Dynamic One-Sided Matching

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Introduction

- Most of real-life matching markets are dynamic.
- Allocation and exchange of indivisible objects in a **dynamic** model, in which agents and objects arrive over time.
- Objects can be private or collectively owned.
- Three important features of our dynamic setting:
  1. Agents face inter-temporal trade-offs.
  2. Consumption and exchange are different choices and, while consumption is irreversible, agents may exchange repeatedly.
  3. Agents make conjectures about how the market may evolve in the future.

The Dynamic Core

- Our primary interest is to propose a notion of stability, the *Dynamic Core*.
- The Dynamic Core builds over the notion of period-t blocking.
- The Dynamic Core has a recursive structure that exhibits three properties.
  1. *strong perfection*: agents in the blocking coalition restrict the set of future allocations that are taken into consideration to those that will be unblocked.
  2. *farsightedness*: agents can trade an object they own for one they like less in order to trade again in the future;
  3. *caution*: only alternatives which give an improvement with certainty are eligible for the blocking coalition.

Main Results (This presentation)

- The Dynamic Core is not empty.
- The Dynamic Core and the Core are unrelated.
- In private (dynamic) economies, the Strong Core is essentially unique and has a non-empty intersection with the Dynamic Core.
- The output of the Intertemporal Top Trading Cycle (ITTC) is in the Dynamic Core.
- The ITTC mechanism is Pareto efficient and group strategy-proof and its output can be supported as a (dynamic) competitive equilibrium.

Literature Review

- Efficient mechanisms in one-sided dynamic matching markets (Unver 2010, Gershkov and Moldovanou 2010, Kurino 2014, Bloch and Cantala 2017, Andersson et a. 2018, Baccara and Yariv 2020, Leshno, 2022).
- Abdulkadiroglu and Loertscher (2007) study a dynamic housing allocation problem over two periods and characterize the set of equilibria of this game.
- Kadam and Kotowski (2018), Kotowski (2019), Liu (2020) and Doval (2022) provide a recursive notion of dynamic stability for two-sided markets with similar perfection requirements than our solution.
- Pereyra (2013) and Kennes (2014) propose solution concepts for dynamic school choice models.

Literature Review

- Our work is also related to the literature on farsighted one-sided matching (Kamijo and Kawasaki (2009), Kawasaki (2010), Klaus et al. (2010), Atay et al. (2022)).

- Our work contributes to the analysis of Core notions in dynamic exchange economies (Gale 1978, Becker 1995).

Example: a 2-period private economy

- Each agent enters endowed with an object.
- Agents 1,2,3 enter at period 1, agent 4 enters at period 2. Preferences are depicted below.

$t=1$			$t=2$
a_1	a_2	a_3	a_4
$h_2, 1$	$h_3, 1$	$h_4, 2$	$h_2, 2$
$h_4, 2$	$h_1, 1$	$h_1, 1$	$h_4, 2$
$h_1, 1$	$h_2, 1$	$h_1, 2$	$\vdots$
$\vdots$	$\vdots$	$h_3, 1$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- Consider the allocation in which agent 1 and 2 exchange their endowment and consume in period 1. Agents 3 and 4 consume their endowment as soon as enter in the market.
- Is this allocation *in the dynamic core*?

Example: a 2-period private economy

- Consider a blocking coalition formed in period 1 by a_2 and a_3 who exchange their endowment.
- Agent 2 consumes h_3 in period 1 while agent 3 (with her new endowment h_2) remains in the market with the aim of exchanging in the next period with a_4.

$t=1$			$t=2$
a_1	a_2	a_3	a_4
$h_2, 1$	$h_3, 1$	$h_4, 2$	$h_2, 2$
$h_4, 2$	$h_1, 1$	$h_1, 1$	$h_4, 2$
$h_1, 1$	$h_2, 1$	$h_1, 2$	$\vdots$
$\vdots$	$\vdots$	$h_3, 1$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- Agent 3 should form conjectures on what will occur in period 2.
- What exchanges will take place in period 2 also depends on a_1's decisions in period 1.

Primitives

- A dynamic one-sided market with indivisible objects; agents and objects arrive over time.
- $n \in \mathbb{N}_+$ **periods**. For any period $t < n$,
  - $A_t \equiv \{a_1, ..., a_m\}$ the set of **agents** and $H_t \equiv \{h_1, ..., h_\ell\}$the set of **objects** entering at t .
  - $A_{\leq t} \equiv \bigcup_{k=1}^{t} A_k \left(H_{\leq t} \equiv \bigcup_{k=1}^{t} H_k \right)$ agents (objects) arrived up to period t; thus, $A \equiv A_{\leq n}$ and $H \equiv H_{\leq n}$ are the entire sets of agents and objects
  - $A_{>t} \equiv A \setminus A_{\leq t} \left(H_{>t} \equiv H \setminus H_{\leq t} \right)$ agents (objects) entering in the market from period $t + 1$ onward.
- A **coalition** S is any non-empty subset of agents.
- A pair (h, t) is **feasible for the agent** a if the agent a and the object h are in the market in period t.
- Each agent $a \in A$ has a **strict preference relation** $\succ_a$over the feasible pairs (h, t).

