On ambiguity-seeking behavior in finance models with smooth ambiguity*

Dmitry Makarov
HSE University, ICEF

February 1, 2023

Abstract

Ambiguity-seeking behavior is universally assumed away in a large growing finance literature incorporating smooth ambiguity preferences. We look at the three common justifications for doing so and find that they are all misguided. First, it is believed that smooth ambiguity models are ill-defined under ambiguity-seeking. Second, it is believed that a representative investor cannot be ambiguity-seeking given the evidence that individuals are, on average, ambiguity-averse. Third, the fact that some calibrations of existing models reveal that a representative investor is ambiguity-averse is taken to indicate that individual traders cannot be, on average, ambiguity-seeking. These three reasons have remained unquestioned by the profession for more than a decade, probably due to their strong intuitive appeal. Our paper is the first to examine each of them rigorously, revealing that, surprisingly, neither of them justifies disregarding ambiguity-seeking. This generates two novel actionable insights. First, a theorist developing a smooth ambiguity model should characterize the admissible levels of ambiguity-seeking behavior such that the model is well-posed, as against the current practice of precluding this behavior by assumption. Second, a researcher using data to calibrate parameters of such a model should allow an ambiguity attitude parameter to take any value from the admissible region, unlike the current practice of disallowing values associated with ambiguity-seeking. Broadly, our point is that letting data to speak for themselves regarding investors' ambiguity attitudes is better than taking a dogmatic stance.

Keywords: ambiguity-seeking, smooth ambiguity, investor heterogeneity, representative investor, limited participation

*I am grateful to Jack Favilukis (the discussant), Michel Verlaine (the discussant), Dimitra Papadovasilaki (the discussant), Nilakshi Borah (the discussant), Anna Bayona (the discussant), Yanyi Wang (the discussant), Suleyman Basak, Wei Bin, Emiliano Catonini, Toshiki Honda, Chris Julliard, Hening Liu, Massimo Marinacci, Anna Pavlova, Marzena Rostek, and seminar participants at ICEF, Moscow Finance International Conference, Econometric Society Winter meetings, Eurofidai Paris Finance conference, WFB Symposium, SWFA, Eastern Finance Association, and 10th Asian Quantitative Finance seminar for their comments. All errors are solely my responsibility. Contact information: email dmakarov@hse.ru; webpage sites.google.com/site/dmmakarov.

1 Introduction

As far as ambiguity-seeking behavior is concerned, there is a large and widening gap between empirical evidence and a burgeoning theoretical finance literature incorporating smooth ambiguity preferences. While the theoretical literature universally disregards ambiguity-seeking, mounting evidence shows that this behavior plays an important role even in the general population. Among participants in financial markets (whose preferences matter most for asset pricing), ambiguity-seeking is expected to be even more pronounced given evidence that ambiguity-averters are more likely to stay away from stock trading. Section 2 reviews the literature.

This paper examines the three main rationales for excluding ambiguity-seeking behavior. First, the absence of ambiguity-seeking can be viewed as a technical condition for models to be well-posed (similarly to the role of no-risk-seeking condition in many models). Second, when one builds a representative-investor model, one may posit that a representative investor cannot be ambiguity-seeking given that an average person is ambiguity-averse (e.g., Ellsberg's experiment). Third, flipping the argument around, because a representative investor is ambiguity-averse according to some model calibrations (e.g., Ju and Miao (2012)), one may surmise that ambiguity-seeking cannot be prevalent among individual traders.[1]

These three rationales appear intuitive and convincing, which is probably why neither of them has been formally examined. Our paper fills this gap. Our key contribution is to argue that each rationale is flawed. Importantly, we do not rely on carefully constructed counterexamples to make this point. On the contrary, we consider several stylized smooth ambiguity models with no unusual features. The

[1] As we discuss in Section 2.2, many papers assume ambiguity-aversion without much discussion, and so we cannot rule out that other justifications might exist, besides the three considered in this paper. However, based on our reading of a large number of papers as well as numerous conversations with researchers in this area, we believe that our list of the three rationales is exhaustive.

key take-away message is that researchers should give the data a chance to speak for themselves on what ambiguity attitude is more empirically relevant, instead of the current approach of postulating ambiguity aversion.

Our main findings are as follows. First, we find that smooth ambiguity models, both considered in this paper and more general ones (see Remark 1), are well-specified if investors are ambiguity-seeking provided that it is not too severe. Second, we find that a representative investor can be ambiguity-seeking even when individual traders whom she "represents" are, on average, ambiguity-averse. Third, we show that ambiguity-seeking can be prevalent among individual traders, and yet a representative investor, when calibrated to the prices generated by the traders, may turn out to be ambiguity-averse.

There are two practical suggestions arising from our analysis. First, researchers should, as part of model analysis, determine the admissible levels of ambiguity-seeking behavior for which their model is well-defined. The result of this analysis is model-specific, and so it cannot be computed once and then used in other settings.[2] Second, when calibrating a model, the ambiguity attitude parameter should be allowed to take values in the whole of the admissible region, and not only in the part corresponding to ambiguity aversion.

Ignoring these suggestions would not be problematic if we could, somehow, be confident that ambiguity-seekers do not play a role in generating any of the observed phenomena in financial markets. However, it is hard to square this conjecture with substantial empirical evidence of ambiguity-seeking behavior (see Section 2.1). Moreover, we know little to nothing about ambiguity attitudes of professional investors

[2] This is notably different from addressing a similar question but for a risk aversion parameter in models without ambiguity. For example, models with CARA (CRRA) utility are typically well-specified if and only if ARA (RRA) coefficient is positive, and this condition does not depend on features such as the number of risky stocks and their characteristics (means, variances, correlations). Under smooth ambiguity, however, these features matter.

(e.g., mutual and hedge fund managers) whose decisions, and hence preferences, play a major role in shaping financial markets. In this situation, it seems best to keep an open mind on what ambiguity attitude is more empirically relevant.

We consider the issues addressed in this paper to be important and timely given that they relate to a flourishing finance and macro-finance literature incorporating smooth ambiguity.[3] Our focus on smooth ambiguity, and not other ambiguity specifications,[4] is motivated by the observation that studies aiming to explain stylized facts *quantitatively* rely predominantly on the smooth ambiguity approach. To give two examples, Ju and Miao (2012) is a pioneering work showing that a smooth ambiguity model can match the equity premium, risk-free rate and equity volatility, and also to generate a variety of other observed regularities. Gallant, R Jahan-Parvar, and Liu (2019) compare several influential models with and without smooth ambiguity and find support for the former models. They also find significant quantitative effects of smooth ambiguity on asset prices. Other quantitatively oriented smooth ambiguity models are listed in footnote 3.

We now turn to describing the details of our analysis. To examine the validity of the first rationale for ambiguity-aversion (that otherwise models would not be well-posed), we consider a setting with a risk-free bond and two risky stocks, one or both of which can be ambiguous. For tractability, we assume that: i) both stock returns and beliefs about ambiguous expected returns are normally distributed, and

[3]The smooth ambiguity approach was initiated by Klibanoff, Marinacci, and Mukerji (2005). Finance works building on this approach include Caskey (2009), Ju and Miao (2012) Chen, Ju, and Miao (2014) Jahan-Parvar and Liu (2014), Backus, Ferriere, and Zin (2015), Thimme and Völkert (2015), Guidolin and Liu (2016), Collard, Mukerji, Sheppard, and Tallon (2018), Wei (2018), Altug, Collard, Akmakli, Mukerji, and Ozsoylev (2019), Gallant, R Jahan-Parvar, and Liu (2019), Miao, Wei, and Zhou (2019), Liu and Zhang (2020), Savor, Wilson, and Puhl (2020).

[4]Other approaches are maxmin model (Gilboa and Schmeidler (1989), α-maxmin model (Ghirardato, Maccheroni, and Marinacci (2004)), variational model (Maccheroni, Marinacci, and Rustichini (2006)), prospect theory (Wakker (2010)), and vector expected utility model (Siniscalchi (2009)). Cubitt, van de Kuilen, and Mukerji (2019), among others, is an experimental study comparing existing approaches.

ii) preferences are represented by a composite of two exponential (CARA) functions. Because of the tractability, this framework is increasingly used in the smooth ambiguity literature.[5] While this literature postulates no ambiguity-seeking, we characterize *endogenously* the condition on investor ambiguity attitude that is necessary and sufficient for the model to be well-posed. Hereafter, we refer to this condition as Allowed Ambiguity-Seeking condition, or briefly *AAS condition*. Essentially, the AAS condition is equivalent to the requirement that investors' portfolio problems have unique solutions.

We derive analytically the AAS condition and find that it is satisfied when investors are moderately ambiguity-seeking. The finding that ambiguity-seeking is admissible is somewhat surprising in light of a natural parallel to the risk-seeking not being admissible in many models. As we know, unconstrained risk-seekers have an infinite demand for risky stocks and, therefore, need to be assumed away.[6] The reason the parallel breaks down is as follows. Investing more in ambiguous stocks makes the portfolio not only more ambiguous but also more risky. An ambiguity-seeking risk-averse[7] investor likes the former and dislikes the latter. When she is moderately ambiguity-seeking, the risk-aversion effect dominates and her demand for ambiguous stocks is bounded.

The second justification for ambiguity-aversion posits that ambiguity aversion of an average individual trader translates into the same attitude of the representative

[5]See, for example, Gollier (2011), Maccheroni, Marinacci, and Ruffino (2013), Hara and Honda (2018), Mukerji, Ozsoylev, and Tallon (2019).

[6]There is a growing number of theoretical works incorporating *local* risk-seeking behavior, in which case there is no problem of an infinite demand. Examples include Cuoco and Kaniel (2011), Basak and Makarov (2014), Sotes-Paladino and Zapatero (2019), Basak, Makarov, Shapiro, and Subrahmanyam (2020).

[7]This paper focuses on one novel aspect of preferences, ambiguity-seeking, and so we abstract from the possibility that investors can be risk-seeking. This behavior is examined, empirically and theoretically, in a number of works, see, for example, Crainich, Eeckhoudt, and Trannoy (2013), Noussair, Trautmann, and van de Kuilen (2014), Halevy, Persitz, and Zrill (2018), Guiso, Sapienza, and Zingales (2018), Brocas, Carrillo, Giga, and Zapatero (2019), Polisson, Quah, and Renou (2020).

investor. This argument considers the average attitude but ignores its dispersion in the population. Our analysis reveals a key role of the dispersion, in that we show that when it is sufficiently high the representative investor can be ambiguity-*seeking*, thus displaying the opposite attitude to that of traders whom she "represents".

We consider the same setting as described above but now with multiple traders with heterogeneous ambiguity attitudes. For simplicity, we consider only two cases: with two and three traders. One trader is set to be ambiguity-seeking and the remaining one or two are ambiguity-averse. We note that the case with one ambiguity-seeking and two ambiguity-averse traders is broadly consistent with the proportion of ambiguity-seekers documented by Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg (2019) (see Section 2.1). We calculate equilibrium asset prices in the multiple-trader settings and then calibrate to these prices the preference parameters of a representative investor.

We show that, regardless of how high the average ambiguity aversion among individual traders is, the representative investor can be ambiguity-seeking provided that traders are sufficiently heterogeneous in ambiguity attitudes. The intuition is as follows. The ambiguity-seeking trader finds ambiguous stocks more attractive and so controls a larger fraction of the supply. As a result, her positive attitude towards ambiguity plays a dominant role in determining the representative investor's attitude. This mechanism is reminiscent of one in models with risk aversion heterogeneity, whereby the property that risky stocks are mostly held by more risk-tolerant traders leads to interesting results (Chan and Kogan (2002), Bhamra and Uppal (2014), Gârleanu and Panageas (2015)).

Turning to the third justification for ambiguity-aversion, recall that it posits that this behavior is more empirically relevant because it is displayed by a representative investor in some models once they are calibrated to the data. Presumably, we are more concerned with preferences of individual traders who actually invest in financial markets rather than of a hypothetical representative investor. The question, then, is whether ambiguity aversion of a representative investor points unequivocally to the same attitude being prevalent among traders. Our analysis reveals that the answer is "no", which is our third key contribution.

We consider a smooth ambiguity setting with limited stock market participation. This feature has been extensively studied under other preferences,[8] but our paper is, to our knowledge, the first smooth ambiguity model incorporating limited participation. We assume that the "true" economy is populated by two traders: the stockholder who invests in both risky and riskless assets and the non-stockholder who invests only in the riskless asset. The stockholder is ambiguity-seeking and the non-stockholder is ambiguity-neutral. Therefore, ambiguity aversion is—by construction—absent in the "true" economy.

We then step into the shoes of a researcher who calibrates a representative-investor model to the observed asset prices. In our case, these are the equilibrium prices obtained in the two-trader "true" economy with limited participation. Interestingly, we find that the representative investor can turn out to be ambiguity-averse even though no individual trader has this attitude. To understand the intuition, recall the general result that limited participation increases the equity premium (see Section 10.2 in Campbell (2017) for a detailed discussion). Given this, the equity premium obtained in the limited-participation economy appears to be relatively high from the standpoint of a representative-investor economy with *full* participation. To match the high premium, we show that the process of model calibration may render the representative investor ambiguity-averse. We conduct comparative static analysis to

[8]Limited participation has been examined under standard CRRA preferences (Basak and Cuoco (1998)), Epstein-Zin preferences (Guvenen (2009)), and maxmin preferences (Cao, Wang, and Zhang (2005), Easley and O'Hara (2009), Hirshleifer, Huang, and Teoh (2017)).

describe configurations of model parameter under which this outcome is likely to occur.

Broadly, our findings suggest that *model risk* is a real, and not just a hypothetical, concern when one relies on representative-agent models to learn about ambiguity attitudes of individual traders. Single-agent models abstract from various types of investor heterogeneity observed in reality, and—as we show by analyzing the heterogeneity in market participation—this can introduce a disconnect between ambiguity attitude prevalent in reality and that of a hypothetical representative investor. Our work complements the literature showing that limited participation can have a substantial effect on estimated levels of risk aversion (Basak and Cuoco (1998), Guvenen (2009)). Given our findings, this feature also has important implications in the context of estimating ambiguity attitude.[9]

The remainder of the paper is organized as follows. Section 2 discusses empirical evidence of ambiguity-seeking behavior and theoretical finance research in which this behavior is disregarded. Section 3 describes the full participation model, and Section 4 analyzes the model. The limited participation model is presented in Section 5 and analyzed in Section 6. Section 7 concludes, and Appendix A presents all proofs.

[9]Besides limited participation, the equity premium puzzle literature has uncovered other factors that have a sizeable effect on estimated level of risk aversion, such as habit formation (Detemple and Zapatero (1991), Constantinides (1990)) and trading frictions (Luttmer (1996)). In future research, it seems interesting to examine whether these factors can, like limited participation, make a representative investor and individual traders to have opposite ambiguity attitudes.

