Existence and Uniqueness of Recursive Equilibria with Aggregate and Idiosyncratic Risk

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Introduction

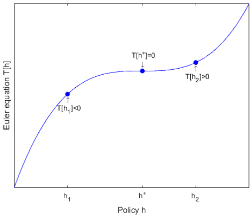
- Models with a continuum of heterogeneous agents facing aggregate risk:
  - **Ex post**: Agents are exposed to idiosyncratic risk like labor income risk $\Rightarrow$ Incompleteness
  - ~~Closed-form solutions~~ $\Rightarrow$ Do solutions exist?
How do we compute them?
- **Existing literature:**
  - Aggregate and idiosyncratic risk:
Brumm et al. (2017), Cao (2020), Miao (2006), Cheridito and Sagredo (2016)
  - Idiosyncratic risk only:
Light (2020), Achdou et al. (2017), Acemoglu and Jensen (2015), Açikgöz (2018), Wang (2003), Toda (2017)

Contribution

- **Open question:** Recursive equilibrium with minimal state space $h\left(z, \varepsilon, x, \mu^{(\varepsilon, x)}\right)$
- **Strategy:**
  1. Show that the recursive equilibrium which depends on the cross-sectional distribution can be written in terms of random variables
  2. Simplify the continuum of Euler equations to finitely many equations using random variables
  3. Apply results from convex analysis to ensure a solution to the simplified Euler and market clearing equations
  4. Sufficiency and uniqueness follows from convexity property

The (Simplified) Monotonicity Argument

- Denote the equilibrium equation depending on $h = (h_c, h_x)$ by $\mathbf{T}[h]$



- Sufficient conditions for each example model allow for the two candidate policies
  - Production economy: Implicit bound on discount factor which decreases with capital depreciation and risk aversion
  - Exchange economy: Finite aggregate endowments

Outline of the Talk

Generic Model

Exogenous variables

$$(z_t)_{t\geq 0} \in \mathbb{R}$$
$$(\varepsilon_t | z_t)_{t\geq 0} \in \mathbb{R}$$

Endogenous variables

$(c_t)_{t\geq 0}$	individual consumption
$(k_t)_{t\geq 0}$	individual capital holdings
$(b_t)_{t\geq 0}$	individual bond holdings
$(R_t^k)_{t\geq 0}$	rental rate of capital
$(R_t^b)_{t\geq 0}$	bond return

- **Individual problem:**

$$\max_{\{c_t, k_t, b_t\}_{t\geq 0}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}\right]$$

$$\text{s.t.} \quad k_t + b_t + c_t = e(z_t, \varepsilon_t) + W_t l(z_t, \varepsilon_t) + (1 + R_t^k)k_{t-1} + (1 + R_t^b)b_{t-1},$$
$$k_t \geq \bar{k},\ b_t \geq \bar{b}\, \forall t \geq 0$$

- **Aggregate variables:**

$$L_t = \int\int\int l(z_t, \varepsilon) d\mu_t(\varepsilon, k, b)$$
$$W_t = \frac{\partial}{\partial L} F(L_t, K_t)$$
$$R_t = \frac{\partial}{\partial K} F(L_t, K_t) - \delta$$

- **Market clearing:**

$$K_t = \int\int\int k d\mu_t(\varepsilon, k, b)$$
$$0 = \int\int\int b d\mu_t(\varepsilon, k, b)$$

- **Recursive equilibrium:** The optimal individual quantities c', k', b' are given by functions $g_x(z', \varepsilon', k, b, \mu')$, whereas, the aggregate equilibrium quantities $R^{k'}, R^{b'}$ are given by functions $g_A(z', \mu')$.

Outline of the talk

From Distributions to Random Variables

- How do we solve convex optimization problem? $\Rightarrow$ Take derivatives
- Derivatives of $f(\mu)$: Change in the function value f when the argument changes in a certain direction from μ^1 to mu^2
- **But, no notion of direction**: Difference of two c.d.f.s $\mu^2 - \mu^1$ is not a c.d.f.
- Use two real valued random variables $m^1 \sim \mu^1$ and $m^1 \sim \mu^1$: $m^2 - m^1$ is a real-valued random variable as well
- Rewriting $f(\mu)$ by $f(m)$ with $m \sim \mu$ ensures that we can take derivatives!

