Credit Rating under Ambiguity[1]

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August 7, 2023

Abstract

We consider the impact of ambiguity on credit rating with feedback effects. A firm signals its quality by surviving phases of apparent distress. A rating agency, whose analysts hold multiple priors about the firm's true asset value, for example, due to the difficulties in the valuation of intangible assets, aims for unbiased ratings. Contrasting classical min-max results, the rating agency selects a dynamically adjusted weighted average of multiple beliefs that overweight uninformative beliefs. The ambiguity impact on ratings hinges on whether the disagreement between the analysts has a common direction: When analysts jointly perceive the firm's value of intangibles as overstated, feedback effects make the firm delay default to benefit from the rating agency's learning.

JEL classification: D83, G24, G33

Keywords: Learning Dynamics, Strategic Interaction, Quantitative Debt Models, Ambiguity

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1 Introduction

Financial information is inherently dynamic, and outsiders generally do not have full information in real time. Previous research indicates the importance of asymmetric information for credit risk (Duffie & Lando 2001). Furthermore, the assessment of credit quality and financing conditions interact and influence the decisions of the rated entity, that is, financial markets and the real economy are interconnected by feedback effects (Manso 2013). Ambiguity captures the empirically well-documented fact that financial information lacks exactly known probabilistic models. It turns out to be a driving factor of credit risk, in particular, it can explain the credit spread increases during the 2007-2008 U.S. financial crisis (Boyarchenko 2012) and significantly influences the pricing of credit default swaps (Augustin & Izhakian 2020).

This paper considers how ambiguity influences the assessment of credit quality by a rating agency that aims for accurate ratings and holds multiple priors over its incomplete information about a firm's cash flow, while the firm maximizes its value by selecting its default strategy. To address the question of how to rate credit under ambiguity with feedback effects, we consider ambiguity similar to recent work on security design (Malenko & Tsoy 2020) and corporate capital structure (Izhakian et al. 2022). Our second building block forms the credit rating with feedback effects. Specifically, the firm's financing costs are connected to the rating agency trough a feedback loop: The rating agency's assessment influences the firm's capital costs, which affects the firm's default considerations, which in turn feed back to the rating. Thus, the firm and the rating agency are interconnected via performance sensitive debt (Manso et al. 2010, Manso 2013), and the rating agency learns the firm's credit quality over time, as in Hilpert et al. (2022).

The rating agency faces a range of plausible distortions of the firm's asset value. The latter is not perfectly observable, for example due to disagreement on how to assess a firm's intangible assets. One interpretation of our setting is that the rating agency needs to aggregate the opinions of several analysts into one rating, as it is the case for all major rating agencies.[1] It is unable to specify exact probabilities for each state of the world; instead, it faces ambiguity in the form of a range of possible scenarios for each plausible distortion, that is, it holds multiple priors (Gilboa & Schmeidler 1989).

The rating agency aims for accurate and time-consistent ratings to maximize its reputation over time. If the estimated default threshold departs from the actual one, we assume that its reputation suffers equally whether an under- or overestimation occurs. Either failure is indeed costly, as underestimating the distance-to-default implies losses for the investors, whereas a too conservative perspective may prevent business. As Kisgen et al. (2020) show, the rating agency's objective function is handed through to the analysts, as accurate analysts are more likely to be promoted and less likely to depart. Contrasting standard approaches to decision making under ambiguity, the rating agency aims to maximize its reputation based on its expectation of the stream of worst-case reputation damages its rating incurs instead of selecting a reputation-optimizing strategy over

[1]For example, Moody's (2009) states: "Moody's ratings are initially determined or subsequently changed through committee. The lead analyst for a given company, industry, country, or asset type frames the discussion, including offering the rating recommendation and its rationale. At minimum, the committee includes a managing director or other designated individual and the lead analyst. The committee may be expanded to include as many perspectives and disciplines as are needed to address all analytical issues relevant to the issuer and the security being rated. Issues affecting the size of the committee may include the size of the issuer, complexity of the security, geography, or whether a transaction of the type has ever been done before."

the worst-case expectation of its reputation under a single prior, as, for example, Such a strategy allows the rating agency to include multiple analyst opinions in its strategy and avoids myopic reputation optimization that induces time-inconsistent rating strategies that occur, e.g. in Garlappi et al. (2017).[2]

The firm's manager-owner maximizes the firm value over its default strategy. As the rating agency observes only a distorted version of its cash flow as a basis for its rating, the firm accounts for the rating, and the subsequent capital costs, in its strategy. In particular, the firm includes the learning of the rating agency over time: Should the observed cash flow imply that the firm is in distress, the rating agency infers that the current situation is costly for the firm. Once the cash flow deteriorates enough, the firm cannot sustain the costs and defaults. Therein lies a learning opportunity for the rating agency. If it observes apparent distress but no default, the rating agency updates its beliefs, for each prior,[3] to exclude distortion levels for which the firm would have defaulted already.

We show that under this learning mechanism, the firm follows, for each possible distortion of the cash flow that the rating agency observes, a threshold strategy. The rating agency's best response then turns out to predict a dynamic updating of the estimated distance-to-default, based on whichever distortion levels it can already eliminate from observing non-default. Hence, for the estimated default threshold, the running minimum of the observed cash flow generates new

[2]This formulation allows us to avoid myopia-caused time inconsistency that from a game-theoretic perspective is challenging to manage. In discrete time, Pahlke (2022) proposes a general approach to rule out dynamic inconsistency given max-min expected utility for games featuring ambiguity.

[3]For tractability, we assume that the priors have identical support, i.e., sure and impossible events are identical.

information because it identifies the distortion levels that the rating agency can rule out. However, although the rating depends on the distance between the observed cash flow to the estimated default threshold, the firm determines the actual default threshold based on the cash flow alone. As in Hilpert et al. (2022), the rating agency can never obtain perfect information because the only time in which it knows the true distortion level is the default time of the firm. The credit rating game generally features a unique equilibrium.

Ambiguity has profound impact on credit rating. First, we argue that rating agency's perceived worst case differs from taking a conservative perspective on the firm's credit risk, that is, its estimated default threshold. This perhaps surprising result roots in the agency's aim to estimate the firm's distance to default as precise as possible. Reputation concerns symmetrically affect the rating agency: Both systematic deviations to over- or underestimate the default threshold are equally unattractive. This preference for precision leads the rating agency to overweight uninformative beliefs over pronounced opinions that are clearly positive or negative.

Second, contrasting classical min-max results rooting in ambiguity aversion in models of multiple priors, the rating agency aggregates multiple prior beliefs in a weighted average. As information unfolds, the rating agency adjusts the weights it places on each prior with the shifts in direction within each prior. For example, if the rating agency considers a pessimistic distribution featuring high probability mass on overvaluation and an optimistic distribution with a lot of probability mass on undervaluation, learning from survival in apparent distress implies a substantial shift in the pessimistic distribution whereas the optimistic belief remains largely stable. The reason lies in the rating agency's learning: Under the pessimistic prior, the rating agency learns more, that is, for the pessimistic distribution, the part of the support that is initially excluded carries more probability mass compared to the optimistic distribution. It turns out that when pessimistic and optimistic beliefs are equally present, the learning raises the informativeness of the pessimistic belief compared to the optimistic one, which in turn causes the rating agency to sharply underweight this particular belief.

Third, we show that the impact of the feedback effect under ambiguity critically hinges on whether rating analysts show a common direction of disagreement. In case of pessimistic and optimistic beliefs being equally present, the rating agency weighs beliefs to avoid deviating from the true default threshold to achieve a balanced estimate. In this case, ambiguity has little impact on either the firm's or the rating agency's aggregated strategy. In contrast, if they are jointly pessimistic, that is, if they place high probability mass on the actual cash flow being below the currently observed one, then the rating agency tends to select the most moderate belief it can justify as its sole prior. This belief poses the worst case from the perspective of a rating agency aiming for accurate ratings because it disperses the belief about the distortion as evenly as possible across the range of possible cash flows.

This extreme response to ambiguity with jointly pessimistic beliefs affects both the rating agency's and the firm's strategy via the feedback loop: As the rating agency now has a belief that is more moderate than a weighted average, the firm delays default compared to the case under asymmetric information, which we take as the average distribution across priors (Halevy 2007), to allow for the rating agency to learn its quality, because the balanced belief reduces the firm's capital costs. In equilibrium, the rating strategy responds accordingly. As the flip side of the coin, the feedback loop with jointly optimistic beliefs accelerates firm defaults because the rating agency takes a more conservative belief as the foundation of its choice, making it more expensive for the firm to wait for the rating agency to learn its distortion. This finding is in line with Bachmann et al. (2020) who document that firms switch between ambiguity beliefs in phases of crises and standard beliefs without ambiguity when there is no prevalent pessimistic sentiment. In a related study, Dicks & Fulghieri (2021) explain corporate innovation waves in a model of investor sentiment under ambiguity, for which investor beliefs switch between optimism and pessimism.

Our paper contributes to the literature of credit rating and credit risk in dynamic settings. Building on structural credit risk models with endogenous default, such as Black & Cox (1976), Leland (1994), and Goldstein et al. (2001), Manso et al. (2010) consider performance-sensitive debt to capture feedback loops. In turn, Manso (2013) considers credit rating with feedback effects. A seminal paper, Duffie & Lando (2001) consider asymmetric information between the firm and its creditors. This allows for learning of credit risk over time. While Duffie & Lando (2001) features exogenous learning, Hilpert et al. (2022) analyze the learning of credit risk from strategic default.

Secondly, our paper contributes to the impact of ambiguity on corporate decisions. Going back to the seminal work of Knight (1921), economists distinguish uncertainty and ambiguity from risk. Originally proposed by Ellsberg (1961) in his famous paradox, ambiguity is a central building block of many financial models. Malenko & Tsoy (2020) consider the optimal security design under ambiguity and show that ambiguity increases the attractiveness of equity over debt to finance new projects. In a related paper, Izhakian et al. (2022) develop a static 2-period model of the trade-off theory of capital structure under ambiguity and show that ambiguity-averse managers tend to increase leverage. Garlappi et al. (2017) focus on investment dynamics and study how multiple prior specifications differ in their investment predictions, emphasizing the importance of ambiguity with heterogeneous decision-makers. Baillon et al. (2018) consider learning for initial public offerings with ambiguous information. Furthermore, it can also explain why investors refrain from investing in seemingly profitable investment opportunities in the first place, as Easley & O'Hara (2009) explain.

This paper proceeds as follows. In Section 2, we present the model of a credit rating game between a firm and a rating agency and detail the information structure and equilibrium concept. Section 3 derives the best responses of the firm and rating agency. In particular, it explains the learning mechanism. In Section 4, we establish the existence of the equilibrium of the rating game and characterize its solution. The economic implications of this equilibrium as well as its empirical implications follow in Section 5.

2 A Model of Credit Rating under Ambiguity

In this section, we build a model of a credit rating agency's learning about a firm's credit risk under ambiguous and dynamic information. We set off by introducing the true and observed cash flow dynamics, followed by the ambiguous information structure and payoffs. The firm's payoff consists of its expected discounted stream of cash flows net of capital costs up to default. The rating agency considers its reputation, which is the discounted stream of reputation costs, which are the deviation of the estimated default threshold to the actual one. Then, we consider the strategy space, consisting of the firm's default strategy and the rating agency's estimated default strategy as well as the belief updating for the rating agency. We discuss economically reasonable strategies to allow for a tractable analysis of ambiguity-extended version of Perfect Bayesian Nash Equilibria (Hanany et al. 2020, Malenko & Tsoy 2020).

2.1 Ambiguity, Dynamics, and Payoffs

We consider a levered firm with one outstanding consol bond, managed by the sole owner of the firm's equity in a structural model of credit risk (Leland 1994, Goldstein et al. 2001). It generates a cash flow $X = (X(t))_{t\geq 0}$ and pays a coupon $C = (C_t)_{t\geq 0}$ on its outstanding debt. The cash flow satisfies

$$\frac{dX(t)}{X(t)} = \mu dt + \sigma dB(t), \tag{1}$$

where $\mu < r$ and σ represents the cash-flow's growth rate and volatility, respectively, with r being the risk-free rate and $B = (B_t)_{t\geq 0}$ being a Wiener process. The cash flow's drift μ, its volatility σ and r are constants and common knowledge. The firm's cash flow X is the firm's manager-owner's private information.

The rating agency observes the firm's cash flow only imperfectly due to a persistent measurement error $\tilde{\theta}$, that is, the observed cash flow follows

$$D_t = \tilde{\theta} X_t \tag{2}$$

for $t \geq 0$ and it has the same dynamics as the true cash flow X.[4] The rating agency is ambiguity averse and has n priors with differing beliefs $\pi_i = (\pi_{i,t})_{t\geq 0}$ about the law of $\tilde{\theta}$ for $i = 1,...,n$, extending the information structure of Grenadier et al. (2016) and Hilpert et al. (2022).[5] The beliefs are common knowledge and initial beliefs are fully described by the densities $\phi^{\pi_{i,0}} := \phi_i$, which we call the priors. All priors are bounded away from zero and above and have the same support $\Theta = [\underline{\theta}, \overline{\theta}]$ with $0 < \underline{\theta} < \overline{\theta} < \infty$. The rating agency for each prior i dynamically updates its beliefs with over $t \geq 0$ and presumes that the density $\phi^{\pi_{i,t}}$characterizes the distribution of the persistent measurement error.[6]

The rating agency determines the rating as the estimated distance to default for its current information. Formally, the rating equals

$$R_t = \frac{D_t}{\hat{D}_t^\star}, \tag{3}$$

[4]As in Hilpert et al. (2022), the persistent measurement error excludes statistical learning as in David (2007) and Pastor & Veronesi (2009).

[5]The special case $n = 1$ delivers the model in Hilpert et al. (2022).

[6] $\tilde{\theta}$ is independent of the Wiener process W. As in Hilpert et al. (2022), the filtrations $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ and $\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}$ capture the information generated by X and D, respectively. Formally, the firm's information set at t is given by $\sigma(\tilde{\theta}) \vee \mathscr{F}_t$, for $t \geq 0$. Since $\tilde{\theta}$ is known to the firm at $t = 0$, we can condition on $\tilde{\theta} = \theta$ and work with $\mathbb{F}$.

the ratio of the current observed cash flow to the predicted default cash flow level $\hat{D}_t^\star$ at each time $t \geq 0$. A rating of $R = 1$ implies immediate default, whereas the rating diverges to infinity as the observed cash flows approach infinity, that is, the default risk vanishes.

The rating agency selects the predicted default level $\hat{D}^\star = (\hat{D}_t^\star)_{t\geq 0}$ as its strategy to minimize the costs of the rating agency under ambiguity aversion, given by

$$U_{RA}^{\pi}(\tau, \hat{D}^\star) = \mathbb{E}\Big[\int_0^\tau e^{-\rho t} \left(\max_{i=1,\dots,n} k_t^{\pi_i}\right) dt\Big], \tag{4}$$

where $\rho > 0$ is the rating agencies internal discount rate, $\tau = (\tau(\theta))_{\theta\in\Theta}$ the default strategy of the firm and $k_t^{\pi_i}$ the cost rate for the predicted default level $\hat{D}^\star$ of belief i. The latter is given by

$$k_t^{\pi_i} = \int_\Theta \big(\hat{D}_t^\star - \mathbb{E}[D_{\tau(\theta)} \mid \mathscr{F}_t]\big)^2 \phi^{\pi_{i,t}}(\theta) d\theta, \tag{5}$$

where we account for $D_{\tau(\theta)}$ being potentially not measurable with respect to the rating agency's filtration $\mathscr{F}$. The given projection $\mathbb{E}[D_{\tau(\theta)} \mid \mathscr{F}_t]$ needs to be applied, however, later in Proposition 2 we show under some mild assumptions that the default time $\tau(\theta)$ is the first hitting time of a default threshold $f(\theta)$, and hence $\mathbb{E}[D_{\tau(\theta)} \mid \mathscr{F}_t] = f(\theta)$.