2 Ambiguity attitudes: empirical evidence and theoretical models

The goal of this Section is to review the literature substantiating the premise of this paper—that ambiguity-seeking behavior plays an important role in reality but is disregarded in the voluminous finance and macro-finance literature incorporating smooth ambiguity preferences. Let us start by summarizing the discussions in Sections 2.1 and 2.2.

As for empirical evidence described in Section 2.1, the key points are: i) ambiguity-seekers make up a sizeable fraction of general population, and ii) ambiguity-seekers are more likely to participate in financial markets than ambiguity-averters, and so the fraction of ambiguity-seekers among investors is likely to be an even higher than in general. Given the evidence, researchers increasingly argue that models "need to be extended beyond the common assumption of universal ambiguity aversion" (Dimmock, Kouwenberg, Mitchell, and Peijnenburg (2015)).

As for the theoretical finance and macro-finance literature discussed in Section 2.2, our key points are: i) all theoretical works we are aware of disregard ambiguity-seeking behavior, and ii) the reasons for doing so are not formally examined. To the best of our knowledge, these points apply not only to the studies discussed below but also to all finance and macro-finance works adopting the smooth ambiguity approach.

2.1 Evidence of ambiguity-seeking behavior

Estimating the share of ambiguity-seekers turns out to be sensitive to the experimental design (participants, method of measuring ambiguity attitude, etc) and so the reported values vary across studies: i) 30% in Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg (2019), ii) 49%, 22%, or 35% depending on event likelihoods of events in Dimmock, Kouwenberg, and Wakker (2016b), iii) 16% in Cubitt, van de Kuilen, and Mukerji (2019), iv) 27% in Kelsey and le Roux (2018). We refer the reader to Trautmann and van de Kuilen (2015) for an excellent, more detailed survey of the evidence.

Another stream of research challenges the view that attitude towards ambiguity is a constant trait defining one's behavior in all situations and argues that the actual picture is more nuanced. For example, as discussed in more detail in Barberis (2018) (Section 8), an individual may be ambiguity-seeking in situations in which she feels competent and ambiguity-averse otherwise (Heath and Tversky (1991), Fox and Tversky (1995)). Presumably, people become investors when they feel competent in matters of stock trading and so, according to the above view, ambiguity-seeking may well be an important behavior among investors. Besides competence, other factors are also shown to determine whether a person is likely to display ambiguity-seeking or -aversion, such as probabilities of gains and losses (Kocher, Lahno, and Trautmann (2018), Dimmock et al. (2015), Baillon and Bleichrodt (2015)). In a recent work, Brenner and Izhakian (2018) also show that ambiguity attitude can be asymmetric, in that "love for ambiguity increases with the expected probability of losses" whereas "aversion to ambiguity increases with the expected probability of gains.'

The share of ambiguity-seekers among investors is expected to be even higher than in general population due to self-selection based on ambiguity attitude. Ambiguity-seekers are more likely to participate in, and ambiguity-averters to stay away from, trading in (ambiguous) stocks, as documented by Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010), Dimmock, Kouwenberg, Mitchell, and Peijnenburg (2016a), and Dimmock, Kouwenberg, and Wakker (2016b), among others.

Finally, we note that large professional investors play a major role in financial markets (Cuoco and Kaniel (2011)), and so, as far as asset pricing models are concerned, the relevant ambiguity attitudes may well be ones displayed by mutual and hedge fund managers. Empirical research on this issue is still in its infancy. However, several findings of existing studies, such as the above-mentioned relation between perceived competence and ambiguity-seeking behavior, can be considered as suggestive evidence that a sizeable share of portfolio managers can be ambiguity-seeking.

2.2 Ambiguity-aversion in finance models

Finance literature considering smooth ambiguity preferences is substantial and growing. In the interest of space, we discuss in some detail only several papers and then we briefly comment on other works.

A growing number of papers consider a tractable CARA-exponential setting, which allows to compute explicitly not only all equilibrium quantities but also the AAS condition, as we do in this paper. Existing works, however, do not derive or discuss this condition. Maccheroni, Marinacci, and Ruffino (2013) present ambiguity-aversion as "a key condition for the paper" (p. 1083); they assume that "the DM [decision-maker] is ambiguity averse" before presenting optimal portfolio analyses. One of our economic settings is the same as analyzed in Maccheroni et al. We get the same results but show they are also valid under moderate ambiguity-seeking (see Remark 3 in the Appendix).

Subsequent works treat ambiguity-seekers in the same way as Maccheroni et al. Hara and Honda (2018) (p. 9) note that they "we will not pay much attention to [ambiguity-seeking] case". Mukerji, Ozsoylev, and Tallon (2019) (p. 8) state they "never consider ambiguity seeking" in their analysis.

Another strand of smooth ambiguity macro-finance literature is initiated by Ju and Miao (2012) who consider preferences with a three-way separation among risk aversion, ambiguity aversion, and intertemporal substitution. Ju and Miao do not discuss explicitly whether their investor is allowed to be ambiguity-seeking or not, however either conclusion can possibly be drawn. On the one hand, ambiguity-seeking seems to be allowed given the formal description of the model.[10] On the other hand, one may conclude otherwise given that ambiguity-seeking scenarios are not considered in an otherwise comprehensive comparative static analysis (see Sections 4.2–4.4 in Ju and Miao).

As an example of the latter conclusion, consider Gallant, R Jahan-Parvar, and Liu (2019) who estimate Ju and Miao's model. Gallant et al. (in footnote 6) state that they "follow Ju and Miao (2012)" in ruling out ambiguity-seeking behavior even though Ju and Miao make no argument to this effect. Such misunderstandings can be avoided if researchers present explicitly the AAS condition under which their results are valid.

Wei (2018) restricts model parameters to ensure ambiguity-aversion and justifies it as "empirically relevant parameter restriction" (p. 14). Many papers disregard ambiguity-seeking behavior with little or no justification, see, for example, Gollier (2011), Jahan-Parvar and Liu (2014), Chen, Ju, and Miao (2014), Backus, Ferriere, and Zin (2015), Thimme and Völkert (2015), Guidolin and Liu (2016), Collard, Mukerji, Sheppard, and Tallon (2018), Altug, Collard, Akmakli, Mukerji, and Ozsoylev (2019), Miao, Wei, and Zhou (2019), and Liu and Zhang (2020).

[10]Looking at expressions (5) and (6) in Ju and Miao, we see that risk and ambiguity attitude parameters γ and η are only constrained to be positive, but the ambiguity-aversion condition $\eta \geq \gamma$ is *not* imposed.

3 Model with full stock market participation

3.1 Setup

We now present assumptions and definitions common to the two cases analyzed later in Sections 4.1 and 4.2, respectively. Specific features are presented in these Sections.

We consider a two-period setup, in which investors trade in the first period and consume in the second period. There are two risky stocks 1 and 2; each stock is in unit supply. The exogenously given risk-free rate is r_f. The future payoffs of stocks 1 and 2, X_1 and X_2, are

$$X_1 = \mu_1 + \varepsilon_1, \quad X_2 = \mu_2 + \varepsilon_2, \tag{1}$$

where $\varepsilon_1 \sim N(0, \sigma_1^2),\ \varepsilon_2 \sim N(0, \sigma_2^2),\ cor(\varepsilon_1, \varepsilon_2) = \rho \in (-1, 1)$.

The final wealth w of an investor is

$$w = \theta_1 X_1 + \theta_2 X_2 + (w_0 - \theta_1 P_1 - \theta_2 P_2) r_f, \tag{2}$$

where w_0 is initial wealth, θ_i is the holding of stock i, and P_i is the price of stock i to be determined in equilibrium.

The investor has smooth ambiguity utility function $V(w)$ given by

$$V(w) = E_\mu v(u^{-1}[E_\varepsilon u(w)]), \tag{3}$$

where E_ε is the expectation over the shocks ε_1 and ε_2, and E_μ is the expectation over the distribution of ambiguous parameter(-s), as made precise later. The risk- and ambiguity-pertinent utility functions $u()$ and $v()$ are given by CARA functions:

$$u(x) = -\exp(-\gamma_r x), \quad v(x) = \begin{cases} -\exp(-\gamma_a x)/\gamma_a, & \gamma_a \neq 0 \\ x, & \gamma_a = 0 \end{cases}, \tag{4}$$

We assume the investor is risk averse, $\gamma_r > 0$, but not necessarily ambiguity-averse (see footnote 7). At the outset, we place no restriction on the ambiguity attitude parameter γ_a, and so the parameter that γ_a of the ambiguity-pertinent utility function $v(x)$ can be any real number, positive or negative. Ambiguity-aversion, -neutrality, and -seeking are obtained under, respectively, $\gamma_a > \gamma_r$, $\gamma_a = \gamma_r$, and $\gamma_a < \gamma_r$.

A key aspect of the ambiguity-seeking case is that not all values of parameter γ_a satisfying $\gamma_a < \gamma_r$ are admissible. In particular, a model may not be well-defined when γ_a is sufficiently low, that is when ambiguity-seeking is sufficiently severe. This brings us to the notion of *Allowed Ambiguity Seeking condition* or, for brevity, *AAS condition*.

DEFINITION 1 (AAS condition): AAS condition is a restriction on the extent of ambiguity-seeking displayed by investors that is necessary and sufficient for their portfolio problems to be well-defined and have unique solutions (the multiplicity issue is discussed in Remark 4).

REMARK 1 (Model extensions): It is for clarity of exposition that we analyze a stylized two-stock setup in the main paper. All our main results remain valid if we consider multiple stocks or other natural extensions of the model.[11]

[11] In the Appendix, we derive the AAS condition for a multiple-stock case (see condition (34)). It is easy to verify that moderate ambiguity-seeking is still allowed by the condition. If we extend the model by considering multiple investors who can be heterogeneous in terms of risk-aversion, ambiguity aversion, and beliefs pertaining to risk- or ambiguity-related stock characteristics, each investor's ambiguity attitude parameter has to satisfy its own AAS condition. The overall AAS condition is, then, given by the requirement each investor-specific AAS condition holds.

3.1.1 Economies I, II and III

We consider economies populated by one, two, and three investors with utility functions of the form (3)–(4). We refer to the economies as, respectively, *Economy-I*, *Economy-II,* and *Economy-III.* One of our exercises is to back out the parameters of the utility function of a single investor in Economy-I from the equilibrium stock prices obtained in multiple-investor Economies-II and III. In the context of this analysis, we refer to the single investor as *representative investor* and to the multiple investors as *individual traders* or *traders.*

Individual traders are assumed to have the same risk aversion $\gamma_r > 0$, because risk aversion heterogeneity is not the focus here. Ambiguity heterogeneity, on the other hand, is a key feature. We introduce parameters A_a and H_a capturing, respectively, the average ambiguity attitude across traders and its heterogeneity. Specifically, for a given A_a and H_a, the ambiguity attitude parameters γ_{1a} and γ_{2a} of traders 1 and 2 in Economy-II are given by $\gamma_{1a} = A_a - H_a$ and $\gamma_{2a} = A_a + H_a$. For Economy-III, the same parameters of investors 1, 2 and 3 are given by $\gamma_{1a} = A_a - H_a$, $\gamma_{2a} = \gamma_{3a} = A_a + H_a/2$. Given this, investor 1 is ambiguity-seeking in both economies, and the other investor or investors are ambiguity-averse.

When examining the second rationale for the ambiguity-aversion assumption, we will consider Economies-II and -III in which the traders are, on average, ambiguity-averse, consistent with empirical evidence. We will account for this by assuming that $A_a > \gamma_r$. In other words, we envision the "average" investor whose ambiguity attitude parameter is equal to A_a and risk aversion parameter is γ_r. Then, the condition that $A_a > \gamma_r$ implies that this "average" investor is ambiguity-averse.

REMARK 2 (Representative investor): In the literature studying investor heterogeneity, the term "representative investor" may refer to different concepts (see a discussion in Section 14 in Shefrin (2008)). In our paper, this term is always used to denote a hypothetical investor who generates the same equilibrium asset prices as individual traders, taking all model parameters as given. Our paper focuses on insights that can apply to the process of model calibration and interpreting the obtained results, and so we do not examine questions of more theoretical interest such as whether a representative investor can be constructed from a weighted sum of individual traders' utility functions.

4 Analysis of full-participation model

4.1 One risky stock, one ambiguous stock

Consider Economy-I in which only stock 2 is ambiguous while stock 1 is purely risky. The setting is essentially the same as analyzed in Maccheroni et al. (2013) (Section 6.3). Their interpretation is that the ambiguous and purely risky stocks correspond to, respectively, a foreign and a domestic stock indices.

The investor is ambiguous about stock 2's mean payoff μ_2, and her belief is represented by a normal distribution

$$\mu_2 \sim N(\mu_{2a}, \sigma_{2a}^2),$$

where σ_{2a}^2 reflects the degree of ambiguity. The expectation E_μ in (3) is taken over this distribution.

Proposition 1 characterizes analytically the AAS condition.

PROPOSITION 1: *The AAS condition in Economy-I with ambiguous stock 2 is*

$$\gamma_a > -\gamma_r \sigma_2^2 (1-\rho^2)/\sigma_{2a}^2. \tag{5}$$

The non-empty region $\gamma_a \in (-\gamma_r\sigma_2^2(1-\rho^2)/\sigma_{2a}^2, \gamma_r)$ corresponds to ambiguity-seeking behavior allowed by AAS condition. The equilibrium risk premia are given by

$$\mu_1 - P_1 r_f = \gamma_r\sigma_1\left(\rho\sigma_2 + \sigma_1\right), \ \mu_{2a} - P_2 r_f = \gamma_a\sigma_{2a}^2 + \gamma_r\sigma_2\left(\rho\sigma_1 + \sigma_2\right). \tag{6}$$

Proposition 1 reveals that moderate ambiguity-seeking is allowed by the AAS condition. Looking at the right-hand side of (5), we see that the threshold $-\gamma_r\sigma_2^2(1-\rho^2)/\sigma_{2a}^2$ decreases in the risk aversion γ_r. The higher is the risk-aversion, the less attractive is stock 2 for the investor, and so she can be even more ambiguity-seeking, i.e., have lower γ_a, without demanding an infinite amount of stock 2. This is why the threshold decreases in γ_r. Similarly, the threshold $-\gamma_r\sigma_2^2(1-\rho^2)/\sigma_{2a}^2$ decreases in the ratio of stock 2's risk to ambiguity, σ_2^2/σ_{2a}^2, because an increase in the ratio makes stock 2 less attractive. Finally, when the two stocks become more positively or negatively correlated, and ρ^2 increases, the risk of stock 2 plays a smaller role as it can be more effectively hedged away via trading in stock 1. Hence, the set of allowed ambiguity-seeking levels shrinks, which is reflected in $-\gamma_r\sigma_2^2(1-\rho^2)/\sigma_{2a}^2$ increasing in ρ^2. The intuition behind the equilibrium risk premia (6) is obvious and so is not presented here. We will use expressions (6) when calculating the preference parameters of the representative investor.