Ownership Structure

- Objects can be owned either by a single agent or by the entire society.
- An object owned by a single agent is called **private** and **common** otherwise. An **ownership structure** establishes who are the owner(s) of an object when it enters the market.

Definition

An **ownership structure** is a map $\omega : H \longrightarrow A \cup \{A\}$ satisfying two properties:

Single Private Object: for all $h, h' \in H$,

$$\omega(h) = \omega(h') \Longrightarrow \omega(h) = A$$

Synchronous Entry: for all $t \leq n$, if $h \in H_t$ and $\omega(h) \neq A$ then

$$\omega(h) \in A_t.$$

A Dynamic Economy

An **economy** is, thus, a tuple

$$\mathcal{E} = \left\langle (A_t, H_t)_{t=1}^{n}, \omega \right\rangle$$

consisting of a collection of agents and objects entering over time together with an ownership structure.

Exchanges

- Objects can be exchanged in more than one periods.

Definition

A **period-t exchange** is a map $\sigma_t : H_{\leq t} \longrightarrow A_{\leq t} \cup \{A\}$ such that

- for all $h, h' \in H$, $\sigma_t(h) = \sigma_t(h') \Longrightarrow \sigma_t(h) = A$

The condition restates the *Single Private Object* property, that is, at the end of any period t, each agent owns at most one private object. $\sigma_t(h)$ specifies the agent (or the society) owning the object h in period t.

- For convention $\sigma_0 = \omega$ and for all $h \in H_{>t}$, $\sigma_t(h) = \omega(h)$. Let Σ_t be the set of all period-t exchanges.

Consumption

- If agent a consumes object h in period t we write $\mu_t(a) = h$;
- If agent a does not consume any object in period t then we write $\mu_t(a) = h_0$.

Definition

A **period-t consumption choice** is a map $\mu_t : A_{\leq t} \longrightarrow H_{\leq t} \cup \{h_0\}$.

We also write $\mu_t(a) = h_0$ whether $t = 0$, or $a \in A_{>t}$. Let M_t be the set of all period-t consumption choices.

Allocations

Definition (Allocation)

Given an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$, an **allocation** is a pair (σ, μ) where $\sigma \equiv (\sigma_1, ..., \sigma_n)$ is a list of period-t exchanges and $\mu \equiv (\mu_1, ..., \mu_n)$ a list of period-t consumption choices such that for all $a \in A_{\leq t}$, if $\mu_t(a) \neq h_0$ then the following conditions hold:

Consumption Rivalry: $\sigma_t \circ \mu_t(a) = a$;

Consumption Irreversibility: for all $t' > t$, $\sigma_{t'} \circ \mu_t(a) = a$ and $\mu_{t'}(a) = \mu_t(a)$.

- The first condition requires that agents can only consume the private object they own, thus objects, private and common, are rivalrous.
- The second condition establishes that consumption is irreversible.

Given the allocation (σ, μ), we write $\mu(a)$ the object that agent a consumes and $t(a, \mu)$ the period when it is consumed by a.

Blocking in a dynamic economy

- Several aspects that are well-defined in a static setting, must be clarified in a dynamic framework.

1. Which coalitions of agents can block at a given period?
2. Which objects a blocking coalition can redistribute among its members?
3. Which are the final consequences of blocking an allocation?

Blocking in a dynamic economy

1. Only agents who are already present in the market in period t, and did not consume yet, can form a blocking coalition: blocking coalition S must belong to $\mathcal{A}_t \equiv \{a \in A_{\leq t} | \mu_{t-1}(a) = h_0\}$.
2. Let $\mathcal{H}_t \equiv \{h \in H_{\leq t} | h \neq \mu_{t-1}(a), \ \forall a \in A_{\leq t}\}$ denote the set of objects that are in the market in period t and have not been previously consumed.