The rating agency's aim to minimize its reputation damage through its estimated default threshold deviating from the correct one for the maximal cost rate at any given time in equation 4 deviates from the standard max-min expected utility framework (Gilboa & Schmeidler 1989). Contrasting the max-min expected utility framework, the above formulation implies a time-consistent rating agency that maximizes its reputation by considering the worst-case reputation costs for every time point. Thus, it avoids a myopic optimization in which the rating agency considers worst-case expectation over the entire horizon that may cause inconsistencies due to a switching worst case over time in the spirit of Pahlke (2022).

The strategies of rating agency and the firm are connected in a game via a feedback loop. Specifically, performance-sensitive debt as in Manso et al. (2010) and Manso (2013) reflects that the firm's capital costs, that is, the coupon reacts to the rating. The coupon follows a non-decreasing function $C : [1, \infty] \mapsto \mathbb{R}^+$ depending on the firm's rating:

$$C_t = C(R_t) \tag{6}$$

We employ the coupon structure by Hilpert et al. (2022), to ensure that C is sufficiently smooth to allow for an equilibrium.[7]

Assumption 1. *Assume that the interest payment rate C satisfies for some* $0 < L_C < 1$ *that*

$$C(z) \leq C(z') \leq (z/z')^{L_C} C(z)\,, \textit{ for } 1 \leq z' \leq z\,. \tag{7}$$

The firm, on the other side, chooses the default strategy τ to maximize its equity value. The firm owner is risk-neutral and maximizes the net present value of the cash flows net the interest

[7]Assumption 1 imposes Lipschitz continuity on the log-log scale; that is, for some L_C, with $0 < L_C < 1$, it holds that $|\log C(z) - \log C(z')| \leq L_C\, |\log(z) - \log z'|$, for all $z, z' \geq 1$.

payments on its outstanding debt. Since the firm is aware of the measurement error at the start, we denote $\tau = (\tau(\theta))_{\theta \in \Theta}$, where $\tau(\theta)$ is a stopping time implemented by type $\theta \in \Theta$. With $X = D/\tilde{\theta}$, the firm's equity value is given by

$$U_F^{(\theta)}(\tau, \hat{D}^\star) = \mathbb{E}\Big[\int_0^{\tau(\theta)} e^{-rt}\big(D_t/\theta - C(D_t/\hat{D}_t^\star)\big)dt\Big]. \tag{8}$$

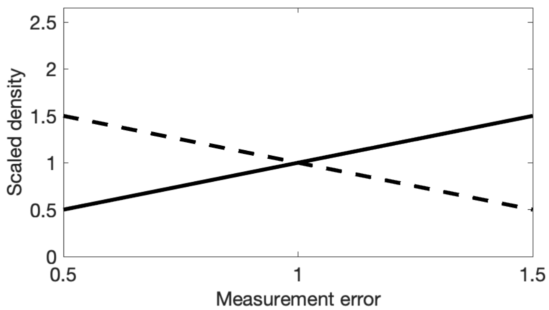
2.2 Markov Strategies and Belief Updating

To obtain a tractable model, we focus on Markov strategies from now own. Because our model, and the beliefs in particular, are formed under ambiguity, Grenadier et al. (2016) and Hilpert et al. (2022), we cannot directly employ Perfect Bayesian Nash equilibria in Markov strategies. Instead, we use the sequential-optimality expansion by Hanany et al. (2020) in the implementation for multiple priors by Malenko & Tsoy (2020).

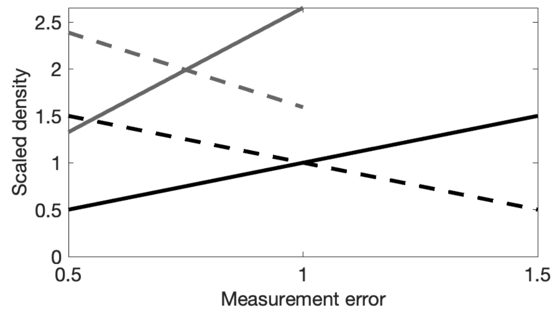
The firm plays a default strategy of the type

$$\tau(\theta) = \inf\{s \geq 0 : D(s) \leq f(\theta)\}. \tag{9}$$

The rating agency's beliefs π_i are updated via Bayes-Rule whenever possible. The rating agency's strategy must be sequentially optimal, and equals the predicted default level which is a function of the observed cash flow's running minimum $E_t = \inf_{s \leq t} D_s$, that is $\hat{D}_t^\star = g(E_t)$ for some measurable $g : \mathbb{R}^+ \to \mathbb{R}^+$. By Equation (3), the firm's rating is then given by $R_t = D_t/g(E_t)$.



***(a)** Initial beliefs*



***(b)** Updated beliefs*

Figure 1: Belief updating: *This figure shows two analysts' beliefs, a pessimistic one (solid) and an optimistic one (dashed). Panel 1a show both analysts' beliefs at time zero in black. Panel 1b shows the same beliefs and the update analysts' beliefs (gray) after measurement errors above one have been ruled out.*

As the rating agency aims to assess the firm's default probability, the running minimum of the observed cash flows E_t carries information value (Hilpert et al. 2022). In combination with the firm's default strategy τ, each consistent belief i equals

$$\phi^{\pi_{i,t}}(\theta) = \frac{1_{\{f(\theta)<E_t\}}}{\int_\Theta 1_{\{f(\theta')<E_t\}} \phi_i(\theta') d\theta'} \phi_i(\theta), \text{ for } \theta \in \Theta, \tag{10}$$

with $0 \leq t < \tau$ and $i \in \{1, ..., n\}$.

Figure 1 illustrates the belief updating from Equation (10). Panel 1a shows the initial situation of two analysts holding initial beliefs about the firm's measurement error. In this case, one pessimistic analyst places more probability mass on high measurement errors (solid line), while an optimistic analyst thinks low measurement errors more likely (dashed line). Panel 1b displays the case in which the minimum of the observed cash flows has decreased sufficiently to allow the rating agency to eliminate measurement errors exceeding $\theta > 1$. Thus, both analysts include this information and update their beliefs according to Equation (10). Panel 1b shows their initial beliefs in black, whereas the updated beliefs are shown in gray.

To obtain a tractable game, we restrict our analysis to pure strategies, and additionally focus on rating strategies that are economically reasonable in the spirit of Hilpert et al. (2022).[8] We call a rating strategy g economically reasonable, if it satisfies the following conditions:

$$g(e) \leq e, \quad g(e) \text{ is non-decreasing} \quad \text{and} \quad g(e)/e \text{ is non-increasing.} \tag{11}$$

The first inequality states that any predicted default level must be below the observed running-minimum of the cash flow. The second condition ensures that firm's survival of an distressed period and recovery to an initial higher cash flow level improves the rating. Furthermore, the third condition ensures that arrival at a new all-time low cash flow does not improve the rating. We denote the set of rating strategies satisfying Equation (11) by $\mathscr{A}_g$. Similarly, a firm's default strategy f is called economically reasonable if it is strictly increasing and continuous. We denote the set of such strategies by $\mathscr{A}_f$.

3 Best Responses

In this section, we consider the best responses of the the firm and the rating agency based on their information. If the rating agency follows an economically reasonable strategy, the firm's best

[8] Note that in our setting, the more involved information structure with ambiguity demands sharper restrictions to the rating agency compared to Hilpert et al. (2022).

response is to default at a type-dependent default barrier. The rating agency's best response is the cost-minimizing expected default threshold given its current information. Depending on the economic situation, the rating agency bases its default threshold prediction on the most moderate prior assessment or a weighted average of beliefs.[9] The rating agency weighs the estimated default thresholds under each prior to obtain its overall estimate, where the weight for a particular prior increases if it features higher residual variance, that is, the prior is less informative.

We characterize the firm's best response default strategy τ for any economically reasonable rating strategy g in the following proposition.

Proposition 1. *For rating strategies $g \in \mathscr{A}_g$, $\theta \in \Theta$, the optimal stopping time of Equation* (8) *is given by*

$$\tau(\theta; g) = \inf\{s \geq 0 : D(s) \leq f(\theta; g)\}, \tag{12}$$

where $f(\theta; g)$ is some positive real constant. Moreover, $f(\cdot; g)$ is strictly increasing and continuous.

Proof. The firm's best response is identical to Hilpert et al. (2022). For the proof, see Proposition 2 in their paper. □

The firm's optimal default strategy is the first hitting time of the imperfectly observed cash flow D with respect to the default barrier $f(\theta; g)$. More structure on the mapping $\theta \mapsto f(\theta; g)$ is provided in Lemma 2 in Appendix I.2.

[9] Formally, we show that for rating strategies $g \in \mathscr{A}_g$, the firm's best response default strategy is indeed of the form (9) and described by a default threshold function $f : \Theta \to \mathbb{R}^+$.

Now, we turn to the dynamic learning and the best response ambiguity averse strategy of the rating agency.

Proposition 2. *Let a firm's default strategy τ be given by some default threshold function $f \in \mathscr{A}_f$, with $\tau(\theta) = \inf\{s \geq 0 : D_s \leq f(\theta)\}$. The rating agency's ambiguity averse best-response to f is given $\hat{D}^\star = g(E; f)$ with*

$$g(e;f) = \underset{D>0}{\arg\min} \left\{ \max_{i=1,\ldots,n} \mathbb{E}^{\pi_{i,0}} \left[(D - f(\tilde{\theta}))^2 \mid f(\tilde{\theta}) < e \right] \right\}. \tag{13}$$

for $e \geq \inf_{\theta \in \Theta} f(\theta)$. Moreover, g is continuous with $g(e) \leq e$ on its domain.

Proof. Continuity and boundedness of g by the identity follows from Lemma 3 and the arguments in Lemma 4 in Appendix I.2. ☐

Focusing on a specific prior $\pi_{i,0}$ in its own right, the best response g_i to a firm strategy f is given by the conditional mean, see Proposition 1 in Hilpert et al. (2022). Denote by v_i the corresponding conditional variance, then

$$g_i(e;f) = E^{\pi_{i,0}} \left[f(\tilde{\theta}) \middle| f(\tilde{\theta}) < e \right], \tag{14}$$

$$v_i^2(e;f) = E^{\pi_{i,0}} \left[\left(g_i(e,f) - f(\tilde{\theta}) \right)^2 \middle| f(\tilde{\theta}) < e \right], \tag{15}$$

for $e > \inf_\theta f(\theta)$, and else $g_i(e;f) = e$ and $v_i^2(g;f) = 0$, $i = 1, \ldots, n$. Then the central equation of

Proposition 2 reads

$$g(e;f) = \arg\min_{D>0} \left\{ \max_{i=1,\ldots,n} (D - g_i(e;f))^2 + \nu_i^2(e;f) \right\}, \tag{16}$$

and accordingly, the rating agency strives to minimize the worst case mean squared error over all priors. It can be shown that the minimizing argument is either the intersection point of two mean squared error curves of two priors, say, i and j, or it is the expected value of a single prior, say, $i = j$, and is given by

$$g(e;f) \quad = \quad \frac{g_i(e;f) + g_j(e;f)}{2} + \mathbb{1}_{i \neq j} \frac{1}{2} \frac{\nu_i^2(e;f) - \nu_j^2(e;f)}{g_i(e;f) - g_j(e;f)}, \tag{17}$$

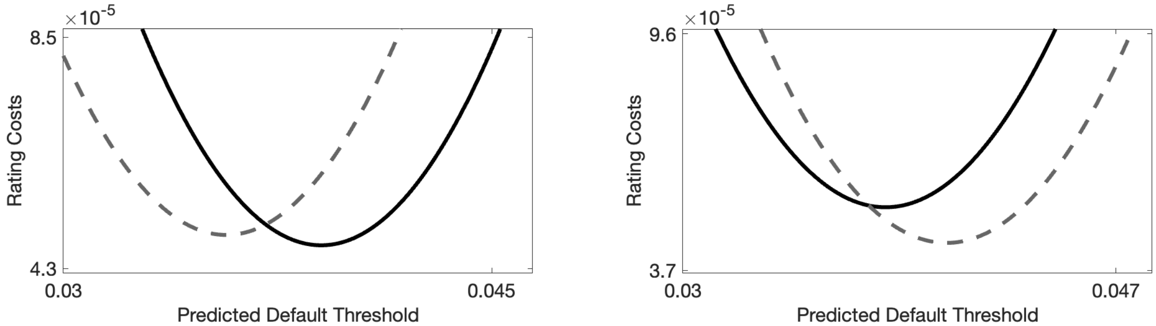
where $(i, j) = (i(e;f), j(e;f))$ depends on the running minimum cash flow e. For better understanding, one may think of i being the most pessimistic and j the most optimistic prior, that is, $g_i(e;f) \geq g_k(e;f) \geq g_j(e;f)$, for all $e \geq 0$ and $k = 1, \ldots, n$.[10] Equation (17) can be restated in terms of weighted average of the best responses $g_i(\cdot;f)$ and $g_j(\cdot;f)$ as follows

$$g(e;f) = w(e;f,i,j)\, g_i(e;f) + (1 - w(e;f,i,j))\, g_j(e;f)\,, \tag{18}$$

$$w(e;f,i,j) = \frac{1}{2} \left(1 + \mathbb{1}_{i \neq j} \frac{\nu_i^2(e;f) - \nu_j^2(e;f)}{\left[g_i(e;f) - g_j(e;f)\right]^2} \right). \tag{19}$$

The weight w for prior i increases (decreases) relative to prior j, if it has a higher (lower) residual

[10]This is not accurate but holds true in case the conditional variances are all identical, that is, $\nu_i^2(e;f) = \nu_j^2(e;f)$, for $1 \leq i, j, \leq n$.



(a) *Balanced analysts* ***(b)*** *Pessimistic analysts*

Figure 2: Cost rates and prior weights: *This figure shows how the rating agency balances the priors of two analysts based on the reputation costs their prior implies. Panel 2a presents an optimistic analyst (solid) and a pessimistic one (dashed). Panel 2b shows a pessimistic analyst (solid) and a super-pessimistic one (dashed).*

variance, that is, it is less (more) informative.

Figure 2 illustrates the above weighting of the analysts' priors by the rating agency. Panel 2a presents a balanced case with a pessimistic analyst (solid line) and an optimistic one (dashed line). The optimistic one predicts a lower default threshold than the pessimistic one. The rating agency now aims to minimize the worst case reputation costs based on both beliefs: to do this, it selects the minimum of the maximal costs it can justify. For example, if the rating agency considers a very low predicted default threshold, the solid line exceeds the dashed one, whereas for a high predicted default threshold the relations change. The overall minimum is obtained by an intermediate predicted default threshold. Note that for this minimum, the rating agency follows neither the optimistic nor the pessimistic analyst alone but its choices falls between their estimates. In contrast, in Panel 2b the rating agency obtains minimal reputation costs by taking the predicted default threshold that minimizes the less pessimistic analysts reputation costs (black), that exceeds the costs from balancing both.

4 Equilibrium

The best responses of the firm and rating agency allow us to discuss perfect Bayesian equilibria in Markov strategies augmented for beliefs under ambiguity via sequential optimality as derived by Hanany et al. (2020) in the implementation for multiple priors by Malenko & Tsoy (2020). Both the rating agency and the firm use Markov strategies characterized by real-valued functions in Propositions 1 and 2. We obtain an equilibrium by the Schauder fixed-point theorem for a feedback loop with sufficiently smooth interest payments, building on Hilpert et al. (2022).