Another main contribution of our paper is to examine whether ambiguity-aversion of an average trader can justify the practice of ruling out ambiguity-seeking behavior by a representative investor. (In the Introduction, we refer to this reasoning as the second rationale for ambiguity-aversion.) Towards this, we first calculate equilibrium stock prices generated by heterogeneous traders who are, on average, ambiguity-averse. Then, we compute the preference parameters of the representative investor generating the same equilibrium prices.

As described in Section 3, we consider Economies-II and -III with, respectively, two and three traders whose average ambiguity attitude is A_a and its heterogeneity is H_a. We denote by γ_r^{II} and γ_a^{II} the risk- and ambiguity attitude parameters of the corresponding representative investor. Analogously, we define parameters γ_r^{III} and γ_a^{III} of the representative investor corresponding to Economy-III. Our interest is not in the individual values of the preference parameters but in whether the investor is ambiguity-averse or -seeking. Given this, Proposition 2 reports the ratios $\gamma_a^{II}/\gamma_r^{II}$ and $\gamma_a^{III}/\gamma_r^{III}$. Ambiguity-seeking (aversion) obtains when the ratio is lower (higher) than one.

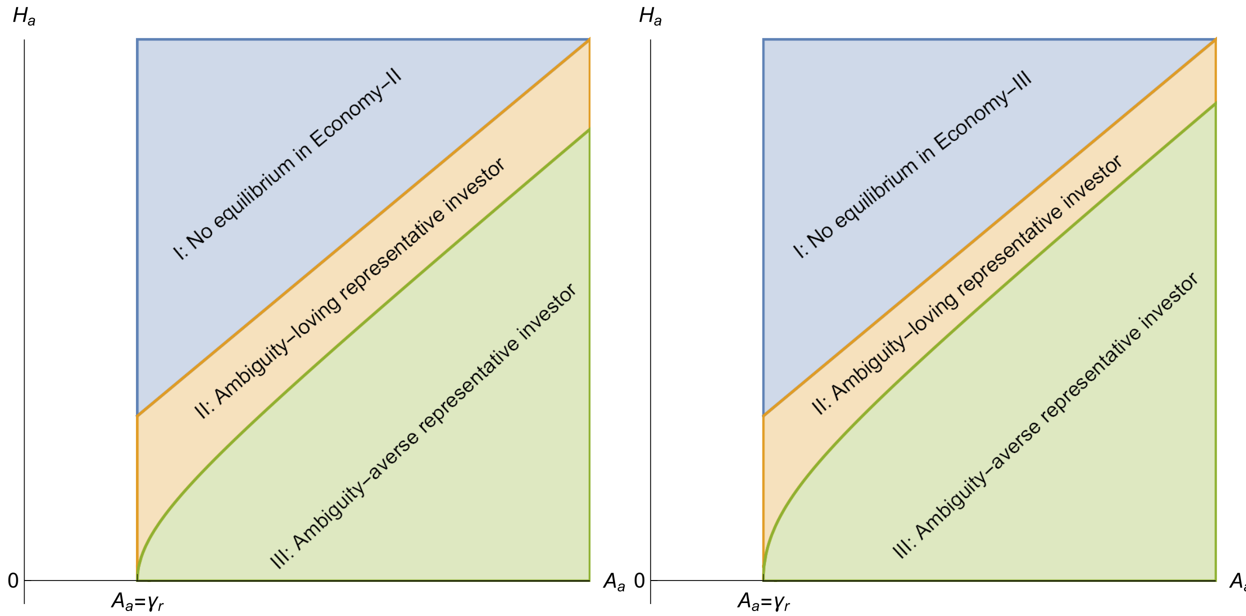
PROPOSITION 2: *Consider Economies II and III with one ambiguous stock in which condition (5) is satisfied for each trader. The preference parameters of the corresponding representative investors are given by:*

$$\begin{aligned} \frac{\gamma_a^{II}}{\gamma_r^{II}} &= \frac{\sigma_{2a}^2\left(A_a^2 - H_a^2\right) + \left(1-\rho^2\right)\sigma_2^2 A_a \gamma_r}{\left(1-\rho^2\right)\sigma_2^2\gamma_r^2 + A_a\gamma_r\sigma_{2a}^2}, \\ \frac{\gamma_a^{III}}{\gamma_r^{III}} &= \frac{2\sigma_2^2 A_a\gamma_r\left(1-\rho^2\right) + \sigma_a^2\left(A_a - H_a\right)\left(2A_a + H_a\right)}{\gamma_r\sigma_a^2\left(2A_a - H_a\right) + 2\left(1-\rho^2\right)\sigma_2^2\gamma_r^2}. \end{aligned} \tag{7}$$

We provide a graphical analysis of expressions (7) because it is more compact and transparent than describing them analytically. We set stock characteristics to plausible value[12], and then consider various combinations of parameters A_a and H_a describing the average ambiguity attitude and its dispersion. We present results under the condition $A_a > \gamma_r$, so that the traders are ambiguity-averse on average. The plots presented in this and the next Sections are typical in a sense that they are not driven by a particular choice of parameter values.

Figure 1 depicts the results for Economy-II in panel (a) and for Economy-III in panel (b). We see that the results in the two panels are qualitatively similar. The

[12] We set parameters to: $\mu_1 = \mu_{2a} = 0.1$, $\sigma_1 = \sigma_2 = \sigma_{2a} = 0.2, \rho = 0.5$, $r_f = 1, \gamma_r = 2$.



(a) Representative investor for "ambiguity-averse" Economy-II

(b) Representative investor for "ambiguity-averse" Economy-III

Figure 1. Attitude towards ambiguity of the representative investor depending on the attitudes of the traders, as measured by $A_a > \gamma_r$ (average ambiguity-aversion) and $H_a > 0$ (dispersion of ambiguity-aversion).

representative investor is ambiguity-seeking when the heterogeneity H_a is sufficiently high (the middle region in both panels). In this case, the ambiguity-seeking trader 1 holds a substantial fraction of the supply of ambiguous stock 2. Therefore, the representative investor's utility in general and ambiguity attitude in particular are largely determined by those of trader 1. Therefore, the representative investor is ambiguity-seeking when individual traders are heterogeneous enough.

The top region in both panels in Figure 1 corresponds to high levels of heterogeneity H_a and, hence, to low values of investor 1's ambiguity attitude parameter $\gamma_{1a} = A_a - H_a$. In these cases, the AAS condition (5) is not satisfied for trader 1 and equilibrium does not exist. Finally, when the heterogeneity is low (bottom regions in both panels in Figure 1), the representative investor's ambiguity attitude is close to the average attitude in the economy, which is negative.

4.2 Two ambiguous stocks

We consider Economy-I in which both stocks are now ambiguous. The goal is to see whether ambiguity-seeking behavior remains to be admissible and, if yes, to understand how the maximum allowed strength of this behavior changes with the number of ambiguous stocks.

Investors are ambiguous about stock 1 and 2's mean payoffs μ_1 and μ_2, and their beliefs are represented by:

$$\mu_1 \sim N(\mu_{1a}, \sigma_{1a}^2), \quad \mu_2 \sim N(\mu_{2a}, \sigma_{2a}^2), \tag{8}$$

and $\rho_a \in (-1, 1)$ denotes the correlation between μ_1 and μ_2.

Proposition 3 reports the AAS condition and equilibrium risk premia.

PROPOSITION 3: *The AAS condition in the setting with two ambiguous stocks is $\gamma_a > \Gamma_a$, where the threshold $\Gamma_a < 0$ is given by*

$$\Gamma_a = \frac{\begin{aligned}&-\left(\sigma_2^2\sigma_{1a}^2 + \sigma_1^2\sigma_{2a}^2 - 2\rho\sigma_1\sigma_2\sigma_{1a}\sigma_{2a}\rho_a\right) + \\ &+\sqrt{2\sigma_1^2\sigma_2^2\sigma_{1a}^2\sigma_{2a}^2\left(2\rho_a^2 + 2\rho^2 - 1\right) - 4\rho\sigma_1\sigma_2^3\sigma_{1a}^3\sigma_{2a}\rho_a - 4\rho\sigma_1^3\sigma_2\sigma_{1a}\sigma_{2a}^3\rho_a + \sigma_2^4\sigma_{1a}^4 + \sigma_1^4\sigma_{2a}^4}\end{aligned}}{2\sigma_{1a}^2\sigma_{2a}^2\left(1 - \rho_a^2\right)/\gamma_r}. \tag{9}$$

The stocks' risk premia are:

$$\begin{aligned}\mu_{1a} - P_1 r_f &= \gamma_r\sigma_1\left(\sigma_1 + \rho\sigma_2\right) + \gamma_a\sigma_{1a}\left(\sigma_{1a} + \rho_a\sigma_{2a}\right), \\ \mu_{2a} - P_2 r_f &= \gamma_r\sigma_2\left(\sigma_2 + \rho\sigma_1\right) + \gamma_a\sigma_{2a}\left(\sigma_{2a} + \sigma_{1a}\rho_a\right).\end{aligned} \tag{10}$$

Proposition 3 establishes that moderate ambiguity-seeking, $\gamma_a \in (\Gamma_a, \gamma_r)$, remains to be admissible with two ambiguous stocks. What changes is the maximum allowed extent of this behavior. Ambiguity attitude parameter γ_a is now constrained by the threshold Γ_a, whereas when only stock 2 is ambiguous the threshold is $-\gamma_r\sigma_2^2(1 -$

$\rho^2)/\sigma_{2a}^2$ (see equation (5)). In general, either of the two thresholds can be the larger one,[13] but in the special case when the beliefs about μ_1 and μ_2 are uncorrelated ($\rho_a = 0$), the thresholds can be ranked.

COROLLARY 1: *In the special case of $\rho_a = 0$, the AAS condition with two ambiguous stocks is more restrictive than with one ambiguous stock. That is, the maximum strength of ambiguity-seeking behavior with one ambiguous stock is higher than with two ambiguous stocks:*

$$\Gamma_a > -\gamma_r \sigma_2^2 (1-\rho^2)/\sigma_{2a}^2. \tag{11}$$

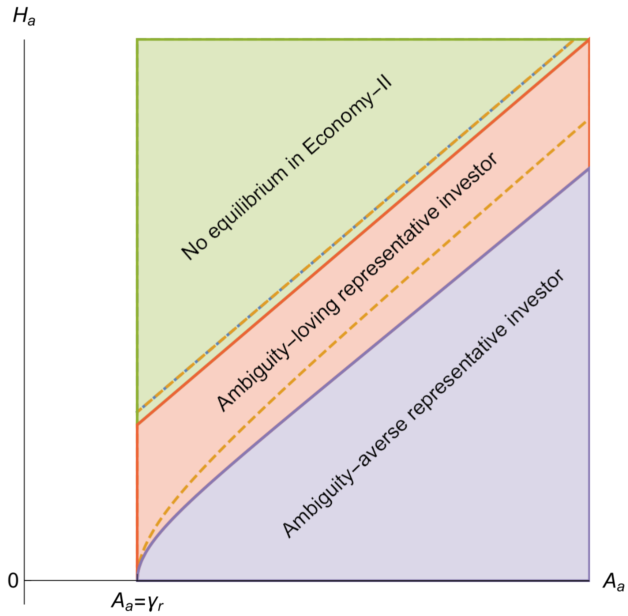
When both stocks are ambiguous, a portfolio of these stocks is, other things equal, more ambiguous and so is more attractive for an ambiguity-seeking investor. Therefore, we need to constrain ambiguity-seeking behavior more with two ambiguous stocks. Condition (11) formalizes this intuition.

We now turn to the question of how ambiguity attitude of the representative investor depends the ambiguity attitudes of individual traders in Economies-II and -II, as measured by parameters A_a and H_a. The problem is the same as described before Proposition 2 in Section 4.1.

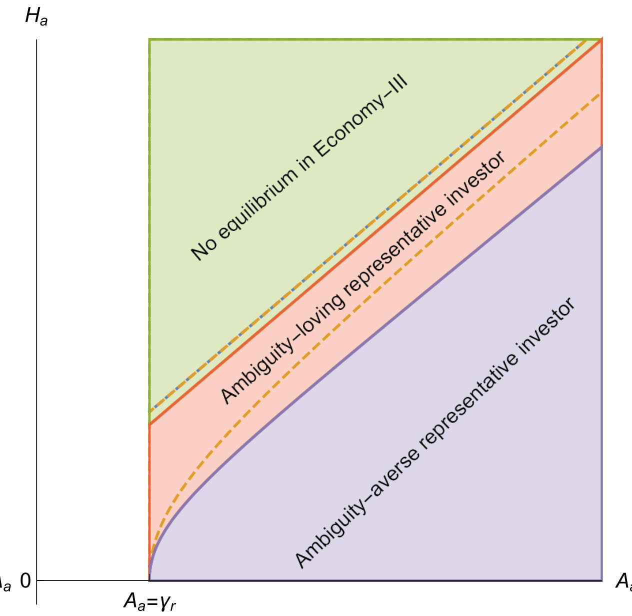
Proposition 4 reports the ratios of the preference parameters of the representative investor corresponding to Economy-II and III. As a reminder, ambiguity aversion (seeking) obtains when the ratio is higher (lower) than one.

PROPOSITION 4: *Consider Economy-II and III with two ambiguous stock. The*

[13] The easiest way to see this is by a numerical example. If $\gamma_r = 2, \rho = 0.5, \rho_a = 0.8$, $\sigma_1 = 0.56, \sigma_2 = 0.1, \sigma_{1a} = 1.5, \sigma_{2a} = 0.5$, we have that $\Gamma_a = -0.0799485$ and $-\gamma_r \sigma_2^2(1-\rho^2)/\sigma_{1a} = -0.06$. On the other hand, if $\sigma_{1a} = 4$ and the other parameters are unchanged, the two thresholds have the opposite order as they are now equal -0.03125 and -0.06.



(a) Representative investor for ambiguity-averse Economy-II



(b) Representative investor for ambiguity-averse Economy-III

Figure 2. *Two ambiguous stocks.* Attitude towards ambiguity of the representative investor depending on the average ambiguity-aversion A_a and its heterogeneity H_a. The dashed lines depict the corresponding boundaries in the case of one ambiguous stock.

ratios characterizing the representative investor's ambiguity attitude are given by

$$\frac{\gamma_a^{II}}{\gamma_r^{II}} = \frac{[\,0 \quad 1] * \boldsymbol{Q}(1)}{[\,1 \quad 0] * \boldsymbol{Q}(1)}, \quad \frac{\gamma_a^{III}}{\gamma_r^{III}} = \frac{[\,0 \quad 1] * \boldsymbol{Q}(2)}{[\,1 \quad 0] * \boldsymbol{Q}(2)}, \tag{12}$$

where the matrix $\boldsymbol{Q}(k)$ is

$$\boldsymbol{Q}(k) = \boldsymbol{M}^{-1}\left((\gamma_r \boldsymbol{\Sigma_r} + (A_a - H_a)\boldsymbol{\Sigma_a})^{-1} + k(\gamma_r \boldsymbol{\Sigma_r} + (A_a + H_a/k)\boldsymbol{\Sigma_a})^{-1}\right)^{-1}\mathbf{1},$$

and the matrices $\boldsymbol{M}, \boldsymbol{\Sigma_r},$ and $\boldsymbol{\Sigma}_a$ are provided in the Appendix.