A Consistent Law of Motion of μ_t

- Summarize $x = (k, b)$.

Distributions	Random Variables
Start with given c.d.f. $\mu(\varepsilon, x)$	Start with given $(\varepsilon, \chi) \sim \mu$
$\downarrow$	$\downarrow$
End-of-period:	End-of-period:
$\tilde{\mu}(\varepsilon, x) =$	$\tilde{\chi} =$
$\int_{\zeta \leq \varepsilon} \int_{\chi \geq \bar{x}} \mathbb{1}_{\{g_x(z,\zeta,\chi,\mu) \leq x\}} d\mu(\zeta, \chi)$	$g_x(z, \varepsilon, \chi, \mu)$
$\downarrow$	$\downarrow$
Beginning-of-next-period:	Beginning-of-next-period:
$\mu'(\varepsilon', x) = \int_{\varepsilon \in \mathcal{Z}^{id}} \tilde{\mu}(\varepsilon, x) \mathbb{P}(\varepsilon' \| d\varepsilon, z, z')$	$(\varepsilon', \tilde{\chi}) = (\varepsilon', g_x(z, \varepsilon, \chi, \mu))$

- It follows by induction that $(\varepsilon_t, \chi_{t-1}) = (\varepsilon_t, f(z^{t-1}, \varepsilon^{t-1}, \chi_{-1}))$ with $f = g_x \circ \ldots \circ g_x$

- There exists a filtered probability space $L(\mathbf{P}, \Sigma_{\mathbf{P}}, \mathbb{P})$ containing the random variables (ε', χ).
- Identify agent-specific quantities with $\omega \in \mathbf{P}$, i.e., $\varepsilon' = \varepsilon(\omega)$ and $x = \chi(\omega)$
- **Rewrite recursive equilibrium:**

$$h_x(z', \omega, m') = g_x(z', \varepsilon(\omega), \chi(\omega), \mu')$$

  where $m' = (\varepsilon', \chi) \sim \mu'$
- Policies are now random fields (generalized stochastic processes) with the index $m' \Rightarrow$ Hilbert space C

Equilibrium Operators

- Simplifying the continuum of Euler equations with the Euler equation operator $\mathbf{T} : C \to C$ with

$$\mathbf{T}^j[h] = -\, c'[h]^{-\gamma} + \mathbb{E}^{(z'',\varepsilon''|z',\varepsilon')}\left[\beta(1+R^j[h])c''[h]^{-\gamma}\right]$$

$$c'[h] = e' + \frac{\partial}{\partial L}F(L', \mathbb{E}[\chi^k])l' + (1+h_{R^k})\chi^k + (1+h_{R^b})\chi^b - h_k - h_b$$

$$h_{R^k} = \frac{\partial}{\partial K}F(L', \mathbb{E}[\chi^k]) - \delta$$

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$$R^j[h] = h_{R^j} \circ h_x$$

- Bond market clearing operator $\mathbf{B} : C \to C$ with

$$\mathbf{B}[h] = -\int h_b d\mathbb{P}(\omega)$$

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$$\mathbf{B}[h] = -\int h_b d\mathbb{P}(\omega)$$

Outline of the talk

Existence and Uniqueness

- **Main idea:** Use a theorem for existence of a root

Rockafellar (1969): If $\mathbf{M}$ is a maximal monotone operator and there exists a subset $B \subset D(\mathbf{M})$ such that $0 \in \operatorname{int}(\operatorname{conv}(\mathbf{M}(B)))$, then there exists a $c \in \mathcal{C}$ such that $0 \in \mathbf{M}(c)$.