In the following, we first provide the existence of an equilibrium candidate in Proposition 3. Then, Proposition 4 characterizes the solution and verifies the existence and uniqueness, provided that specific technical conditions hold. To establish our equilibrium, we apply the Schauder fixed-point theorem to the best responses. To do this, we must ensure that the best response $g(;f)$ is an economically reasonable strategy by a suitable transform $\mathscr{R}$. In particular, the transform $\mathscr{R}$ is defined on the set of measurable functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ bounded by id, that is, $g(e) \leq e$, for all $e \geq 0$, by

$$\mathscr{R}(g)(e) = \begin{cases} e \inf \{\sup \{g(s)/z : 0 < s \leq z\} : 0 < z \leq e\}, & \text{for } e > 0\,, \\ 0, & \text{for } e = 0\,. \end{cases} \tag{20}$$

Then $\mathscr{R}(g)$ is economically reasonable as desired, see Lemma 1. As in Hilpert et al. (2022), we apply the Schauder fixed-point theorem to the mapping $T : (f,g) \mapsto (f(;g), \mathscr{R}(g(;f)))$, where $f(;g)$

is the firm's best response given in Proposition 1 and $g(;f)$ is the rating agency's best response given in Proposition 2.

Proposition 3. *Suppose Assumption 1 holds. Then* $T : (f,g) \mapsto (f(\cdot;g), \mathscr{R}(g(\cdot;f)))$ *has at least one fixed point in the space of economically reasonable strategies. Let* $(f^\star, g^\star)$ *be such a fixed point, if* $\mathscr{R} \circ g(\cdot;f^\star) = g(\cdot;f^\star)$*, then* $(f^\star, g^\star)$ *is an equilibrium.*

Proof. See Appendix. □

Proposition 3 is an existence result. It ensures that at least one candidate for an equilibrium exists and this candidate is indeed an equilibrium if the condition $\mathscr{R} \circ g(\cdot;f^\star) = g(\cdot;f^\star)$ holds. However, no particular guidance is given on how to actually compute such an equilibrium candidate. Proposition 4 characterizes an equilibrium candidate as the solution to a $2 \times (n+1)$-dimensional ordinary differential equation (ODE) under some technical assumptions. The strategies f and g depend on g_i and v_i^2 the conditional mean and variance of each prior, see (17), explaining the dimension of the setup. For the subsequent Proposition 4, the following transforms are useful $\hat{g}_i(\theta;f) = g_i(f(\theta);f)$ and $\hat{v}_i^2(\theta;f) = v_i^2(f(\theta);f)$, for $\theta \in \Theta$, $i = 1,...,n$. Both transforms are collected in a column vector each, that is, $\hat{G}(\cdot;f) = (\hat{g}_1(\cdot;f),...,\hat{g}_n(\cdot;f))^\top$ and $\hat{V}^2(\cdot;f) = (\hat{v}_1^2(\cdot;f),...,\hat{v}_n^2(\cdot;f))^\top$, and denote by $\odot$ the pointwise product of vectors and matrices and by $\mathbf{1}_n = (1,...,1)^\top$ the one vector of dimension n. Define $\eta = \frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2} + \sqrt{\left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$, $\Phi_i(\theta) = \int_{\underline{\theta}}^{\theta} \phi_i(t)\,\mathrm{d}t$, $i = 1,...,n$, then set $\Pi = (\frac{\phi_1}{\Phi_1},...,\frac{\phi_n}{\Phi_n})^\top$, and denote by $b(\cdot;\theta,g)$ the boundary that separates the default region from the nondefault region depending on measurement error θ and rating agency strategy g. Finally, define the function

$$\mathscr{L}(f,\hat{g},\hat{g}_i,\hat{g}_j,\hat{v}_i^2,\hat{v}_j^2,i,j,\theta)=\frac{1}{2}\frac{\phi_i(\theta)}{\Phi_i(\theta)}(f-\hat{g}_i)+\frac{1}{2}\frac{\phi_j(\theta)}{\Phi_j(\theta)}(f-\hat{g}_j) \tag{21}$$
$$\frac{\mathbb{1}_{i\neq j}}{2(\hat{g}_i-\hat{g}_j)}\left(\frac{\phi_i(\theta)}{\Phi_i(\theta)}\left[(f-\hat{g}_i)^2-\hat{v}_i^2-\frac{\hat{v}_i^2-\hat{v}_j^2}{\hat{g}_i-\hat{g}_j}(f-\hat{g}_i)\right]-\frac{\phi_j(\theta)}{\Phi_j(\theta)}\left[(f-\hat{g}_j)^2-\hat{v}_j^2-\frac{\hat{v}_i^2-\hat{v}_j^2}{\hat{g}_i-\hat{g}_j}(f-\hat{g}_j)\right]\right).$$

Proposition 4. *Given the setting of Proposition 3, denote by* $(f^\star,g^\star)$ *a fixed point of* T. *Suppose* $f^\star,g^\star$, $(\phi_i)_i$ *are continuously differentiable, and the collection of the firm's equity values, denoted by* $(v(\cdot,\cdot;\theta,g^\star))_{\theta\in\Theta}$, *is sufficiently differentiable. Denote by* $(f,\hat{g},\hat{G},\hat{V}^2)$ *the solution of the ODE*

$$\begin{pmatrix} f'(\theta) \\ \hat{g}'(\theta) \\ \hat{G}'(\theta) \\ \hat{V}^{2\prime}(\theta) \end{pmatrix}=\begin{pmatrix} \frac{(1+\eta)\sigma^2}{2(r-\mu)}\frac{(f(\theta)/\theta)^2}{C(f(\theta)/\hat{g}(\theta))-f(\theta)/\theta}+\frac{\partial\hat{b}(\theta,\theta;g)}{\partial\hat{\theta}} \\ \mathscr{L}(f(\theta),\hat{g}(\theta),\hat{g}_{i(\theta)}(\theta),\hat{g}_{j(\theta)}(\theta),\hat{v}^2_{i(\theta)}(\theta),\hat{v}^2_{j(\theta)}(\theta),i(\theta),j(\theta),\theta) \\ \Pi(\theta)\odot(f(\theta)\mathbf{1}_n-\hat{G}(\theta)) \\ \Pi(\theta)\odot\left([f(\theta)\mathbf{1}_n-\hat{G}(\theta)]\odot[f(\theta)\mathbf{1}_n-\hat{G}(\theta)]-\hat{V}^2(\theta)\right) \end{pmatrix} \tag{22}$$

on $(\underline{\theta},\overline{\theta})$ *with initial condition* $f(\underline{\theta})=\underline{\theta}\,f_1^\star$, $\hat{g}(\underline{\theta})=\underline{\theta}\,g_1^\star$, $\hat{G}(\underline{\theta})=\mathbf{1}_n\,\underline{\theta}\,g_1^\star$, *and* $\hat{V}^2(\underline{\theta})=0$, *where* $(f_1^\star,g_1^\star)$ *denotes the unique equilibrium of the case* $\Theta=\{1\}$ *with* $f_1^\star=g_1^\star$, $\hat{b}(\hat{\theta},\theta;g)=b(f^\star(\hat{\theta}),\theta;g^\star)$, *for* $\hat{\theta}\leq\theta$, *and* $(i(\theta),j(\theta))$ *are the minimizing indices of* (16) *as given in* (17). *If*

$$0\leq\hat{g}'(\theta)\leq f'(\theta)\frac{\hat{g}(\theta)}{f(\theta)}\,,\ \textit{for all}\ \theta\in(\underline{\theta},\overline{\theta})\,, \tag{23}$$

then the fixed point is characterized by $(f^\star, g^\star) = (f, \hat{g} \circ f^{-1})$ *and is moreover an equilibrium.*

Proof. See Appendix. □

The ODE in Equation (22) is explicit and admits a unique solution $(f, \hat{g})$ that can be used for computations. If Condition (23) holds, then $(f^\star, g^\star) = (f, \hat{g} \circ f^{-1})$ is the unique equilibrium satisfying the differentiability conditions in Proposition 4.

5 Economic Implications of Credit Rating with Ambiguity

In this section, we shed light on the implications of ambiguity for the equilibrium strategies of the firm and rating agency with an emphasis on learning. In the first part, we provide general implications on the firm's equilibrium equity value and its optimal default strategy. In our subsequent numerical equilibrium analysis, we initially focus on a market with pessimistic and optimistic beliefs being equally present. Thereafter, we shift the analysis to cases in which rating analysts show a common direction of disagreement, where the ambiguity evolves around a either a pessimistic or an optimistic distribution.

Throughout the numerical analysis, we use the parametrization by Hilpert et al. (2022). Specifically, the firm's debt face value is scaled to unity; the cash flow features the expected growth rate of $\mu = 0$ and a volatility of $\sigma = 0.30$; the risk-free rate equals $r = 0.0211$; the interest payment rate C follows the specification in Assumption 1. Both the risk-free rate and the interest payment rate are calibrated as in Hilpert et al. (2022).

5.1 General Implications of Equilibrium

In this section, we provide general implications on the firm's equity value and its default strategy. We present general implications that hold under information asymmetry (Hilpert et al. (2022)) and are further valid in our setting under the consideration of ambiguity. We denote by $(f^\star, g^\star)$ a unique equilibrium and a corresponding firm's equity value by $v^\star(d,e;\theta) = v^\star(d,e;\theta,g^\star)$ for $\theta \in \Theta$ in compliance with Proposition 4.

The first implication states that a firm benefits from surviving distressed periods: The firm signals its quality by surviving an apparently distressed period and thereby improves its future financing prospects. Formally, let $D^\star(\theta) = f^\star(\theta)$ be the firm's observed cash flow based default threshold, d the currently observed cash flow and e, e' two minimum observed cash flows with $D^\star(\theta) \leq e' \leq e \leq d$. Then, it holds that

$$v^\star(d,e';\theta) \geq v^\star(d,e;\theta). \tag{24}$$

Surviving at a new all time low cash flow level e' and recovering to the initial observed cash flow level d increases the firm's equity value.

Next, note that the observed cash flow of a firm that is subject to a high distortion parameter is above its true cash flow. Thus, the rating of a firm with a high distortion parameter is too optimistic and the cost of capital lower than accurate. A firm with a higher distortion parameter thereby delays default compared to firm with a lower distortion parameter, since the latter faces less favorable cost of capital. On the other side, consider a situation of two firms with similar observed cash flows but differing distortion parameters. Before one of them defaults, both firms have the same rating and pay the same coupon, while the firm with a lower distortion parameter has higher true cash flow and therefore will choose to survive longer. Let $X^\star(\theta) = D^\star(\theta)/\theta$ be the firm's default threshold in terms of its true cash flow. Then, it holds for $\theta, \theta' \in \Theta$ with $\theta' \leq \theta$:

$$X^\star(\theta') \geq X^\star(\theta) \quad \text{and} \quad D^\star(\theta') \leq D^\star(\theta). \tag{25}$$

Now, consider a fixed observed cash flow d and a fixed observed running minimum e. Let us regard two firms with different distortion parameters: By the previous equation, the firm with the higher distortion parameter defaults at a higher observed cash flow level. As before, up to the time when latter firm defaults and for any observed cash flow level, both firms are going to have the same rating and the same cost of capital. However, the firm with the lower distortion parameter constantly has a higher true cash flow. Therefore, due to standard monotony arguments, equity value must be a decreasing function in the distortion parameter. That is, for $\theta' \leq \theta$, the firm's equity value satisfies

$$v^\star(d, e; \theta') \geq v^\star(d, e; \theta). \tag{26}$$

Finally, let us consider a firm's fixed true cash flow level x and a fixed true running minimum y. A higher distortion parameter leads to a higher equity value, since a higher observed cash flow is associated with cheaper cost of capital. Formally, the observed cash flow is given by $d = \theta x$ and the observed running minimum by $e = \theta y$ for any distortion parameter θ. The previous argument translates to:

$$v^{\star}(\theta' x, \theta' y; \theta') \leq v^{\star}(\theta x, \theta y; \theta) \tag{27}$$

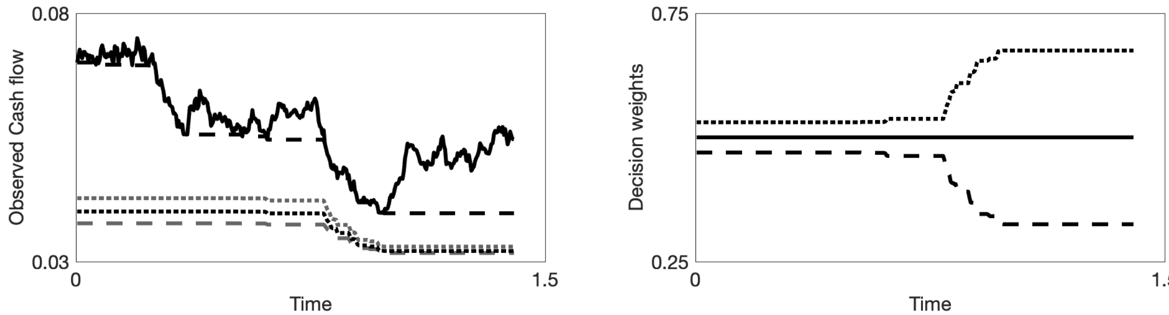
for $\theta' \leq \theta$. For formal and detailed proofs, see Hilpert et al. (2022). The following numerical analysis captures the specific effect of ambiguity on the unique market equilibrium.

5.2 Learning of Credit Quality under Ambiguity

In this section, we discuss how ambiguity affects the rating agency's learning of the firm's default threshold and rating over time. Consider the case of two priors ($n = 2$) having symmetric initial beliefs. One prior, the pessimist, assigns high likelihood to cases in which the true cash flow is upward distorted, that is, the firm is worse off than it looks. Conversely, the likelihood for the downward distorted cases is low, that is, those in which the firm is better off than it looks. In contrast, the other prior, called optimist, assigns high likelihood to cases in which the true cash flow is downward distorted, and low likelihood to the upward distorted cases. Formally, the priors are described by the densities ϕ_1 and ϕ_2 given by

$$\phi_1(\theta) = \mathbb{1}_{[\frac{1}{2},\frac{3}{2}]}(\theta)\theta, \quad \text{and} \quad \phi_2(\theta) = \mathbb{1}_{[\frac{1}{2},\frac{3}{2}]}(\theta)(2-\theta). \tag{28}$$

The rating agency places weights ω_i, with $i = 1, 2$, on each prior, where the weights are given by Equation (19).

