the consistency conditions that preference maximization imposes on choices across menus. To make comparative judgments of rationality, she should thus comparatively check for the existence of these violations.

A different way to compare the rationality of choices is to assess the relative predictive performance of the preference maximization model. That is, suppose the analyst had access to Alice and Bob's choices from menus in a collection $\mathcal{B}$ and based on these choices and the assumption of rationality tries to predict what Alice and Bob would choose from menus not in $\mathcal{B}$. If the analyst makes a mistake when predicting Bob's choices whenever she makes a mistake predicting Alice's choices, she would declare that Alice's choices are more compatible with preference maximization than Bob's, at least given their choices in $\mathcal{B}$.

If this were true for any collection $\mathcal{B}$, then she would be forced to conclude that Alice's choices are more compatible with preference maximization than Bob's. This perspective provides an equivalent way of interpreting $\succsim_{\text{rat}}$ that leads to an intuitive characterization of when an index of incompatibility is (weakly) consistent with $\succsim_{\text{rat}}$. To formalize the argument, I need the following definition.

Definition 6. Given a choice correspondence $c \in \mathcal{C}(X, \mathcal{A})$, a sub-collection $\mathcal{B} \subseteq \mathcal{A}$ and a menu $A \in \mathcal{A}$, we say that that A is a **$\mathcal{B}$-predictive error on** c if c is not rationalizable on $\mathcal{B} \cup \{A\}$. If, in addition, c is rationalizable on $\mathcal{B}$, we say that A is a **proper $\mathcal{B}$-predictive error** on c. The **predictive error map** $\Theta : \mathcal{C}(X, \mathcal{A}) \to \left(2^{\mathcal{A}}\right)^{2^{\mathcal{A}}}$ is defined as

$$\Theta(c)(\mathcal{B}) \coloneqq \{A \in \mathcal{A} : A \text{ is a } \mathcal{B}\text{-predictive error on } c\}\,.$$

The next result shows that comparatively checking for violations of rationality and comparatively checking for (proper) predictive errors are equivalent ways of interpreting $\succsim_{\text{rat}}$.

Proposition 13. *For any $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$, the following statements are equivalent:*

(i) $c_1 \succsim_{rat} c_2$;

(ii) $\Theta(c_1)(\mathcal{B}) \subseteq \Theta(c_2)(\mathcal{B})$ *for all* $\mathcal{B} \subseteq \mathcal{A}$;

(iii) $\Theta(c_1)(\mathcal{B}) \subseteq \Theta(c_2)(\mathcal{B})$, *for all* $\mathcal{B} \subseteq \mathcal{A}$ *such that* c_1 *and* c_2 *are rationalizable in it.*

For expositional convenience, define the partial order $\sqsubseteq$ on $\Theta(\mathcal{C}(X, \mathcal{A}))$ by $\Theta(c_1) \sqsubseteq \Theta(c_2)$ iff $\Theta(c_1)(\mathcal{B}) \subseteq \Theta(c_2)(\mathcal{B})$, for all $\mathcal{B} \subseteq \mathcal{A}$. Proposition 13 then says that $c_1 \succsim_{\text{rat}} c_2$ if, and only if, $\Theta(c_1) \sqsubseteq \Theta(c_2)$, which leads to a characterization of consistency with $\succsim_{\text{rat}}$.

Corollary 1. *Given a choice space $(X, \mathcal{A})$, an index of incompatibility $I \in \mathcal{I}(X, \mathcal{A})$ is consistent (resp., weakly consistent) with $\succsim_{rat}$ if, and only if, there is a function $g : \Theta(\mathcal{C}(X, \mathcal{A})) \to \mathbb{R}$ such that:*

(i) $I = g \circ \Theta$;

(ii) *For all* $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$, $\Theta(c_1) \sqsubseteq \Theta(c_2)$ *and* $\Theta(c_1) \neq \Theta(c_2)$ *implies* $g(\Theta(c_1)) < g(\Theta(c_2))$ *(resp.,* $g(\Theta(c_1)) \leqslant g(\Theta(c_2))$*);*

(iii) *For all* $c \in \mathcal{C}(X, \mathcal{A})$, $g(\Theta(c)) = 0$ *if, and only if,* c *is rational.*

Condition (i) of Corollary 1 says that any index of incompatibility (weakly) consistent with $\succsim_{\text{rat}}$ is an aggregator of predictive errors. Condition (ii) is the key property that this aggregator must satisfy, namely it must increase as the collections of predictive errors increase. Condition (iii) is just a normalization given our definition of an index of incompatibility.

Remark 5. When c is not rationalizable on a sub-collection $\mathcal{B} \subseteq \mathcal{A}$, $\Theta(c)(\mathcal{B}) = \mathcal{A}$. In this case, interpreting $\Theta(c)(\mathcal{B})$ as a collection of predictive errors is inappropriate. The equivalence between (i) and (iii) in Proposition 13 says that we can ignore these sub-collections when using $\succsim_{\text{rat}}$ to compare the rationality of two choice correspondences. However, Corollary 1 implies that we cannot restrict attention to proper predictive errors to get a characterization of (weak) consistency with $\succsim_{\text{rat}}$. In fact, there are indices of incompatibility that are consistent with $\succsim_{\text{rat}}$ but that use information about non-proper predictive errors (e.g., the index obtained by summing $|\Theta(c)(\mathcal{B})|$ for all $\mathcal{B} \subseteq \mathcal{A}$). There are also indices that are strictly increasing in the cardinality of the sets of proper predictive errors of a choice correspondence c but that are not consistent with $\succsim_{\text{rat}}$ (e.g., the index obtained by summing $|\Theta(c)(\mathcal{B})|$ for all $\mathcal{B} \subseteq \mathcal{A}$ where c is rationalizable).

Remark 6. We cannot always construct indices of incompatibility consistent with $\succsim_{\text{rat}}$ (while, of course, $\succsim_{\text{rat}}$ remains applicable). We might run into trouble if $\mathcal{C}(X, \mathcal{A})$ is sufficiently rich, because $\succsim_{\text{rat}}$ might have "too many" indifference classes. In Appendix C, I show that this happens in the standard environment of consumer theory.

5.2 Why Do Existing Indices Disagree with $\succsim_{\text{rat}}$?

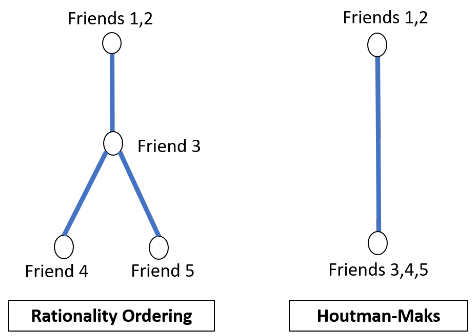
In this section, I discuss several incompatibility indices that apply to the choice environment of this paper. I show how these indices rank Friends 1 to 5 in the leading example (Table 1) and discuss their consistency with $\succsim_{\text{rat}}$.

Example 2. (The Houtman-Maks Index) Houtman and Maks (1985) propose to measure incompatibility with rational choice by the minimum number of menus one needs to eliminate so that choices become rationalizable in the remaining menus. Formally, the Houtman-Maks index is defined as

$$I_{HM}(c) := \min \left\{ |\mathcal{B}| : \ \mathcal{B} \subseteq \mathcal{A} \text{ with } c|_{\mathcal{A} \setminus \mathcal{B}} \text{ is rationalizable} \right\}.$$

Figure 2 displays how I_{HM} ranks the rationality of the choices of Friends 1 to 5. Among the friends that $\succsim_{\text{rat}}$ can compare, I_{HM} disagrees with the rationality assessments of $\succsim_{\text{rat}}$ in the rankings of Friends 3 and 4 and Friends 3 and 5. The issue in both cases is that I_{HM} does not increase with the introduction of a new violation of rationality in choices. For Friends 3 and 4, when Friend 4 chooses b instead of a from $\{a, b\}$, a new violation of rationality emerges in the sub-collection $\{\{a, b, c\}, \{a, b\}\}$,

Figure 2: Comparison $\succsim_{\text{rat}}$ and I_{HM}



but I_{HM} ignores it. For Friends 3 and 5, when Friend 5 chooses b instead of c from $\{b, c\}$, a new violation of rationality emerges in the sub-collection of pairwise menus, but, again, I_{HM} ignores this information.

Nevertheless, the Houtman-Maks index is weakly consistent with $\succsim_{\text{rat}}$. In fact, this follows from Corollary 1 once we realize that

$$I_{HM}(c) = \min_{\mathcal{B} \in \text{MCC}(c)} |\Theta(c)(\mathcal{B})|.$$

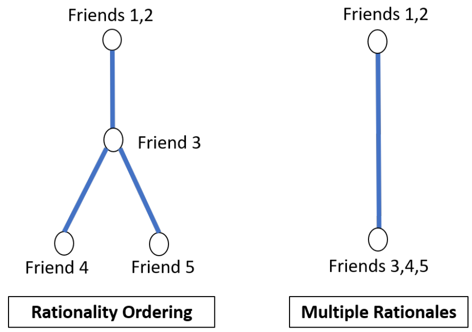
This representation and Corollary 1 also shed light on why I_{HM} is not, in general, consistent with $\succsim_{\text{rat}}$. I_{HM} only considers the number of predictive errors of the maximal compatible collections with the fewest predictive errors. Given that $\Theta(c)(\mathcal{B}) = \mathcal{A} \backslash \mathcal{B}$ for all $\mathcal{B} \in \text{MCC}(c)$, if the introduction of new violations of rationality in choices does not affect the maximum cardinality of the collections in $\text{MCC}(c)$, the Houtman-Maks index does not change.

Example 3. (The Multiple Rationales Index) In Section 4.1, we introduced the model of Rationalization by Multiple Rationales (RMR) proposed by Kalai et al. (2002). The *multiple rationales index,* I_{MR}, measures the rationality of $c \in \mathcal{C}(X, \mathcal{A})$ by the minimum cardinality of a RMR-representation of c. Formally,

$$I_{MR}(c) \coloneqq \min \{|\mathcal{R}| : \mathcal{R} \text{ is a RMR-representation of } c\} - 1.$$

Figure 3 displays how I_{MR} ranks the rationality of the choices of Friends 1 to 5. The ranking of I_{MR} is thus the same as that of I_{HM} in our leading example. Moreover, the reason behind the disagreement between $\succsim_{\text{rat}}$ and I_{MR} is the same as the reason behind the disagreements between $\succsim_{\text{rat}}$ and I_{HM}, namely that I_{MR} need not react to

Figure 3: Comparison $\succsim_{\text{rat}}$ and I_{MR}



the introduction of new violations of rationality in choices. Nevertheless, it follows from Proposition 3 that I_{MR} is weakly consistent with $\succsim_{\text{rat}}$.

Example 4. (The Swap Index) Apesteguia and Ballester (2015) propose to measure the rationality of choices by the minimum (ordinal) welfare loss incurred under the assumption of preference maximization. To make this precise, assume X to be finite, and fix a total order $\succcurlyeq$ on X. Apesteguia and Ballester (2015) measure the welfare loss associated with $\succcurlyeq$ by the total number of swaps one needs to make to account for the choices, where the number of swaps needed to account for an alternative x being chosen from a menu A is $|\{y \in A : y \succ x\}|$.[15] Formally, the Swap index is defined as

$$I_S(c) := \min \sum_{A \in \mathcal{A}} |\{x \in A : x \succ c(A)\}|,$$

where the minimum is taken over all total orders on X. Any total order that minimizes the sum in this definition is called a *swaps preference.*

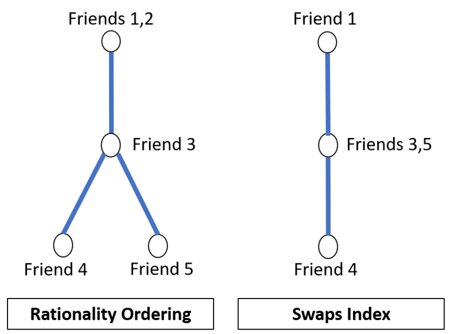
Figure 4 displays how I_S ranks the rationality of the choices of Friends 1, 3, 4, and 5.[16] In contrast to the Houtman-Maks and the Multiple Rationales indices, the Swaps index agrees with $\succsim_{\text{rat}}$ in the rankings of Friends 3 and 4, reacting to the fact that the choice of a in the menu $\{a, b, c\}$ is a "smaller" violation for Friend 3 than for Friend 4. However, I_S disagrees with $\succsim_{\text{rat}}$ in the rankings of Friends 3 and 5, failing to react to the introduction of a new violation of rationality in choices.

The Swaps index may not even be weakly consistent with $\succsim_{\text{rat}}$. To see this, let

[15] Apesteguia and Ballester (2015) take as primitives frequency distributions f over pairs (x, A) with $x \in A$, that is, $f(x, A)$ is the fraction of times we observe x being chosen from A. Therefore, the Swaps index can deal with representations of choices that $\succsim_{\text{rat}}$ cannot. In the choice environment of this paper, however, the Swaps index can only compare choice functions whereas $\succsim_{\text{rat}}$ applies to choice correspondences as well.

[16] See the previous footnote.

Figure 4: Comparison $\succsim_{\text{rat}}$ and I_S



$X = \{x, y, z, w\}$, and let $\mathcal{A}$, c_1 and c_2 be given by

$\mathcal{A}$	$\{x, y, z, w\}$	$\{x, y, w\}$	$\{x, z, w\}$	$\{x, w\}$
c_1	w	y	z	w
c_2	x	w	w	w

It is easily checked that $c_1 \succ_{\text{rat}} c_2$, and that $I_S(c_2) = 1$. Nevertheless, we contend that $I_S(c_1) = 2$. Fix a swaps preference $\succ_S$ for c_1. Clearly, x cannot be the most preferred nor the second most preferred element of $\succ_S$. If w is the most preferred element of $\succ_S$, we need 2 swaps to account for c_1. If either y or z is the most preferred element of $\succ_S$, we need at least 2 swaps to account for c_1, establishing that $I_S(c_1) = 2$. This example suggests that the Swaps index can conflate homogeneity in choice with rationality.