- Generalizes the intermediate value theorem: There exists a root of a continuous real function f if there are two points $b_1, b_2 \in \mathbb{R}$ with $f(b_1) > 0$ and $f(b_2) < 0$.
- **Uniqueness:** follows from strict monotonicity
- **Strategy:** Show maximal (strict) monotonicity of

$$\mathbf{M}[h, y] = \left\{ \begin{bmatrix} \mathbf{T}[h] + y \\ \mathbf{B}[h] \end{bmatrix}, (h_x - \bar{x}) \perp y \geq 0 \right\}$$

First: Model without Borrowing Constraints

- **Unconstrained case:** Maximal monotonicity of $\mathbf{M}$ requires various assumptions:

1. square-integrability of exogenous shocks, e, l, the index set m'
2. Admissible set $h \in \mathcal{H}$:
    - positive consumption and returns
    - consumption increases on average with higher cash on hand
    - bond return decreases with higher aggregate bond demand
    - continuity and differentiability in the index m'
    - concavity of consumption and convexity of bond return in m'
    - transversality condition

Second: Attaching the Borrowing Constraints

- There exists an objective function which is optimized at the root of the maximal monotone operator $\mathbf{M}$

$$L_{\mathbf{M}}(h) = \sup_{g \in \mathcal{H}} \langle \mathbf{M}[g], h - g \rangle$$

- Attaching the borrowing constraint

$$L(h, y) = L_{\mathbf{M}}(h) + \langle \bar{x} - h, y \rangle$$

  preserves maximal monotonicity for the first-order condition.
- Interpretation of the Lagrangian: Benevolent social planner

Outline of the talk

The Iterative Algorithm

- We want a policy updating rule $h^n = F(h^{n-1})$ such that $h^n \xrightarrow{n\to\infty} h$ with $\mathbf{M}[h] = 0$, where $\mathbf{M}$ denotes the equilibrium equation
- Standard value function iteration is not guaranteed to converge
- Monotonicity guarantees convergence of a **modified iteration**:

$$\mathbf{M}[h] = 0 \Leftrightarrow \mathbf{M}[h] + h = h \Leftrightarrow (\mathbf{M} + \mathbf{Id})[h] = h \Leftrightarrow h = (\mathbf{M} + \mathbf{Id})^{-1}[h],$$

⇒ Iterating on the resolvent is equivalent to damped fixed-point iteration and thus, $h^{n+1} = (\mathbf{M} + \mathbf{Id})^{-1}[h^n]$ converges.

- The resolvent can be computed by maximizing the Lagrangian augmented by some regularization terms

Outline of the talk

Capital Only: The Aiyagari-Bewley Model

- **Model specifics:**
  - Aggregate risk affects productivity, idiosyncratic risk the employment status
  - Wage and rental rate follow from a Cobb-Douglas production function
  - Exogenous labor supply features unemployment insurance

$$l(z, \varepsilon) = \varepsilon\pi(1-\tau) + (1-\varepsilon)\nu$$

  - Borrowing constraint $k \geq 0$
- **Sufficient conditions:**
  (i) $\gamma \geq 1$ and $\beta(1-\delta)^{1-\gamma} < 1$, OR,
  (ii) $\gamma < 1$, minimum productivity such that

$$\frac{F\left(z''_{\text{min}}, 1\right)}{\max_{z'} \mathbb{E}^{(z''|z')} F\left(z'', 1\right)} > \alpha \frac{\delta\beta^{\frac{1}{1-\gamma}}}{\delta\beta^{\frac{1}{1-\gamma}} + \left(1 - \beta^{\frac{1}{1-\gamma}}\right)}.$$

  and minimum initial capital $K_0 > K_{\text{min}}$ with K_{min} given by

$$\left(1 + \max_{z'} \mathbb{E}^{(z''|z')} R\left(z'', K_{\text{min}}\right)\right)^{1-\gamma} = \frac{1}{\beta}$$

Bonds Only: The Huggett Economy

- **Model specifics:**
  - No firm/capital, only exogenous endowments
  - Aggregate and idiosyncratic risk affect the endowments
  - Borrowing constraint $b \geq \bar{b}$
- **Sufficient conditions:** Minimum and maximum average endowment

$$0 \leq \min_z \mathbb{E}\left[e(\varepsilon, z)|z\right] \leq \max_z \mathbb{E}\left[e(\varepsilon, z)|z\right] < \infty$$

Conclusions

- Describe the cross-sectional distribution and its law of motion in terms of random variables
- Reduce the continuum of Euler equation to an operator by substituting the individual security holdings with the corresponding random variable
- Use results from convex analysis to derive existence and uniqueness

Thank you for your attention!