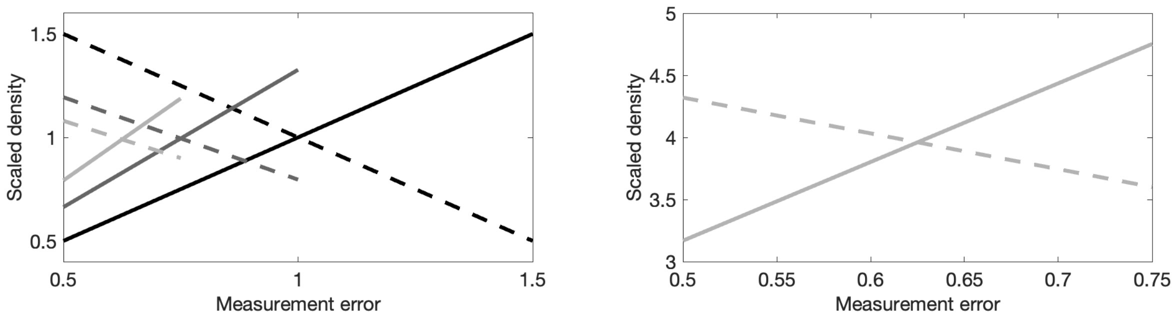


(a) Trajectory *(b) Weights*

***Figure 3: Learning for sample path**. This figure illustrates the rating agency's learning of the firms default threshold (Panel 3a) and the evolution of its weights (Panel 3b) for a sample trajectory of the observed cash flow. Panel 3a shows the path of an observed cash flow (solid black line) and its observed running-minimum (dashed black line). Based on the observed running minimum, the expected default threshold for the pessimistic prior and an optimistic prior follow the grey dashed and dotted lines, respectively. The agency's estimated default threshold is captured by the black dotted line. Panel 3b displays the weights the rating agency places on the pessimistic (dashed line) and optimistic (dotted line) priors. The black line shows an equal weighting for comparison.*

To grasp the intuition of the rating agency's learning, consider Figure 3. Figure 3 shows a trajectory of the observed cash flow (Panel 3a) and the rating agency's weights (Panel 3b). Panel 3a shows the path of an observed cash flow (solid black line) and its observed running-minimum (dashed black line). Based on the observed running minimum, the rating agency postulates an expected default level for an optimistic (grey dotted line) and a pessimistic prior (grey dashed line). Its strategy equals its expected default threshold (black dotted line). Panel 3b displays the weights the rating agency places on the pessimistic (dotted line) and optimistic (dashed line) priors.

For this trajectory of the observed cash flow, each prior individually implies a default threshold for the firm optimizing the stream of the agency's expected reputation damage conditional on their prior, which it updates as information unfolds. The optimistic prior postulates a lower expected default threshold because the probability mass on the measurement errors for which the actual cash



***(a)** Belief updating*

***(b)** Updated belief*

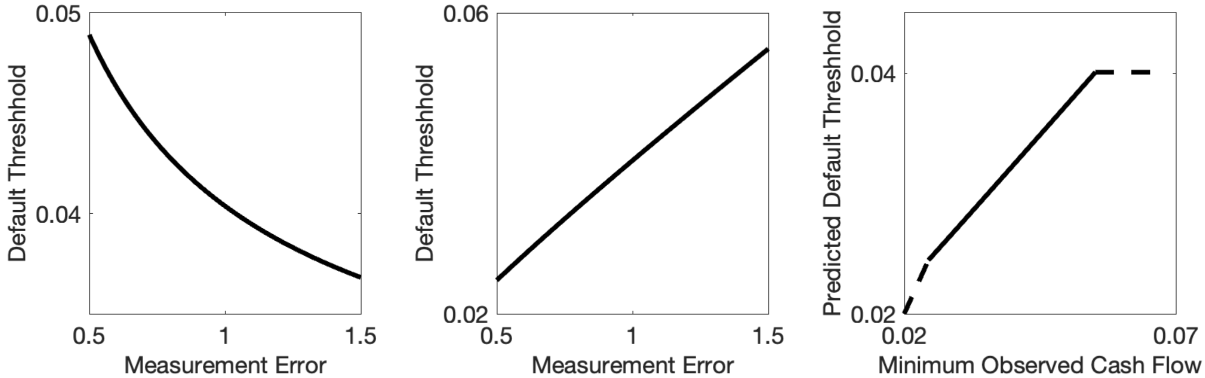
***Figure 4: Belief updating**. This figure illustrates the updating of the beliefs for a pessimistic (solid lines) and optimistic prior (dashed lines). Panel 4a shows the initial belief, i.e., the prior (black) and after two phases of learning (dark and light grey). Panel 4b shows the updated belief after both phases on a larger scale. Note that the densities are rescaled to the original scale to be comparable.*

flow exceeds the currently observed one are lower. Conversely, the pessimistic prior implies a firm default for a higher observed cash flow. Initially, the rating agency overweights the optimistic prior slightly (Panel 3b).

Approximately at time 0.6, the observed cash flow deteriorates sufficiently for both priors to discard the highest measurement error as implausible (recall that both priors have the same support). Consequently, both adjust their expectation of the firm's default threshold downwards. The learning affects the pessimistic prior's information stronger because the eliminated measurement errors carry high weight in its belief, causing a higher relative shift in its belief. As the increasing perceived distress of the firm deepens, the rating agency updates its expectation of the default threshold based on both priors. However, its weights for the analysts shift with this updating: After the learning, the rating agency considerably overweights the optimistic belief in forming the overall assessment.

This surprising result originates in the rating agency's cost function. Figure 4 illustrates how the rating agency's learning shifts its prior beliefs over time. Panel 4a shows the initial belief, that is, the prior (black) and after two phases of learning (blue and red), for the pessimistic (solid lines) and optimistic (dashed lines) priors, rescaled to be comparable. Panel 4b zooms in on the updated beliefs after both phases. Because the rating agency aims to optimize its reputation costs under ambiguity, it minimizes the maximal reputation damage, that is, the reputation damage with the highest reputation cost rate. From this perspective, as the perceived distress worsens and more and more measurement errors are excluded, the optimist's belief is becomes increasingly uninformative: it distributes the remaining probability mass substantially more evenly across the remaining measurement errors. In contrast, the pessimistic belief shifts to a considerably more extreme position over time and places most of its probability mass on default in the immediate future. Panel 4b shows a much steeper slope for the pessimist. Consequently, the optimistic prior is more challenging from the rating agency's perspective because the beliefs allows for meaningful and likely deviations of the actual default threshold from the estimate. The pessimistic prior, on the opposite, allows for high confidence for the default to happen in a narrow corridor, allowing for a precise estimate with low reputation cost. Hence, the optimistic prior presents actually the worst case for a rating agency that aims for accurate ratings.

Continuing from Figure 3 that illustrates the rating strategy for a single trajectory of observed cash flows for a firm that does not default during the displayed time interval, in contrast, Figure 5 presents the credit rating equilibrium in general for the state space; in particular, it illustrates the firm and rating strategies in equilibrium independent of a concrete realization of the observed cash flow. In Panels 5a and 5b, we show how a firm with a given measurement error responds to the



***(a)** Firm strategy* ***(b)** Firm strategy* ***(c)** Rating strategy, plain*

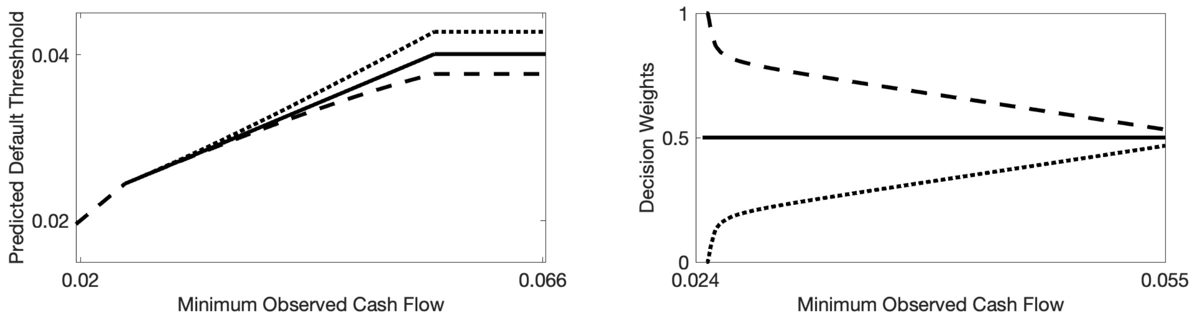
Figure 5: Rating Game Equilibrium under Ambiguity. *This figure illustrates the equilibrium strategies for both the firm and the rating agency. Panel 5a and 5b shows firm's corresponding equilibrium default strategy in terms of the firm's actual and observed cash flow, respectively. Panel 5c presents the rating agency's corresponding equilibrium rating strategy. In this panel, the solid line represents the area for which the rating agency learns. The dashed line indicates the rating strategy before learning (right part) and in case all possible firm types already defaulted (left part).*

rating agency's judgment in terms of its actual and the observed cash flow, respectively. Panel 5c shows how the rating strategy, that is, how the estimated default threshold changes once rating agency observes a lower minimum cash flow (distorted by the measurement error).

Consider the firm strategy first. In equilibrium, as Panels 5a and Panel 5b show, the firm default at a cut-off threshold that depends on the distortion level. The default threshold decreases in the distortion with which the rating agency observes the actual cash flow in terms of the true cash flow and rises for the observed one. If the rating agency perceives the firm to have a higher cash flow, a high distortion implies that the observed cash flow exceeds the actual one, that is, the firms true cash flow decreases. In consequence, the firm defaults at a higher cash flow to mitigate losses from high capital costs via the feedback loop. The flip side of the coin is that the firm profits from an overvaluation of its cash flow under a high distortion for the same actual cash flow; hence the rising shape in Panel 5b because the firm decides to delay default to enjoy the lower capital costs the feedback loop implies via the improved rating. Structurally, the firm strategy is similar, but not identical, to Hilpert et al. (2022), however, the ambiguity implies equilibrium effects which we discuss below.

Considering the rating agency, Panel 5c shows that its strategy consist of three distinct parts. Initially, the running minimum of the observed cash flow is high. Without a phase of apparent distress, the rating agency cannot learn anything; its expected default threshold remains constant (dashed part on the right). Once the running minimum deteriorates, the rating agency learns from survival that it can discard the highest part of the distribution as implausible because at these cash flow levels, the firm would have defaulted already. Hence, the expected default threshold decreases along the decreasing running minimum (solid line). The dashed part on the left presents the strategy for the hypothetical case that the rating agency observes cash flow level for which the firm should have defaulted regardless of the type. In this case, the rating agency expects immediate default. As for the firm strategy, the rating strategy is structurally similar but not identical to Hilpert et al. (2022).

To highlight the impact of ambiguity on the equilibrium of the rating game, Figure 6 presents the rating strategy in more detail. Panel 6a shows the rating strategy (solid line) as well as the expected default levels under the pessimistic (dashed line) and optimistic (dotted line) priors, respectively. Panel 6b presents the weights the rating agency places on the pessimistic (dotted line) and optimistic (dashed line) priors. The solid line indicates an equal weighting.

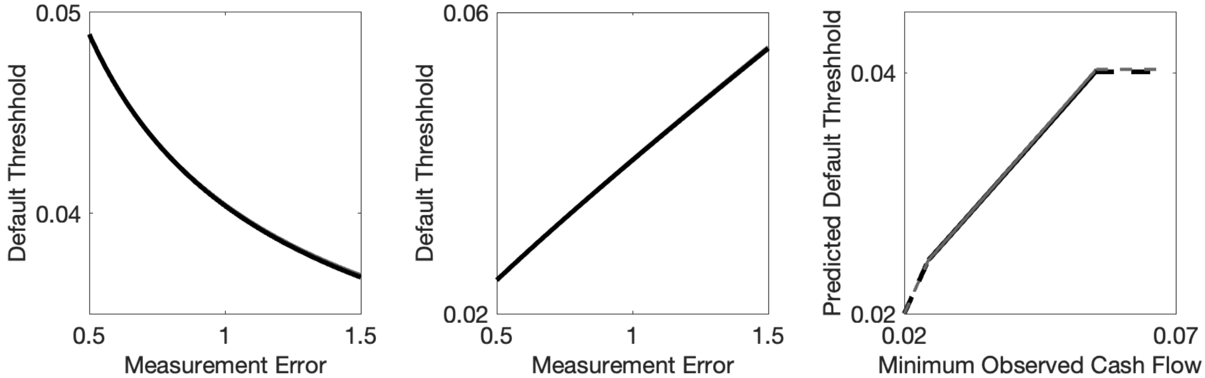


(a) *Rating strategy, plain* ***(b)*** *Weights*

Figure 6: Ambiguity impact on rating strategy. *This figure illustrates the impact of ambiguity on the rating agency. Panel 6a presents the rating agency's equilibrium rating strategy. The pessimistic prior expects the default threshold indicated by the dotted line, whereas the optimistic prior expects default at the dashed line. Panel 6b displays the weights the rating agency places on the pessimistic (dotted line) and optimistic (dashed line) prior, whereas the solid line indicates equal weights for comparison.*

As our initial example for the sample trajectory indicates, rating under ambiguity does not lead to what might be expected as a classical "worst-case" behavior. That is, in the equilibrium above, the first intuition might be that a rating agency that minimizes the costs of the worst case should adapt the worst case prior, that is, the prior giving the most pessimistic estimate of the firm's prospect. After all, this seems to minimize the costs of deviating from the true default threshold, if the most pessimistic prior indeed has the most appropriate assessment of the measurement error distribution.

As is obvious in the example above, this is not the case (Panel 6b). For the given case, the rating agency does not follow a strategy that assigns a single worst-case prior. Rather, the rating agency chooses a strategy that considers the expectations under each prior. Generalizing our previous example, the rating agency does not place equal weight on each prior but balances them, and, as a result of learning on both beliefs, considerably shifts the weights over time. The reason is that the rating agency's objective function is symmetric in nature, and the rating agency's objective is to



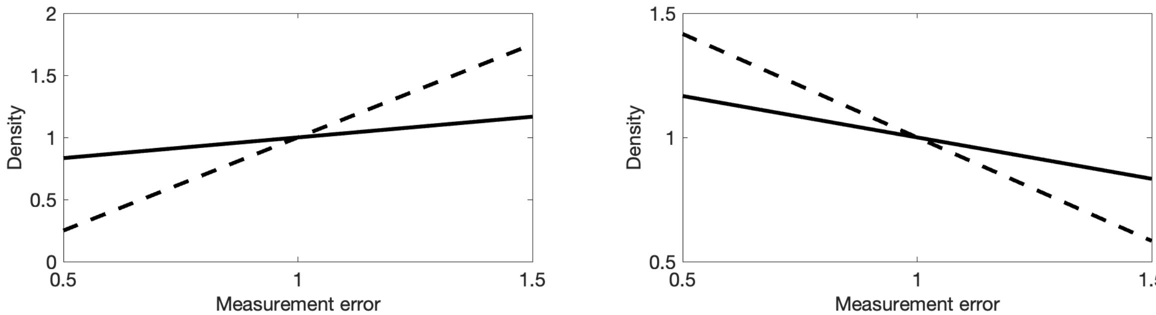
***(a)** Firm strategy*　　***(b)** Firm strategy (observed CF)*　　***(c)** Rating strategy*

Figure 7: Ambiguity impact on equilibrium. *This figure illustrates the impact of ambiguity on the equilibrium. Panel 7a presents the firm's equilibrium strategy in terms of the true cash flow, whereas Panel 7b displays it in terms of the observed cash flow. Panel 7c shows the rating agency's equilibrium rating strategy. The solid line represents the area for which the rating agency learns. The dashed line indicates the rating strategy before learning (right part) and in case all possible firm types already defaulted (left part). The case under ambiguity is indicated in black. The case without ambiguity is captured by the grey lines.*

produce a rating that is as accurate as possible.

That is why from the point of view of the rating agency, the worst case is not that the firm's cash flow distortion has a lot of weight on upward measurement errors. Rather, the worst case is to deliver an estimate that is far away from the true situation – regardless whether it is too optimistic or pessimistic. Thus, avoiding the worst case is effectively achieved by taking a middle position between the possible extremes, rather than following the most pessimistic estimate of the default threshold. The intuition of Figure 4 carries over to the equilibrium: The learning moderates the optimistic prior, whereas the pessimistic becomes increasingly extreme in its predictions. Consequently, the rating agency overweights the optimistic prior in this case.