Example 5. (Counting Violations) I now discuss three ways of counting violations of the preference maximization model. The first way is to count the sub-collections where choices are not compatible with preference maximization. Clearly, this will lead to an index that is consistent with $\succsim_{\text{rat}}$. To the best of my knowledge, no one has proposed counting violations in this way.[17]

The second way, adopted by Swofford and Whitney (1986) and Famulari (1995), is to count the violations of an axiom known to characterize the rationality of choices. In the choice framework I use, their proposals translate to measuring the number of violations of rationality of $c \in \mathcal{C}(X, \mathcal{A})$ by the number of pairs of alternatives $(x, y) \in X^2$ such that $x \, \text{tran}(R(c)) \, y$,[18] but $y \, P(c) \, x$. Formally, let

$$I_{\text{pair}}(c) \coloneqq \left| \left\{ (x, y) \in X^2 : \, x \, \text{tran}(R(c)) \, y \text{ and } y \, P(c) \, x \right\} \right| .$$

[17] I define an index that does that in Example 7.

[18] $\text{tran}(R(c))$ is the transitive closure of $R(c)$., i.e., the smallest transitive relation that contains $R(c)$.

In the beer example, Friend 3 has two such violations, namely (a, c), and (c, a). Friend 4 has four such violations, (a, c), (c, a), (a, b), and (b, a). Friend 5 has four such violations, namely (a, c), (c, a), (b, a), and (c, b). Therefore, I_{pair} delivers a ranking that is consistent with the ranking of $\succsim_{\text{rat}}$ in the beer example. Consider, however, the choices of a new friend, Friend 6:

	$\{a, b, c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, c\}$
Friend 6	a, b, c	a	b	c

According to $\succsim_{\text{rat}}$, Friend 6 is the least rational of all friends because his choices violate rationality in any sub-collection where rationality can be violated. However, I_{pair} only counts three violations for Friend 6, namely (b, a), (c, b), and (a, c). Therefore, I_{pair} declares Friend 6 as more rational than Friends 4 and 5, which shows that it is not consistent with $\succsim_{\text{rat}}$.

The third way to count violations is to count the number of revealed cycles in the data.[19] Although different ways of counting revealed cycles[20] will lead to different conclusions about the rationality of choices, counting cycles is not even weakly consistent with $\succsim_{\text{rat}}$, unless one counts cycles by counting the number of sub-collections that reveal them. Disagreements can happen because different sub-collections can reveal the same cycle whereas the same sub-collection can reveal multiple cycles.

Example 6. (The Irrationality Kernel) Caradonna (2019) proposes a different way of counting (proper) revealed cycles in choices by noting that some revealed cycles are unavoidable given the existence of other revealed cycles. Caradonna (2019) then argues one should not count these unavoidable revealed cycles when counting the number of cycles in choices.

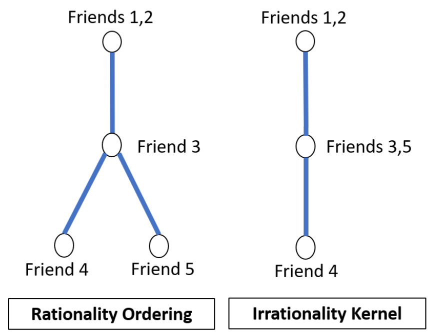
Formally, fix a choice correspondence $c \in \mathcal{C}(X, \mathcal{A})$ and assume $\mathcal{A}$ is finite. Given a proper revealed cycle $(x_1, \ldots, x_n)$ of c, a collection $\mathcal{B}_{(x_1,\ldots,x_n)} \subseteq \mathcal{A}$ is said to be a *generator* for $(x_1, \ldots, x_n)$ if $\mathcal{B}_{(x_1,\ldots,x_n)}$ is a minimal incompatible collection of c and $(x_1, \ldots, x_n)$ is a proper revealed cycle of $c|_{\mathcal{B}_{(x_1,\ldots,x_n)}}$. Given a proper revealed cycle $\boldsymbol{a}$ and a generator $\mathcal{B}_{\boldsymbol{a}}$ for $\boldsymbol{a}$, we say that a menu A *covers* $\mathcal{B}_{\boldsymbol{a}}$ when the existence of the revealed cycle $\boldsymbol{a}$ implies a proper revealed cycle involving A no matter what is chosen from the menu A.[21] Let $\mathcal{C}(\mathcal{B}_{\boldsymbol{a}})$ be the collection of menus that cover $\mathcal{B}_{\boldsymbol{a}}$ and

[19] For the definition of proper revealed cycles, see Section 2.

[20] Even how we define revealed cycles matters when counting them.

[21] Formally, given a proper revealed cycle $(x_1, \ldots, x_n)$ of c and a generator $\mathcal{B}_{(x_1,\ldots,x_n)}$, we say that A *covers* $\mathcal{B}_{(x_1,\ldots,x_n)}$ if $A \subseteq \bigcup \mathcal{B}_{(x_1,\ldots,x_n)}$ and either of the two conditions hold: (i) $\{x_1, \ldots, x_n\} \subseteq A$; (ii) for some $i, j \in \{1, \ldots, n\}$ such that $1 < |i - j| < n - 1$, $\{x_i, x_j\} \subseteq A$.

Figure 5: Comparison $\succsim_{\text{rat}}$ and I_{IK}



define $\mathcal{C}ov(\mathcal{B}_{\boldsymbol{a}}) := \mathcal{B}_{\boldsymbol{a}} \cup \mathcal{C}(\mathcal{B}_{\boldsymbol{a}})$. We then say that a proper revealed cycle $\boldsymbol{a}$ *covers a proper revealed cycle* $\boldsymbol{b}$ if there exist generators $\mathcal{B}_{\boldsymbol{a}}$ and $\mathcal{B}_{\boldsymbol{b}}$ such that $\mathcal{B}_{\boldsymbol{b}} \subseteq \mathcal{C}ov(\mathcal{B}_{\boldsymbol{a}})$. Intuitively, when a proper revealed cycle $\boldsymbol{a}$ covers a proper revealed cycle $\boldsymbol{b}$, $\boldsymbol{b}$ is unavoidable given $\boldsymbol{a}$.

A sub-collection of proper revealed cycles $\mathcal{I}$ of c is an irrationality kernel of c if for every proper revealed cycle $\boldsymbol{b}$ of c there exists a proper revealed cycle $\boldsymbol{a} \in \mathcal{I}$ and proper revealed cycles $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ of c such that $\boldsymbol{a}_1 = \boldsymbol{a}$, $\boldsymbol{a}_n = \boldsymbol{b}$, and $\boldsymbol{a}_i$ covers $\boldsymbol{a}_{i+1}$ for $i \in \{1, \dots, n-1\}$. Therefore, an irrationality kernel is a subset of proper revealed cycles that (indirectly) explains all other proper revealed cycles in choices. Caradonna (2019) proposes we measure incompatibility with preference maximization by

$$I_{IK}(c) := \min\{|\mathcal{I}| : \mathcal{I} \text{ is an irrationality kernel of } c\}.$$

Figure 5 displays how I_{IK} ranks the rationality of the choices of Friends 1 to 5. This ranking disagrees with $\succsim_{\text{rat}}$ in the rankings of Friends 3 and 5. Friends 3 and 5 share the revealed cycle (a, c). However, the cycle (a, b, c) in the choices of Friend 5 covers this cycle. Thus, the Irrationality Kernel counts the cycle (a, c) as evidence of irrationality for Friend 3 but not for Friend 5.

The Irrationality Kernel is not (weakly) consistent with $\succsim_{\text{rat}}$. To see this, consider Friend 6 in Example 5. We know that $\succsim_{\text{rat}}$ ranks Friend 6 as strictly less rational than Friends 1 to 5. Because, however, Friend 6 has only one independent cycle, I_{IK} ranks Friend 6 as equally rational to Friends 3 and 5 and more rational than Friend 4.

5.3 A Method for Adjusting Weakly Consistent Indices

Although no index of incompatibility (that I am aware of) is consistent with $\succsim_{\text{rat}}$, some of them are weakly consistent. The problem with weakly consistent indices is that they can fail to increase as we modify choices in a way that existing violations are preserved and new violations of rationality are introduced. Therefore, weakly consistent indices are not discerning enough.

I now propose a method that corrects this lack of discernment, delivering consistency with $\succsim_{\text{rat}}$, while allowing we incorporate domain-specific knowledge to evaluate the rationality of choices. The idea is simple: we calculate the weakly consistent index in each sub-collection and then aggregate these values using some weights. The method relies on the following Lemma, whose proof follows immediately from Definition 5.

Lemma 2. *Let $(X, \mathcal{A})$ be a choice space, and, for every $\mathcal{B} \subseteq \mathcal{A}$, let $I_{\mathcal{B}}$ be an index of incompatibility on $(X, \mathcal{B})$ that is weakly consistent with $\succsim_{rat}$. Then, for every $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$,*

$$c_1 \succsim_{rat} c_2 \text{ if, and only if, for all non-empty } \mathcal{B} \subseteq \mathcal{A},\ I_{\mathcal{B}}(c_1|_{|\mathcal{B}}) \leqslant I_{\mathcal{B}}(c_2|_{|\mathcal{B}}).$$

The next result, which introduces the method I propose, is an easy consequence of Lemma 2.

Proposition 14. *Let $(X, \mathcal{A})$ be a choice space where $\mathcal{A}$ is finite. For every $\mathcal{B} \subseteq \mathcal{A}$, let $I_{\mathcal{B}}$ an index of incompatibility on $(X, \mathcal{B})$ that is weakly consistent with $\succsim_{rat}$ and $w : 2^{\mathcal{A}} \backslash \{\emptyset\} \to [0, 1]$ be a weighting function with $w(\mathcal{B}) > 0$ whenever there is a $c \in \mathcal{C}(X, \mathcal{B})$ that is not rationalizable. Then, the index $\mathcal{I}_{I,w(B)}$ defined as*

$$\mathcal{I}_{I,w(B)}(c) := \sum_{\emptyset \neq \mathcal{B} \subseteq \mathcal{A}} w(\mathcal{B}) I_{\mathcal{B}}(c|_{\mathcal{B}})$$

is consistent with $\succsim_{rat}$.

Remark 7. The weighting function w in Proposition 14 allows we attribute weights to different types of violations of rationality. In this way, we can incorporate our judgments about the importance of different violations of rationality in a way that is consistent with $\succsim_{\text{rat}}$. For example, we might want to attribute lower weights to sub-collections with many menus or to sub-collections composed of menus with many alternatives.

Proposition 14 provides a method to build consistent indices from weakly consistent ones. To illustrate the method, I use it to construct two new indices of incompatibility.

Example 7. (The Probability Index) Define the index of incompatibility I_{binary} on $(X, \mathcal{B})$ by

$$I_{\mathcal{B}\text{-binary}}(c) := \begin{cases} 0 & \text{, if } c \text{ is rationalizable on } \mathcal{B} \\ 1 & \text{, otherwise} \end{cases},$$

and note that it is weakly consistent with $\succsim_{\text{rat}}$. The **Probability Index** is defined as

$$I_P(c) := \frac{\sum_{\emptyset \neq \mathcal{B} \subseteq \mathcal{A}} I_{\mathcal{B}\text{-binary}}(c|_{\mathcal{B}})}{2^{|\mathcal{A}|} - 1}.$$

By Proposition 14, I_P is consistent with $\succsim_{\text{rat}}$. The index can be interpreted as the probability of drawing a sub-collection of menus where choices are not rationalizable, if the distribution we use is the uniform distribution over non-empty collections.

Example 8. (The Average Houtman-Maks Index) The Houtman-Maks index is weakly consistent with $\succsim_{\text{rat}}$ for every choice space $(X, \mathcal{B})$ (Example 2). The **average Houtman-Maks Index** is defined as:

$$I_{AHM}(c) := \frac{\sum_{\emptyset \neq \mathcal{B} \subseteq \mathcal{A}} I_{HM}(c|_{\mathcal{B}})}{2^{|\mathcal{A}|} - 1}.$$

Again, By Proposition 14, I_{AHM} is consistent with $\succsim_{\text{rat}}$.

6 $\succsim_{\text{rat}}$ and Rationality Tests

Rationality tests are field or lab experiments that study to what extent people's choices in a domain are compatible with the maximization of a preference relation. Two choice environments, with their corresponding definitions of preference maximization, are used in these tests.[22]

The first environment is the one used in the seminal Afriat (1967).[23] Fix $n, k \in \mathbb{N}$. The primitives of this environment are collections $\{(\boldsymbol{p}^i, \boldsymbol{x}^i) \in \mathbb{R}^{2n}_{+} : i \in \{1, \ldots, k\}\}$ with the interpretation that, for every $i \in \{1, \ldots, k\}$, we observe the bundle $\boldsymbol{x}^i$ being chosen from the budget set with prices $\boldsymbol{p}^i$ and income $\boldsymbol{p}^i \cdot \boldsymbol{x}^i$ denoted by $B(\boldsymbol{p}^i, \boldsymbol{p}^i \cdot \boldsymbol{x}^i)$.[24]

[22] For a discussion of the relationship between these environments, see Nishimura et al. (2017).

[23] For instance, see Mattei (2000), Harbaugh et al. (2001), Andreoni and Miller (2002), Février and Visser (2004), Choi et al. (2014).

[24] Here, $\cdot$ denotes the dot product in $\mathbb{R}^n$.

The underlying choice space is then given by $(\mathbb{R}^n, \mathcal{A}_B)$, where

$$\mathcal{A}_B = \left\{ B(\boldsymbol{p}^i, \boldsymbol{p}^i \cdot \boldsymbol{x}^i) : i \in \{1, \ldots, k\} \right\}.$$

The collection of observed price-bundle pairs naturally induce a choice correspondence $c \in \mathcal{C}(\mathbb{R}^n, \mathcal{A}_B)$ by

$$c(B(\boldsymbol{p}^i, y)) \coloneqq \left\{ \boldsymbol{x}^i \in \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k\} : \boldsymbol{p}^i \cdot \boldsymbol{x}^i = y \right\}.$$

In this environment, choices are *rational* if there is a complete and transitive binary relation $\succsim$ that is strictly increasing with respect to the standard ordering of $\mathbb{R}^n$ and such that $c(A) \subseteq \max(A, \succsim)$, for every $A \in \mathcal{A}_B$.

In this paper, however, I use a different choice environment. In it, the choice space $(X, \mathcal{A})$ is arbitrary, and choices are given by choice correspondences on $(X, \mathcal{A})$. Moreover, choices are *rational* if there is a complete and transitive binary relation $\succsim$ such that $c(A) = \max(A, \succsim)$ for every $A \in \mathcal{A}$.