Figure 2 depicts the regions in which the representative investor is ambiguity-seeking (middle regions in both panels), ambiguity-averse (bottom regions in both panels), and when equilibrium in Economy-II or III does not exist (top regions). We see that our qualitative insights remain the same as with one ambiguous stock—the representative investor is ambiguity-seeking when the heterogeneity of traders is relatively high (but not too high). Quantitative implications are different, however, which we should not ignore given that the increasingly quantitative focus of finance models. We see from Figure 2 that the solid lines (region boundaries with two ambiguous stocks) differ from the dashed lines (the boundaries with one ambiguous stock).

5 A model with limited stock market participation

A well-recognized concern in finance and economic models is *model risk*. Conclusions derived from a model may no longer be valid once we make the model more realistic by introducing additional realistic ingredients. The general goal of this Section is to show that this concern is relevant when one tries to learn about ambiguity attitudes of multiple heterogeneous traders from results of representative-investor models that do not account for heterogeneities.

Our key contribution is to show that one may find a representative investor to be ambiguity-averse after calibrating her preferences to match asset prices, even when the stock traders who have actually generated the prices are ambiguity-seeking.

We modify the earlier setting in several ways, but most importantly we incorporate heterogeneity in stock market participation leading to *limited stock market participation*. The reason to introduce this feature is two-fold. First, limited participation is a widely-documented pattern of investor behavior (see Guiso and Sodini (2013) and Gomes, Haliassos, and Ramadorai (2020) for excellent reviews of the evidence). Second, studying limited participation has generated a number of valuable insights under other utility specifications (Basak and Cuoco (1998), Guvenen (2009), Cao, Wang, and Zhang (2005), Easley and O'Hara (2009), Hirshleifer, Huang, and Teoh (2017)), but to our knowledge this feature has not yet been studied under smooth ambiguity preferences. In this paper, we do not attempt to endogenize the non-participation decision of some investors.[14]

5.1 *Assets, ambiguity, and investors*

The economy has two dates, $t = 1$ and $t = 2$. The investment opportunities are given by a risky stock in positive supply $x > 0$ and a risk-free bond in zero supply paying the rate of return r_f. The risk-free rate is determined endogenously in equilibrium (in the earlier analysis, it was exogenously given).

The stock pays dividends X_1 and X_2 (per unit of stock) at dates 1 and 2, respectively. Date 1 is taken to be the ex-dividend date, meaning that buying the stock at date 1 gives the holder a claim to dividend X_2 but not to $X_1 > 0$.[15] Date-2 dividend is distributed normally:

$$X_2 \sim N(\mu, \sigma^2). \tag{13}$$

The expected value of date-2 dividend μ is ambiguous, and investors belief is represented by

$$\mu \sim N(\mu_a, \sigma_a^2). \tag{14}$$

[14] Examples of mechanisms that can generate non-participation include participation costs (Vissing-Jørgensen (2002), Gomes and Michaelides (2005), Fagereng, Gottlieb, and Guiso (2017)), disappointment aversion (Ang, Bekaert, and Liu (2005)), loss aversion (Dimmock and Kouwenberg (2010), Gomes (2005)), uninsurable wealth shocks (Gormley, Liu, and Zhou (2010)), narrow framing (Barberis, Huang, and Thaler (2006)), inattention arising under "news utility" (Pagel (2018)), rank-dependent preferences (Chapman and Polkovnichenko (2009)), cointegration of stock and labor markets (Benzoni, Collin-Dufresne, and Goldstein (2007)), and social interactions (Hong, Kubik, and Stein (2004)).

[15] Note the difference in notation between the current and the previous models. In this section, X_1 and X_2 are the dividends of the same asset occurring at two different dates. In the earlier analyses, X_1 and X_2 are the dividends of the two assets occurring at the same terminal date.

There are two investors in the economy, one who can trade in both the stock and the bond and the other invests only in the bond. As in Guvenen (2009), we refer to the former as the *stockholder* and to the latter as the *non-stockholder*.

The investors' endowments are as in Basak and Cuoco (1998): the non-stockholder's endowment consists of a positive position in the bond and the stockholder's endowment consists of x units of the stock (total stock supply) and a short position in the bond of the same size as the non-stockholder's long position (ensuring that the total bond endowment is zero). The value of the non-stockholder's endowment is denoted by $e^{ns} > 0$, and the value of the stockholder's endowment e^s is then given by

$$e^s = xX_1 + xP - e^{ns}, \tag{15}$$

where P is the stock price.

The investors consume at both dates $t = 1$ and 2, and c_t^s and c_t^{ns} denote date-t consumptions of the stockholder and non-stockholder, respectively. The stockholder's stock position is denoted by θ. The intertemporal budget constraints of the two investors are

$$c_2^s = \theta X_2 + (e^s - c_1^s - \theta P)r_f, \tag{16}$$

$$c_2^{ns} = (e^{ns} - c_1^{ns})r_f. \tag{17}$$

5.2 Preferences of investors

The non-stockholder is not exposed to ambiguity because she, by assumption, does not invest in the ambiguous stock. Hence, her preferences towards ambiguity are irrelevant and do not affect any of the results. For simplicity, we consider the non-stockholder to be ambiguity-neutral. Her optimization problem is a standard intertemporal utility maximization with respect to date-1 consumption c_1^{ns}:

$$\max_{c_1^{ns}} \quad u\left(c_1^{ns}\right) + \beta u\left(c_2^{ns}\right), \tag{18}$$

subject to the budget constraint (17). In equation (18), the utility function $u(\cdot)$ is as given in equation (4), and $\beta \in (0, 1)$ is the time discount factor.

The stockholder, on the other hand, is exposed to ambiguity (in equilibrium, she cannot have zero stock holding), and so her ambiguity attitude jointly with risk attitude affect her decision-making. Formally, she chooses date-1 consumption c_1^s and stock investment θ to maximize the following objective function:

$$\max_{c_1^s, \theta} \quad u\left(c_1^s\right) + \beta \phi^{-1}\left(E_\mu[\phi\left(E_{X_2} u(c_2^s)\right)]\right), \tag{19}$$

subject to the budget constraint (16).

In equation (19), the utility function $u(\cdot)$ is given by (4) (same as for the non-stockholder), and the function $\phi(\cdot)$ capturing the stockholder's sensitivity to ambiguity is specified, as in Section 3, as a composite function

$$\phi(y) = v(u^{-1}(y)), \tag{20}$$

where function $v(\cdot)$ is as defined in (4). Computing the composite function, we get

$$\phi(y) = -\frac{(-y)^{\frac{\gamma_a}{\gamma_r}}}{\gamma_a}. \tag{21}$$

Importantly, given the goal of this paper, we do *not* assume that $\phi(\cdot)$ is a concave function, that is, we do not rule out ambiguity-seeking behavior by assuming that $\gamma_a > \gamma_r$. Instead, we, as in the previous section, seek to determine endogenously the allowed extent of ambiguity-seeking behavior that is necessary and sufficient for the model to be well-posed.

5.3 Equilibrium

An equilibrium in the economy is defined by a pair (P, r_f), the stock price and the risk-free rate, and a triple of the investors' choices $(c_1^{ns}, c_1^s, \theta)$, the consumptions of the two investors and the stockholder's stock investment, such that:

i) taking as given r_f, c_1^{ns} is a solution of (18),

ii) taking as given P and r_f, c_1^s and θ is a solution of (19),

iii) good and asset markets clear at date 1: $c_1^{ns} + c_1^s = X_1$, $\theta = x$, and $e^{ns} - c_1^{ns} = -(e^s - c_1^s - \theta P)$,

iv) good market clears at date 2: $c_2^{ns} + c_2^s = xX_2$ where c_2^{ns} and c_2^s are given by equations (16)–(17).

6 Analysis of the limited-participation model

6.1 Characterizing optimal behavior and equilibrium

The first-order condition for the non-stockholder's optimization problem (18) is

$$\gamma_r e^{-\gamma_r c_1^{ns}} - \beta r_f \gamma_r e^{-r_f \gamma_r \left(e^{ns} - c_1^{ns}\right)} = 0, \tag{22}$$

solving which yields the optimal consumption of the non-stockholder

$$c_1^{ns} = \frac{r_f \gamma_r e^{ns} - \log\left(\beta r_f\right)}{\left(1 + r_f\right)\gamma_r}. \tag{23}$$

Before describing the optimal behavior of the stockholder, we characterize the AAS condition under which the stockholder's objective function is strictly concave in the two choice variables. Proposition 5 presents the objective function and the condition.

PROPOSITION 5: *The stockholder's maximization problem* (19) *is equivalent to*

$$\max_{c_1^s,\theta} \quad -e^{-\gamma_r c_1^s} - \beta e^{-\gamma_r\left(-\theta\left(\theta\gamma_a{\sigma_a}^2-2\mu_a+\theta\sigma^2\gamma_r\right)/2+(e^s-c_1^s-\theta P)r_f\right)}. \tag{24}$$

The AAS condition is given by the condition

$$\gamma_a > -\sigma^2\gamma_r/\sigma_a^2. \tag{25}$$

Proposition 5 confirms that our key insight—that smooth ambiguity models can be well-posed under moderate ambiguity-seeking—is not somehow specific to the settings examined earlier. Indeed, we see that the non-empty region $(-\sigma^2\gamma_r/\sigma_a^2, \gamma_r)$ describes the allowed levels of ambiguity-seeking behavior. Comparing the (25) and (5), we see that the two are similar and so the intuition provided when discussing (5) is relevant for the result in this Section. Briefly, when condition (25) holds, the risk from investing a large amount in the stock harms the stockholder more than the resulting ambiguity benefits her. As a result, she has a finite demand for the stock, implying that the model is well-specified under moderate ambiguity-seeking behavior.

In the remainder of the analysis, we assume that AAS condition (25) is satisfied. Proposition 6 presents the solution of the stockholder's maximization problem (24).

PROPOSITION 6: *The stockholder's optimal date-1 consumption* c_1^s *and stock* *investment θ are given by*

$$c_1^s = (2r_f\gamma_r e^s(\gamma_a\sigma_a^2 + \gamma_r\sigma^2) - 2\log(\beta r_f)(\gamma_a\sigma_a^2 + \gamma_r\sigma^2) - 2P\mu_a r_f\gamma_r + \mu_a^2\gamma_r$$
$$+ P^2 r_f^2\gamma_r)/(2\,(r_f+1)\,\gamma_r\left(\gamma_a\sigma_a^2 + \sigma^2\gamma_r\right)), \tag{26}$$
$$\theta = \frac{\mu_a - Pr_f}{\gamma_a\sigma_a^2 + \gamma_r\sigma^2} \tag{27}$$

We see from Proposition 6 that expression (27) for the optimal stock investment admits a clear interpretation, while the optimal consumption (26) depends on model parameters in a more involved way and so, to save space, is not discussed here. Given condition (25), the denominator in (27) is positive and so the stockholders holds a long, zero, or a short position in the stock when the stock's expected return μ_a/P is, respectively, higher, the same, or lower than the risk-free rate r_f. The intuition behind the dependence of the optimal stock investment on each of the parameters in the denominator of (27) is straightforward.

In equilibrium, the investors' date-1 consumptions c_1^{ns} and c_1^s, given by (23) and (26), has to sum up to X_1 (dividend realization at date 1). Moreover, the stockholder's stock position θ given in (27) has to be equal to the stock supply x. Proposition 7 presents the equilibrium stock price P and the risk-free rate r_f such that these two conditions are satisfied. In the proof of the Proposition, we verify that all the remaining equilibrium conditions (presented in Section 5.3) are also satisfied.

PROPOSITION 7: *In equilibrium, the risk-free rate r_f is given by*

$$\log r_f = (2\gamma_r x\mu_a - \gamma_r x^2(\gamma_a\sigma_a^2 + \gamma_r\sigma^2) - 2x\gamma_r X_1)/4 - \log(\beta), \tag{28}$$

and the equilibrium stock price P is

$$P = \frac{\mu_a - x\gamma_a\sigma_a^2 - x\gamma_r\sigma^2}{r_f}, \tag{29}$$

in which r_f is as given in equation (28).

6.2 Ambiguity attitude of representative investor

We now turn to the main question of this Section. We consider a scenario in which the above model *with* limited participation is the "true" description of how asset prices are formed. Then, the prices that researchers would use to calibrate a model are given by (28) and (29). The model to be calibrated has a single representative investor *without* limited participation, as is typically done in existing works.

The utility function of the representative investor is assumed to the same as that of the stockholder, and therefore so is her optimization problem meaning that it is as presented in (19). The representative investor's risk- and ambiguity-attitude parameters are denoted by $\tilde{\gamma}_r$ and $\tilde{\gamma}_a$, respectively.

We envision the following process of model calibration. A researcher computes the representative investor's parameters $\tilde{\gamma}_r$ and $\tilde{\gamma}_a$ such that the equilibrium prices generated by this investor match those observed in the data. As explained above, the observed prices are given by (28) and (29). We are interested in whether the ratio $\tilde{\gamma}_r/\tilde{\gamma}_a$ is higher or lower than one: when higher (lower), the representative investor is ambiguity-seeking (-averse).

Proposition 8 characterizes analytically the ratio $\tilde{\gamma}_r/\tilde{\gamma}_a$ as a function of parameters of the "true" economy.

PROPOSITION 8: *Suppose we compute parameters $\tilde{\gamma}_r$ and $\tilde{\gamma}_a$ of the representative investor such that the resulting equilibrium prices match the prices in the "true" model* *with limited participation. The ratio of the two parameters is given by*

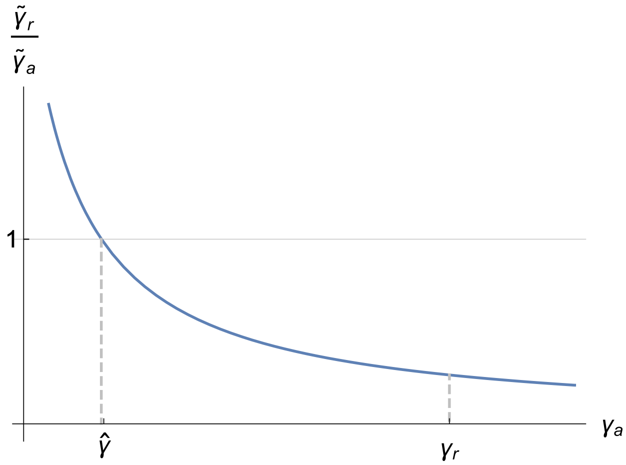
$$\tilde{\gamma}_r/\tilde{\gamma}_a = \frac{\sigma_a^2\gamma_r}{2\gamma_a\sigma_a^2 + \gamma_r\sigma^2}. \tag{30}$$

To show that Proposition 8 provides support for our main message, consider a calibrated example.