Secondly, the impact of ambiguity on both the firm and the rating agency in this case of a symmetric distribution of ambiguity, where the optimistic and pessimistic priors are balanced,



***(a)** Jointly pessimistic priors* ***(b)** Jointly optimistic priors*

***Figure 8: Jointly pessimistic and optimistic priors**. This figure displays the prior beliefs for jointly pessimistic priors (Panel 8a) and jointly optimistic priors (Panel 8b). The black line indicates a moderate belief, whereas the dashed line displays a more pronounced belief.*

implies that the priors overall do not have a clear common direction.

This results in a structure of ambiguity for which the average distribution is uninformative, that is, all measurement errors are effectively plausible and approximately equally likely. Figure 7 compares the equilibrium strategies for the firm (Panels 7a and 7b) and the rating agency (Panel 7c) under ambiguity (solid line) to the case without ambiguity (dashed line), which we take as the average distribution over both priors. The no-ambiguity case coincides with Hilpert et al. (2022). In this case, ambiguity has no influence on either the firm or the rating agency as the information is too dispersed for the rating agency to move its aggregated belief. Hence, the rating agency, and, in turn, the firm ignore the ambiguity in their strategies.

5.3 Credit Rating under Ambiguity with Common Direction of Disagreement

In this section, we analyze how the firm and rating agency are influenced by ambiguity if the disagreement between the analysts has a common direction. Yet, the analysts' beliefs about how high the value of the firm's intangible assets actually is can remain heterogeneous.

Consider the case when analysts are pessimistic and jointly perceive the firm's value of intangibles as overstated. In this case, both priors imply that the firm features cash flows that are lower than the currently observed ones although they have more nuanced beliefs about how bad the firm's prospect actually is. Likewise, when analysts are jointly optimistic and perceive the firm's value of intangibles as understated, all priors may agree that a firm has a good outlook and form beliefs that underweight the likelihood that the firm has lower than observed cash flows.

Staying in the two-prior case ($n = 2$), we start with the case of both priors having a pessimistic belief. Specifically, we consider initial prior beliefs follow the densities ϕ_1 and ϕ_2, given by

$$\phi_1(\theta) = \mathbb{1}_{[\frac{1}{2},\frac{3}{2}]}(\theta)\left(\frac{2}{3}+\frac{1}{3}\theta\right), \quad \text{and} \quad \phi_2(\theta) = \mathbb{1}_{[\frac{1}{2},\frac{3}{2}]}(\theta)\left(\frac{1}{6}+\frac{5}{6}\theta\right). \tag{29}$$

The solid prior is more moderate than the dashed prior. For the jointly optimistic case, we consider the two prior beliefs with the densities ϕ_1 and ϕ_2, given by

$$\phi_1(\theta) = \mathbb{1}_{[\frac{1}{2},\frac{3}{2}]}(\theta)\left(\frac{4}{3}-\frac{1}{3}\theta\right), \quad \text{and} \quad \phi_2(\theta) = \mathbb{1}_{[\frac{1}{2},\frac{3}{2}]}(\theta)\left(\frac{11}{6}-\frac{5}{6}\theta\right). \tag{30}$$

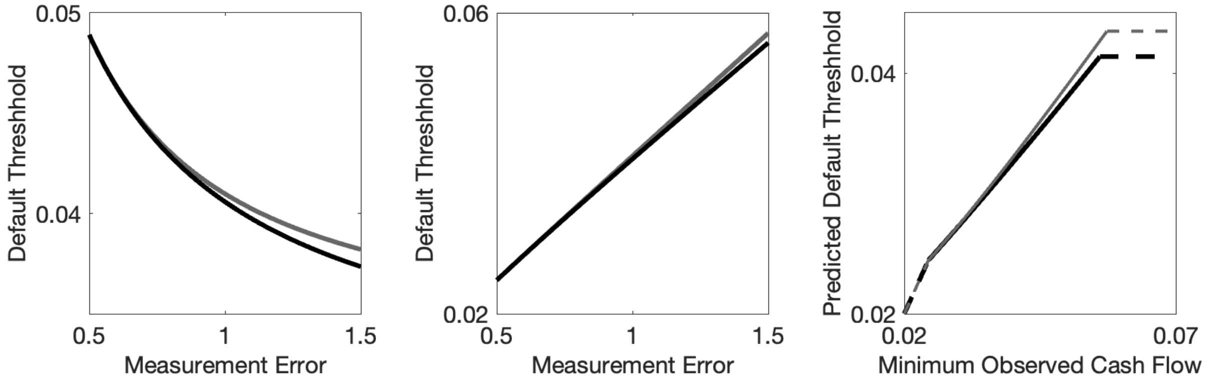
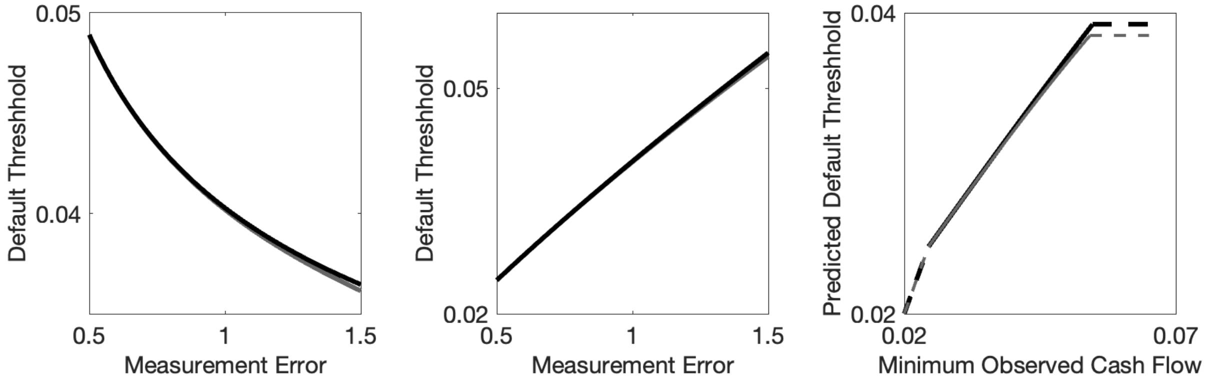


Figure 9: Ambiguity impact on equilibrium, pessimistic priors. *This figure illustrates the impact of ambiguity on the equilibrium for pessimistic priors. Panel 9a presents the firm's equilibrium strategy in terms of the true cash flow, whereas Panel 9b displays it in terms of the observed cash flow. Panel 9c shows the rating agency's equilibrium rating strategy. The solid line represents the area for which the rating agency learns. The dashed line indicates the rating strategy before learning (right part) and in case all possible firm types already defaulted (left part). The case under ambiguity is indicated in black. The case without ambiguity is captured by the grey lines.*

Figure 9 shows the equilibrium strategies for this rating game, that is, we consider the priors from Figure 8a. It features the same information as Figure 7. It highlights the impact of ambiguity on the feedback loop: The rating agency, as in the previous section, takes the conservative view. In this case, following the more nuanced prior is the prudent strategy, so in this case the rating agency ignores the very pessimistic belief of the first prior. As the perspective, and subsequently the strategy, of the rating agency is now deviating from a best response to an average distribution, the firm responds by delaying its default compared to the case without ambiguity to benefit from the rating that addresses the less pessimistic perspective on the market. In equilibrium, the feedback loop implies that ambiguity in combination with an overall jointly pessimistic beliefs systematically delays firm default.

Similar to Figure 9, Figure 10 shows the equilibrium results for the case of jointly optimistic



***(a)** Firm strategy* ***(b)** Firm strategy (observed CF)* ***(c)** Rating strategy*

Figure 10: Ambiguity impact on equilibrium, optimistic priors. *This figure illustrates the impact of ambiguity on the equilibrium for optimistic priors. Panel 10a presents the firm's equilibrium strategy in terms of the true cash flow, whereas Panel 10b displays it in terms of the observed cash flow. Panel 10c shows the rating agency's equilibrium rating strategy. The solid line represents the area for which the rating agency learns. The dashed line indicates the rating strategy before learning (right part) and in case all possible firm types already defaulted (left part). The case under ambiguity is indicated in black. The case without ambiguity is captured by the grey lines.*

priors, that is, the priors displayed in Figure 8b. As expected, the feedback effect flips the impact of ambiguity on both equilibrium strategies: The firm accelerates the default because the rating agency places a relatively heavier weight on high distortion parameters and rates accordingly, that is, more conservatively. The feedback loop thus causes higher capital costs than without ambiguity, raising the opportunity costs of waiting for the rating agency to learn higher in this case.

6 Conclusion

In this paper, we consider the impact of ambiguity, or Knightian uncertainty, on credit rating in presence of a feedback loop. The feedback loop connects a rating agency, aiming for an accurate and unbiased estimate of a firm's distance-to-default, and a firm that maximizes its equity value over its default strategy, via performance-sensitive debt.

We show that the rating agency, learning the firm's true quality from observing survival of periods of apparent distress, deviates from a classical min-max response to ambiguity. Instead, to satisfy its aim of accurate ratings, it balances different priors to a moderate, that is, uninformative, aggregate distribution.

When rating analysts show a common direction of disagreement, the impact of ambiguity on the equilibrium of the credit rating game has important implications for the rated firm's default policy. For jointly pessimistic priors, the firm strategically delays default compared to the results under classical asymmetric information without ambiguity to profit from the feedback loop in form of lower capital costs, as the rating agency rapidly shifts its priors. In contrast, for jointly optimistic priors, the firm strategically accelerates default to cut costs from rising capital costs, as the rating agency sluggishly updates its prior.

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Credit Rating under Ambiguity
Appendix

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August 7, 2023

List of Appendices

Appendix I Main Results, Proofs and Auxiliary Results

I.1 Proofs of Main Results

In this section we prove the main results and corresponding auxiliary results. We start with the proof of the fixed point theorem:

Proof of Theorem 3

Proof. The proof is conducted with Schauder's fixed point theorem. For the existence of a fixed point, we identify a sufficiently rich subset $\mathscr{K}$ of the space of economically reasonable strategies $\mathscr{A}_g \times \mathscr{A}_f$. Now, Lemma 2 states that for all $g \in \mathscr{A}_g$, the best response firm strategy $f(\cdot;g)$ is in $\mathscr{K}_f$ with

$$\mathscr{K}_f = \big\{ f \in C(\Theta,\mathbb{R}_+) : l_f(\theta-\theta') \leq f(\theta') - f(\theta) \leq L_f(\theta-\theta'), \\ \underline{\theta}\,\underline{f} \leq f(\theta) \leq \overline{\theta}\,\overline{f},\ \text{for } \theta,\theta' \in \Theta \text{ with } \theta' \leq \theta \big\}, \tag{I.1}$$

where $0 < l_f < L_f$ and $0 < \underline{f} < \overline{f}$. On the other side, for $f \in \mathscr{K}_f$, the best response rating strategy then satisfies $g(\cdot;f) = id$ on $(0, \underline{\theta}\,\underline{f}]$ and $g(e;f) = g(\overline{\theta}\,\overline{f};f)$ for $e \geq \overline{\theta}\,\overline{f}$. This property is preserved when considering the transform $\mathscr{R} \circ g(\cdot;f)$ and the set $[\underline{e}, \overline{e}] := [\underline{\theta}\,\underline{f}, \overline{\theta}^2\,\overline{f}^2/(\underline{\theta}\,\underline{f})]$ due to Lemma 5, that is,

$$\mathscr{R} \circ g(e;f) = \mathscr{R} \circ g(\overline{e};f) \text{ for } e \geq \overline{e} \text{ and } \mathscr{R} \circ g(\cdot;f) = id \text{ on } (0, \underline{e}].$$

Therefore, it is sufficient to constrain the domain of the rating strategy from originally $\mathbb{R}_+$ to $[\underline{e}, \overline{e}]$. We define

$$\mathscr{K}_g = \big\{ g \in C([\underline{e}, \overline{e}], \mathbb{R}_+) : \underline{e} \leq g \leq id, \tag{I.2}$$

$$g \text{ is non-decreasing and } g/id \text{ is non-increasing} \big\} \tag{I.3}$$

We $\mathscr{K} = \mathscr{K}_g \times \mathscr{K}_f$. We have argued that $T(\mathscr{K}) \subset \mathscr{K}$. In "The Information Value of Distress" it was proven that $\mathscr{K}$ is a non-empty, convex and compact subset of a Banach space, here $\mathscr{V} = C(\Theta, \mathbb{R}) \times C([\underline{e}, \overline{e}], \mathbb{R})$, endowed with the sup-norm. For the fixed point theorem to hold it is sufficient to show that $g \mapsto f(\cdot;g)$ and $f \mapsto \mathscr{R}(g(\cdot;f))$ are both continuous. Continuity of the first mapping was shown in "The Information Value of Distress". It thereby remains to show that the mapping

$$\mathscr{K}_f \to \mathscr{K}_g \quad \text{with} \quad f \mapsto \mathscr{R}(g(\cdot;f))|_{[\underline{e}, \overline{e}]}$$

is continuous under the sup-norm. We begin by showing that the mapping $f \mapsto g(\cdot;f)$ is continuous with respect to the sup-norm: Note, that from that from the proof in "The Information Value of

Distress" it directly follows that

$$f \mapsto a_i(\cdot; f) \tag{I.4}$$

is continuous with the respect to the sup-norm. We have that $a_i(\cdot; f)$ is continuous, non-decreasing, bounded by id and with $f(\Theta) \subset [\underline{\theta}\,\underline{f}, \overline{\theta}\,\overline{f}]$ for all $f \in \mathscr{K}_f$, it holds that

$$a_i(e; f) = e \text{ for } 0 \leq e \leq \underline{\theta}\,\underline{f} \quad \text{and} \quad a_i(e; f) = a_i(\overline{\theta}\,\overline{f}; f) \text{ for } e \geq \overline{\theta}\,\overline{f}. \tag{I.5}$$

We denote the set of continuous, non-decreasing functions that are bounded by id and satisfy (I.5) by $\mathscr{A}$. By Lemma 4 it follows that

$$f \mapsto b_i(\cdot; f) \tag{I.6}$$

is continuous with respect to the sup-norm, too. For all $f \in \mathscr{K}_f$ it holds that $b_i(\cdot; f)$ is continuous, non-decreasing, bounded by id^2 with

$$b_i(e; f) = e^2 \text{ for } 0 \leq e \leq \underline{\theta}\,\underline{f} \quad \text{and} \quad b_i(e; f) = b_i(\overline{\theta}\,\overline{f}; f) \text{ for } e \geq \overline{\theta}\,\overline{f}. \tag{I.7}$$

Now, we denote the set of continuous, non-decreasing functions that are bounded by id^2 and satisfy (I.7) by $\mathscr{B}$. We define

$$\mathscr{G} := \{ \left(v_i, w_i\right)_{i=1,\ldots,n} \mid v_i \in \mathscr{A} \quad \text{and} \quad w_i \in \mathscr{B} \}. \tag{I.8}$$

Now, the mapping

$$Q: \mathscr{K}_f \to \mathscr{G} \quad \text{with} \quad f \mapsto \big(a_i(\cdot;f), b_i(\cdot;f)\big)_{i=1,\dots,n} \tag{I.9}$$

is continuous with respect to the sup-norm. Therefore, it is sufficient to show that the mapping $Z: \mathscr{G} \to C(\mathbb{R}^+, \mathbb{R}^+)$ with

$$\big(v_i(\cdot), w_i(\cdot)\big)_{i=1,\dots,n} \mapsto \arg\min_{D>0} \Big\{ \max_{i=1,\dots,n} \Big(D^2 - 2Dv_i(\cdot) + w_i(\cdot)\Big) \Big\} \tag{I.10}$$

is continuous regarding the respective sup-norms. This then implies continuity of $Z \circ Q$ and noting that $Z \circ Q(f) = g(\cdot;f)$ will then finish the proof. Let us show continuity of the function Z in (I.10). Since all $\big(v_i(\cdot), w_i(\cdot)\big)_i \in \mathscr{G}$ have values in the set $\big([0, \overline{\theta}\,\overline{f}] \times [0, (\overline{\theta}\,\overline{f})^2]\big)^n$, uniform continuity of the mapping $\big([0, \overline{\theta}\,\overline{f}] \times [0, (\overline{\theta}\,\overline{f})^2]\big)^n \to \mathbb{R}^+$

$$(x_i, y_i)_{i=1,\dots,n} \mapsto \arg\min_{D>0} \Big\{ \max_{i=1,\dots,n} \Big(D^2 - 2Dx_i + y_i\Big) \Big\} \tag{I.11}$$

will suffice. However, the domain of that function is a compact subset of $\mathbb{R}^{2n}$ and uniform continuity and continuity are equivalent. The mapping is continuous by Lemma 3.