Therefore, the choice environment in Afriat (1967) dispenses with the assumption that we observe all the alternatives the decision-maker would choose from a menu. But to do so, the set of alternatives must be endowed with a dominance relation, which can be a restrictive requirement (e.g., what is a dominance relation for preferences over schools?). The choice environment I use applies even when no such dominance relation exists.[25] But to do so, we need to elicit all the alternatives the decision-maker would choose from a menu.

Bouacida (2021) proposes one method to do so,[26] which he then uses to elicit the choice correspondences of 189 subjects in the lab. The subjects in the experiment choose from all possible non-singleton menus composed of the following four alternatives: an addition task (a); a spellcheck task (s); a memory task (m); and a copy task (c). Each task yields a monetary reward conditional on the subject's performance.[27] Therefore, the choice space of the experiment consists of the set of alternatives $X = \{a, s, m, c\}$ and the collection of menus $\mathcal{A} = 2^X \setminus \{\emptyset, \{a\}, \{s\}, \{m\}, \{c\}\}$.

I use the choice correspondences elicited by Bouacida (2021) to first address how

[25] Despite its greater flexibility, I could only find four papers that elicit choices in this choice environment: Manzini and Mariotti (2010), Qiu and Ong (2017), Costa-Gomes et al. (2020), and Bouacida (2021).

[26] For a different method of eliciting choice correspondences, see Balakrishnan et al. (2021).

[27] If a subject chooses more than one alternative from a menu selected for payment, then a random draw determines what alternatives she gets among the selected ones.

Table 2: Summary of $\succsim_{\text{rat}}$

Number of Choice Correspondences	189
Number of Possible Comparabilities of $\succsim_{\text{rat}}$	$17,766$
Number of $\succsim_{\text{rat}}$-comparable pairs	$12,161$
Number of $\succsim_{\text{rat}}$-incomparable pairs	$5,605$
Number of Rational Choices	81

incomplete is $\succsim_{\text{rat}}$ in the data, and whether it has any bite in assessing the rationality of choices. I then compare how the following indices of incompatibility rank the rationality of choices in the experiment: Houtman-Maks (Example 2), Multiple Rationales (Example 3), Probability (Example 7), and Average Houtman-Maks (Example 8). In particular, I calculate how much these indices disagree with each other and how they react to the increase in three types of violations of rationality. Finally, I study why the choices of some subjects fail to be rational by (i) calculating how prevalent the violations of some well-known properties of choice correspondences are, and (ii) checking in what sub-collections of menus the choices of subjects more often violate rationality.

6.1 $\succsim_{\text{rat}}$ in the data

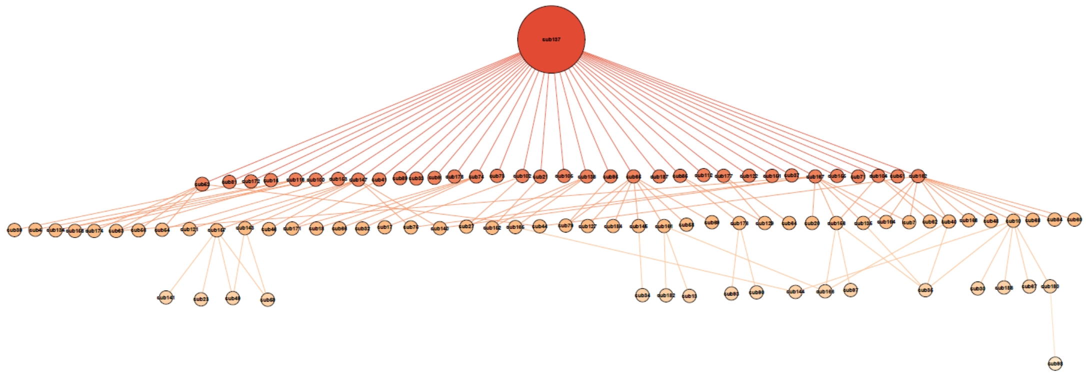
Table 2 presents the summary of $\succsim_{\text{rat}}$ on the choice correspondences elicited by Bouacida (2021). Many of the comparabilities of $\succsim_{\text{rat}}$ in Table 2 are introduced by the 81 rationalizable choice correspondences. In fact, they account for 11,988 of $\succsim_{\text{rat}}$-comparable pairs.[28] These comparisons are trivial because any measure of rationality must agree in them. Therefore, $\succsim_{\text{rat}}$ makes 173 non-trivial comparisons between choice correspondences. Since there are 108 non-rationalizable choice correspondences, each non-rationalizable choice correspondence is compared by $\succsim_{\text{rat}}$ to, on average, 3.2 other non-rationalizable choice correspondences.

Figure 6 displays the Hasse diagram of the partial order $\succcurlyeq_{\text{rat}}$ that $\succsim_{\text{rat}}$ induces by passing to the quotient.[29] There are five levels of rationality in the data. The first level, at the top of Figure 6, consists of the indifference class of rational choices. All indifference classes directly dominated by the indifference class of rational choices are in the second level. All indifference classes directly dominated by an indifference class in the second level are in the third level, and so on.

[28]Each rationalizable choice correspondence is comparable to (i) every other rationalizable choice correspondence, which accounts for $3,240$ $\succsim_{\text{rat}}$-comparable pairs; (ii) to every other non-rationalizable choice correspondence, which accounts for $8,748$ $\succsim_{\text{rat}}$-comparable pairs.

[29]Given two indifference classes I_1 and I_2 of $\succsim_{\text{rat}}$, $I_1 \succcurlyeq_{\text{rat}} I_2$ iff $c_1 \succsim_{\text{rat}} c_2$ for all $(c_1, c_2) \in I_1 \times I_2$.

Figure 6: The Rationality Hierarchy



Note: the size of a node in the diagram is proportional to the number of choices in the indifference class.

The edges between the first and second levels represent the trivial comparisons I alluded to. The comparisons between the second and third levels, third and fourth levels, fourth and fifth levels, and the indirect comparisons these direct comparisons imply are non-trivial. Now, do these comparisons matter in assessing the rationality of choices in the experiment?

When assessing the rationality of choices, experimenters use indices of incompatibility to set a threshold for the "acceptable" degree of violation of rationality. If the degree of violation is below the threshold, preference maximization provides a "sufficiently" good description of choices and should not be rejected.

Suppose we declare that subjects whose choices have a Houtman-Maks index of at most 1 are "sufficiently" rational. In the data, 113 (60%) subjects satisfy this criterion. The Houtman-Maks index, however, is only weakly consistent with $\succsim_{\text{rat}}$. In the data, $\succsim_{\text{rat}}$ ranks 17 of these 32 subjects as less rational than one of the remaining 15 subjects. If we correct the Houtman-Maks index by eliminating these 17 subjects, only 96 (51%) subjects are "sufficiently" rational, a 15% decrease in the proportion of "sufficiently" rational subjects.

6.2 Comparing Different Indices of Incompatibility

I now study how different indices of incompatibility assess the rationality of choices in the data. We focus on the disagreements between the indices, and how they react to the increase in violations of rationality.

Table 3: Disagreement Between Different Rationality Orderings

	$\succsim_{\text{rat}}$	Houtman-Maks	Multiple Rationales	Probability	Average Houtman-Maks
$\succsim_{\text{rat}}$	-	40	87	0	0
Houtman-Maks	23%	-	2,672	1,278	1,278
Multiple Rationales	51%	46%	-	3,411	3,359
Probability	0%	22%	59%	-	64
Average Houtman-Maks	0%	22%	58%	1%	-

Notes:
(i) Above diagonal cells display the (absolute) number of disagreements between the orderings
(ii) Below diagonal cells display the disagreements between the orderings as a percentage of possible disagreements

6.2.1 The Disagreements Between Indices

Every index of incompatibility I on $(X, \mathcal{A})$ induces a (complete) rationality ordering $\succsim_I$ on $\mathcal{C}(X, \mathcal{A})$, namely, for all $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$,

$$c_1 \succsim_I c_2 \text{ if, and only if, } I(c_1) \leqslant I(c_2).$$

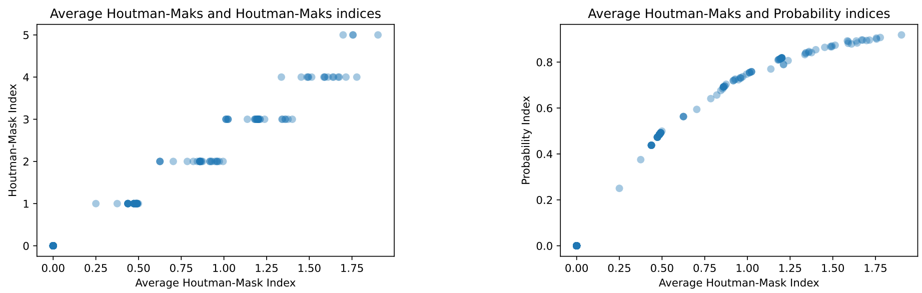
Given two rationality orderings $\succsim_{R_1}$ and $\succsim_{R_2}$ on $\mathcal{C}(X, \mathcal{A})$, we say that they *disagree on a pair* $\{c, c'\} \subseteq \mathcal{C}(X, \mathcal{A})$ if, for every $c_1, c_2 \in \{c, c'\}$,either $c_1 \succsim_{R_1} c_2$ and $c_2 \succ_{R_2} c_1$, or $c_2 \succ_{R_1} c_1$ and $c_1 \succsim_{R_2} c_2$.

By definition of an index of incompatibility, two rationality orderings cannot disagree on pairs in which at least one of the choice correspondences is rational. Therefore, when comparing the orderings generated by two indices of incompatibility I_1 and I_2 in the data of Bouacida (2021), the number of possible disagreements is the number of pairs of non-rational choice correspondences, namely $5,778$. Moreover, the definition of disagreement implies that pairs that one rationality ordering cannot compare do not count as disagreements. Therefore, when comparing the disagreements between the rationality ordering induced by an index I and $\succsim_{\text{rat}}$, the number of possible disagreements in the data reduces to 171, which is the number of pairs $\succsim_{\text{rat}}$ can compare.

Table 3 summarizes the number of disagreements between the different rationality orderings. The above diagonal elements are the number of disagreements between the orderings in the corresponding row and column, and the below diagonal elements present this disagreement as a fraction of possible disagreements.

Three facts are noteworthy in Table 3. First, the ordering induced by the Houtman-Maks index disagrees less than the one induced by the Multiple Rationales index with both $\succsim_{\text{rat}}$ and the orderings induced by the Probability and Average Houtman-Maks indices.

Figure 7: Scatter Plots Between Indices of Incompatibility



Left: I_{AHM} and I_{HM} | **Right**: I_{AHM} and I_P

Second, the Houtman-Maks index and the average Houtman-Maks index disagree in 1,278 pairs. Interestingly, however, only 10 of these disagreements are strict in the sense that for a pair $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$, we have $I_{HM}(c_1) < I_{HM}(c_2)$ and $I_{AHM}(c_2) < I_{AHM}(c_1)$. In the other disagreements, the Houtman-Maks index declares a pair to be equally rational whereas the Average Houtman-Maks index can strictly rank them. The scatter plot on the left of Figure 7 exhibits the relationship between these indices, which shows that the Average Houtman-Maks index is much more discerning than the Houtman-Maks index.

Third, the Probability and the Average Houtman-Maks indices only disagree in 1% of pairs in which they can disagree. Therefore, these indices deliver a similar assessment of the rationality of choices in the data. The scatter plot on the right of Figure 7 exhibits the relationship between these two indices.

6.2.2 Reacting to the Increase in Violations of Rational Choice

Ideally, measures of incompatibility with rationality should increase when the violations of rationality in choices increase. In this spirit, I now analyze how the different indices react to the increase in three types of violations of rationality.

The first type of violation is the number of proper revealed cycles in choices. Given $c \in \mathcal{C}(X, \mathcal{A})$, let $N_{\text{cycles}}(c)$ be the number of such cycles in c. In the choice space of the experiment, the maximum number of proper revealed cycles is 20, once we discard redundancies.[30]

The second type of violation is the number of menus where preferences are not

[30] For example, if (x, y, z) is a proper revealed cycle, then (y, z, x) and (z, x, y) are also proper revealed cycles by our definition. To avoid counting this cycle three times, I count it only once. The number of non-redundant revealed cycles with n elements is $(n-1)!$.

Table 4: Disagreements Between the Evidence and the Indices

	$\succsim_{\text{rat}}$	Houtman-Maks	Multiple Rationales	Probability	Average Houtman-Maks
N_{cycles}	1	2,293	3,380	1,396	1,433
$N_{\max}$	0	166	642	141	141
N_{rev}	0	1,244	2532	476	480

maximized. This provides a test of the validity of the maximization hypothesis. To implement it, however, we need to choose how to infer preferences from choices. We make the standard assumption that preferences are revealed in choices from pairwise menus.[31] Given $c \in \mathcal{C}(X, \mathcal{A})$ such that $c|_{\mathcal{M}_2(X)}$ is rationalizable, let $N_{\max}(c)$ be the number of menus $A \in \mathcal{A}$ such that $c(A) \neq \max(A, B_c)$. In the choice space of the experiment, there are 5 menus with more than two alternatives.

The third type of violation is the number of pairs of alternatives that have menu-dependent revealed rankings. That is, given $x, y \in X$, if in a menu that both are available a subject chooses x but not y, while in a different menu that both are available she either chooses both or chooses y but not x, then x and y have menu-dependent revealed rankings.[32] The number of such pairs measures the instability of a subject's pairwise rankings of alternatives. Given $c \in \mathcal{C}(X, \mathcal{A})$, let $N_{\text{Rev}}(c)$ be the number of pairs that have menu-dependent revealed rankings in c. In the choice space of the experiment, the maximum number of such pairs is 6.

Each of these three types of violation induces an ordering on $\mathcal{C}(X, \mathcal{A})$. We can then calculate the number of disagreements between these orderings and the rationality orderings induced by the indices we have been considering. [33] Table 4 summarizes the disagreements between these three types of violation and the different rationality orderings.

Two things are noteworthy in Table 4. First, $\succsim_{\text{rat}}$ only disagrees with the ordering induced by the counting of proper revealed cycles in one pair. Both choice correspondences in this pair have the same number of proper revealed cycles whereas $\succsim_{\text{rat}}$ strictly ranks them. Second, the Probability and Average Houtman-Maks indices outperform the Houtman-Maks and the Multiple Rationales indices in reacting to all

[31]Since the choices of 60 subjects are not rationalizable in pairwise menus, we exclude them when dealing with this violation.

[32]Formally, pairs that exhibit menu-dependent revealed rankings are those that are incomparable according to the relation $R_c^\star$ defined in Section 4.2.6.