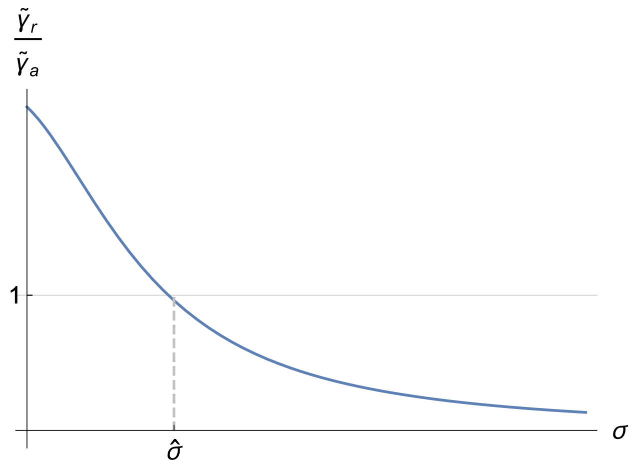
Calibrated Example. We assume that the stockholder's risk- and ambiguity attitudes in the "true" economy are, respectively, $\gamma_r = 10$ and $\gamma_a = 2$, and so the stockholder is *ambiguity-seeking* given that $\gamma_r > \gamma_a$. The volatility of date-2 dividend is $\sigma = 0.2$, and the extent of ambiguity about the dividend is $\sigma_a = 0.15$. Substituting these values into (30) yields $\tilde{\gamma}_r/\tilde{\gamma}_a \approx 0.46$. Therefore, the representative investor is *ambiguity-averse* because $\tilde{\gamma}_r < \tilde{\gamma}_a$.

To gain further insight into equation (30), we examine in Figure 3 how the ratio $\tilde{\gamma}_r/\tilde{\gamma}_a$ changes when we vary each parameter in the right-hand side of equation (30). Figure 3(a) looks at how the stockholder's ambiguity attitude γ_a affects $\gamma_r/\tilde{\gamma}_a$, and shows that whether the stockholder in the "true" model and the representative investor have the same or opposite ambiguity attitudes depends on the level of γ_a. If the stockholder is ambiguity-averse, $\gamma_a > \gamma_r$, the representative investor is also ambiguity-averse given that, as seen from the plot, $\tilde{\gamma}_r/\tilde{\gamma}_a < 1$. For intermediate values of γ_a (specifically, $\gamma_a \in (\hat{\gamma}, \gamma_r)$), the stockholder is ambiguity-seeking but the representative investor is ambiguity-averse as $\tilde{\gamma}_r/\tilde{\gamma}_a < 1$. The earlier calibrated example illustrates this case. Finally, for low values of γ_a (specifically, $\gamma_a < \hat{\gamma}$), the stockholder is ambiguity-seeking and so is the representative investor, $\tilde{\gamma}_r/\tilde{\gamma}_a > 1$.

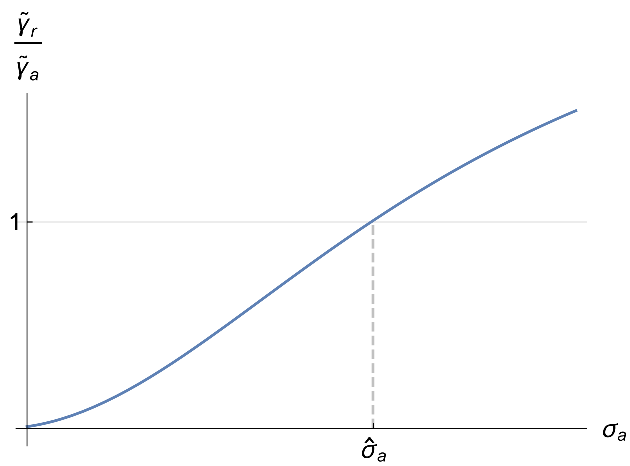
Looking at panel (b) of Figure 3, we see that the representative investor and the stockholder in the "true" model may have opposite attitudes towards ambiguity provided that the dividend volatility σ is higher than a threshold $\hat{\sigma}$ (where $\hat{\sigma}$ is marked



(a) Varying stockholder's ambiguity attitude



(b) Varying dividend volatility



(c) Varying ambiguity about dividend

Figure 3. Representative investor's ambiguity attitude for varying model parameters. The Figure plots the ratio $\tilde{\gamma}_r/\tilde{\gamma}_a$ as a function of the stockholder's ambiguity attitude γ_a (panel (a)), dividend volatility σ (panel (b)), and ambiguity about dividend σ_a (panel (c)). In all panels, when the plot is above (below) the thin grey line the representative investor is ambiguity-seeking (averse). The parameter values are as in calibrated example presented after Proposition 8.

on x-axis)—the former is ambiguity-averse and the latter is ambiguity-seeking. Otherwise, when $\sigma < \hat{\sigma}$ both are ambiguity-seeking. From panel (c), the representative investor and the stockholder have opposite attitudes towards ambiguity provided that the extent of ambiguity is lower than a threshold $\sigma_a > \hat{\sigma}_a$ (where $\hat{\sigma}_a$ is marked on x-axis); otherwise, they have the same ambiguity attitude.

6.2.1 "True" model with full participation

One may wonder whether limited participation indeed plays a key role in our main result—that ambiguity aversion of a representative investor (a common finding in existing works) does not imply that individual traders are also ambiguity averse. To address this question, we assume that the new data-generating model is as presented in Section 4 but with one change: the non-stockholder is replaced by an ambiguity-neutral trader who can invest in both available assets.

In the interest of space, we here do not discuss properties of equilibrium in this model and go straight to characterizing the ratio of the representative investor's preference parameters, $\tilde{\gamma}_r/\tilde{\gamma}_a$.

PROPOSITION 9: *The ratio of the preference parameters of the representative investor corresponding to the "true" model with full participation is*

$$\tilde{\gamma}_r/\tilde{\gamma}_a = \frac{\sigma_a^2\left(\gamma_a+\gamma_r\right)+2\sigma^2\gamma_r}{2\gamma_a\sigma_a^2+\sigma^2\left(\gamma_a+\gamma_r\right)}. \tag{31}$$

Recalling one of the individual traders is ambiguity-neutral, the average ambiguity attitude in the "true" economy is the same as that of the other trader, which is determined by parameter γ_a. If the average attitude is neutral, we substitute $\gamma_a = \gamma_r$ into (31) and obtain that $\tilde{\gamma}_r/\tilde{\gamma}_a = 1$. Hence, the representative investor is also ambiguity-neutral.

Differentiating expression (31) with respect to γ_a yields

$$\frac{\partial(\tilde{\gamma}_r/\tilde{\gamma}_a)}{\gamma_a} = -\frac{2\left(\sigma_a^2+\sigma^2\right)^2\gamma_r}{\left(2\gamma_a\sigma_a^2+\sigma^2\left(\gamma_a+\gamma_r\right)\right)^2} < 0, \tag{32}$$

and the ratio $\tilde{\gamma}_r/\tilde{\gamma}_a$ is decreasing in γ_a. Accordingly, if the ambiguity-sensitive trader is ambiguity-seeking $\gamma_a < \gamma_r$ then so is the representative investor because $\tilde{\gamma}_r/\tilde{\gamma}_a$ is higher than one. Analogously, if the former is ambiguity-averse then so is the latter. This establishes a crucial role of limited participation behind our surprising results.

7 Conclusion

Assuming away ambiguity-seeking behavior is a universal practice in voluminous finance literature incorporating smooth ambiguity preferences. We examine the three main rationales for disregarding ambiguity-seeking behavior and show that each of them is flawed. First, we show that imposing ambiguity-aversion is not necessary for smooth ambiguity models to be well-posed. Second, we show that when, in line with evidence, heterogeneous individual traders are on average ambiguity-averse, the corresponding representative investor can be ambiguity-seeking if the heterogeneity is high enough. Third, we show that ambiguity-aversion of a representative investor does not imply that individual traders are on average ambiguity-averse. The practical message of our work is that researchers should characterize explicitly the levels of ambiguity-seeking for which their model is well-defined. This result should then be used when calibrating the model, allowing the data to speak for themselves about what ambiguity attitude is most consistent with evidence.

A Proofs

General Results.

We start by computing equilibrium stock prices in the general case with an arbitrary number of stocks. We later specialize the analysis to the two-stock settings described in the main text.

Using bold symbols to denote vectors and matrices containing respective scalar variables written in normal font, the final wealth (2) is given by:

$$w = \boldsymbol{\theta'}(\boldsymbol{X} - \boldsymbol{P} r_f) + w_0 r_f. \tag{33}$$

Substituting this expression into (3), and using the result that if x is $N(\mu, \sigma^2)$ then $E[e^x] = e^{\mu+\sigma^2/2}$, we get that the inner expectation in (3) is

$$E_\varepsilon u(w) = -\exp\left(-\gamma_r \boldsymbol{\theta'}(\boldsymbol{\mu} - \boldsymbol{P} r_f) + \frac{1}{2}\gamma_r^2 \boldsymbol{\theta'}\boldsymbol{\Sigma_r}\boldsymbol{\theta}\right),$$

where we dropped $w_0 r_f$ as it does not affect optimal choices. The associated certainty equivalent $CE_\varepsilon \equiv u^{-1}[E_\varepsilon u(w_)]$ is

$$CE_\varepsilon = \boldsymbol{\theta'}(\boldsymbol{\mu} - \boldsymbol{P} r_f) - \frac{1}{2}\gamma_r \boldsymbol{\theta'}\boldsymbol{\Sigma_r}\boldsymbol{\theta}.$$

We now treat $\boldsymbol{\mu}$ as an uncertain vector with distribution $N(\boldsymbol{\mu_a}, \boldsymbol{\Sigma_a})$, and compute the outer expectation in (3) and then the ambiguity-adjusted certainty equivalent $CE_\mu \equiv v^{-1}[E_\mu v(u^{-1}[E_\varepsilon u(w_)])]$:

$$CE_\mu = \boldsymbol{\theta'}(\boldsymbol{\mu_a} - \boldsymbol{P} r_f) - \frac{1}{2}\gamma_r \boldsymbol{\theta'}\boldsymbol{\Sigma_r}\boldsymbol{\theta} - \frac{1}{2}\gamma_a \boldsymbol{\theta'}\boldsymbol{\Sigma_a}\boldsymbol{\theta}.$$

For the function $CE_\mu(\theta_1, \theta_2)$ to have a unique global maximum, its Hessian $\mathcal{H}$ must be negative definite (see, e.g., Lemma 2.41 in Beck (2014)):

$$\mathcal{H} = -\gamma_r \boldsymbol{\Sigma_r} - \gamma_a \boldsymbol{\Sigma_a} \prec 0. \tag{34}$$

When this condition holds, the optimal stock holdings are computed from the first-order condition:

$$\boldsymbol{\mu_a} - \boldsymbol{P} r_f = \gamma_r \boldsymbol{\Sigma_r} \boldsymbol{\theta} + \gamma_a \boldsymbol{\Sigma_a} \boldsymbol{\theta} \;\Rightarrow\; \boldsymbol{\theta} = (\gamma_r \boldsymbol{\Sigma_r} + \gamma_a \boldsymbol{\Sigma_a})^{-1} (\boldsymbol{\mu_a} - \boldsymbol{P} r_f). \tag{35}$$

Equilibrium vector of risk premia in Economy-I is computed by substituting the market clearing condition $\boldsymbol{\theta} = \mathbf{1}$ into (35):

$$\boldsymbol{\mu_a} - \boldsymbol{P} r_f = (\gamma_r \boldsymbol{\Sigma_r} + \gamma_a \boldsymbol{\Sigma_a}) \mathbf{1}. \tag{36}$$

Representative and individual investors. To find an equilibrium in Economy-II, we substitute $\gamma_{1a} = A_a - H_a$ and $\gamma_{2a} = A_a + H_a$ in (35) and sum up the resulting optimal stock holdings. The outcome must be a vector of ones for markets to clear:

$$\begin{aligned} (\gamma_r \boldsymbol{\Sigma_r} + (A_a - H_a)\boldsymbol{\Sigma_a})^{-1} (\boldsymbol{\mu_a} - \boldsymbol{P} r_f) + (\gamma_r \boldsymbol{\Sigma_r} + (A_a + H_a)\boldsymbol{\Sigma_a})^{-1} (\boldsymbol{\mu_a} - \boldsymbol{P} r_f) &= \mathbf{1}, \\ \boldsymbol{\mu_a} - \boldsymbol{P} r_f = ((\gamma_r \boldsymbol{\Sigma_r} + (A_a - H_a)\boldsymbol{\Sigma_a})^{-1} + (\gamma_r \boldsymbol{\Sigma_r} + (A_a + H_a)\boldsymbol{\Sigma_a})^{-1})^{-1} \mathbf{1}&. \end{aligned} \tag{37}$$

The utility parameters γ_r^{II} and γ_a^{II} of the representative investor in the equivalent Economy-I are computed by equating the risk premia given in (36) in which we substitutes γ_r^{II} and γ_a^{II} and the risk premia given in (37):

$$(\gamma_r^{II} \boldsymbol{\Sigma_r} + \gamma_a^{II} \boldsymbol{\Sigma_a}) \mathbf{1} = \left((\gamma_r \boldsymbol{\Sigma_r} + (A_a - H_a)\boldsymbol{\Sigma_a})^{-1} + (\gamma_r \boldsymbol{\Sigma_r} + (A_a + H_a)\boldsymbol{\Sigma_a})^{-1} \right)^{-1} \mathbf{1}. \tag{38}$$

Analogously, for Economy-III: we substitute $\gamma_{1a} = A_a - H_a$ and $\gamma_{2a} = \gamma_{3a} = A_a + H_a/2$ into (35), obtain the market clearing condition, solve for the equilibrium risk premia, and then equate the result to the risk premia in Economy-I with a single investor with the preference parameters sγ_r^{III} and γ_a^{III}. This gives us:

$$(\gamma_r^{III}\boldsymbol{\Sigma_r} + \gamma_a^{III}\boldsymbol{\Sigma_a})\mathbf{1} = \left((\gamma_r\boldsymbol{\Sigma_r} + (A_a - H_a)\boldsymbol{\Sigma_a})^{-1} + 2(\gamma_r\boldsymbol{\Sigma_r} + (A_a + H_a/2)\boldsymbol{\Sigma_a})^{-1}\right)^{-1}\mathbf{1}. \tag{39}$$

Proof of Proposition 1.