So far, we have shown that the mapping $f \mapsto g(\cdot;f)$ is continuous with respect to the sup-norm. The continuity of the mapping $f \mapsto \mathscr{R}(g(\cdot;f))$ then follows from Lemma 6. Including the restriction to $[\underline{e}, \overline{e}]$ does no harm and therefore $f \mapsto \mathscr{R}(g(\cdot;f))|_{[\underline{e},\overline{e}]}$ is continuous. $\square$

Proposition 5. *Given a continuous and strictly increasing default strategy f, the ambiguity averse best response rating strategy satisfies*[11]

$$\hat{g}'(\theta) = H\Big((\phi_i(\theta))_i, (\Phi_i(\theta))_i, (\hat{a}_i(\theta))_i, (\hat{b}_i(\theta))_i, \theta, f(\theta)\Big) \tag{I.12}$$

on $(\underline{\theta}, \overline{\theta})$ *with initial conditions*

$$\hat{g}'(\underline{\theta}) = \frac{1}{2} f'(\underline{\theta}) \quad \textit{and} \quad \hat{g}(\underline{\theta}) = f(\underline{\theta}) \tag{I.13}$$

and some specific function $H : \mathbb{R}^{+^{4n+2}} \to \mathbb{R}$*. Additionally, it holds that*

$$\hat{a}_i'(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)} \big(f(\theta) - \hat{a}_i(\theta)\big) \quad \textit{and} \quad \hat{b}_i'(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)} \big(f(\theta)^2 - \hat{b}_i(\theta)\big) \tag{I.14}$$

for $i = 1, \ldots, n$ *on* $(\underline{\theta}, \overline{\theta})$ *with initial conditions*

$$\hat{a}_i(\underline{\theta}) = f(\underline{\theta}), \quad \hat{a}_i'(\underline{\theta}) = \frac{1}{2} f'(\underline{\theta}), \quad \hat{b}_i(\underline{\theta}) = f(\underline{\theta})^2 \;\; \textit{and} \quad \hat{b}_i'(\underline{\theta}) = f(\underline{\theta}) f'(\underline{\theta}). \tag{I.15}$$

Proof. Recall that $\hat{g}(\theta)$ is the argument that minimizes the upper envelope of quadratic functions of the form

$$D^2 - 2D\hat{a}_i(\theta) + \hat{b}_i(\theta) \quad \text{with} \quad i = 1, \ldots, n.$$

[11] The derivative of $\hat{g}$ exists by assumption everywhere except on a set of finite cardinality.

Consider the function

$$\hat{a}_{i,j}(\theta) = \begin{cases} \frac{1}{2}\frac{\hat{b}_i(\theta)-\hat{b}_j(\theta)}{\hat{a}_i(\theta)-\hat{a}_j(\theta)} & \text{if} \quad i \neq j \quad \text{and} \quad \hat{a}_i(\theta) \neq \hat{a}_j(\theta) \\ \\ \hat{a}_i(\theta) & \text{else.} \end{cases} \tag{I.16}$$

Note that for $i \neq j$, $\hat{a}_{i,j}(\theta)$ is the argument of the intercept of the quadratic functions i and j, if such exists. If $i = j$, $\hat{a}_{i,j}(\theta)$ is just given by the argument of the minimum of the respective quadratic function, which is $\hat{a}_i(\theta)$. It can easily be shown that for each θ

$$\hat{g}(\theta) \in \left\{ \hat{a}_{i,j}(\theta) \mid i \leq j \text{ and } i,j \in \{1,...,n\} \right\}. \tag{I.17}$$

That is because the minimum of the upper envelope is either the minimum of a single quadratic function or an intercept of two quadratic functions. We assume that up to set of finite cardinality, when

$$\hat{g}(\theta) = \hat{a}_{i,j}(\theta), \quad \text{for some } i \leq j \text{, then} \quad \hat{g}'(\theta) = \hat{a}'_{i,j}(\theta).$$

By the Laws of l'Hopital it follows that on $(\underline{\theta}, \overline{\theta})$

$$\hat{a}'_i(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)}\big(f(\theta) - \hat{a}_i(\theta)\big) \quad \text{and} \quad \hat{b}'_i(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)}\big(f(\theta)^2 - \hat{b}_i(\theta)\big) \tag{I.18}$$

for $i = 1,...,n$ (proof as in the "Information Value of Distress"). For $i \neq j$ with $\hat{a}_i(\theta) \neq \hat{a}_j(\theta)$,

$\hat{a}'_{i,j}(\theta)$ can be calculated from the Quotient rule and is given by

$$\hat{a}'_{i,j}(\theta) = \frac{1}{2}\frac{\left(\hat{b}'_i(\theta) - \hat{b}'_j(\theta)\right)\left(\hat{a}_i(\theta) - \hat{a}_j(\theta)\right) - \left(\hat{b}_i(\theta) - \hat{b}_j(\theta)\right)\left(\hat{a}'_i(\theta) - \hat{a}'_j(\theta)\right)}{\left(\hat{a}_i(\theta) - \hat{a}_j(\theta)\right)^2}.$$

Putting things together, it follows that the derivative of $\hat{g}$ can be represented as

$$\hat{g}'(\theta) = H\Big((\phi_i(\theta))_i, (\Phi_i(\theta))_i, (\hat{a}_i(\theta))_i, (\hat{b}_i(\theta))_i, \theta, f(\theta)\Big). \tag{I.19}$$

It remains to show the initial conditions. The initial conditions for the ambiguity averse strategy $\hat{g}(\theta)$ follow from the initial conditions in Equation (I.15), since for all $\theta \in (\underline{\theta}, \overline{\theta})$:

$$\hat{g}(\theta) \in [\min_i\{\hat{a}_i(\theta)\}, \max_i\{\hat{a}_i(\theta)\}]. \tag{I.20}$$

For demonstrating purposes we only show the last condition in (I.15):

$$\begin{aligned}\hat{b}'_i(\underline{\theta}) &= \lim_{\theta \searrow \underline{\theta}} \hat{b}'_i(\theta) = \lim_{\theta \searrow \underline{\theta}} \frac{\phi_i(\theta)}{\Phi_i(\theta)}\left(f(\theta)^2 - \hat{b}_i(\theta)\right) = \phi_i(\underline{\theta}) \lim_{\theta \searrow \underline{\theta}} \frac{f(\theta)^2 - \hat{b}_i(\theta)}{\Phi_i(\theta)} \\ &= \phi_i(\underline{\theta}) \lim_{\theta \searrow \underline{\theta}} \frac{(f(\theta)^2)' - \hat{b}'_i(\theta)}{\phi_i(\theta)} = 2f'(\underline{\theta})f(\underline{\theta}) - \hat{b}'_i(\underline{\theta}).\end{aligned}$$

The proof for the remaining initial conditions is already given in the "Information Value of Distress".

□

Proposition 6. *Let $f \in \mathscr{K}_f$. For the ambiguity averse best response $g = g(\cdot; f)$ and its transforma-*

tion $\mathscr{R}(g) = \mathscr{R}(g(\cdot; f))$ denote

$$\hat{g} = g \circ f \quad \textit{and} \quad \tilde{g} = \mathscr{R}(g) \circ f. \tag{I.21}$$

Then, $\tilde{g}$ is Lebesgue almost everywhere differentiable and satisfies on $(\underline{\theta}, \overline{\theta})$:

$$\tilde{g}' = f' \frac{\tilde{g}}{f} \mathbf{1}_{\{\tilde{g} \neq \hat{g}\}} + \hat{g}' \mathbf{1}_{\{\tilde{g} = \hat{g}\}}. \tag{I.22}$$

The initial conditions are given by

$$\tilde{g}(\underline{\theta}) = \hat{g}(\underline{\theta}) = f(\underline{\theta}) \quad \textit{and} \quad \tilde{g}'(\underline{\theta}) = \hat{g}'(\underline{\theta}) = f'(\underline{\theta})/2. \tag{I.23}$$

Proof. With $\hat{g}^*(\theta) := \sup\{\hat{g}(\theta) : \underline{\theta} \leq \theta' \leq \theta\}$ it holds

$$\begin{aligned} \tilde{g}(\theta) = \mathscr{R}(g)(f(\theta) &= e \inf_{0<z\leq e} \frac{\sup\{g(s) \mid 0 < s \leq z\}}{z} \Big|_{e=f(\theta)} \\ &= f(\theta) \inf_{f(\underline{\theta})<z\leq f(\theta)} \frac{\sup\{g(s) \mid 0 < s \leq z\}}{z} = f(\theta) \inf_{\underline{\theta}<\theta'\leq\theta} \frac{\hat{g}^*(\theta')}{f(\theta')}. \end{aligned} \tag{I.24}$$

Therefore, the initial value is given by $\tilde{g}(\underline{\theta}) = \hat{g}(\underline{\theta}) = f(\underline{\theta})$, where the initial value for $\hat{g}$ comes from Proposition 5. By Lemma 1, $\tilde{g}$ is continuous, non-decreasing and bounded by f. It holds for

$\theta \geq \theta'$:

$$0 \leq \tilde{g}(\theta) - \tilde{g}(\theta') = f(\theta) \inf_{\underline{\theta} < z \leq \theta} \frac{\hat{g}^*(z)}{f(z)} - f(\theta') \inf_{\underline{\theta} < z \leq \theta'} \frac{\hat{g}^*(z)}{f(z)}$$
$$\leq (f(\theta) - f(\theta')) \inf_{\underline{\theta} < z \leq \theta} \frac{\hat{g}^*(z)}{f(z)} \leq (f(\theta) - f(\theta')) \leq L_f |\theta - \theta'| \tag{I.25}$$

Then, $\tilde{g}$ is Lipschitz continuous and has a derivative Lebegues almost everywhere on Θ. By the same argument, f is differentiable almost everywhere. We denote $E_f \subset \Theta$ the set where $\tilde{g} = \hat{g}$, that is $E_f = \{\theta \in \Theta \mid \hat{g}(\theta) = \tilde{g}(\theta)\}$. Since $\hat{g}$ and $\tilde{g}$ are both continuous and Θ bounded, E_f is compact. On the interior of E_f it holds almost everywhere that $\tilde{g}' = \hat{g}'$. Since the boundary of E_f hast Lebegues measure 0, we have:

$$\tilde{g}' = \hat{g}', \quad \text{almost everywhere on } E_f. \tag{I.26}$$

Next, consider $(\underline{\theta}, \overline{\theta}) \setminus E_f$, which is open. Take $\theta \in (\underline{\theta}, \overline{\theta}) \setminus E_f$ and by continuity of $\tilde{g}, \hat{g}$ there exists some $\varepsilon > 0$, such that either $\tilde{g} < \hat{g}$ or $\tilde{g} > \hat{g}$ everywhere on $B_\varepsilon(\theta) \subset (\underline{\theta}, \overline{\theta}) \setminus E_f$. In either case, there exists some $\theta^\star < \theta$ with

$$\tilde{g}(\theta) = f(\theta) \inf_{\underline{\theta} < \theta' \leq \theta} \frac{\hat{g}^*(\theta')}{f(\theta')} = f(\theta) \frac{\hat{g}^*(\theta^\star)}{f(\theta^\star)} \tag{I.27}$$

and with continuity of $f, \hat{g}^*$ some $\varepsilon^\star < \varepsilon$, such that

$$\tilde{g}(\theta') = f(\theta') \frac{\hat{g}^*(\theta^\star)}{f(\theta^\star)} \quad \text{for} \quad \theta' \in B_{\varepsilon^\star}(\theta). \tag{I.28}$$

Therefore, if f is differentiable in θ, which holds almost surely on Θ, we have with (I.27):

$$\tilde{g}'(\theta) = f'(\theta)\frac{\tilde{g}(\theta)}{f(\theta)}. \tag{I.29}$$

Putting all together, it holds almost everywhere on $(\underline{\theta}, \overline{\theta})$:

$$\tilde{g}' = f'\frac{\tilde{g}}{f}\mathbf{1}_{\{\tilde{g}\neq\hat{g}\}} + \hat{g}'\mathbf{1}_{\{\tilde{g}=\hat{g}\}}. \tag{I.30}$$

It remains to show the initial condition for the derivative. Observe that $\tilde{g}(\underline{\theta}) = \hat{g}(\underline{\theta}) = f(\underline{\theta})$ and $\hat{g}'(\underline{\theta}) = f'(\underline{\theta})/2 > 0$. We show that $\hat{g}^*(\underline{\theta})' = f'(\underline{\theta})/2$:

Let $(\varepsilon_n)_n$ be any sequence that converges to zero from above. Now, there exists some sequence $0 < \varepsilon_n' \leq \varepsilon_n$ with

$$\hat{g}(\underline{\theta} + \varepsilon_n') = \hat{g}^*(\underline{\theta} + \varepsilon_n) \geq \hat{g}(\underline{\theta} + \varepsilon_n). \tag{I.31}$$

We show that $\varepsilon_n'/\varepsilon_n \to 1$ must hold true: Assume that $\varepsilon_n'/\varepsilon_n$ does not converge to 1. The sequence is bounded and there mus an exist an $0 \leq d < 1$, such that a subsequence $\varepsilon_{n_k}'/\varepsilon_{n_k}$ converges to d. Therefore,

$$\begin{aligned}
\hat{g}'(\underline{\theta}) &= \lim_{k\to\infty}\frac{\hat{g}(\underline{\theta}+\varepsilon_{n_k}) - \hat{g}(\underline{\theta})}{\varepsilon_{n_k}} \leq \lim_{k\to\infty}\frac{\hat{g}^*(\underline{\theta}+\varepsilon_{n_k}) - \hat{g}^*(\underline{\theta})}{\varepsilon_{n_k}} \\
&= \lim_{k\to\infty}\frac{\hat{g}(\underline{\theta}+\varepsilon_{n_k}') - \hat{g}(\underline{\theta})}{\varepsilon_{n_k}'}\frac{\varepsilon_{n_k}'}{\varepsilon_{n_k}} = \hat{g}'(\underline{\theta})\lim_{k\to\infty}\frac{\varepsilon_{n_k}'}{\varepsilon_{n_k}} = \hat{g}'(\underline{\theta})\,d
\end{aligned} \tag{I.32}$$