[33]Given that $N_{\max}$ and N_{rev} only provide partial evidence of incompatibility with rationality, I exclude pairs $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ with $N_{\max}(c_1) = N_{\max}(c_2)$ (resp., $N_{\text{rev}}(c_1) = N_{\text{rev}}(c_2)$) when counting disagreements between $N_{\max}$ (resp., N_{rev}) and the rationality orderings.

Table 5: Properties of Choice Correspondences

	Statement
Property α	$\forall A, B \in \mathcal{A}$ with $A \subseteq B$, $c(B) \cap A \subseteq c(A)$
Property β	$\forall A, B \in \mathcal{A}$ with $A \subseteq B$, if $c(B) \cap A \neq \emptyset$, then $c(A) \subseteq c(B) \cap A$
Aizerman's Property	$\forall A, B \in \mathcal{A}$ with $A \subseteq B$, if $c(B) \subseteq A$, then $c(B) \subseteq c(A)$
Property γ	$\forall A, B \in \mathcal{A}$ and $x \in A$ if $x \in c(A) \cap c(B)$, then $x \in c(A \cup B)$
Always Chosen	$\forall A \in \mathcal{A}$ and $x \in A$, if $x \in c(\{x, y\})$ $\forall y \in A$, then $x \in c(A)$
Strict Pairwise Transitivity	$\forall x, y, z \in X$ if $c(\{x, y\}) = \{x\}$ and $c(\{y, z\}) = \{y\}$, then $c(\{x, z\}) = \{x\}$
Weak Pairwise Transitivity	$\forall x, y, z \in X$ if $x \in c(\{x, y\})$ and $y \in c(\{y, z\})$, then $x \in c(\{x, z\})$
Strict Pairwise Acyclicity	$\forall n > 3$ and distinct $x_1, \ldots, x_n \in X$ if $x_1 B_c^{>} \cdots B_c^{>} x_n$, then $x_1 \in c(\{x_1, x_n\})$
Weak Pairwise Acyclicity	$\forall n > 3$ and distinct $x_1, \ldots, x_n \in X$ if $x_1 B_c \cdots B_c x_n$, then $x_1 \in c(\{x_1, x_n\})$

three types of violations, a point that can be seen graphically in Figures 8 to 10.

The greater responsiveness of the average Houtman-Maks and the Probability indices to the increase in the three types of violations when compared to the Houtman-Maks and Multiple Rationales indices translate to the rationality orderings induced by the first pair having more indifference classes than the rationality orderings induced by the second pair. Whereas the ordering induced by the Houtman-Maks and Multiple Rationales indices have 6 and 4 indifference classes in the data, the Probability and the Average Houtman-Maks indices have 62 and 64 indifference classes.

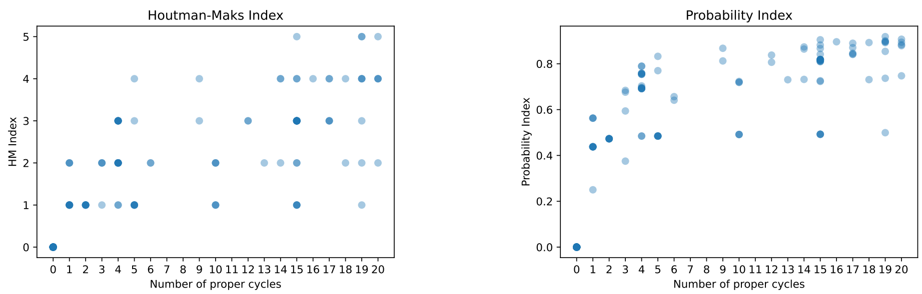
6.3 What Are the Most Common Violations of Rationality in the Data?

To understand how subjects violate rationality, I first check for the violation of the properties listed in Table 5. These are properties that many models of boundedly rational choice satisfy and, hence, checking for their violation can help to identify what models would explain the choices that are not rational.

For each property in Table 5, Table 6 displays the percentage of non-rational subjects that violate the property and the average number of violations per non-rational subject.

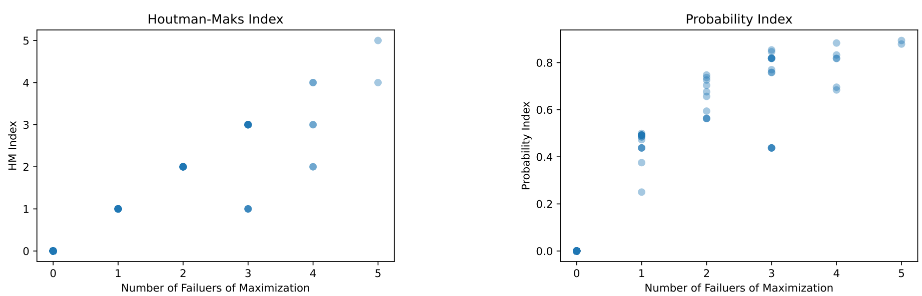
Since properties α and β characterize compatibility with preference maximization in the choice space of the experiment, choices in the experiment that violate rationality must violate at least one of them. Interestingly, although property α is considered to be more normatively appealing than property β, the fraction of non-rational subjects that violate them and the average number of violations per non-rational subject are similar.

Figure 8: Scatter Plots - Proper Revealed Cycles



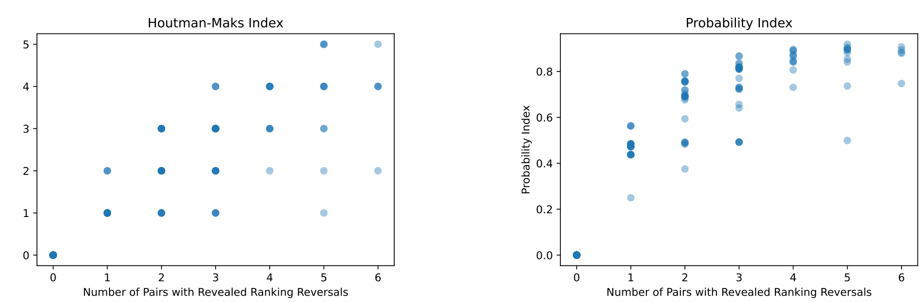
Left: Houtman-Maks | **Right**: Probability

Figure 9: Scatter Plots - Failures of Maximization



Left: Houtman-Maks | **Right**: Probability

Figure 10: Scatter Plots - Menu-dependent Rankings



Left: Houtman-Maks | **Right**: Probability

Table 6: Summary of Violations of the Properties

	Violations as % of Non-Rational Subjects	Average Number of Violations per Subject
Property α	84%	3.23
Property β	83%	3.44
Aizerman's Property	54%	1.05
Property γ	72%	2.31
Always Chosen	59%	1.07
Strict Binary Transitivity	0.9%	0.01
Weak Binary Transitivity	55%	0.95
Strict Binary Acyclicity	0%	0
Weak Binary Acyclicity	32%	0.58

54% of non-rational subjects violate the Aizerman's property, a weakening of α, and the average number of violations is roughly 33% of that of property α. 72% of non-rational subjects violate Property γ, a weakening of property β, and the average number of violations is roughly 67% of that of property β. 59% of non-rational subjects violate the property Always Chosen, and the average number of violations is "small," namely 1 violation per non-rational subject.

Strict cycles, including violations of strict pairwise transitivity, are rare in the data. If we assume that choices from pairwise menus reveal preferences, this suggests that quasi-transitivity and acyclicity are descriptively accurate assumptions about preferences. Weak cycles, including violations of weak pairwise transitivity, are, however, frequent in the data. If we assume that choices from pairwise menus reveal preferences, then either indifferences are intransitive or preferences are incomplete. Because, however, choices compatible with either the maximization of an incomplete preference or the maximization of a quasi-transitive preference must satisfy α and Always Chosen, these models alone cannot account for the irrationality of choices in the experiment.

To further understand why choices violate rationality, Table 7 lists the ten most prevalent minimal incompatible collections in the data. The top eight consist of nested menus, and the top four consist of nested menus with more than two elements. One possible explanation is that the number of comparisons that subjects need to make increases as the number of alternatives in the menu increase. Some subjects might then adopt context-dependent heuristics to shrink the number of comparisons they need to make, leading to more violations of rational choice in collections with larger menus.

The two remaining minimal incompatible collections among the top ten display a

Table 7: Top 10 Minimal Incompatible Collections

Collection	% of Non-Rational Subjects that Violate Rationality in the Collection
$\{\{a,s,m\},\{a,s,m,c\}\}$	45%
$\{\{a,m,c\},\{a,s,m,c\}\}$	44%
$\{\{a,s,c\},\{a,s,m,c\}\}$	36%
$\{\{s,m,c\},\{a,s,m,c\}\}$	35%
$\{\{s,c\},\{a,s,c\}\}$	35%
$\{\{a,s\},\{s,c\},\{a,m,c\}\}$	35%
$\{\{a,c\},\{m,c\},\{a,s,m\}\}$	30%
$\{\{s,c\},\{a,s,m,c\}\}$	29%
$\{\{s,m\},\{a,s,m,c\}\}$	29%
$\{\{a,s\},\{a,s,m\}\}$	29%

violation that is not captured by any of the properties listed in Table 5. These are collections of the form $\{\{x,y\},\{y,z\},\{x,w,z\}\}$, where x,y,z,w are different alternatives. For the sake of concreteness, consider the collection $\{\{a,s\},\{s,c\},\{a,m,c\}\}$. A subject that violates rationality in this collection cannot choose only the alternative s from both $\{a,s\}$ and $\{s,c\}$, nor only a from $\{a,s\}$ and only c from $\{s,c\}$. Hence, the choices from $\{a,s\}$ and $\{s,c\}$ reveal a ranking between a and c mediated by s. Since choices are not rationalizable on $\{\{a,s\},\{s,c\},\{a,m,c\}\}$, this ranking is contradicted by the choices from the menu $\{a,m,c\}$.

Of the 38 subjects that violate rationality in this sub-collection, 15 are rationalizable on $\{\{a,s\},\{s,c\},\{a,c\}\}$, which suggests that, for these subjects, the violation of rationality in the collection $\{\{a,s\},\{s,c\},\{a,m,c\}\}$ cannot be due to a and c being directly compared in the menu $\{a,m,c\}$. One possibility is that a change of the reference point from s to m causes the violation. The remaining 23 subjects are not rationalizable on $\{\{a,s\},\{s,c\},\{a,c\}\}$, which suggests that the direct comparison of a and c is responsible for the violation of rationality in the collection $\{\{a,s\},\{s,c\},\{a,m,c\}\}$.

7 Related Literature

This paper contributes to three strands of the literature on revealed preference theory. The first strand deals with the measurement of incompatibility with preference maximization and dates back to Afriat (1973), who proposes the first index of incompatibility with preference maximization in the standard environment of consumer theory with a finite number of observations. Houtman and Maks (1985), Swofford and Whitney (1986), Varian (1990), Echenique et al. (2011), and Dean and Martin (2016) propose other indices of incompatibility to this choice environment. For an axiomatic study of some of these indices, see Mononen (2021). Different approaches to the measurement of incompatibility in consumer theory include Aguiar and Serrano (2018) and de Clippel and Rozen (2021).

The indices of incompatibility that apply to the choice environment I study were discussed in Section 5.2 with one exception. Ambrus and Rozen (2013) propose a particular way of counting violations of property α to measure the degree of incompatibility of choices with preference maximization. However, this measure is only appropriate for choice functions.[34] To this first strand, this paper contributes by introducing a novel approach to the measurement of incompatibility with rational choice.[35]

Reacting to the evidence that choices need not be rational, choice theorists have proposed models of choice that subsume the preference model and can account for some of the violations of rational choice. Several of these models were discussed in Section 4, and other contributions include Rubinstein and Salant (2008), Tyson (2008), Bossert and Suzumura (2009), Masatlioglu et al. (2012), Apesteguia and Ballester (2013), Yildiz (2016), Dietrich and List (2016), and Frick (2016). This paper contributes to this literature by showing that, in some of these models, we can find a representation of choices by the model that approximates the preference maximization representation as choices become more rational according to $\succsim_{\text{rat}}$. Since these models subsume the preference maximization model, selecting a representation of choices by the model that "converge" to preference maximization with the decrease in the violations of rationality seems like an appropriate selection criterion, which would help to alleviate the characteristic non-uniqueness of the representation of choices in models of boundedly rational choice.

The third strand of literature deals with how to make welfare inferences from non-rationalizable choices (see Section 4). We contribute to this literature by showing that, as violations of rationality become pervasive, the leading model-free approach to choice-based welfare analysis proposed by Bernheim and Rangel (2009) becomes uninformative.

[34]Even then, we need to make assumptions about the collection menus we observe. For instance, if we only observe choices from pairwise menus, property α is (vacuously) satisfied by all choice functions.

[35]Ok and Tserenjigmid (2021) adopt a similar approach to the one I do to compare the rationality of stochastic choice functions.

8 Conclusion

There are many different reasons why choices violate preference maximization, and it is not always easy to rank the severity of these violations without further information about the choice domain. In this paper, I introduce a novel and intuitive criterion to make comparative judgments of rationality, which induces an incomplete rationality ordering. Its incompleteness reflects the difficulty of comparing the rationality of choices.

The approach to the measurement of incompatibility I adopt here is, in a sense, more foundational than the approaches that rely on indices of incompatibility. When we begin with an index, we make comparative judgments indirectly, through the numbers the index assigns to choices. Since the number of possible choices from a collection of menus increases exponentially with the number of menus in it, indices of incompatibility can end up making counter-intuitive assessments over some pairs that have an "obvious" ranking when directly compared.

I am not, however, suggesting that indices be replaced by incomplete orderings of rationality. Indices play an important role in controlling for the possibility of incorrectly rejecting the preference maximization model, a role that incomplete partial orderings cannot fulfill. But normatively appealing orderings of rationality can guide the researcher in deciding what indices are appropriate to a particular application.

Finally, my approach equates violations of rationality to the collections of menus where these violations take place. One could instead equate them to proper revealed cycles. These approaches are non-nested and have different merits and drawbacks (see Example 5). I advance, however, that taking both revealed cycles and the collections where they are revealed into account when comparing the rationality of choices is a promising avenue for future research. This would result in a rationality ordering that, although more incomplete than $\succsim_{\text{rat}}$, can deliver even more "unambiguous" comparisons than the ones that my rationality ordering delivers.