In the proofs of Propositions 1 and 2, we specialize the above general results to the setting presented in Section 4.1 by setting $\boldsymbol{\mu_a}$ to $[\mu_1, \mu_{2a}]'$ and assuming that:

$$\boldsymbol{\Sigma}_r = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \boldsymbol{\Sigma_a} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{2a}^2 \end{bmatrix}, \; \boldsymbol{M} \equiv \begin{bmatrix} \sigma_1\left(\rho\sigma_2 + \sigma_1\right) & 0 \\ \sigma_2\left(\rho\sigma_1 + \sigma_2\right) & \sigma_{2a}^2 \end{bmatrix}. \tag{40}$$

With these assumptions, the AAS condition (34) becomes

$$\mathcal{H} = \begin{bmatrix} -\gamma_r\sigma_1^2 & -\gamma_r\rho\sigma_1\sigma_2 \\ -\gamma_r\rho\sigma_1\sigma_2 & -\gamma_r\sigma_2^2 - \gamma_a\sigma_a^2 \end{bmatrix} \prec 0, \tag{41}$$

which is equivalent to two restrictions: $-\gamma_r\sigma_1^2 < 0$ and $det(\mathcal{H}) > 0$. The first one is always satisfied given the assumed risk aversion, $\gamma_r > 0$, and the second one expands to

$$\gamma_r\sigma_1^2(\gamma_r\sigma_2^2 + \gamma_a\sigma_a^2) - (\gamma_r\rho\sigma_1\sigma_2)^2 > 0,$$

which after rearranging leads to condition (5), as reported in the Proposition.

Substituting (40) into (36) and writing the outcome in the scalar form, we obtain (6).

REMARK 3: *Maccheroni, Marinacci, and Ruffino (2013)'s optimal portfolio.*

As a consistency check, let us verify that our optimal portfolios are the same as in Maccheroni et al.'s analysis of the same setting with two stocks one of which is ambiguous. Consider their results on pp. 1092-1093 for dollar investments[16] in stocks 1 and 2:

$$\theta_1 P_1 = \frac{BD - HA}{CD - H^2}, \quad \theta_2 P_2 = \frac{CA - HB}{CD - H^2}. \tag{42}$$

In our notation, the five constants parameters are:

$$\begin{aligned} A &= \frac{\mu_{2a}}{P_2} - r_f, \quad B = \frac{\mu_1}{P_1} - r_f, \quad C = \frac{\sigma_1^2 \gamma_r}{P_1^2}, \\ D &= \gamma_r \frac{\sigma_{2a}^2 + \sigma_2^2}{P_2^2} + (\gamma_a - \gamma_r) \frac{\sigma_{2a}^2}{P_2^2}, \quad H = \frac{\rho \sigma_1 \sigma_2 \gamma_r}{P_1 P_2}. \end{aligned}$$

Substituting these values into (42) and rearranging, we get the optimal holding of stock 1

$$\theta_1 = \frac{(\gamma_a \sigma_{2a}^2 + \sigma_2^2)(\mu_1 - P_1 r_f) - \rho \sigma_1 \sigma_2 (\mu_2 - P_2 r_f)}{\sigma_1^2 \gamma_r \left((1 - \rho^2)\, \sigma_2^2 + \gamma_a \sigma_{2a}^2\right)}.$$

It is straightforward to verify that substituting (40) into our optimal portfolio formula (35) produces the same θ_1; analogously, for θ_2.

REMARK 4: *Multiple optimal portfolios.*

If $det(\mathcal{H}) = 0$, where $\mathcal{H}$ is given in (41), then the investor's maximization problem either does not have a solution, or there are infinitely many solutions. The latter occurs when the rank of the augmented matrix $[\mathcal{H} \quad \boldsymbol{\mu_a} - \boldsymbol{P} r_f]$ is one, which is the rank of $\mathcal{H}$. Equilibrium stock premia in this case are given by the same expressions (6), because (36) does not rely on the invertibility of $\mathcal{H}$. Because the investor finds a

[16] Maccheroni et al. refer to these quantities as portfolio weights, but they are also equal to the dollar amounts invested in the stocks given that their investor is endowed with a unit of wealth.

continuum of other portfolios equally attractive to the equilibrium (market-clearing) choice $\theta_1 = \theta_2 = 1$, we view this equilibrium outcome as unstable and rule it out via having a strict inequality in (5). Equivalently, our condition (5) allows only strictly concave utility function functions with respect to portfolio variables θ_1 and θ_2. $\square$

Proof of Proposition 2.

Substituting (40) into (38) and performing three matrix inversion operations on the right-hand side, we get:

$$\boldsymbol{M}\begin{bmatrix}\gamma_r^{II}\\ \gamma_a^{II}\end{bmatrix} = \begin{pmatrix} \frac{1}{2}\sigma_1^2\gamma_r & \frac{1}{2}\rho\sigma_1\sigma_2\gamma_r \\ \frac{1}{2}\rho\sigma_1\sigma_2\gamma_r & \frac{\sigma_2 a^4\left(H_a^2-A_a^2\right)+\left(\rho^2-2\right)\sigma_2^2\sigma_{2a}^2A_a\gamma_r+\left(\rho^2-1\right)\sigma_2^4\gamma_r^2}{2(\rho^2-1)\sigma_2^2\gamma_r-2\sigma_{2a}^2A_a} \end{pmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}, \tag{43}$$

where $\boldsymbol{M}$ is defined in (40). The first equation in (7) is then obtained by premultiplying both sides of (43) by $\boldsymbol{M}^{-1}$. The second equation in (7) is derived analogously by solving (39).

What remains to be shown is that the two computed representative investors are not "too ambiguity seeking," i.e., that their utilities' parameters satisfy the AAS condition (5). Otherwise, there would be no equilibrium in Economy-I with the same stock prices as in Economy-II or III.

Given that we consider such Economies II and III in which equilibrium exists, it must be the case that the utility of investor 1, who is the most ambiguity-tolerant, satisfy (5):

$$A_a - H_a > -\gamma_r\sigma_2^2(1-\rho^2)/\sigma_{2a}^2 \Rightarrow H_a < A_a + \gamma_r\sigma_2^2(1-\rho^2)/\sigma_{2a}^2. \tag{44}$$

Substituting the upper bound $H_a = A_a + \gamma_r\sigma_2^2(1-\rho^2)/\sigma_{2a}^2$ into (7), we get after simple algebra

$$\frac{\gamma_a^{II}}{\gamma_r^{II}} = \quad \frac{\gamma_a^{III}}{\gamma_r^{III}} = -\frac{(1-\rho^2)\,\sigma_2^2}{\sigma_a^2}. \tag{45}$$

From (7), we immediately see that $\gamma_a^{II}/\gamma_r^{II}$ is decreasing in H_a, and the same is true for $\gamma_a^{III}/\gamma_r^{III}$, though showing this requires additional manipulations that we do not present here. Therefore, when (44) holds as strict inequality, $\gamma_a^{II}/\gamma_r^{II}$ and $\gamma_a^{III}/\gamma_r^{III}$ are both strictly larger than the right-hand side in (45), and thus both representative investors' preferences satisfy the AAS condition. □

Proof of Proposition 3.

In the proofs of Propositions 3 and 4, we adopt the results presented at the beginning of the Appendix to the setting of Section 4.2 by assuming

$$\begin{aligned}
&\boldsymbol{\mu_a} = [\mu_{1a}, \mu_{2a}]',\ \boldsymbol{\Sigma}_r = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \boldsymbol{\Sigma_a} = \begin{bmatrix} \sigma_{1a}^2 & \rho_a\sigma_{1a}\sigma_{2a} \\ \rho_a\sigma_{1a}\sigma_{2a} & \sigma_{2a}^2 \end{bmatrix}, \\
&\boldsymbol{M} = \begin{bmatrix} \sigma_1\left(\rho\sigma_2+\sigma_1\right) & \sigma_{1a}(\sigma_{1a}+\rho_a\sigma_{2a}) \\ \sigma_2\left(\rho\sigma_1+\sigma_2\right) & \sigma_{2a}(\sigma_{2a}+\rho_a\sigma_{1a}) \end{bmatrix}
\end{aligned} \tag{46}$$

Under (46), the AAS condition (34) becomes

$$\mathcal{H} = \begin{bmatrix} -\sigma_{1a}^2\gamma_a - \sigma_1^2\gamma_r & -\rho\sigma_1\sigma_2\gamma_r - \rho_a\sigma_{1a}\sigma_{2a}\gamma_a \\ -\rho\sigma_1\sigma_2\gamma_r - \rho_a\sigma_{1a}\sigma_{2a}\gamma_a & -\sigma_{2a}^2\gamma_a - \sigma_2^2\gamma_r \end{bmatrix} \prec 0, \tag{47}$$

which is equivalent to two restrictions: $\gamma_a > -\sigma_1^2\gamma_r/\sigma_{1a}^2$ and $det(\mathcal{H}) > 0$, where

$$det(\mathcal{H}) = \gamma_a^2\left(1-\rho_a^2\right)\sigma_{2a}^2\sigma_{2a}^2 + \gamma_a\gamma_r\left(\sigma_2^2\sigma_{2a}^2 + \sigma_1^2\sigma_{2a}^2 - 2\rho\sigma_1\sigma_2\rho_a\sigma_{2a}\sigma_{2a}\right) + \left(1-\rho^2\right)\sigma_1^2\sigma_2^2\gamma_r^2. \tag{48}$$

We see that $det(\mathcal{H})$ as a function of γ_a is an upward opening parabola, and its value at the threshold pertaining to the first restriction, $\gamma_a = -\sigma_1^2\gamma_r/\sigma_{1a}^2$, is equal to (after some algebra):

$$det(\mathcal{H})\Big|_{\gamma_a=-\sigma_1^2\gamma_r/\sigma_{1a}^2} = -\frac{\sigma_1^2\gamma_r^2\left(\rho\sigma_2\sigma_{1a} - \sigma_1\rho_a\sigma_{2a}\right)^2}{\sigma_{1a}^2} < 0. \tag{49}$$

Therefore, some part of the quadratic function $det(\mathcal{H})$ is negative, and so the equation $det(\mathcal{H}) = 0$ has two solutions. Then, if we set Γ_a to the larger one, matrix $\mathcal{H}$ will be negative definite when $\gamma_a > \Gamma_a$. Solving the quadratic equation, the larger solution is given in (9). To establish that Γ_a is negative, we combine the inequalities (49) and

$$det(\mathcal{H})\Big|_{\gamma_a=0} = (1-\rho^2)\sigma_1^2\sigma_2^2\gamma_r^2 > 0,$$

implying that the point Γ_a at which the function $det(\mathcal{H})$ changes sign belongs to the interval $(-\sigma_1^2\gamma_r/\sigma_{1a}, 0)$, and so Γ_a is negative.

From (34) and (36), the risk premium vector can be written as $-\mathcal{H}\mathbf{1}$. Computing this quantity for $\mathcal{H}$ given by (47) and expressing the outcome in the scalar form yields (10). □

Proof of Corollary 1.

Substituting $\rho_a = 0$ into (9), we obtain the threshold

$$\Gamma_a = \frac{-\left(\sigma_2^2\sigma_{1a}^2 + \sigma_1^2\sigma_{2a}^2\right) + \sqrt{2\sigma_1^2\sigma_2^2\sigma_{1a}^2\sigma_{2a}^2\left(2\rho^2-1\right) + \sigma_2^4\sigma_{1a}^4 + \sigma_1^4\sigma_{2a}^4}}{2\sigma_{1a}^2\sigma_{2a}^2}, \tag{50}$$

which we treat as a univariate function $\Gamma_a(\sigma_{1a})$. To prove the required result, it is sufficient to establish two properties of $\Gamma_a(\sigma_{1a})$: a) $\lim\limits_{\sigma_{1a}\to 0}\Gamma_a(\sigma_{1a}) = -\sigma_2^2(1-\rho^2)/\sigma_{2a}^2$, and b) $\Gamma_a(\sigma_{1a})$ is increasing over the interval $\sigma_{1a} > 0$.

As for a), both the numerator and denominator in (50) tend to zero when $\sigma_{1a} \to 0$, and so we apply L'Hospital rule to get

$$\lim_{\sigma_{1a}\to 0}\Gamma_a(\sigma_{1a}) = \frac{-2\sigma_2^2\sigma_{1a} + 0.5(4\sigma_1^2\sigma_2^2\sigma_{1a}\sigma_{2a}^2\left(2\rho^2-1\right) + 4\sigma_2^4\sigma_{1a}^3)(2\sigma_1^2\sigma_2^2\sigma_{1a}^2\sigma_{2a}^2\left(2\rho^2-1\right) + \sigma_2^4\sigma_{1a}^4 + \sigma_1^4\sigma_{2a}^4)^{-1/2}}{4\sigma_{1a}\sigma_{2a}^2}. \tag{51}$$

Again, the fraction is of 0/0 type and so we apply L'Hospital rule for the second time. The resulting expressions is rather bulky and is omitted here; evaluating it at $\sigma_{1a} = 0$ yields the above property a).

As for property b), differentiating (50) with respect to σ_{1a} we get after some algebra

$$\frac{d\Gamma_a}{d\sigma_{1a}} = \frac{\sigma_1^2\gamma_r\left(\overbrace{\sqrt{2\left(2\rho^2-1\right)\sigma_1^2\sigma_2^2\sigma_{1a}^2\sigma_{2a}^2 + \sigma_2^4\sigma_{1a}^4 + \sigma_1^4\sigma_{2a}^4}}^{X} - \overbrace{\left(\left(2\rho^2-1\right)\sigma_2^2\sigma_{1a}^2 + \sigma_1^2\sigma_{2a}^2\right)}^{Y}\right)}{\sigma_{1a}^3\sqrt{2\left(2\rho^2-1\right)\sigma_1^2\sigma_2^2\sigma_{1a}^2\sigma_{2a}^2 + \sigma_2^4\sigma_{1a}^4 + \sigma_1^4\sigma_{2a}^4}}, \tag{52}$$

which is positive when $X - Y$ in the numerator is positive. This is automatically true when $Y < 0$, and is also true when $Y \geq 0$ because

$$sgn(X - Y) = sgn(X^2 - Y^2) = sgn(4\rho^2\left(1-\rho^2\right)\sigma_2^4\sigma_{1a}^4) = 1. \tag{53}$$

$\square$

Proof of Proposition 4.

Substituting 46 into (38) and rearranging the left-hand side so that it becomes a matrix equation for the utility parameters, we get:

$$\boldsymbol{M}\begin{bmatrix}\gamma_r^{II}\\ \gamma_a^{II}\end{bmatrix} = \left((\gamma_r\boldsymbol{\Sigma_r} + (A_a - H_a)\boldsymbol{\Sigma_a})^{-1} + (\gamma_r\boldsymbol{\Sigma_r} + (A_a + H_a)\boldsymbol{\Sigma_a})^{-1}\right)^{-1}\mathbf{1}, \tag{54}$$

where matrices $\boldsymbol{M}, \boldsymbol{\Sigma}_r, \boldsymbol{\Sigma}_a$ are given in (46). Premultiplying both sides by $\boldsymbol{M}^{-1}$ (assuming that $\boldsymbol{M}$ is invertible), we find the vector $[\gamma_r^{II}, \gamma_a^{II}]$ and then the first equation in (12) represents the ratio of the vector's second element over the first element. The second equation is obtained analogously by solving (39) under 46.

Unlike the earlier setting with one ambiguous stock, with two ambiguous stocks it is a rather tedious exercise to show analytically that the utilities of the two representative investors satisfy the AAS condition whenever the condition is satisfied by the utilities of the individual traders in Economies II and III. We have verified that this is true for a large number of calibrated versions of the model, including those analyzed in Figure 2. We have not pursued this analysis further to verify that this is true in general, because even if it is not the key message of our paper would not change.