This is a contradiction and with $\varepsilon_n'/\varepsilon_n \to 1$ the same argument gives $\hat{g}^*(\underline{\theta})' = \hat{g}'(\underline{\theta}) = f'(\underline{\theta})/2$. From that, it follows that $(\hat{g}^*/f)'(\underline{\theta}) < 0$. Now, let $(\varepsilon_n)_n$ be another sequence that converges to zero from above and $0 < \varepsilon_n \leq \varepsilon$ such that

$$\tilde{g}(\underline{\theta}+\varepsilon_n) = f(\underline{\theta}+\varepsilon_n)\frac{\hat{g}^*(\underline{\theta}+\varepsilon_n')}{f(\underline{\theta}+\varepsilon_n')}. \tag{I.33}$$

Next,

$$\begin{aligned}
&\frac{\hat{g}^*(\underline{\theta}+\varepsilon_n)-\hat{g}^*(\underline{\theta})}{\varepsilon_n} \geq \frac{\tilde{g}(\underline{\theta}+\varepsilon_n)-\hat{g}^*(\underline{\theta})}{\varepsilon_n} = \frac{\tilde{g}(\underline{\theta}+\varepsilon_n)-\tilde{g}(\underline{\theta})}{\varepsilon_n} \\
=&\frac{f(\underline{\theta}+\varepsilon_n)/(f(\underline{\theta}+\varepsilon_n'))\hat{g}^*(\underline{\theta}+\varepsilon_n)-\hat{g}^*(\underline{\theta})}{\varepsilon_n} \\
=&\frac{\varepsilon_n'}{\varepsilon_n}\frac{f(\underline{\theta}+\varepsilon_n)/(f(\underline{\theta}+\varepsilon_n'))\hat{g}^*(\underline{\theta}+\varepsilon_n')-\hat{g}^*(\underline{\theta})}{\varepsilon_n'} \geq \frac{\varepsilon_n'}{\varepsilon_n}\frac{\hat{g}^*(\underline{\theta}+\varepsilon_n')-\hat{g}^*(\underline{\theta})}{\varepsilon_n'}.
\end{aligned}$$

Showing that $\varepsilon_n'/\varepsilon_n$ must converge to 1 is sufficient to finish the proof. Then $(\tilde{g}(\underline{\theta}+\varepsilon_n)-\tilde{g}(\underline{\theta}))/\varepsilon_n$ will converge to $g^*(\underline{\theta})' = f'(\underline{\theta})/2$, as required. Now, if $\varepsilon_n'/\varepsilon_n$ does not converge to 1, there exists a subsequence $\varepsilon_{n_k}'/\varepsilon_{n_k}$ that converges to some $0 \leq p < 1$. However, this is a contradiction to $(\hat{g}^*/f)'(\underline{\theta}) < 0$: By definition of the sequence $(\varepsilon_n')_n$ it holds that

$$(\hat{g}^*/f)(\underline{\theta}+\varepsilon_{n_k}) \geq (\hat{g}^*/f)(\underline{\theta}+\varepsilon_{n_k}'). \tag{I.34}$$

It holds for $p > 0$:

$$(\hat{g}^*/f)'(\underline{\theta}) = \lim_{k\to\infty} \frac{(\hat{g}^*/f)(\underline{\theta}+\varepsilon'_{n_k}) - (\hat{g}^*/f)(\underline{\theta})}{\varepsilon'_{n_k}}$$
$$\leq \lim_{k\to\infty} \frac{(\hat{g}^*/f)(\underline{\theta}+\varepsilon_{n_k}) - (\hat{g}^*/f)(\underline{\theta})}{\varepsilon_{n_k}} \frac{\varepsilon_{n_k}}{\varepsilon'_{n_k}} = (\hat{g}^*/f)'(\underline{\theta})\, p^{-1}$$

This is a contradiction, since $(\hat{g}^*/f)'(\underline{\theta})$ is negative. Analogously, for $p = 0$, the contradictory result is $(\hat{g}^*/f)'(\underline{\theta}) \leq \infty$. Therefore $\tilde{g}'(\underline{\theta}) = f'(\underline{\theta})/2$.

□

Proposition 7. *Let $g \in \mathscr{A}_g$ be an economically reasonable rating strategy, which is almost everywhere continuously differentiable. The firms best response strategy $f = f(\cdot; g)$ satisfies*

$$f'(\theta) = \frac{(1+\eta)\sigma^2}{2(r-\mu)} \frac{(f(\theta)/\theta)^2}{C(f(\theta)/\hat{g}(\theta)) - f(\theta)/\theta} + \frac{\partial \tilde{b}(\theta,\theta;g)}{\partial \tilde{\theta}} \tag{I.35}$$

*for $\theta \in (\underline{\theta}, \overline{\theta})$, where η is defined in Equation (**??**), $\hat{g} = g \circ f$ and the partial derivative of the boundary describing function $\tilde{b}$ in (θ,θ) is a function of $f(\theta), \hat{g}'(\theta)$ and $\hat{g}(\theta)$, that is*

$$\frac{\partial \tilde{b}(\theta,\theta;g)}{\partial \tilde{\theta}} = h(\theta, f(\theta), \hat{g}(\theta), \hat{g}'(\theta)) \tag{I.36}$$

for some function $h : \Theta \times \mathbb{R}^3_+ \to \mathbb{R}_{\geq 0}$. Moreover, the initial starting value for the derivative of f is given by

$$f'(\underline{\theta}) = k(\theta, f(\theta), \hat{g}(\theta)) \tag{I.37}$$

for some specific real-valued function k. Especially it does not depend on $\hat{g}'$.

Proof. see "Information Value of Distress"

□

Proposition 8. *Let* $(f^\star, g^\star)$ *be a fixed point of* T. *Suppose* $f^\star, g^\star$, $(\phi_i)_i$ *and the collection of the equity value function are sufficiently differentiable. Then, the fixed point is fully characterized, via* $f^\star = f$ *and* $g^\star = \tilde{g} \circ f^{-1}$, *by the solution* $(f, \tilde{g}, \hat{g})$ *of the differential equation:*

$$\begin{pmatrix} f'(\theta) \\ \tilde{g}'(\theta) \\ \hat{g}'(\theta) \end{pmatrix} = \begin{pmatrix} \frac{(1+\eta)\sigma^2}{2(r-\mu)} \frac{(f(\theta)/\theta)^2}{C(f(\theta)/\tilde{g}(\theta)) - f(\theta)/\theta} + \frac{\partial \tilde{b}(\theta,\theta;g^\star)}{\partial \tilde{\theta}} \\ f'(\tilde{g}/f)\mathbf{1}_{\{\tilde{g} \neq \hat{g}\}} + \hat{g}'\mathbf{1}_{\{\tilde{g} = \hat{g}\}} \\ H\Big((\phi_i(\theta))_i, (\Phi_i(\theta))_i, (\hat{a}_i(\theta))_i, (\hat{b}_i(\theta))_i, \theta, f(\theta)\Big) \end{pmatrix} \tag{I.38}$$

on $(\underline{\theta}, \overline{\theta})$ *with* $\hat{g}(\theta) = g \circ f$, *where*

$$\frac{\partial \tilde{b}(\theta, \theta; g^\star)}{\partial \tilde{\theta}} = h(\theta, f(\theta), \tilde{g}(\theta), \tilde{g}'(\theta)) \tag{I.39}$$

for specific real-valued functions H and h. In the differential Equation (22), *it holds that*

$$\hat{a}_i'(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)}\big(f(\theta) - \hat{a}_i(\theta)\big) \quad \textit{and} \quad \hat{b}_i'(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)}\big(f(\theta)^2 - \hat{b}_i(\theta)\big) \tag{I.40}$$

for $i = 1, \dots, n$ *on* $(\underline{\theta}, \overline{\theta})$*. The initial condition is*

$$\begin{pmatrix} f(\underline{\theta}) \\ \tilde{g}(\underline{\theta}) \\ \hat{g}(\underline{\theta}) \end{pmatrix} = \underline{\theta} \begin{pmatrix} f_1^\star \\ f_1^\star \\ f_1^\star \end{pmatrix}, \tag{I.41}$$

where $(f_1^\star, g_1^\star)$ *denotes the equilibrium of the perfect information case, that is* $\Theta = \{1\}$*; hence* $D = X$ *with* $f_1^\star = g_1^\star$*, which exists and is unique under the given assumptions. The remaining initial conditions for the derivatives are given in Proposition 5, 6 and 7.*

Proof. The differential equation is composed from Propositions 5, 6 and 7. By those Propositions the best responses satisfy $f(\underline{\theta}) = \hat{g}(\underline{\theta}) = \tilde{g}(\underline{\theta})$ and $\hat{g}'(\underline{\theta}) = \tilde{g}'(\underline{\theta})$. Now, for $\theta \searrow \underline{\theta}$ the information asymmetry vanishes and the limit must be the unique equilibrium of the perfect information case of Manso (2013). □

I.2 Auxiliary Results

Lemma 1. *Let* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *be any function that is bounded by id. For the transformed strategy it holds that* $\mathscr{R}(g) \in \mathscr{A}_g$*, i.e.*

$$\mathscr{R}(g)(e) \leq e,\ \mathscr{R}(g)(e) \text{ is non-decreasing and } \mathscr{R}(g)(e)/e \text{ is non-increasing}.$$

Proof. Since

$$\mathscr{R}(g)(e) = e \inf_{0<z\le e} \frac{\sup\{g(s) \mid 0 < s \le z\}}{z}$$

it follows from the boundedness by the identity that

$$\mathscr{R}(g)(e) \le e \inf\{e/z \mid 0 < z \le e\} = e. \tag{I.42}$$

We define $g^*(e) = \sup\{g(s) \mid 0 < s \le e\}$ and observe that $\mathscr{R}(g) \le g^* \le id$. To show that $\mathscr{R}$ is non-decreasing take $e' \ge e > 0$. Then

$$\begin{aligned}
\mathscr{R}(g)(e') &= e' \inf\{g^*(z)/z \mid 0 < z \le e'\} \\
&= e'\Big(\inf\{g^*(z)/z \mid 0 < z \le e\} \wedge \inf\{g^*(z)/z \mid e < z \le e'\}\Big) \\
&\ge \Big(\frac{e'}{e}\mathscr{R}(g)(e)\Big) \wedge \Big(e' \inf\{g^*(e)/z \mid e < z \le e'\}\Big) \\
&= \Big(\frac{e'}{e}\mathscr{R}(g)(e)\Big) \wedge g^*(e) \\
&= \mathscr{R}(g)(e) + \Big(\frac{e'-e}{e}\mathscr{R}(g)(e) \wedge (g^*(e) - \mathscr{R}(g)(e))\Big) \\
&\ge \mathscr{R}(g)(e).
\end{aligned}$$

The property that $\mathscr{R}(g)(e)/e$ is non-decreasing holds by definition. □

The following Lemma characterizes the firm's best response default strategy $f(\cdot;g)$ to an economically reasonable rating strategy $g \in \mathscr{A}_g$:

Lemma 2. *There exist* $0 < \underline{f} < \overline{f} < \infty$ *and* $0 < l_f < L_f < \infty$ *such that for all* $\theta, \theta' \in \Theta$ *with* $\theta' \leq \theta$*:*

$$f(\theta';g) \leq f(\theta;g) \quad \textit{and} \quad \underline{f} \leq \frac{f(\theta;g)}{\theta} \leq \frac{f(\theta';g)}{\theta'} \leq \overline{f} \tag{I.43}$$

uniformly in $g \in \mathscr{A}_g$*. In particular,* $f(\cdot;g)$ *is Lipschitz continuous:*

$$|f(\theta;g) - f(\theta';g)| \leq L_f|\theta - \theta'| \quad \textit{for} \quad \theta, \theta' \in \Theta, \tag{I.44}$$

where $L_f = \overline{f} > 0$ *is the uniform Lipschitz constant for all g. Moreover, assume that Assumption 2 holds, then for* $\theta, \theta' \in \Theta$ *with* $\theta' \leq \theta$ *it holds that*

$$f(\theta;g) - f(\theta';g) \geq l_f(\theta - \theta'), \tag{I.45}$$

where $l_f = (1 - L_C)\underline{f} > 0$ *is the uniform constant for all* $g \in \mathscr{A}_g$*.*

Proof. in "Information Value of Distress" □

Note that $\underline{f}, \overline{f}$ in the previous proof are the default thresholds of the perfect information boundary cases of constant coupons $\underline{C}, \overline{C}$, respectively.

Lemma 3. *The mapping* $Z^* : \mathbb{R}^{2n} \to \mathbb{R}$

$$(x_i, y_i)_{i=1,\ldots,n} \mapsto \arg\min_{D>0} \Big\{ \max_{i=1,\ldots,n} \Big(D^2 - 2Dx_i + y_i \Big) \Big\}$$

is continuous.

Proof. For $(x,y) = (x_i, y_i)_i \in \mathbb{R}^{2n}$ consider

$$\overline{B}_1((x,y)) = \big\{(x^*,y^*) \mid \text{ for all } i : |x_i^* - x_i| < 1 \text{ and } |y_i^* - y_i| < 1\big\}.$$

We define $a := \min_i\{x_i\} - 2$ and $b = \max_i\{x_i\} + 2$. The mapping

$$\overline{B}_1((x,y)) \to C([a,b]) \quad \text{with} \quad (x_i^*, y_i^*)_i \mapsto c_{(x^*,y^*)}(D)$$

where

$$c_{(x^*,y^*)}(D) = \max_{i=1,\dots,n}(D^2 - 2Dx_i^* + y_i^*) \tag{I.46}$$

is continuous under the respective sup-norms. The functions $D \mapsto c_{(x^*,y^*)}(D)$ are strictly convex and the argument of their global minimum is an element of (a,b) for all $(x_i^*, y_i^*) \in \overline{B}_1((x,y))$. Let $z_{(x,y)}$ denote the argument of the minimum from the function $D \mapsto c_{(x,y)}(D)$. For $0 < \varepsilon < 1$, choose

$$\delta' = \frac{\min\{c_{(x,y)}(z_{(x,y)} - \varepsilon), c_{(x,y)}(z_{(x,y)} + \varepsilon)\} - c_{(x,y)}(z_{(x,y)})}{3} > 0$$

However, there exists some $0 < \delta < 1$ with

$$||(x,y) - (x^*,y^*)|| < \delta \quad \Rightarrow \quad ||c_{(x,y)} - c_{(x^*,y^*)}|| < \delta'.$$

By the choice of δ' and the strict convexity of $c_{(x^*,y^*)}$, it follows that the argument $z_{(x^*,y^*)}$, which minimizes $D \mapsto c_{(x^*,y^*)}(D)$, must fulfil $|z_{(x^*,y^*)} - z_{(x,y)}| < \varepsilon$.

$\square$

Lemma 4. *The mapping*

$$\mathscr{K}_f \to \mathscr{B} \quad \textit{with} \quad f \mapsto b_i(\cdot; f)$$

is continuous with respect to the sup-norm.