Appendix

A Proofs

A.1 Proposition 2

We begin by proving two facts:

(a) *For every $c \in \mathcal{C}(X, \mathcal{A})$ and $\mathcal{B} \subseteq \mathcal{A}$, if c is not rationalizable on $\mathcal{B}$, then there is a $\mathcal{B}' \in$ MIC(c) such that $\mathcal{B}' \subseteq \mathcal{B}$*: by Richter's Theorem (Proposition 1), if c is not rationalizable on $\mathcal{B}$, then c must reveal a proper cycle of smallest cardinality on $\mathcal{B}$, say $(x_1, \ldots, x_n)$. Let $\mathcal{B}' \subseteq \mathcal{B}$ be a sub-collection of smallest cardinality such that c reveals the proper cycle $(x_1, \ldots, x_n)$ on $\mathcal{B}'$. Clearly, $\mathcal{B}' \in \mathrm{MIC}(c)$.

(b) *For every $c \in \mathcal{C}(X, \mathcal{A})$ and $\mathcal{B} \subseteq \mathcal{A}$, if c is rationalizable on $\mathcal{B} \subseteq \mathcal{A}$, then there is a $\mathcal{B}' \in$ MCC(c) such that $\mathcal{B} \subseteq \mathcal{B}'$*: to show this, fix $\mathcal{B} \subseteq \mathcal{A}$ and define

$$\boldsymbol{\mathcal{B}}_c \coloneqq \{\mathcal{C} \subseteq \mathcal{A} : \mathcal{B} \subseteq \mathcal{C} \text{ and } c|_{\mathcal{C}} \text{ is rationalizable}\}.$$

$\boldsymbol{\mathcal{B}}_c$ is a nonempty partially ordered set under the the set inclusion relation $\supseteq$. Moreover, any $\supseteq$-chain[36] has an $\supseteq$-upper-bound[37] in $\mathcal{B}_c$, because, again by Richter's Theorem (Proposition 1) and the finiteness of revealed cycles, the union of a *nested* collection of menus where c is rationalizable in each menu must itself be a collection of menus where c is rationalizable. Zorn's Lemma[38] then implies that there must be a $\mathcal{B}' \in \boldsymbol{\mathcal{B}}_c$ such that $\mathcal{B}'$ is not contained in any other elements of $\boldsymbol{\mathcal{B}}_c$. Hence, $\mathcal{B}' \in \mathrm{MCC}(c)$.

We are now ready to prove the result. We show that (i) and (ii) are equivalent. Suppose $c_1 \succsim_{\text{rat}} c_2$ and suppose that $\mathcal{B} \in \mathrm{MIC}(c_1)$. Since $c_1 \succsim_{\text{rat}} c_2$, c_2 cannot be rationalized on $\mathcal{B}$. By (a), we can then find $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathcal{B}' \in \mathrm{MIC}(c_2)$. This proves one direction. To prove the converse, suppose that $c_1 \succsim_{\text{rat}} c_2$ does not hold. Then, there must exist a sub-collection $\bar{\mathcal{B}}$ such that c_2 is rationalizable on $\bar{\mathcal{B}}$, but c_1 is not. By (a), there must exist a sub-collection $\mathcal{B}' \in \mathrm{MIC}(c_1)$ such that $\mathcal{B}' \subseteq \bar{\mathcal{B}}$. Since

[36] Given a partial order $\geq$ on an arbitrary set Y, a $\geq$-*chain* C is a subset of Y such that $\geq|_C$ is complete.

[37] Given a partial order $\geq$ on an arbitrary set Y and a subset $Y' \subseteq Y$, we say that x is an $\geq$-*upper-bound* of Y' if $x \geq y$, for all $y \in Y'$.

[38] Let $\geq$ be partial order on Y. Zorn's Lemma states that if every $\geq$-chain has an $\geq$-upper-bound on Y, then there must exist an element $x \in Y$ such that, for every $y \in Y \backslash \{x\}$, $y \geq x$ does not hold.

c_2 is rationalizable on $\mathcal{B}'$, it is rationalizable in any $\mathcal{B} \subseteq \mathcal{B}'$. Thus, no $\mathcal{B} \subseteq \mathcal{B}'$ can belong to $\mathrm{MIC}(c_2)$. The equivalence between (i) and (iii) is proved analogously using (b).

A.2 Proposition 3

Let $\mathcal{R}_2$ be a minimum RMR of c_2. For each $\succsim \in \mathcal{R}_2$, let $\mathcal{A}_{\succsim}$ be the collection of menus A such that $c_2(A) = \max(A, \succsim)$. By the minimality of $\mathcal{R}_2$, we have $\mathcal{A}_{\succsim} \neq \emptyset$. Moreover, $\mathcal{A} = \bigcup_{\succsim \in \mathcal{R}_2} \mathcal{A}_{\succsim}$. Since $c_1 \succsim_{\text{rat}} c_2$, c_1 is rationalizable on $\mathcal{A}_{\succsim}$ for every $\succsim \in \mathcal{R}_2$. We can then find a RMR $\mathcal{R}_1$ of c_1 with the same cardinality as $\mathcal{R}_2$, concluding the proof.

A.3 Proposition 4

Assume that $\mathcal{M}_2(X) \subseteq \mathcal{A}$ and take $c_1, c_2 \in \mathcal{C}(X, \mathcal{A})$ with $c_1 \succsim_{\text{rat}} c_2$. For each $i \in \{1, 2\}$, let c_i be rationalizable by the complete relations R_i with R_i-maximizer Γ_{R_i}. Since $\mathcal{M}_2(X) \subseteq \mathcal{A}$, the definition of a R_i-maximizer implies that xR_iy if, and only if, $x \in c_i(\{x, y\})$. Now, assume that R_2 satisfies transitivity on $\{x, y, z\}$. Then, c_2 is rationalizable on $\{\{x, y\}, \{x, z\}, \{y, z\}\}$, and, hence, so is c_1. This implies that R_1 satisfies transitivity in $\{x, y, z\}$, concluding the proof.

A.4 Lemma 1

Fix $c \in \mathcal{C}(X, \mathcal{A})$. Since $\mathcal{M}_2(X) \subseteq \mathcal{A}$ and c is rationalizable by an incomplete preference relation, then, for any preorder $\succsim$ that rationalizes c, $x \succ y$ if, and only if, $c(\{x, y\}) = \{x\}$. Thus, the strict part of any $\succsim$ that rationalizes choices must be the same as $B_c^{>}$. Therefore, two different preorders that rationalize c can only differ in pairs $(x, y) \in X^2$ with $c(\{x, y\}) = \{x, y\}$. They can either declare them to be indifferent or incomparable. The key insight is that some of these pairs cannot be declared indifferent and that there is a condition that characterizes when this is so.

First, fix distinct $x, y \in X$ with $c(\{x, y\}) = \{x, y\}$. If there exists a $z \in X$ such that either $|c(\{x, z\})| = 2$ and $|c(\{y, z\}) = 1|$, or $|c(\{x, z\})| = 1$ and $|c(\{y, z\})| = 2$, we cannot have $x \sim y$ for any preorder $\succsim$ that rationalizes c. For instance, assume that $|c(\{x, z\})| = 2$ and $|c(\{y, z\})| = 1$. If we declare $x \sim y$, given that $|c(\{y, z\})| = 1$, I would conclude that either $x \succ z$ or $z \succ x$, contradicting $|c(\{x, z\})| = 2$.

Conversely, suppose that, for every $z \in X$, neither $|c(\{x, z\})| = 2$ and $|c(\{y, z\})| = 1$, nor $|c(\{x, z\})| = 1$ and $|c(\{y, z\})| = 2$. One can then check that, for every preorder

$\succsim$ that rationalizes the data, we must have $xB_c^{>}z \iff yB_c^{>}z$ and $yB_c^{>}z \iff yB_c^{>}z$, for every $z \in X$. Thus, x and y share the same upper and lower contour sets with respect to the strict part of any preorder that rationalizes c. We can then declare x and y to be indifferent.

Define then $x \sim^\star y$ if, and only if, $c(\{x,y\}) = \{x,y\}$ and there is no $z \in X$ such that either $|c(\{x,z\})| = 2$ and $|c(\{y,z\}) = 1|$, or $|c(\{x,z\})| = 1$ and $|c(\{y,z\})| = 2$. Finally, let $\succsim^\star = B_c^{>} \cup \sim^\star$. It is now easily checked that $\succsim^\star$ is the most complete preorder that rationalizes c, and that, for every $x, y \in X$, $(x,y) \in \mathrm{Inc}(\succsim^\star)$ if, and only if, there exists a $z \in X$ such that either $|c(\{x,z\})| = 2$ and $|c(\{y,z\})| = 1$, or $|c(\{x,z\})| = 1$ and $|c(\{y,z\})| = 2$.

A.5 Proposition 5

To prove (i), assume that, for every $x, y \in X$, $|c_1(\{x,y\})| = 2$ implies $|c_2(\{x,y\})| = 2$. By Lemma 1, given $(x,y) \in \mathrm{Inc}(\succsim_1)$, there is a z such that c_1 reveals incomparability in $\{x,y,z\}$ and, hence, so must c_2. But since $c_1(\{x,y\}) = \{x,y\}$, we have that $c_2(\{x,y\}) = \{x,y\}$ and, hence, by Lemma 1 again, that $(x,y) \in \mathrm{Inc}(\succsim_2)$.

To prove (ii), assume that, for $A \in \mathcal{M}_2(X)$, if A is a c_1-revealer, then it is a c_2-revealer. By Lemma 1, given $(x,y) \in \mathrm{Inc}(\succsim_1)$, there is a z such that c_1 reveals incomparability in $\{x,y,z\}$ and, hence, so does c_2. Therefore, there is a menu $A \in \{\{x,z\},\{y,z\}\}$ such that A is a c_1-revealer and, thus, a c_2-revealer, which implies that $|c_2(A)| = 1$. Because c_2 reveals incomparability in $\{x,y,z\}$, $c_2(\{x,y\}) = \{x,y\}$, implying that $(x,y) \in \mathrm{Inc}(\succsim_2)$.

To prove (iii), fix $i \in \{1,2\}$, and let $\mathcal{R}_{c_i}$ be the collection of c_i-revealers. Define also

$$\mathcal{A}_{c_i} \coloneqq \{\{x,y\} \in \mathcal{M}_2(X) : \exists z \in X \text{ s.t. } c_i \text{ reveals an incomparability at } \{x,y,z\}\}.$$

Since $\succsim_i$ is the most complete relation that rationalizes c_i, Lemma 1 implies that

$$\mathcal{A}_{c_i} = \mathcal{R}_{c_i} \cup \{\{x,y\} : (x,y) \in \mathrm{Inc}(\succsim_i)\},$$

with the union is disjoint. Hence,

$$|\mathcal{A}_{c_i}| = \frac{|\mathrm{Inc}(\succsim_i)|}{2} + |\mathcal{R}_{c_i}|$$

Since $c_1 \succsim_{\mathrm{rat}} c_2$, then $\mathcal{A}_{c_1} \subseteq \mathcal{A}_{c_2}$ and, hence, $|\mathcal{A}_{c_2}| \geqslant |\mathcal{A}_{c_1}|$. By assumption, we also have that $|\mathcal{R}_{c_1}| \geqslant |\mathcal{R}_{c_2}|$. We must then have $|\text{Inc}(\succsim_2)| \geqslant |\text{Inc}(\succsim_1)|$.

A.6 Proposition 7

For each $i \in \{1, 2\}$, define

$$x \succ^1_{c_i} y \text{ if, and only if, for all } A \in \mathcal{A} \text{ with } x, y \in A,\ x \in A \text{ implies } c(A) \neq y$$

$$\succ^2_{c_i} := B_{c_i} \setminus \succ^1_{c_i}$$

It follows from the proof of Theorem 1 in Manzini and Mariotti (2007) that $(\succ^1_{c_i}, \succ^2_{c_i})$ is a RSM-pair for c_i. To show that $\succ^1_{c_i}$ is acyclic for each $i \in \{1, 2\}$, notice that if there existed a cycle $x_1 \succ^1_{c_i} \cdots \succ^1_{c_i} x_n \succ^1_{c_i} x_1$, then we would have $c_i(\{x_1, \ldots, x_n\}) = \emptyset$, a contradiction.

Moreover, $\succ^1_{c_i} \subseteq B_{c_i}$ and B_{c_i} is complete and asymmetric. Thus, if we can find an injection from $\succ^1_{c_2}$ to $\succ^1_{c_1}$, we will have proved the first part of assertions (ii) and (iii). Since $c_1 \succsim_{\text{rat}} c_2$, we contend that, for every $x, y \in X$,

$$x \succ^1_{c_2} y \text{ implies that either } x \succ^1_{c_1} y \text{ or } x \succ^1_{c_1} y.$$

In fact, $x \succ^1_{c_2} y$ if, and only if, $c_2(\{x, y\}) = \{x\}$ and c_2 is rationalizable on $\{\{x, y\}, A\}$ for every $A \in \mathcal{A}$. Then, c_1 is also rationalizable on $\{\{x, y\}, A\}$, for every $A \in \mathcal{A}$. Hence, if $c_1(\{x, y\}) = \{x\}$, then $x \succ^1_{c_1} y$; and if $c_1(\{x, y\}) = \{y\}$, then $y \succ^1_{c_1} x$. Therefore, if we define $\psi_1 : \succ^1_{c_2} \to \succ^1_{c_1}$ as

$$\psi_1(x, y) := \begin{cases} (x, y) & , \ c_1(\{x, y\}) = x \\ (y, x) & , \ c_1(\{x, y\}) = y \end{cases},$$

we have that ψ_1 is well-defined and injective. Now, since $(x, y) \in \succ^1_{c_2}$ implies that either $(x, y) \in \succ^1_{c_1}$ or $(y, x) \in \succ^1_{c_1}$, it follows that $\text{Inc}(\succ^1_{c_1}) \subseteq \text{Inc}(\succ^1_{c_2})$, proving the second part of assertion (ii). But then we must also have $\text{Inc}(\succ^2_{c_2}) \subseteq \text{Inc}(\succ^2_{c_1})$, proving the second part of assertion (iii).

Finally, if we assume that c_1 and c_2 are the same when restricted to pairwise menus, i.e, $B_{c_1} = B_{c_2}$, then $\psi_1(x, y) = (x, y)$, for all $(x, y) \in \succ^1_{c_2}$. Hence, $\succ^1_{c_2} \subseteq \succ^1_{c_1}$ and $\succ^2_{c_1} \subseteq \succ^2_{c_2}$. Finally, since both $\succ^1_{c_1}$ and $\succ^1_{c_2}$ are acyclic and $\succ^1_{c_2} \subseteq \succ^1_{c_1}$, then $\succ^1_{c_1}$ must be at least as transitive as $\succ^1_{c_2}$, which proves the last part of the proposition.