□

Proof of Proposition 5.

Substituting (4) and (16) into (19), we get the maximization problem:

$$\max_{c_1^s, \theta} \quad -\exp(-\gamma_r c_1^s) + \beta \phi^{-1} \left(E_\mu [\phi \left(E_{X_2} [-\exp(-\gamma_r (\theta X_2 + (e^s - c_1^s - \theta P) r_f))] \right)] \right), \tag{55}$$

Using (14), we compute the inner expectation of the second term in the above expression:

$$E_{X_2}[-\exp(-\gamma_r(\theta X_2 + (e^s - c_1^s - \theta P) r_f))]] = -e^{-\gamma_r \left((e^s - c_1^s - \theta P) r_f + \mu\theta - \theta^2 \sigma^2 \gamma_r / 2 \right)}. \tag{56}$$

Evaluating the function $\phi()$ given in (20) at the value given in the above equation and computing the expectation E_μ (with respect to (14)) of the resulting expression, we get

$$E_\mu[\phi\,(eqn.(56)])=-\frac{\exp\left(-\gamma_a\left(-\theta\left(\theta\gamma_a\sigma_a^2-2\mu_a+\theta\sigma^2\gamma_r\right)/2+(e^s-c_1^s-\theta P)r_f\right)\right)}{\gamma_a}. \tag{57}$$

The inverse function of $\phi()$ given in (20) is

$$\phi^{-1}(x)=-\left(-x\gamma_a\right)^{\frac{\gamma_r}{\gamma_a}}. \tag{58}$$

Evaluating this function at the value given in (57) and substituting the result into (55) we get the objective function (24) presented in Proposition 5.

Towards characterizing the appropriate AAS condition, we compute the Hessian of the function (24). We obtain 2×2 matrix $H=\{h_{ij}\}$, $i,j=1,2$, with the following elements

$$h_{11}=-\gamma_r^2e^{-c_1^s\gamma_r}-\beta r_f^2\gamma_r^2e^{-\gamma_r\left(-\theta\left(\theta\gamma_a\sigma_a^2-2\mu_a+\theta\sigma^2\gamma_r\right)/2+(e^s-c_1^s-\theta P)r_f\right)}, \tag{59}$$

$$h_{12}=h_{21}=\beta r_f\gamma_r^2\left(\mu_a-\theta\left(\gamma_a\sigma_a^2+\sigma^2\gamma_r\right)-Pr_f\right)e^{-\gamma_r\left(-\theta\left(\theta\gamma_a\sigma_a^2-2\mu_a+\theta\sigma^2\gamma_r\right)/2+(e^s-c_1^s-\theta P)r_f\right)}, \tag{60}$$

$$\begin{aligned}h_{22}=&-\frac{1}{2}\beta\gamma_r^2\left(-\mu_a+\theta\left(\gamma_a\sigma_a^2+\sigma^2\gamma_r\right)+Pr_f\right)^2e^{-\gamma_r\left(-\theta\left(\theta\gamma_a\sigma_a^2-2\mu_a+\theta\sigma^2\gamma_r\right)/2+(e^s-c_1^s-\theta P)r_f\right)}\\&-\beta\gamma_r\left(\gamma_a\sigma_a^2+\sigma^2\gamma_r\right)e^{-\gamma_r\left(-\theta\left(\theta\gamma_a\sigma_a^2-2\mu_a+\theta\sigma^2\gamma_r\right)/2+(e^s-c_1^s-\theta P)r_f\right)}.\end{aligned} \tag{61}$$

The function (24) is strictly concave if H is negative definite. The first condition for this to be the case is that $h_{11}<0$, which, as we see from (59), is always true. The second condition is that $det(H)=h_{11}h_{22}-h_{12}h_{21}>0$. Substituting (59)–(61) into this condition and dividing the resulting expression by the positive term $e^{-\gamma_r\left(-\theta\left(\theta\gamma_a\sigma_a^2-2\mu_a+\theta\sigma^2\gamma_r\right)/2+(e^s-c_1^s-\theta P)r_f\right)}$, we get, after some algebra, that $det(H)>0$ is equivalent to the condition that

$$\beta r_f^2\left(\gamma_a\sigma_a^2+\gamma_r\sigma^2\right)e^{-\gamma_r\left(-\theta\left(\theta\gamma_a\sigma_a^2-2\mu_a+\theta\sigma^2\gamma_r\right)/2+(e^s-c_1^s-\theta P)r_f\right)}+$$
$$+\left(\gamma_a\sigma_a^2+\gamma_r\sigma^2\right)+\gamma_r\left(\theta(\gamma_a\sigma_a^2+\gamma_r\sigma^2)-\mu_a+Pr_f\right)^2>0. \tag{62}$$

When $\gamma_a\sigma_a^2+\gamma_r\sigma^2>0$—which is equivalent to condition (25)—it is easy to see that the left-hand side of (62) is a sum of three non-negative terms and the first two are strictly positive, implying that (62) is satisfied. Therefore, when (25) holds the objective function (24) is strictly concave. If $\gamma_a\sigma_a^2+\gamma_r\sigma^2<0$, we can find such investment strategies θ that the last (quadratic) term in (62) is zero or close to zero so that the two other negative terms dominate. In this case, we have the opposite inequality to that in (62) implying that the objective function (24) is not strictly concave over its domain. Finally, if $\gamma_a\sigma_a^2+\gamma_r\sigma^2=0$ we can see directly from the objective function (24) that it is either does not have a bounded maximum point θ (if $-2\mu_a+2Pr_f\neq 0$) or is not sensitive to θ (if $-2\mu_a+2Pr_f=0$). In either case, the model is not well-posed and so this case is not allowed by condition (25).

□

Proof of Proposition 6.

Given the assumed condition (25), the objective function (24) is strictly concave and so there can be at most one critical (stationary) point of function (24); if it exists, it is the point of global maximum. The first-order conditions for a critical point with respect to c_1^s and θ are given by, respectively:

$$\gamma_r \left(e^{-c_1^s \gamma_r} - \beta r_f e^{\frac{1}{2}\gamma_r \left(\theta\left(\theta\gamma_a\sigma_a^2 - 2\mu_a + 2Pr_f + \theta\sigma^2\gamma_r\right) + 2c_1^s r_f - 2r_f e^s\right)} \right) = 0, \tag{63}$$

$$-\beta\gamma_r \left(\theta\gamma_a\sigma_a^2 - \mu_a + Pr_f + \theta\sigma^2\gamma_r\right) e^{\frac{1}{2}\gamma_r \left(\theta\left(\theta\gamma_a\sigma_a^2 - 2\mu_a + 2Pr_f + \theta\sigma^2\gamma_r\right) + 2c_1^s r_f - 2r_f e^s\right)} = 0. \tag{64}$$

For equation (64) to hold, the term in brackets has to zero, leading to equation (27) in the Proposition. Manipulating equation (63), we get

$$e^{-c_1^s \gamma_r} = \beta r_f e^{\frac{1}{2}\gamma_r \left(\theta\left(\theta\gamma_a\sigma_a^2 - 2\mu_a + 2Pr_f + \theta\sigma^2\gamma_r\right) + 2c_1^s r_f - 2r_f e^s\right)}, \tag{65}$$

$$\text{(take logs)} \quad -c_1^s\gamma_r = \log(\beta r_f) + \frac{1}{2}\gamma_r \left(\theta\left(\theta\gamma_a\sigma_a^2 - 2\mu_a + 2Pr_f + \theta\sigma^2\gamma_r\right) + 2c_1^s r_f - 2r_f e^s\right), \tag{66}$$

$$\text{(rearrange)} \quad -c_1^s\gamma_r - c_1^s\gamma_r r_f = \log(\beta r_f) + \frac{1}{2}\gamma_r\theta\left(\theta\gamma_a\sigma_a^2 - 2\mu_a + 2Pr_f + \theta\sigma^2\gamma_r\right) - r_f\gamma_r e^s, \tag{67}$$

$$\text{(solve for } c_1^s\text{)} \quad c_1^s = \frac{r_f\gamma_r e^s - \log(\beta r_f) - \frac{1}{2}\gamma_r\theta\left(\theta\gamma_a\sigma_a^2 - 2\mu_a + 2Pr_f + \theta\sigma^2\gamma_r\right)}{\gamma_r(1 + r_f)} \tag{68}$$

Looking at the numerator of (68), we see that the term in brackets is a sum of $(\theta\gamma_a\sigma_a^2 - \mu_a + Pr_f + \theta\sigma^2\gamma_r)$, which is zero according to (27), and $(-\mu_a + Pr_f)$. Therefore, the term in brackets can be replaced by $(-\mu_a + Pr_f)$. Making this replacement, and then substituting θ with (27) in the resulting expression we get:

$$c_1^s = \frac{r_f\gamma_r e^s - \log(\beta r_f) + \frac{1}{2}\gamma_r\theta(Pr_f - \mu_a)^2/(\gamma_a\sigma_a^2 + \gamma_r\sigma^2)}{\gamma_r(1 + r_f)}. \tag{69}$$

Multiplying the numerator and denominator by $2(\gamma_a\sigma_a^2 + \gamma_r\sigma^2)$ yields the required equation (26). □

Proof of Proposition 7.

The equilibrium stock price (29) is obtained by equating the stockholder's optimal stock demand (27) to the supply x and solving the resulting equation. We can also rearrange (29) to get:

$$Pr_f = \mu_a - x(\gamma_a \sigma_a^2 + \gamma_r \sigma^2), \tag{70}$$

which we will use later.

Substituting the value of the stockholder's endowment (15) into her optimal consumption (26) yields

$$\frac{\frac{(\mu_a - Pr_f)^2}{\gamma_a \sigma_a^2 + \sigma^2 \gamma_r} - 2r_f e^{ns} + 2xr_f(P + X_1) - 2\log(\beta r_f)/\gamma_r}{2(r_f + 1)}, \tag{71}$$

and adding this expression to the non-stockholder's optimal consumption (15) yields the total demand for date-1 consumption:

$$\frac{\frac{(\mu_a - Pr_f)^2}{\gamma_a \sigma_a^2 + \sigma^2 \gamma_r} + 2xr_f(P + X_1) - 4\log(\beta r_f)/\gamma_r}{2(r_f + 1)}. \tag{72}$$

Substituting (70) into (72), the total demand is

$$\begin{aligned} &\frac{x^2(\gamma_a \sigma_a^2 + \gamma_r \sigma^2) + 2x(\mu_a - x(\gamma_a \sigma_a^2 + \gamma_r \sigma^2)) + 2xr_f X_1 - 4\log(\beta r_f)/\gamma_r}{2(r_f + 1)} = \\ &= \frac{-x^2(\gamma_a \sigma_a^2 + \gamma_r \sigma^2) + 2x\mu_a + 2xr_f X_1 - 4\log(\beta r_f)/\gamma_r}{2(r_f + 1)}. \end{aligned} \tag{73}$$

Equating (73) to the total supply of the consumption good xX_1 and rearranging yields the equilibrium condition (28).

In the above analysis, we explicitly imposed the market clearing conditions in the markets to date-1 consumption and stock investment. As a consistency check, let us now verify that the two other market clearing conditions presented in Section 5.3 are also satisfied in equilibrium.

As for the bond market at date 1, in equilibrium the stockholder's date-1 investment in the bond $e^s - c_1^s - \theta P$ is given by—using equations (15), (29), and the market clearing conditions $c_1^s = xX_1 - c_1^{ns}$ and $\theta = x$:

$$e_-^s c_1^s - \theta P = (xX_1 + xP - e^{ns}) - (xX_1 - c_1^{ns}) - xP = c^{ns} - e^{ns},$$

which is the opposite of the non-stockholder's bond investment $e^{ns} - c^{ns}$. Therefore, the total bond investment is zero and so the bond market clears.

As for the date-2 consumption, the total consumption in equilibrium is obtained by summing up (16) and (17):

$$c_2^s + c_2^{ns} = \theta X_2 + (e_-^s c_1^s - \theta P) r_f + (e^{ns} - c_1^{ns}) r_. \tag{74}$$

Given that the bond market clears, the two expressions in brackets in equation (74) are opposite of each other and so the total consumption is θX_2. Therefore, the market for date-2 consumption clears. □

Proof of Proposition 8. The representative investor's stock demand $\tilde{\theta}$ is obtained by substituting her preference parameters $\tilde{\gamma}_r$ and $\tilde{\gamma}_a$ into equation (27), which yields

$$\tilde{\theta} = \frac{\mu_a - P r_f}{\tilde{\gamma}_a \sigma_a^2 + \tilde{\gamma}_r \sigma^2}. \tag{75}$$

Because we want to find such $\tilde{\gamma}_r$ and $\tilde{\gamma}_a$ that the equilibrium quantities P and r_f are the same in the two models, "true" and representative-investor, the quantity $P r_f$ in the numerators in (75) and (27) is the same and so, given that the stock demands in the two economies are equal in equilibrium, $\tilde{\theta} = \theta$, we have the first condition that must be satisfied by $\tilde{\gamma}_r$ and $\tilde{\gamma}_a$:

$$\tilde{\gamma}_a \sigma_a^2 + \tilde{\gamma}_r \sigma^2 = \gamma_a \sigma_a^2 + \gamma_r \sigma^2. \tag{76}$$

Analogously, we equate the total demand for date-1 dividend in the "true" economy, given by (73), to the representative investor's demand, obtained by substituting substituting $\tilde{\gamma}_r$ and $\tilde{\gamma}_a$ into the stockholder's demand function (71) and setting $e^{ns} = 0$ in the resulting equation (because there is no non-stockhoder). This yields

$$
\begin{aligned}
&-x^2(\gamma_a \sigma_a^2 + \gamma_r \sigma^2) + 2x\mu_a + 2xr_f X_1 - 4\log(\beta r_f)/\gamma_r = \frac{(\mu_a - Pr_f)^2}{\tilde{\gamma}_a \sigma_a^2 + \tilde{\gamma}_r \sigma^2} + 2xr_f(P + X_1) - 2\log(\beta r_f)/\tilde{\gamma}_r,\\
&-x^2(\gamma_a \sigma_a^2 + \gamma_r \sigma^2) + 2x(\mu_a - Pr_f) - 4\log(\beta r_f)/\gamma_r = x^2(\tilde{\gamma}_a \sigma_a^2 + \tilde{\gamma}_r \sigma^2) - 2\log(\beta r_f)/\tilde{\gamma}_r,\\
&x^2(\gamma_a \sigma_a^2 + \gamma_r \sigma^2) - 4\log(\beta r_f)/\gamma_r = x^2(\tilde{\gamma}_a \sigma_a^2 + \tilde{\gamma}_r \sigma^2) - 2\log(\beta r_f)/\tilde{\gamma}_r,\\
&\tilde{\gamma}_r = \gamma_r/2. \qquad (77)
\end{aligned}
$$

In the above calculations, the second equation is obtained by cancelling $2xr_f X_1$ from both sides, moving $2xr_f P$ to the left-hand side, and using condition (70) to transform the first term on the right-hand side. The third equation, again, makes use of (70) to transform $-x^2(\gamma_a \sigma_a^2 + \gamma_r \sigma^2) + 2x(\mu_a - Pr_f)$ to $x^2(\gamma_a \sigma_a^2 + \gamma_r \sigma^2)$. The final expression is obtained by cancelling the first terms on both sides of the third equation, which as are equal given (70).