Proof. Take $f_1, f_2 \in \mathscr{K}_f$ with $||f_1 - f_2||_\infty \leq \varepsilon$ for some $\varepsilon > 0$. Consider $e \in [0, f_1(\underline{\theta}) \wedge f_2(\underline{\theta}))$. Then $b_i(e; f_1) = b_i(e; f_2) = e^2$ and

$$|b_i(e; f_1) - b_i(e; f_2)| = 0, \quad \text{for} \quad e \in [0, f_1(\underline{\theta}) \wedge f_2(\underline{\theta})). \tag{I.47}$$

Now, let $e \in [f_1(\underline{\theta}) \wedge f_2(\underline{\theta}), f_1(\underline{\theta}) \vee f_2(\underline{\theta}))$. Without loss of generality, assume $f_1(\underline{\theta}) < f_2(\underline{\theta})$ and thus $f_1(\underline{\theta}) \leq e < f_2(\underline{\theta})$. From this, we see by Equation (**??**) that $f_1(\underline{\theta})^2 \leq b_i(e; f_1) \leq e^2$. Note that $b_i(e; f_2) = e^2 < f_2(\underline{\theta})^2$. By the uniform boundedness of all $f \in \mathscr{K}_f$ it follows from $f_1^2 - f_2^2 = (f_1 - f_2)(f_1 + f_2)$ holds that there exists some uniform constant $2\overline{f} = K > 0$, such that

$$\forall \varepsilon > 0: \quad ||f_1 - f_2||_\infty \leq \varepsilon \Rightarrow ||f_1^2 - f_2^2||_\infty \leq K\varepsilon.$$

Therefore, it follows that

$$|b_i(e;f_1) - b_i(e;f_2)| \leq K\varepsilon, \quad \text{for} \quad e \in [f_1(\underline{\theta}) \wedge f_2(\underline{\theta}), f_1(\underline{\theta}) \vee f_2(\underline{\theta})). \tag{I.48}$$

Consider $e \in [f_1(\underline{\theta}) \vee f_2(\underline{\theta}), f_1(\underline{\theta}) \vee f_2(\underline{\theta}) + \varepsilon^{\frac{1}{2}})$. Then, we have

$$f_1(\underline{\theta})^2 \wedge f_2(\underline{\theta})^2 \leq b_i(e;f_1), b_i(e;f_2) \leq e^2 \tag{I.49}$$

and

$$e^2 \leq \big((f_1(\underline{\theta})^2 + 2f_1(\underline{\theta})\varepsilon^{\frac{1}{2}}) \vee (f_2(\underline{\theta})^2 + 2f_2(\underline{\theta})\varepsilon^{\frac{1}{2}})\big) + \varepsilon. \tag{I.50}$$

Note that all $f \in \mathscr{K}_f$ are positive and bounded by $\overline{\theta}\,\overline{f} < \infty$. Thereby it holds that for $e \in [f_1(\underline{\theta}) \vee f_2(\underline{\theta}), f_1(\underline{\theta}) \vee f_2(\underline{\theta}) + \varepsilon^{\frac{1}{2}})$:

$$|b_i(e;f_1) - b_i(e;f_2)| \leq K\varepsilon + 2\overline{\theta}\,\overline{f}\varepsilon^{\frac{1}{2}} + \varepsilon. \tag{I.51}$$

Consider $e \geq (f_1(\underline{\theta}) \vee f_2(\underline{\theta})) + \varepsilon^{\frac{1}{2}}$. Note that by $f_1, f_2 \in \mathscr{K}_f$ both functions are continuous and strictly increasing with minimum slope l_f and thus their respective inverse functions exist and are well-defined. Without loss of generality, assume $f_1^{-1}(e) \leq f_2^{-1}(e)$. Using the definition of $b_i(e;f)$

in Equation (**??**), we can write

$$b_i(e; f_2) = \frac{\int_{\underline{\theta}}^{f_2^{-1}(e)} f_2(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta}.$$

This equation can be rearranged to

$$\begin{aligned}
b_i(e; f_2) &= \frac{\int_{\underline{\theta}}^{f_1^{-1}(e)} f_2(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} + \frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} f_2(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} \\
&= \frac{\int_{\underline{\theta}}^{f_1^{-1}(e)} f_1(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} \\
&+ \frac{\int_{\underline{\theta}}^{f_1^{-1}(e)} (f_2(\theta)^2 - f_1(\theta)^2) \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} + \frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} f_2(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} \\
&= \frac{\int_{\underline{\theta}}^{f_1^{-1}(e)} f_1(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_1^{-1}(e)} \phi_i(\theta) d\theta} \frac{\int_{\underline{\theta}}^{f_1^{-1}(e)} \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} \\
&+ \frac{\int_{\underline{\theta}}^{f_1^{-1}(e)} (f_2(\theta)^2 - f_1(\theta)^2) \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} + \frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} f_2(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} \\
&= b_i(e; f_1) - b_i(e; f_1) \frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} \\
&+ \frac{\int_{\underline{\theta}}^{f_1^{-1}(e)} (f_2(\theta)^2 - f_1(\theta)^2) \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta} + \frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} f_2(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta) d\theta}.
\end{aligned}$$

Hence,

$$|b_i(e;f_2) - b_i(e;f_1)| \leq b_i(e;f_1)\frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} \phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta)d\theta} + \frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} f_2(\theta)^2\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta)d\theta} + \frac{\int_{\underline{\theta}}^{f_1^{-1}(e)} |f_2(\theta)^2 - f_1(\theta)^2|\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta)d\theta}. \tag{I.52}$$

We will show that all three equations on the right-hand side become small for $\varepsilon \searrow 0$. Observe that $0 \leq f_2^{-1}(e) - f_1^{-1}(e) \leq l_f^{-1}\varepsilon$, where the first inequality follows by assumption and the second by the fact that f_1, f_2 are strictly increasing with a minimum slope $l_f > 0$ and $||f_1 - f_2|| < \infty$. Also f_1, f_2 are bounded by $\overline{\theta}\,\overline{f}$ and uniformly Lipschitz continuous with Lipschitz constant L_f. Since ϕ_i is bounded away from zero and bounded from above by assumption, we see that

$$\begin{aligned}\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta)d\theta &\geq \int_{\underline{\theta}}^{f_2^{-1}(e)} \underline{\phi}d\theta \geq \int_{\underline{\theta}}^{f_2^{-1}(f_2(\underline{\theta})+\varepsilon^{\frac{1}{2}})} \underline{\phi}d\theta \\ &= \underline{\phi}\big(f_2^{-1}(f_2(\underline{\theta}) + \varepsilon^{\frac{1}{2}}) - \underline{\theta}\big) \geq \underline{\phi}\frac{1}{L_f}\varepsilon^{\frac{1}{2}}.\end{aligned} \tag{I.53}$$

Using this, we can bound the first expression on the right-hand side of Equation (I.52):

$$b_i(e;f_1)\frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} \phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta)d\theta} \leq \overline{\theta}^2\overline{f}^2 L_f \frac{\overline{\phi}l_f^{-1}\varepsilon}{\underline{\phi}\varepsilon^{\frac{1}{2}}} = \frac{\overline{\theta}^2\overline{f^2}L_f\overline{\phi}}{\underline{\phi}l_f}\varepsilon^{\frac{1}{2}}. \tag{I.54}$$

For the second expression in Equation (I.52) the same arguments apply and we receive

$$\frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} f_2(\theta)^2\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta)d\theta} \leq \overline{\theta}^2\overline{f}^2 \frac{\int_{f_1^{-1}(e)}^{f_2^{-1}(e)} \phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta)d\theta} = \frac{\overline{\theta}^2\overline{f^2}L_f\overline{\phi}}{\underline{\phi}l_f}\varepsilon^{\frac{1}{2}}. \tag{I.55}$$

The third expression in Equation (I.52) can be estimated as follows:

$$\frac{\int_{\underline{\theta}}^{f_1^{-1}(e)} |f_2(\theta)^2 - f_1(\theta)^2|\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f_2^{-1}(e)} \phi_i(\theta)d\theta} \leq L_f\frac{K||f_1-f_2||_\infty}{\underline{\phi}\varepsilon^{\frac{1}{2}}} \leq L_fK\frac{\varepsilon^{\frac{1}{2}}}{\underline{\phi}}. \tag{I.56}$$

This implies that $f \mapsto b_i(\cdot;f)$ is continuous with respect to the sup-norm. □

Lemma 5. *There exist* $0 < \underline{e} < \overline{e} < \infty$ *such that for all* $f \in \mathscr{K}_f$ *the ambiguity averse best response rating strategy* $g = g(\cdot;f)$ *satisfies*

$$\forall e \in (0,\underline{e}] : \; g(e) = e \quad \textit{and} \quad \forall e > \overline{e} : \; \mathscr{R}^*(g)(e) = \sup\{g(s) \mid 0 < s \leq \overline{e}\}.$$

Proof. Since $\underline{\theta}\,\underline{f} > 0$ is a uniform constant and lower boundary for $f \in \mathscr{K}_f$, such that $a_i(e;f) = e$ and $b_i(e;f) = e^2$ for $e \leq \underline{\theta}\,\underline{f}$, we can conclude that $g(e;f) = e$ for $0 < e < \underline{\theta}\,\underline{f}$. Therefore, we can set $\underline{e} = \underline{\theta}\,\underline{f}$.

Now, denote $g^*(e) = \sup\{g(s) \mid 0 < s \leq e\}$. From Equations (I.5) and (I.7), we can conduct that there exists some uniform constant $\overline{\theta}\,\overline{f} > 0$, such that

$$\forall e > \overline{\theta}\,\overline{f} : g(e) = g(\overline{\theta}\,\overline{f}) \tag{I.57}$$

and then it directly follows

$$\forall e > \overline{\theta}\,\overline{f} : g^*(e) = g^*(\overline{\theta}\,\overline{f}). \tag{I.58}$$

Moreover, since $\overline{\theta}\,\overline{f}$ is a uniform boundary for f, it follows that

$$\forall e \in (0,\infty) : \quad g(e) \le \overline{\theta}\,\overline{f}. \tag{I.59}$$

This last equation holds, since from the definition of the best response ambiguity averse rating strategy it can be conducted that $g(e) \le \max_i(a_i(e;f))$ and $a_i(e;f)$ is bounded by $\overline{\theta}\,\overline{f}$. It follows that

$$\forall e \in (0,\infty) : \quad g^*(e) \le \overline{\theta}\,\overline{f}. \tag{I.60}$$

Furthermore, it holds that with $k := \underline{\theta}\,\underline{f}(\overline{\theta}\,\overline{f})^{-1}$ that

$$\forall e \in (0,\overline{\theta}\,\overline{f}] : \quad \frac{g^*(e)}{e} \ge k > 0. \tag{I.61}$$

The last three equations imply that for $e > 0$ with $\overline{\theta}\,\overline{f}/e < k$, both $g^*(e)/e < k$ and $\inf\{g^*(z)/z \mid 0 < z \le e\} = g^*(e)/e$ must follow. Therefore it holds

$$\forall e > \frac{\overline{\theta}\,\overline{f}}{k} : \quad \mathscr{R}(g^*)(e) = g^*(e) = g^*(\overline{\theta}\,\overline{f}). \tag{I.62}$$

The claim holds with $\overline{e} = \overline{\theta}\,\overline{f}/k$. □

Lemma 6. *There exist* $0 < \underline{e} < \overline{e} < \infty$ *such that for all* $f_1, f_2 \in \mathscr{K}_f$ *it holds*

$$\forall \varepsilon > 0: \quad ||g_1 - g_2||_\infty < \varepsilon \Rightarrow ||\mathscr{R}(g_1) - \mathscr{R}(g_2)||_\infty < (1 + \overline{e}\underline{e}^{-1})\varepsilon,$$

where the best response rating strategy is given by $g_i(e) = g(e, f_i)$ *for* $i \in \{1, 2\}$.

Proof. We denote $g_i^*(e) = \sup\{g_i(s) \mid 0 < s \leq e\}$ for $i = 1, 2$. It holds that

$$||g_1 - g_2||_\infty < \varepsilon \Rightarrow ||g_1^* - g_2^*||_\infty < \varepsilon. \tag{I.63}$$

Choose $0 < \underline{e} < \overline{e} < \infty$ from Lemma 5. Then it holds that $g_1^*(e) = g_2^*(e) = e$ on $(0, \underline{e}]$. With

$$\mathscr{R}(g) := e \inf\{g(z)/z \mid 0 < z \leq e\} \tag{I.64}$$

we receive the equation $\mathscr{R}(g_j) = \mathscr{R}(g_j^*)$ for $j \in \{1, 2\}$. To finish the proof we will show

$$||\mathscr{R}(g_1^*) - \mathscr{R}(g_2^*)||_\infty < (1 + \overline{e}\underline{e}^{-1})\varepsilon. \tag{I.65}$$

It follows from $||g_1^* - g_2^*||_\infty < \varepsilon$ that $|\frac{g_1^*(e)}{e} - \frac{g_2^*(e)}{e}| < \frac{\varepsilon}{e}$ for all $e > 0$. With $g_1^*(e) = g_2^*(e) = e$ on $(0, \underline{e}]$ we have

$$|\frac{g_1^*(e)}{e} - \frac{g_2^*(e)}{e}| < \underline{e}^{-1}\varepsilon \quad \text{for all} \quad e > 0. \tag{I.66}$$

Let $e \in (0, \overline{e})$ be arbitrary. By the continuity of g_1^*, g_2^* there exist some $e_1, e_2 \in (0, \overline{e}]$ with $\mathscr{R}(g_1^*)(e) =$

$e\frac{g_1^*(e_1)}{e_1}$ and $\mathscr{R}(g_2^*)(e) = e\frac{g_2^*(e_2)}{e_2}$ Without loss of generality, we assume $\frac{g_1^*(e_1)}{e_1} \leq \frac{g_2^*(e_2)}{e_2}$. Since $\frac{g_2^*(e_1)}{e_1} \geq \frac{g_2^*(e_2)}{e_2}$ per definition of $\mathscr{R}$, it follows from (I.66) that $|\frac{g_1^*(e_1)}{e_1} - \frac{g_2^*(e_2)}{e_2}| < \underline{e}^{-1}\varepsilon$. Now, it so far follows that $|\mathscr{R}(g_1^*)(e) - \mathscr{R}(g_2^*)(e)| < \overline{e}\underline{e}^{-1}\varepsilon$ on $(0,\overline{e})$. On $[\overline{e},\infty)$ the claim holds by Lemma 5. □

We here state the inequality, which implies that the solution to the ODE to be economically reasonable:

Lemma 7. *Let f be a strictly increasing, differentiable and non-negative firm‘s strategy and g a rating strategy, which is almost everywhere differentiable. With $\hat{g} = g \circ f$ it holds that*

$$\forall \theta \in (\underline{\theta}, \overline{\theta}) : 0 \leq \hat{g}'(\theta) \leq f'(\theta)\frac{\hat{g}(\theta)}{f(\theta)} \quad \Longrightarrow \quad g = \mathscr{R}(g).$$

Proof. The inequality $\hat{g}'(\theta) \geq 0$ and strict monotony of f imply that g is non-decreasing as well. Thereby, it holds that

$$\mathscr{R}(g)(e) = e \inf_{0<z\leq e} \frac{g(z)}{z}.$$

We denote $\tilde{g}(\theta) = \mathscr{R}(g) \circ f$ and then we have

$$\tilde{g}(\theta) = f(\theta) \inf_{f(\underline{\theta})\leq z \leq f(\theta)} \frac{g(z)}{z} = f(\theta) \inf_{\underline{\theta}\leq\theta'\leq\theta} \frac{g(f(\theta'))}{f(\theta')} = f(\theta) \inf_{\underline{\theta}\leq\theta'\leq\theta} \frac{\hat{g}(\theta')}{f(\theta')}. \tag{I.67}$$

Now,

$$\hat{g}'(\theta) \leq f'(\theta)\frac{\hat{g}(\theta)}{f(\theta)} \quad \text{implies} \quad \frac{\hat{g}'f - \hat{g}f'}{f^2} = \left(\frac{\hat{g}}{f}\right)' \leq 0.$$

This, together with Equation (I.67), implies

$$\tilde{g}(\theta) = f(\theta) \inf_{\underline{\theta} \leq \theta' \leq \theta} \frac{\hat{g}(\theta')}{f(\theta')} = \hat{g}(\theta). \tag{I.68}$$

Since f is strictly increasing it furthermore follows that $g = \mathscr{R}(g)$. □