A.7 Proposition 8

For each $i \in \{1,2\}$, define $\succ_{c_i} := B_{c_i}$, and, for each $A, B \in \mathcal{A}\backslash\{\emptyset\}$, define $A \succ_s^{c_i} B$ if, and only if, there exists a $C \in \mathcal{A}$ such that $A = \{c_i(C)\} \cup \{x \in C: \ c_i(C) \succ_{c_i} x\}$, and $B = \{x \in C: \ x \succ_{c_i} c_i(C)\}$ is *non-empty*. It follows from the proof of Theorem 1 in Manzini and Mariotti (2012) that $(\succ_s^{c_1}, \succ_{c_1})$ and $(\succ_s^{c_2}, \succ_{c_2})$ are CTC-representations of c_1 and c_2.

Part (ii) of the statement follows from the proof of Proposition 4. Therefore, we only need to prove part (i). For ease of exposition, define, for each $C \in \mathcal{A}$ and $i \in \{1,2\}$,

$$A_C^i \coloneqq \{c_i(C)\} \cup \{x \in C: \ c_i(C) \succ_{c_i} x\}$$

$$B_C^i \coloneqq \{x \in C: \ x \succ_{c_i} c_i(C)\}.$$

Define $\psi : \succ_s^{c_1} \to \succ_s^{c_2}$ by

$$\psi(A_C^1, B_C^1) \coloneqq (A_C^2, B_C^2).$$

If ψ is well-defined, then it is injective, because, for each $i \in \{1,2\}$ and $C \in \mathcal{A}$, $\{A_C^i, B_C^i\}$ is a partition of C. To show that ψ is well-defined, fix $C \in \mathcal{A}$ with $B_C^1 \neq \emptyset$ so that $A_C^1 \succ_s^{c_1} B_C^1$. We need to show that $B_C^2 \neq \emptyset$, because this means that $A_C^2 \succ_s^{c_2} B_C^2$.

Since $c_2(C) \in C$ and $\{A_C^1, B_C^1\}$ is a partition of C, either $c_2(C) \in A_C^1$ or $c_2(C) \in B_C^1$. If the latter holds, then $c_2(C) \succ_{c_1} c_1(C)$ and, hence, c_1 is not rationalizable on $\{\{c_1(C), c_2(C)\}, C\}$. Thus, c_2 is also not rationalizable in this sub-collection, which implies $c_1(C) \succ_{c_2} c_2(C)$. Therefore $c_1(C) \in B_C^2$ and, hence, $B_C^2 \neq \emptyset$.

Suppose now that $c_2(C) \in A_C^1$, but, by contradiction, that $B_C^2 = \emptyset$. It follows that $c_2(C) \succ_{c_2} x$, for all $x \in C\backslash\{c_2(C)\}$. Since $B_C^1 \neq \emptyset$, there is an $a \in C\backslash\{c_1(C), c_2(C)\}$ such that $a \succ_{c_1} c_1(C)$, which implies that C has more than two alternatives. Given that $c_2(C) \succ_{c_2} a$ and $a \succ_{c_1} c_1(C)$, c_1 is not rationalizable on $\{\{c_1(C), a\}, C\}$ whereas c_2 is, contradicting $c_1 \succsim_{\text{rat}} c_2$. Thus, $B_C^2 \neq \emptyset$, completing the proof.

A.8 Proposition 9

Define $\succ \coloneqq B_{c_1} (= B_{c_2})$. For each $i \in \{1,2\}$, let φ_{c_i} be the largest constraint such that $(\varphi_{c_i}, \succ)$ is an OR-representations of c_i. Fix $A \in \mathcal{A}$. If $|A| = 2$, then, by assumption $c_1(A) = c_2(A)$, and it follows from the proof of Proposition 3 in Cherepanov et al.

(2013), that we must have $\varphi_{c_1}(A) = \varphi_{c_2}(A) = A$. Thus,

$$\varphi_{c_1}(A)\backslash\{c_1(A)\} = \varphi_{c_2}(A)\backslash\{c_2(A)\}.$$

So assume that $|A| \geqslant 3$ and take $x \in \varphi_{c_2}(A)\backslash\{c_2(A)\}$. Since $(\varphi_{c_2}, \succ)$ is a OR-representation of c_2, we must have $c_2(A) \succ x$. By contradiction, assume that $x \notin \varphi_{c_1}(A)\backslash\{c_1(A)\}$. If $x = c_1(A)$, then $c_2(A) \succ c_1(A)$, implying that c_2 is rationalizable on $\{\{c_2(A), c_1(A)\}, A\}$ whereas $c_1$is not, contradicting $c_1 \succsim_{\text{rat}} c_2$. If $x \neq c_1(A)$, then $x \notin \varphi_{c_1}(A)$, and because $\varphi_{c_1}(A)$ is the largest constraint consistent with the order rationalization of c_1, either $x \succ c_1(A)$ or, for some $B \subseteq A\backslash\{c_1(A)\}$ with $x \in B$, $x \succ c_1(B)$. In fact, if neither were true, we could include x in $\varphi_{c_1}(B)$ for all $B \subseteq A$ with $x \in B$ to get to a larger constraint that OR-maximizes c_1, contradicting the maximality of φ_{c_1}.

Now, if $x \succ c_1(A)$, by the transitivity of $\succ$, we get that $c_2(A) \succ c_1(A)$, again contradicting $c_1 \succsim_{\text{rat}} c_2$. Therefore, we must have $x \succ c_1(B)$ for some $B \subseteq A\backslash\{c_1(A)\}$ with $x \in B$. Given the definition of $\succ$, B must have more than two alternatives. Since $x \in \varphi_{c_2}(A)$ and φ_{c_2} satisfies α, $x \in \varphi_{c_2}(B)$ and, hence, either $c_2(B) = x$ or $c_2(B) \succ x$. In either case, we conclude that $c_2(B) \succ c_1(B)$, again contradicting $c_1 \succsim_{\text{rat}} c_2$. Therefore, we must have $x \in \varphi_{c_1}(A)\backslash\{c_1(A)\}$, completing the proof.

A.9 Proposition 10

For each $i \in \{1, 2\}$, take any RDC-representation $(\succsim_{c_i}, \mathbf{r}_{c_i}, \mathcal{U}_{c_i})$. Then, $\succsim_{c_i} = B_{c_i}$, and notice that it is without loss of generality to assume that $\mathbf{r}_{c_i}(A) = \Diamond$ if, and only if, $c_i(A) = \max(A, \succsim_{c_i})$. Otherwise, for each $i \in \{1, 2\}$, redefine $\mathbf{r}_{c_i}(A)$ to $\Diamond$ in the sets $A \in \mathcal{A}$ such that $\mathbf{r}_{c_i}(A) \neq \Diamond$ and $c_i(A) = \max(A, \succsim_{c_i})$ to get to a reference function $\mathbf{r}'_{c_i}$ that still RDC-represents c_i.

Now, fix $A \in \mathcal{A}$ and assume that $\mathbf{r}_{c_2}(A) = \Diamond$. Then, $c_2(A) = \max(A, \succsim_{c_2})$ and, since $\succsim_{c_2} = B_{c_2}$, c_2 is rationalizable on $\mathcal{M}_2(X) \cup \{A\}$. Given that $c_1 \succsim_{\text{rat}} c_2$, c_1 is also rationalizable in this sub-collection, which implies

$$c_1(A) = \max(A, B_{c_1}) = \max(A, \succsim_{c_1}).$$

Thus, $\mathbf{r}_{c_1}(A) = \Diamond$, completing the proof.

A.10 Proposition 11

Part (i) follows from the proof of Proposition 4. We just show parts (ii) and (iii). For each $i \in \{1, 2\}$, define the following relations:

$$x \succ_{c_i} y \iff y \notin c(A), \text{ for every } A \in \mathcal{A} \text{ with } x \in A$$

$$x \sim_{c_i} y \iff \text{For every } A \in \mathcal{A} \text{ and } z \in A, \begin{cases} x \in c(A \cup \{x\}) \iff y \in c(A \cup \{y\}) \\ z \in c(A \cup \{x\}) \iff z \in c(A \cup \{y\}) \end{cases}$$

$$\succsim_{c_i} := \succ_{c_i} \cup \sim_{c_i}$$

As shown in Nishimura and Ok (2020), $(\succsim_{c_i}, B_{c_i})$ is a preference structure that CPS-represents c_i and $\succsim_{c_i}$ is the most complete relation among the relations $\succsim$ such that $(\succsim, B_{c_i})$ is a preference structure that CPS-rationalizes c_i.

Assume that $|c_1(\{x.y\})| = |c_2(\{x, y\})|$, for all $x, y \in X$, and fix $(x, y) \in \succsim_{c_2}$. If $|c_2(\{x, y\})| = 1$, then either $x \succ_{c_2} y$ or $y \succ_{c_2} x$, and also $|c_1(\{x, y\})| = 1$. This implies that, for every $A \in \mathcal{A}$, c_2 is rationalizable on $\{\{x, y\}, A\}$. Since $c_1 \succsim_{\text{rat}} c_2$, then so is c_1 and, hence, if $|c_1(\{x, y\})| = 1$, we must have $x \succ_{c_1} y$ or $y \succ_{c_1} x$. Thus, (x, y) or (y, x) belong to $\succsim_{c_1}$. If, in addition, $c_1(\{x, y\}) = c_2(\{x, y\})$, we conclude that $\succ_{c_2} \subseteq \succ_{c_1}$.

If $|c_2(\{x, y\})| = 2$, then $x \sim_{c_2} y$ and, hence, c_2 is rationalizable on every sub-collection of the form $\{\{x, y\}, A \cup \{x\}, A \cup \{y\}\}$. Since $c_1 \succsim_{\text{rat}} c_2$, then so is c_1 and, hence, if $|c_1(\{x, y\})| = 2$, we must have $x \sim_{c_1} y$, proving part (iii).

The two previous paragraphs established that when $|c_1(\{x.y\})| = |c_2(\{x, y\})|$ for all $x, y \in X$, whenever a pair (x, y) is comparable according to $\succsim_{c_2}$, then it is also comparable according to $\succsim_{c_1}$. It follows that $\text{Inc}(\succsim_{c_1}) \subseteq \text{Inc}(\succsim_{c_2})$, concluding the proof of part (ii).

A.11 Proposition 12

The proposition is a consequence of the following observation. When all finite subsets of X are in $\mathcal{A}$, given any $c \in \mathcal{C}(X, \mathcal{A})$, $xR^\star_c y$ if, and only if, c is rationalizable in$\{\{x, y\}, A\}$ for every $A \in \mathcal{A}$. That is, $xR^\star_c y$ if, and only, there are not direct revealed ranking reversals between x and y.

A.12 Proposition 13

It follows immediately from the definitions that (i) implies (ii), while it is obvious that (ii) implies (iii). It remains to show that (iii) implies (i). Suppose, then, that $c_1 \succsim_{\text{rat}} c_2$ does not hold. Then, there exists $\mathcal{B} \subseteq \mathcal{A}$ such that c_2 is rationalizable on $\mathcal{B}$, but c_1 is not rationalizable on $\mathcal{B}$. By the proof of Proposition 2, there is a $\mathcal{B}' \in \text{MIC}(c_1)$ with $\mathcal{B}' \subseteq \mathcal{B}$. Fix $A \in \mathcal{B}'$ and note that because $\mathcal{B}' \in \text{MIC}(c_1)$, c_1 is rationalizable on $\mathcal{B}' \backslash \{A\}$, but not on $\mathcal{B}'$. Therefore, $A \in \Theta(c_1)(\mathcal{B}')$, but $A \notin \Theta(c_2)(\mathcal{B}')$, completing the proof.

A.13 Corollary 1

Suppose I is (weakly) consistent with $\succsim_{\text{rat}}$ and for each $\boldsymbol{\mathcal{M}} \in \Theta(\mathcal{C}(X, \mathcal{A}))$, define $g(\boldsymbol{\mathcal{M}}) \coloneqq I(c)$, where $\Theta(c) = \boldsymbol{\mathcal{M}}$. One can check that g is well-defined. By construction, (i) holds, and (ii) follows from the definition of consistency with $\succsim_{\text{rat}}$ and the equivalence between (i) and (ii) in Proposition 13. Since $\Theta(c)(\mathcal{B}) = \emptyset$, for all $\mathcal{B} \subseteq \mathcal{A}$, if, and only if, c is rationalizable on $\mathcal{A}$, we get (iii). This proves the "only if" part of the assertion. The converse follows readily from the equivalence between (i) and (ii) in Proposition 13.

B Extensions of $\succsim_{\text{rat}}$

In some applications, there might be comparisons that $\succsim_{\text{rat}}$ cannot make despite being "obvious" in the applications. We might thus want to extend $\succsim_{\text{rat}}$ to account for these comparisons. In this appendix I discuss three types of extensions of $\succsim_{\text{rat}}$: invariance to the relabeling of alternatives, hierarchies of violations, and restrictions on the set of test sub-collections.

B.1 Invariance to Relabeling of Alternatives[39]

$\succsim_{\text{rat}}$ is not invariant to the relabeling of alternatives. To illustrate, consider you have a Friend 7 whose choices are in Table 8, where I also repeat the choices of Friend 3. Friends 3 and 7 are $\succsim_{\text{rat}}$-incomparable because they violate rationality in different sub-collections. Nevertheless, the choices of your Friend 7 are derived from the choices

[39] This type of extension was suggested by Efe Ok.

Table 8: Invariance to Relabeling

$\mathcal{A}$	$\{a,b,c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$
Friend 3	a	a	c	c
Friend 7	c	a	c	a

of Friend 3 by labeling beer a as beer c and beer c as beer a. It is then natural to judge that Friends 3 and 7 are equally rational.

I refrain to do so in general, because alternatives can be different from one another in some applications. For example, some can be simple to understand while others very complex. Violations of rationality that involve complex alternatives can be a consequence of a faulty understanding of the alternatives rather than a failure of rationality *per se*. More generally, alternatives in an application can have attributes that are relevant in judging the severity of a violation of rationality but that my framework abstracts from.