Substituting (77) into (76) and solving for $\tilde{\gamma}_a$, we obtain

$$\tilde{\gamma}_a = \frac{2\gamma_a \sigma_a^2 + \gamma_r \sigma^2}{2\sigma_a^2}. \tag{78}$$

Dividing (77) by (78) yields the required expression (30). □

Proof of Proposition 9. The overall demand for date-1 dividend in the "true" economy is obtained by summing up the demands of the two investors, each given by (71). For the ambiguity-neutral investor, we set $\gamma_a = \gamma_r$ in (71). This demand must equal to the demand in the representative-investor economy, which is given by (71) in which we replace γ_a and γ_r by, respectively, $\tilde{\gamma}_a$ and $\tilde{\gamma}_r$. This leads the first equation:

$$\frac{\frac{\left(\mu_a - P r_f\right)^2}{(\sigma_a^2+\sigma^2)\gamma_r} + \frac{\left(\mu_a - P r_f\right)^2}{\gamma_a\sigma_a^2+\sigma^2\gamma_r} + 2 x r_f (P + X_1) - 4\log(\beta r_f)/\gamma_r}{2\left(r_f+1\right)} =$$
$$\frac{\frac{\left(\mu_a - P r_f\right)^2}{\tilde{\gamma}_a\sigma_a^2+\tilde{\gamma}_r\sigma^2} + 2 x r_f (P + X_1) - 2\log(\beta r_f)/\tilde{\gamma}_r}{2\left(r_f+1\right)}. \tag{79}$$

Similarly, equating the stock demands in the two economies, we obtain the second equation

$$\frac{\mu_a - P r_f}{\gamma_a\sigma_a^2 + \gamma_r\sigma^2} + \frac{\mu_a - P r_f}{\gamma_r(\sigma_a^2+\sigma^2)} = \frac{\mu_a - P r_f}{\tilde{\gamma}_a\sigma_a^2 + \tilde{\gamma}_r\sigma^2}. \tag{80}$$

Solving the system of equations (79)–(80) for $\tilde{\gamma}_a$ and $\tilde{\gamma}_r$ yields

$$\tilde{\gamma}_r = \gamma_r/2, \qquad \tilde{\gamma}_a = \frac{\gamma_r\left(\sigma^2\gamma_a + 2\gamma_a\sigma_a^2 + \sigma^2\gamma_r\right)}{2\left(\gamma_a\sigma_a^2 + \sigma_a^2\gamma_r + 2\sigma^2\gamma_r\right)},$$

and computing the ratio of these two values yields (31). □

References

ALTUG, S., F. COLLARD, C. C. AKMAKLI, S. MUKERJI, AND H. OZSOYLEV (2019): "Ambiguous Business Cycles: A Quantitative Assessment," Working paper.

ANANTANASUWONG, K., R. KOUWENBERG, O. MITCHELL, AND K. PEIJNENBERG (2019): "Ambiguity Attitudes about Investments: Evidence from the Field," Working paper.

ANG, A., G. BEKAERT, AND J. LIU (2005): "Why stocks may disappoint," *Journal of Financial Economics*, 76, 471–508.

BACKUS, D., A. FERRIERE, AND S. ZIN (2015): "Risk and ambiguity in models of business cycles," *Journal of Monetary Economics*, 69, 42–63.

BAILLON, A. AND H. BLEICHRODT (2015): "Testing Ambiguity Models through the Measurement of Probabilities for Gains and Losses," *American Economic Journal: Microeconomics*, 7, 77–100.

BARBERIS, N. (2018): "Psychology-based Models of Asset Prices and Trading Volume," in *Handbook of Behavioral Economics: Applications and Foundations 1*, ed. by D. Bernheim, S. DellaVigna, and D. Laibson, 79–175.

BARBERIS, N., M. HUANG, AND R. THALER (2006): "Individual Preferences, Monetary Gambles, and Stock Market Participation: A Case for Narrow Framing," *American Economic Review*, 96, 1069–1090.

BASAK, S. AND D. CUOCO (1998): "An Equilibrium Model with Restricted Stock Market Participation," *Review of Financial Studies*, 11, 309–341.

Basak, S. and D. Makarov (2014): "Strategic Asset Allocation in Money Management," *The Journal of finance*, 69, 179–217.

Basak, S., D. Makarov, A. Shapiro, and M. Subrahmanyam (2020): "Security Design with Status Concerns," .

Beck, A. (2014): *Introduction to Nonlinear Optimization: Theory, Algorithms, and Applications with MATLAB*, SIAM.

Benzoni, L., P. Collin-Dufresne, and R. S. Goldstein (2007): "Portfolio Choice over the Life-Cycle when the Stock and Labor Markets Are Cointegrated," *Journal of Finance*, 62, 2123–2167.

Bhamra, H. S. and R. Uppal (2014): "Asset Prices with Heterogeneity in Preferences and Beliefs," *Review of Financial Studies*, 27, 519–580.

Bossaerts, P., P. Ghirardato, S. Guarnaschelli, and W. Zame (2010): "Ambiguity in Asset Markets: Theory and Experiment," *Review of Financial Studies*, 23, 1325–1359.

Brenner, M. and Y. Izhakian (2018): "Asset pricing and ambiguity: Empirical evidence," *Journal of Financial Economics*, 130, 503–531.

Brocas, I., J. D. Carrillo, A. Giga, and F. Zapatero (2019): "Risk Aversion in a Dynamic Asset Allocation Experiment," *Journal of Financial and Quantitative Analysis*, 54, 2209–2232.

Campbell, J. Y. (2017): *Financial Decisions and Markets: A Course in Asset Pricing*, Princeton University Press.

Cao, H. H., T. Wang, and H. H. Zhang (2005): "Model Uncertainty, Limited Market Participation, and Asset Prices," *Review of Financial Studies*, 18, 1219–1251.

Caskey, J. A. (2009): "Information in Equity Markets with Ambiguity-Averse Investors," *The review of financial studies*, 22, 3595–3627.

Chan, Y. L. and L. Kogan (2002): "Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices," *Journal of Political Economy*, 110, 1255–1285.

Chapman, D. A. and V. Polkovnichenko (2009): "First-Order Risk Aversion, Heterogeneity, and Asset Market Outcomes," *Journal of Finance*, 64, 1863–1887.

Chen, H., N. Ju, and J. Miao (2014): "Dynamic asset allocation with ambiguous return predictability," *Review of Economic Dynamics*, 17, 799–823.

Collard, F., S. Mukerji, K. Sheppard, and J.-M. Tallon (2018): "Ambiguity and the historical equity premium," *Quantitative economics*, 9, 945–993.

Constantinides, G. M. (1990): "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy*, 98, 519–543.

Crainich, D., L. Eeckhoudt, and A. Trannoy (2013): "Even (Mixed) Risk Lovers Are Prudent," *The American economic review*, 103, 1529–1535.

Cubitt, R., G. van de Kuilen, and S. Mukerji (2019): "Discriminating Between Models of Ambiguity Attitude: a Qualitative Test," *Journal of the European Economic Association.*

Cuoco, D. and R. Kaniel (2011): "Equilibrium prices in the presence of delegated portfolio management," *Journal of Financial Economics*, 101, 264–296.

Detemple, J. B. and F. Zapatero (1991): "Asset Prices in an Exchange Economy with Habit Formation," *Econometrica*, 59, 1633–1657.

Dimmock, S. G. and R. Kouwenberg (2010): "Loss-aversion and household portfolio choice," *Journal of Empirical Finance*, 17, 441–459.

Dimmock, S. G., R. Kouwenberg, O. S. Mitchell, and K. Peijnenburg (2015): "Estimating ambiguity preferences and perceptions in multiple prior models: Evidence from the field," *Journal of Risk and Uncertainty*, 51, 219–244.

——— (2016a): "Ambiguity aversion and household portfolio choice puzzles: Empirical evidence," *Journal of Financial Economics*, 119, 559–577.

Dimmock, S. G., R. Kouwenberg, and P. P. Wakker (2016b): "Ambiguity Attitudes in a Large Representative Sample," *Management Science*, 62, 1363–1380.

Easley, D. and M. O'Hara (2009): "Ambiguity and Nonparticipation: The Role of Regulation," *Review of Financial Studies*, 22, 1817–1843.

Fagereng, A., C. Gottlieb, and L. Guiso (2017): "Asset Market Participation and Portfolio Choice over the Life-Cycle," *Journal of Finance*, 72, 705–750.

Fox, C. R. and A. Tversky (1995): "Ambiguity Aversion and Comparative Ignorance," *Quarterly Journal of Economics*, 110, 585–603.

Gallant, A. R., M. R Jahan-Parvar, and H. Liu (2019): "Does Smooth Ambiguity Matter for Asset Pricing?" *Review of Financial Studies*, 32, 3617–3666.

Gârleanu, N. and S. Panageas (2015): "Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing," *Journal of Political Economy*, 123, 670–685.

Ghirardato, P., F. Maccheroni, and M. Marinacci (2004): "Differentiating ambiguity and ambiguity attitude," *Journal of Economic Theory*, 118, 133–173.

Gilboa, I. and D. Schmeidler (1989): "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 18, 141–153.

Gollier, C. (2011): "Portfolio Choices and Asset Prices: The Comparative Statics of Ambiguity Aversion," *Review of economic studies*, 78, 1329–1344.

Gomes, F., M. Haliassos, and T. Ramadorai (2020): "Household Finance," *Journal of Economic Literature*, forthcoming.

Gomes, F. and A. Michaelides (2005): "Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence," *Journal of Finance*, 60, 869–904.

Gomes, F. J. (2005): "Portfolio Choice and Trading Volume with LossAverse Investors," *Journal of Business*, 78, 675–706.

Gormley, T., H. Liu, and G. Zhou (2010): "Limited participation and consumption-saving puzzles: A simple explanation and the role of insurance," *Journal of Financial Economics*, 96, 331–344.

Guidolin, M. and H. Liu (2016): "Ambiguity Aversion and Underdiversification," *Journal of Financial and Quantitative Analysis*, 51, 1297–1323.

Guiso, L., P. Sapienza, and L. Zingales (2018): "Time varying risk aversion," *Journal of financial economics*, 128, 403–421.

Guiso, L. and P. Sodini (2013): "Chapter 21 - Household Finance: An Emerging Field," in *Handbook of the Economics of Finance*, ed. by G. M. Constantinides, M. Harris, and R. M. Stulz, Elsevier, vol. 2, 1397–1532.

Guvenen, F. (2009): "A Parsimonious Macroeconomic Model for Asset Pricing," *Econometrica*, 77, 1711–1750.

Halevy, Y., D. Persitz, and L. Zrill (2018): "Parametric Recoverability of Preferences," .

Hara, C. and T. Honda (2018): "Implied Ambiguity: Mean-Variance Efficiency and Pricing Errors," Working paper.

Heath, C. and A. Tversky (1991): "Preference and Belief: Ambiguity and Competence in Choice under Uncertainty," *Journal of Risk and Uncertainty*, 4, 5–28.

Hirshleifer, D., C. Huang, and S. H. Teoh (2017): "Model Uncertainty, Ambiguity Aversion and Market Participation," Working paper.

Hong, H., J. D. Kubik, and J. C. Stein (2004): "Social Interaction and Stock-Market Participation," *Journal of Finance*, 59, 137–163.

Jahan-Parvar, M. R. and H. Liu (2014): "Ambiguity Aversion and Asset Prices in Production Economies," *Review of Financial Studies*, 27, 3060–3097.

Ju, N. and J. Miao (2012): "Ambiguity, Learning, and Asset Returns," *Econometrica*, 80, 559–591.

Kelsey, D. and S. le Roux (2018): "Strategic ambiguity and decision-making: an experimental study," *Theory and decision*, 84, 387–404.

Klibanoff, P., M. Marinacci, and S. Mukerji (2005): "A Smooth Model of Decision Making under Ambiguity," *Econometrica*, 73, 1849–1892.

Kocher, M. G., A. M. Lahno, and S. T. Trautmann (2018): "Ambiguity aversion is not universal," *European Economic Review*, 101, 268–283.

Liu, H. and Y. Zhang (2020): "Financial Uncertainty with Ambiguity and Learning," Working paper.

Luttmer, E. G. J. (1996): "Asset Pricing in Economies with Frictions," *Econometrica*, 1439–1467.

Maccheroni, F., M. Marinacci, and D. Ruffino (2013): "Alpha as Ambiguity: Robust Mean-Variance Portfolio Analysis," *Econometrica*, 81, 1075–1113.

Maccheroni, F., M. Marinacci, and A. Rustichini (2006): "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica*, 74, 1447–1498.

Miao, J., B. Wei, and H. Zhou (2019): "Ambiguity Aversion and the Variance Premium," *Quarterly Review of Economics and Finance*, 09, 1950003.

Mukerji, S., H. N. Ozsoylev, and J.-M. Tallon (2019): "Trading ambiguity: a tale of two heterogeneities," Working paper.

Noussair, C. N., S. T. Trautmann, and G. van de Kuilen (2014): "Higher Order Risk Attitudes, Demographics, and Financial Decisions," *The Review of economic studies*, 81, 325–355.

Pagel, M. (2018): "A News-utility Theory for Inattention and Delegation in Portfolio Choice," *Econometrica*, 86, 491–522.

Polisson, M., J. Quah, and L. Renou (2020): "Revealed preferences over risk and uncertainty," *American Economic Review*, forthcoming.

Savor, P. G., M. I. Wilson, and M. Puhl (2020): "Uncertainty Premia for Small and Large Risks," Working paper.

Shefrin, H. (2008): *A Behavioral Approach to Asset Pricing*, Academic Press/Elsevier.

Siniscalchi, M. (2009): "Vector Expected Utility and Attitudes Toward Variation," *Econometrica*, 77, 801–855.

Sotes-Paladino, J. and F. Zapatero (2019): "Riding the Bubble with Convex Incentives," *The review of financial studies*, 32, 1416–1456.

Thimme, J. and C. Völkert (2015): "Ambiguity in the Cross-Section of Expected Returns: An Empirical Assessment," *Journal of business & economic statistics: a publication of the American Statistical Association*, 33, 418–429.

Trautmann, S. T. and G. van de Kuilen (2015): *Ambiguity Attitudes*, The Wiley Blackwell Handbook of Judgment and Decision Making.

Vissing-Jørgensen, A. (2002): "Limited Asset Market Participation and the Elasticity of Intertemporal Substitution," *Journal of Political Economy*, 110, 825–853.

Wakker, P. P. (2010): *Prospect Theory: For Risk and Ambiguity*, Cambridge University Press.

Wei, B. (2018): "Ambiguity, long-run risks, and asset prices," Working paper.