If invariance to relabeling is desirable in an application, we can extend $\succsim_{\text{rat}}$ to account for it. Formally, fix a choice space $(X, \mathcal{A})$, a partition $\mathcal{X}$ of X and a bijection $\gamma : X \to X$. We say that γ is $\mathcal{X}$-*permissible* in $(X, \mathcal{A})$ if:

(i) $\gamma(B) = B$, for every $B \in \mathcal{X}$;

(ii) $\gamma(A), \gamma^{-1}(A) \in \mathcal{A}$, for every $A \in \mathcal{A}$.

We think of the alternatives that belong to the same element of the partition $\mathcal{X}$ as being comparable (e.g., having similar costs, or complexity, etc.). Condition (i) says that we can only relabel comparable alternatives. Condition (ii) is a technical condition imposed in the collection of menus whose role will become clear shortly.

To illustrate this definition, let X be a set of products. Each product $x \in X$ costs c_x. For each $c \geqslant 0$, let $X_c := \{x : c_x = c\}$ and define $\mathcal{X} := \{X_{c_x} : x \in X\}$. If a bijection $\gamma : X \to X$ is $\mathcal{X}$-permissible in $(X, \mathcal{A})$, then it can only relabel products that cost the same.

Given $c \in \mathcal{C}(X, \mathcal{A})$, a partition $\mathcal{X}$ of X and a $\mathcal{X}$-permissible γ in $(X, \mathcal{A})$, define $c_\gamma(\cdot) := \gamma(c(\gamma^{-1}(\cdot)))$. Condition (ii) guarantees that c_γ is a well-defined choice correspondence on $(X, \mathcal{A})$. Moreover, for any bijection γ on X, define $\gamma(\mathcal{B}) := \{\gamma(B) : B \in \mathcal{B}\}$. To formalize the idea of invariance to relabeling, I introduce a preorder $\succsim_{\text{rat}}^{\mathcal{X}}$ defined as: for every $c, c' \in \mathcal{C}(X, \mathcal{A})$, $c \succsim_{\text{rat}}^{\mathcal{X}} c'$ if, and only if, there exists a $\mathcal{X}$-permissible γ such that $c_\gamma \succsim_{\text{rat}} c'$.

To prove that, when X is finite, $\succsim_{\text{rat}}^{\mathcal{X}}$ extends $\succsim_{\text{rat}}$, we need the following lemma.

Lemma 3. *Given $c \in \mathcal{C}(X, \mathcal{A})$, a partition $\mathcal{X}$ of X and a $\mathcal{X}$-permissible γ in $(X, \mathcal{A})$, $(x_1, \ldots, x_n)$ is a revealed cycle of c on $\mathcal{B} \subseteq \mathcal{A}$ if, and only if, $(\gamma(x_1), \ldots, \gamma(x_n))$ is a revealed cycle of c_γ on $\gamma(\mathcal{B})$. Symmetrically, $(x_1, \ldots, x_n)$ is a revealed cycle of c_γ on $\mathcal{B}$ if, and only if, $(\gamma^{-1}(x_1), \ldots, \gamma^{-1}(x_n))$ is a revealed cycle of c on $\gamma^{-1}(\mathcal{B})$.*

Proof. Condition (ii) guarantees that $\gamma(\mathcal{B}) \subseteq \mathcal{A}$ and, hence, that $R_{\gamma(\mathcal{B})}$ and $P_{\gamma(\mathcal{B})}$ are well-defined. Moreover,

$$\begin{array}{llc} xR_{\mathcal{B}}(c)y & \text{if, and only if,} & x \in c(A) \text{ and } y \in A \\ & \text{if, and only if,} & \gamma(x) \in \gamma(c(A)) \text{ and } \gamma(y) \in \gamma(A) \\ & \text{if, and only if,} & \gamma(x) \in c_\gamma(\gamma(A)) \text{ and } \gamma(y) \in \gamma(A) \\ & \text{if, and only if,} & \gamma(x) R_{\gamma(\mathcal{B})}(c_\gamma) \gamma(y) \end{array}.$$

Similarly, we can show that $x\,P(c)\,y$ if, and only if, $\gamma(x)\,P_{\gamma(\mathcal{B})}(c_\gamma)\,\gamma(y)$. The first part of the result follows from these observations. The second part is proved analogously. □

Proposition 15. *Given a choice space $(X, \mathcal{A})$ and a partition $\mathcal{X}$ of X, $\succsim_{rat}^{\mathcal{X}}$ is a preorder that extends $\succsim_{rat}$ when X is finite.*

Proof. Fix a partition $\mathcal{X}$ of X. We first show that $\succsim_{\text{rat}}^{\mathcal{X}}$ is a preorder. Since reflexivity is trivial, we only show transitivity. Let $c, c', c'' \in \mathcal{C}(X, \mathcal{A})$ satisfy $c \succsim_{\text{rat}}^{\mathcal{X}} c' \succsim_{\text{rat}}^{\mathcal{X}} c''$. Then, for some $\mathcal{X}$-permissible bijections γ_1, γ_2, we have that $c_{\gamma_1} \succsim_{\text{rat}} c'$ and $c'_{\gamma_2} \succsim_{\text{rat}} c''$. By the transitivity of $\succsim_{\text{rat}}$, we will be done if we show that $c_{\gamma_2 \circ \gamma_1} \succsim_{\text{rat}} c'_{\gamma_2}$. Suppose $c_{\gamma_2 \circ \gamma_1}$ is not rationalizable on $\mathcal{B} \subseteq \mathcal{A}$. Therefore, by Proposition 1, we can find a proper cycle of $c_{\gamma_2 \circ \gamma_1}$ on $\mathcal{B}$. Lemma 3 now implies that c_{γ_1} must exhibit a cycle on $\gamma_2^{-1}(\mathcal{B})$. Since $c_{\gamma_1} \succsim_{\text{rat}} c'$, c' must also exhibit a cycle on $\gamma_2^{-1}(\mathcal{B})$. Then, again by Lemma 3, c'_{γ_2} must exhibit a cycle on $\mathcal{B}$. Therefore, c'_{γ_2} is not rationalizable on $\mathcal{B}$. Since $\mathcal{B}$ was arbitrary, we conclude that $c_{\gamma_2 \circ \gamma_1} \succsim_{\text{rat}} c'_{\gamma_2}$.

We now show that, when X is finite, $\succsim_{\text{rat}}^{\mathcal{X}}$ extends $\succsim_{\text{rat}}$. Clearly, $\succsim_{\text{rat}} \subseteq \succsim_{\text{rat}}^{\mathcal{X}}$, so we just need to show that $\succ_{\text{rat}} \subseteq \succ_{\text{rat}}^{\mathcal{X}}$. By contradiction, suppose there are $c, c' \in \mathcal{C}(X, \mathcal{A})$ such that $c \succ_{\text{rat}} c'$, but that for some $\mathcal{X}$-permissible γ, $c'_\gamma \succsim_{\text{rat}} c$. If this were the case, we would have $c'_\gamma \succ_{\text{rat}} c'$. But this cannot be the case, because X is finite. In fact, $c'_\gamma \succ_{\text{rat}} c'$ implies that there is a collection $\mathcal{B}$ where c'_γ is rationalizable on $\mathcal{B}$, but not c'. Then, c' must have a cycle on $\mathcal{B}$. By Lemma 3, c'_γ must have a cycle on $\gamma(\mathcal{B})$. Since $c'_\gamma \succ_{\text{rat}} c'$ so must c' and, again by Lemma 3, c'_γ must have a cycle on $\gamma^2(\mathcal{B})$. Proceeding inductively, we conclude that c'_γ has a cycle on $\gamma^m(\mathcal{B})$ for every $m \in \mathbb{N}$. Since X is finite, $\gamma^m(\mathcal{B}) = \mathcal{B}$ for some $m \in \mathbb{N}$, contradicting that c'_γ is rationalizable on $\mathcal{B}$. □

In Proposition 15, we assumed that X is finite. To understand what can go wrong when X is not finite, let $X \coloneqq \{\dots, x_{-2}, x_{-1}, x_0 x_1, x_2, \dots\}$ and define

$$\mathcal{A} \coloneqq \{\{x_i, x_{i+1}\} : i \in \mathbb{Z}\} \cup \{\{x_i, x_{i+2}\} : i \in \mathbb{Z}\}.$$

Let $\mathcal{X} \coloneqq \{X\}$ and $\gamma : X \to X$ as $\gamma(x_i) = x_{i+1}$, and notice that γ is $\mathcal{X}$-permissible. Define the binary relation $\succ$ on X by:

- $x_i \succ x_{i+1}$, for every $i \in \mathbb{Z}$;
- $x_i \succ x_{i+2}$, for every $i \in \{\dots, -1, 0, 1\}$;
- $x_{i+2} \succ x_i$, for every $i \in \{2, 3, \dots\}$.

Let $c_1 \in \mathcal{C}(X, \mathcal{A})$ be the choice correspondence defined in such a way that $B_{c_1} = \succ$. Therefore, the minimal incompatible collections of c_1 are

$$\mathrm{MIC}(c_1) = \{\{\{x_i, x_{i+1}\}, \{x_{i+1}, x_{i+2}\}, \{x_i, x_{i+2}\}\} : i = 2, 3, \dots\}.$$

If we define $c_2(\cdot) \coloneqq \gamma^{-1}(c_1(\gamma(\cdot)))$, then:

- $x_i B_{c_2} x_{i+1}$, for every $i \in \mathbb{Z}$;
- $x_i B_{c_2} x_{i+2}$, for every $i \in \{\dots, -1, 0\}$;
- $x_{i+2} B_{c_2} x_i$, for every $i \in \{1, 2, 3, \dots\}$.

Intuitively, c_2 anticipates the "first" cycle of c_1. Therefore,

$$\mathrm{MIC}(c_2) = \{\{\{x_i, x_{i+1}\}, \{x_{i+1}, x_{i+2}\}, \{x_i, x_{i+2}\}\} : i = 1, 2, \dots\}$$

and, by Proposition 2, $c_1 \succ_{\text{rat}} c_2$.

B.2 Hierarchy of Violations[40]

Another way to extend $\succsim_{\text{rat}}$ is to introduce a hierarchy over nonempty sub-collections of menus that reflects our judgments about the relative importance of violating rationality in different sub-collections. Going back to our leading example (Table 1), if we believe that cycles over pairwise menus constitute a more "serious" violation than

[40]This type of extension was suggested by Debraj Ray.

violations in other collections, then we would say that Friend 4 is more rational than Friend 5.

Formally, fix a choice space $(X, \mathcal{A})$ and let $\geq$ be a partial order on $\mathcal{A}$. We interpret $\geq$ as a *hierarchy of violations*. Therefore, $\mathcal{B} \geq \mathcal{B}'$ means that inconsistencies in the sub-collection $\mathcal{B}$ are more "serious" than inconsistencies in the sub-collection $\mathcal{B}'$. Given $c, c' \in \mathcal{C}(X, \mathcal{A})$, an *exclusive violation of c with respect to c'* is a nonempty sub-collection of menus $\mathcal{B}$ in which c is not rationalizable, but c' is.

For every hierarchy of violations $\geq$, we say that c_1 *is at least as rational as c_2 given* $\geq$ if for every exclusive violation $\mathcal{B}_1$ of c_1 with respect to c_2, there exists an exclusive violation $\mathcal{B}_2$ of c_2 with respect to c_1 such that $\mathcal{B}_2 \geq \mathcal{B}_1$. In this case, we write $c_1 \succsim_{\geq\text{-rat}} c_2$. One can check that $\succsim_{\geq\text{-rat}}$ extends $\succsim_{\text{rat}}$.

B.3 Restricting the Domain of Test Sub-collections[41]

Since $\succsim_{\text{rat}}$ makes a sub-collection by sub-collection comparison of rationality, it is natural to refer to a nonempty sub-collection of menus $\mathcal{B} \subseteq \mathcal{A}$ as a *test sub-collection*. In some applications, we might want to impose restrictions on the space of test sub-collections.

For instance, if we are interested only in cycles in pairwise menus, we should only use test sub-collections composed of pairwise menus. Or if we think that choices from some menus do not really reveal preferences, e.g., from menus with many elements, we can exclude sub-collections that include these menus.

Formally, fix a choice space $(X, \mathcal{A})$ and a nonempty $\boldsymbol{\mathcal{M}} \subseteq 2^{\mathcal{A}} \backslash \{\emptyset\}$ of test sub-collections. We interpret $\boldsymbol{\mathcal{M}}$ as the domain of valid test sub-collections. Define the binary relation $\succsim_{\text{rat}}^{\boldsymbol{\mathcal{M}}}$ on $\mathcal{C}(X, \mathcal{A})$ by $c_1 \succsim_{\text{rat}}^{\boldsymbol{\mathcal{M}}} c_2$ if, and only if, for every $\mathcal{B} \in \boldsymbol{\mathcal{M}}$, whenever c_2 is rationalizable on $\mathcal{B}$, then so is c_1. Although $\succsim_{\text{rat}}^{\boldsymbol{\mathcal{M}}}$ contains $\succsim_{\text{rat}}$, $\succsim_{\text{rat}}^{\boldsymbol{\mathcal{M}}}$ might not extend $\succsim_{\text{rat}}$, because we can eliminate some strict comparisons $\succsim_{\text{rat}}$ makes when we eliminate test sub-collections.

C An Impossibility Result

We cannot always construct an index of incompatibility that is consistent with $\succsim_{\text{rat}}$, as the next Lemma shows.

Lemma 4. *Let $(X, \mathcal{A})$ be a choice space that satisfies two conditions:*

[41]This type of extension was suggested by Ariel Rubinstein.

(a) $\bigcap \mathcal{A} \neq \emptyset$;

(b) *There is a* $\mathcal{A}' \subseteq \mathcal{A}$ *such that card*$(\bigcap \mathcal{A}') \geqslant$ *card*$(\mathcal{A}') \geqslant$ *card*$(\mathbb{R})$.[42]

Then, there is no index of incompatibility consistent with $\succsim_{rat}$.[43]

Proof. The basic idea of the proof is to construct an uncountable sequence of strict comparabilities using conditions (a) and (b). We proceed by contradiction. Suppose there is an index of incompatibility $I : \mathcal{C}(X, \mathcal{A}) \to \mathbb{R}$ consistent with $\succsim_{\text{rat}}$.

Let $\trianglerighteq$ be a well-order[44] of $\mathcal{A}'$ and, for every $A \in \mathcal{A}'$, $\text{succ}(A)$ be the immediate successor of A according to $\trianglerighteq$. Define also, for every $A \in \mathcal{A}'$,

$$A^{\uparrow,\trianglerighteq} := \{A' \in \mathcal{A} : A' \trianglerighteq A\} \text{ and } A^{\downarrow\downarrow,\trianglerighteq} := \{A' \in \mathcal{A} : A \trianglerighteq A' \text{ and } A' \neq A\}.$$

By conditions (a) and (b), there is an element $a \in \bigcap \mathcal{A}$ and an injection $f : \mathcal{A}' \to (\bigcap \mathcal{A}') \setminus \{a\}$. Define, for each $A \in \mathcal{A}'$, the choice function $c_A \in \mathcal{C}(X, \mathcal{A})$ as follows:

- If $B \in (\mathcal{A} \setminus \mathcal{A}') \cup \left(A^{\uparrow,\trianglerighteq}\right)$, define $c_A(B) := \{a\}$;
- If $B \in A^{\downarrow\downarrow,\trianglerighteq}$, define $c_A(B) := \{f(B)\}$.

We contend that $c_A \succ_{\text{rat}} c_{\text{succ}(A)}$ for every $A \in \mathcal{A}'$. Fix $A \in \mathcal{A}'$. By construction, c_A is rationalizable on $A^{\uparrow,\trianglerighteq}$ (c_A always chooses $\{a\}$), but $c_{\text{succ}(A)}$ is not, because $c_{\text{succ}(A)}(\text{succ}(A)) = \{a\}$, $c_{\text{succ}(A)}(A) = \{f(A)\}$, $a \neq f(A)$, and $\{a, f(A)\} \subseteq A \cap \text{succ}(A)$. Therefore, to show that $c_A \succ_{\text{rat}} c_{\text{succ}(A)}$, we just need to show that for any $\mathcal{B} \subseteq \mathcal{A}$, if $c_{\text{succ}(A)}$ is rationalizable on $\mathcal{B}$, then c_A is rationalizable on $\mathcal{B}$. Since c_A and $c_{\text{succ}(A)}$ only differ on the choice they make from the menu A, we can focus on collections $\mathcal{B} \subseteq \mathcal{A}$ with $A \in \mathcal{B}$. So fix one such collection and assume that $c_{\text{succ}(A)}$ is rationalizable in $\mathcal{B}$. Then, by construction of f, $\mathcal{B} \cap A^{\downarrow\downarrow,\trianglerighteq} = \emptyset$. But $\mathcal{B} \cap A^{\downarrow\downarrow,\trianglerighteq} = \emptyset$ implies that $c_A(B) = \{a\}$, for every $B \in \mathcal{B}$ and c_A is thus rationalizable in $\mathcal{B}$. This shows that $c_A \succ_{\text{rat}} c_{\text{succ}(A)}$. By the contradiction assumption, $I(c_{\text{succ}(A)}) > I(c_A)$ for every $A \in \mathcal{A}'$.

We now contend that $c_{\text{succ}(A)} \succsim_{\text{rat}} c_D$, for every $A, D \in \mathcal{A}'$ with $D \trianglerighteq A$ and $A \neq D$. In fact, fix such $A, D \in \mathcal{A}'$ and let $\mathcal{B}$ be a sub-collection such that $c_{\text{succ}(A)}$ is not rationalizable on $\mathcal{B}$. We cannot have $\mathcal{B} \subseteq (\mathcal{A} \setminus \mathcal{A}') \cup \left(D^{\uparrow,\trianglerighteq}\right)$, because from $D^{\uparrow,\trianglerighteq} \subseteq$

[42] Given two arbitrary sets X and Y, $\text{card}(X) \geqslant \text{card}(Y)$ means that either $Y = \emptyset$ or there is an injection $f : Y \to X$.

[43] The proof of the Lemma makes it clear that the result holds even if we restrict attention to the set of choice *functions* on $(X, \mathcal{A})$.

[44] A *well-order* $\succsim$ on a arbitrary set Y is a total order that for every nonempty subset A of Y there exists an element $a \in A$ such that $x \succsim a$, for every $x \in A$.

$\mathrm{succ}(A)^{\uparrow,\trianglerighteq}$, we would get $\mathcal{B} \subseteq (\mathcal{A} \backslash \mathcal{A}') \cup \mathrm{succ}(A)^{\uparrow,\trianglerighteq}$ and, hence, $c_{\mathrm{succ}(A)}(B) = \{a\}$, for every $B \in \mathcal{B}$, contradicting the fact that $c_{\mathrm{succ}(A)}$ is not rationalizable on $\mathcal{B}$. It follows that $\mathcal{B} \cap D^{\Downarrow,\trianglerighteq} \neq \emptyset$. If $\mathcal{B} \cap D^{\Downarrow,\trianglerighteq}$ has more than two elements, then c_D is not rationalizable on $\mathcal{B}$ by construction, and we are done. Therefore, we must have $\mathcal{B} \cap D^{\Downarrow,\trianglerighteq} = \{C\}$ and $c_D(C) = \{f(C)\}$. Given that $D^{\uparrow,\trianglerighteq} \subseteq \mathrm{succ}(A)^{\uparrow,\trianglerighteq}$, it follows that $c_D(B) = \{a\} = c_{\mathrm{succ}(A)}(B)$, for every $B \in \mathcal{B} \backslash \{C\}$. Given that $c_{\mathrm{succ}(A)}$ is not rationalizable on $\mathcal{B}$, we must have $c_{\mathrm{succ}(A)}(C) = \{f(C)\}$ and, hence, that $c_{\mathrm{succ}(A)}|_{\mathcal{B}} = c_D|_{\mathcal{B}}$ implying that c_D is not rationalizable on $\mathcal{B}$, as required. By the contradiction assumption and the completeness of $\trianglerighteq$, for any $A, D \in \mathcal{A}'$ with $A \neq D$, either $I(c_D) \geqslant I(c_{\mathrm{succ}(A)})$ or $I(c_A) \geqslant I(c_{\mathrm{succ}(D)})$.

Putting together what we learned, we see that, for every $A, B \in \mathcal{A}'$ with $A \neq B$, $(I(c_A), I(c_{\mathrm{succ}(A)})) \neq \emptyset$, $(I(c_B), I(c_{\mathrm{succ}(B)})) \neq \emptyset$ and $(I(c_A), I(c_{\mathrm{succ}(A)})) \cap (I(c_B), I(c_{\mathrm{succ}(B)})) = \emptyset$. But this implies that we can construct an injection from $\mathcal{A}'$ into $\mathbb{Q}$, contradicting the fact that $\mathrm{card}(\mathcal{A}') \geqslant \mathrm{card}(\mathbb{R})$. □

As corollary of Lemma 4, we get the following result:

Proposition 16. *Fix $n \in \mathbb{N}$. Let $X = \mathbb{R}^n_+$ and $\mathcal{A}$ be the collection of budget sets on $\mathbb{R}^n_+$, i. e. $A \in \mathcal{A}$ iff there is a vector $\boldsymbol{p} \in \mathbb{R}^n_+$ and $w \in \mathbb{R}_+$ such that $A = B(\boldsymbol{p}, w) \coloneqq \{\boldsymbol{x} \in X : \boldsymbol{p} \cdot \boldsymbol{x} \leqslant w\}$. Then, there is no index of incompatibility consistent with $\succsim_{rat}$.*

Proof. We have that $\mathbf{0} \in \bigcap \mathcal{A}$ and if we define

$$\mathcal{A}' \coloneqq \{B(\boldsymbol{p}, 1) \in \mathcal{A} : p_i \leqslant 1 \text{ for all } i \in \{1, \ldots, n\}\},$$

both conditions of Lemma 4 are satisfied. □

References

Afriat, S. N. (1967). The construction o f a utility function from expenditure data. *International Economic Review 8*(1), 67–77.

Afriat, S. N. (1973). On a system of inequalities in demand analysis: An extension of the classical method. *International Economic Review 14*(2), 460–472.

Aguiar, V. H. and R. Serrano (2018). Classifying bounded rationality in limited data sets: a slutsky matrix approach. *SERIEs* (9), 389–421.

Ambrus, A. and K. Rozen (2013). Rationalizing choice with multi-self models. *Unpublished Paper*.

Andreoni, J. and J. Miller (2002). Giving according to garp: An experimental test of the consistency of preferences for altruism. *Econometrica 70*(2), 737–753.

Apesteguia, J. and M. Ballester (2015). A measure of rationality and welfare. *Journal of Political Economy 123*(6), 1278–1310.

Apesteguia, J. and M. A. Ballester (2013). Choice by sequential procedures. *Games and Economic Behavior*, forthcoming.

Aumann, R. J. (1962). Utility theory without the completeness axiom. *Econometrica 30*(3), 445–462.

Balakrishnan, N., E. A. Ok, and P. Ortoleva (2021). Inferential choice theory. *Unpublished paper*.

Bernheim, B. D. (2016). The good, the bad, and the ugly: A unified approach to behavioral welfare economics. *Journal of Benefit-Cost Analysis 7*(1), 12–68.

Bernheim, D. and A. Rangel (2009). Beyond revealed preference: Choice-theoretic foundations for behavioral welfare economics. *Quarterly Journal of Economics 124*(1), 51–104.

Bordes, G. (1976). Rationality and collective choice. *The Review of Economic Studies 43*(3), 451–457.

Bossert, W., Y. Sprumont, and K. Suzumura (2005). Maximal-element rationalizability. *Theory and Decision* (58), 325350.

Bossert, W. and K. Suzumura (2009). External norms and rationality of choice. *Economics Philosophy 25*(2), 139–152.

Bouacida, E. (2021). Identifying choice correspondences. *Unpublished Paper*.

Brocas, I., J. D. Carrilos, T. D. Combos, and N. Kodaverdian (2019). The development of consistent decision-making across economic domains. *Games and Economic Behavior* (116), 217–240.

Caradonna, P. (2019). How strong is the weak axiom. *Unpublished paper*.

Cherepanov, V., T. Feddersen, and A. Sandroni (2013). Rationalization. *Theoretical Economics 8*, 775–800.

Choi, S., S. Kariv, W. Müller, and D. Silverman (2014). Who is (more) rational? *American Economic Review 104*(6), 1518–1550.

Costa-Gomes, M., C. Cueva, G. Gerasimou, and M. Tejiscak (2020). Choice, deferral and consistency. *School of Economics and Finance Discussion Paper* (1416).

de Clippel, G. and K. Rozen (2021). Relaxed optimization: How close is a consumer to satisfying first-order conditions? *The Review of Economics and Statistics*.

Dean, M. and D. Martin (2016). Measuring rationality with the minimum cost of revealed preference relations. *The Review of Economics and Statistics 98*(3), 524534.

Dietrich, F. and C. List (2016). Reason-based choice and context-dependence: an explanatory framework. *Economics Philosophy 32*(2), 175–229.

Echenique, F., S. Lee, and M. Shum (2011). The money pump as a measure of revealed preference violations. *Journal of Political Economy 119*(6), 12011223.

Eliaz, K. and E. A. Ok (2006). Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences. *Games and Economic Behavior 56*, 61–86.

Famulari, M. (1995). A household-based, nonparametric test of demand theory. *The Review of Economics and Statistics 77*(2), 372–382.

Frick, M. (2016). Monotone threshold representations. *Theoretical Economics 11*(3), 757–772.

Février, P. and M. Visser (2004). A study of consumer behavior using laboratory data. *Experimental Economics 7*, 93–114.

Green, J. and D. Hojman (2007). Choice, rationality and welfare measurement. *Unpublished Paper*.

Harbaugh, W. T., K. Krause, and T. R. Berry (2001). Garp for kids: On the development of rational choice behavior. *American Economic Review 91*(5), 1539–1545.

Houtman, M. and J. Maks (1985). Determining all maximal data subsets consistent with revealed preference. *Kwantitatieve Methoden 6*(19), 89–104.

Kalai, G., A. Rubinstein, and R. Spiegler (2002). Rationalizing choice functions by multiple rationales. *Econometrica 70*(6), 2481–2488.

Laslier, J.-F. (1997). *Tournament Solutions and Majority Voting.* Springer.

Manzini, P. and M. Mariotti (2007). Sequentially rationalizable choice. *American Economic Review 97*(5), 1824–1839.

Manzini, P. and M. Mariotti (2010). Revealed preferences and boundedly rational choice procedures: an experiment. *Unpublished Paper*.

Manzini, P. and M. Mariotti (2012). Categorize then choose: Boundedly rational choice and welfare. *Journal of the European Economics Association 10*(5), 1141–1165.

Masatlioglu, Y., D. Nakajima, and E. Y. Ozbay (2012). Revealed attention. *American Economic Review 102*(5), 2183–2205.

Mattei, A. (2000). Full-scale real tests of consumer behavior using experimental data. *Journal of Economic Behavior Organization 43*(4), 487–497.

Mononen, L. (2021). The foundations ofmeasuring (cardinal) rationality. *Unpublished paper*.

Nielsen, K. and J. Rehbeck (2020). When choices are mistakes. *Unpublished paper*.

Nishimura, H. and E. A. Ok (2020). Preference structures. *mimeo*, New York University.

Nishimura, H., E. A. Ok, and J. K.-H. Quah (2017). A comprehensive approach to revealed preference theory. *American Economic Review 107*(4), 1239–1263.

Ok, E. A., P. Ortoleva, and G. Riella (2015). Revealed (p)reference theory. *American Economic Review 105*(1), 299–321.

Ok, E. A. and G. Tserenjigmid (2021). Comparative rationality ofrandom choice behaviors. *Unpublished paper*.

Qiu, J. and Q. Ong (2017). Indifference or indecisiveness: a strict discrimination. *Unpublished Paper*.

Ribeiro, M. and G. Riella (2017). Regular preorders and behavioral indifference. *Theory and Decision 82*(1), 1–12.

Richter, M. K. (1966). Revealed preference theory. *Econometrica 34*(3), 635–645.

Rubinstein, A. and Y. Salant (2008). (a,f): Choice with frames. *Review of Economic Studies 75*, 1287–1296.

Swofford, J. L. and G. A. Whitney (1986). Flexible functional forms and the utility approach to the demand for money: A nonparametric analysis: A note. *Journal of Money, Credit and Banking 18*(3), 383–389.

Tyson, C. J. (2008, January). Cognitive constraints, contraction consistency, and the satiscing criterion. *Journal of Economic Theory 138*(1), 51–70.

Varian, H. R. (1990). Goodness-of-fit in optimizing models. *Journal of Econometrics 46*(1-2), 125–140.

Yildiz, K. (2016). List-rationalizable choice. *Theoretical Economics 11*(2), 587–599.