Definition

A **period-t endowment** $\omega_t : 2^{\mathcal{A}_t} \longrightarrow 2^{\mathcal{H}_t}$ is a map such that for all $S \in 2^{A_{\leq t}} \setminus \emptyset$,

$$\omega_t(S) \equiv \begin{cases} \mathcal{H}_t & \text{if } S = \mathcal{A}_t, \\ \{h \in \mathcal{H}_t | \sigma_{t-1}(h) \in S\} & \text{if } S \subsetneq \mathcal{A}_t. \end{cases}$$

Blocking in a dynamic economy

- At any period t, a coalition $S \subseteq \mathcal{A}_t$ can block an allocation (σ, μ) by proposing a period-t exchange τ_t and a period-t consumption choice ν_t.
- The set of admissible period-t exchanges for S consists of

$$\Sigma_t(S) \equiv \left\{ \tau_t \in \Sigma_t \;\middle|\; \begin{array}{ll} \tau_t(h) \in S \vee \tau_t(h) = A, & \forall h \in \omega_t(S), \\ \tau_t(h) = \sigma_{t-1}(h), & \forall h \in H_{\leq t} \setminus \omega_t(S). \end{array} \right\}$$

- The set of admissible period-t consumption choice for S is restricted to

$$M_t(S) \equiv \{\nu_t \in M_t | \nu_t(a) = \mu_{t-1}(a),\ \forall a \in A_{\leq t} \setminus S\}.$$

Continuation Economy

- Agents can form a blocking coalition without necessarily consuming the objects that they get at the period they block, in order to further exchange them in the future.
- It follows that blocking agents should form conjectures about which will be the final consequences of their block.
- This will depend not only on how contemporary agents outside the coalition will react, but also on the behavior of those agents who enter in the market in the following periods.
- The notion of continuation economy frames this idea.

Let $S^w \equiv \{a \in s | \nu_t(a) = h_0\}$ be the agents in the blocking coalition S who stay in the market without consuming. A **continuation economy** $\mathcal{E}_{\geq t}(S, \tau_t, \nu_t)$ consists of

$$\left\langle (\mathcal{A}_t \backslash S, \mathcal{H}_t \backslash \omega_t(S), (A_{t+1} \cup S^w, H_{t+1} \cup \omega_t(S) \backslash \nu_t(S)), ..., (A_n, H_n), \tau_t|_{H_{\leq t} \backslash \nu_t(A_{\leq t})}\right)$$

The Dynamic Core

Definition (Period-t Blocking)

Let (σ, μ) be an allocation of the economy $\mathcal{E} = \langle (A_t, H_t,)_{t=1}^n, \omega \rangle$. A coalition $S \subseteq \mathcal{A}_t$ can **period-t block** (σ, μ) if there exists a pair $(\nu_t, \tau_t) \in \Sigma_t(S) \times M_t(S)$ such that:

- $(\nu_t(a), t) \succ_a (\sigma, \mu)$ for all $a \in S \setminus S^w$
- $(\upsilon, \xi) \succ_a (\sigma, \mu)$ for all $a \in S^w$ and all allocations (υ, ξ) of the economy $\mathcal{E}_{\geq t}$ that cannot be period-t' blocked at any $t' \geq t$.

The **Dynamic Core** of an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ is the set of allocations that cannot be period-t blocked in any period, by any coalition.

Example

A 2-period private economy with 4 agents and 4 objects.

$t=1$			$t=2$
a_1	a_2	a_3	a_4
$h_2, 1$	$h_3, 1$	$h_4, 2$	$h_2, 2$
$h_4, 2$	$h_1, 1$	$h_1, 1$	$h_4, 2$
$h_1, 1$	$h_2, 1$	$h_1, 2$	$\vdots$
$\vdots$	$\vdots$	$h_3, 1$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure: Agents' preferences; in red the outcome of the allocation (σ, μ)

$$(\sigma,\mu) = \left(\begin{array}{cc} a_1 = \sigma_1(h_2) & \mu_1(a_1) = h_2 \\ a_2 = \sigma_1(h_1) & \mu_1(a_2) = h_1 \\ a_3 = \sigma_1(h_3) & \mu_1(a_3) = h_3 \\ \hline a_4 = \sigma_2(h_4) & \mu_2(a_4) = h_4 \end{array} \right)$$

An allocation NOT in the Dynamic Core

- The allocation (σ, μ) does not belong to the Dynamic Core.
- Consider a blocking by the coalition $\{a_2, a_3\}$ such that:

$$(\tau_1, \nu_1) = \left(\begin{array}{ll} a_1 = \tau_1(h_1) & \nu_1(a_1) = h_0 \\ a_2 = \tau_1(h_3) & \nu_1(a_2) = h_3 \\ a_3 = \tau_1(h_2) & \nu_1(a_3) = h_0 \end{array} \right)$$

The continuation economy $\mathcal{E}_{\geq t}(\{a_2, a_3\}, \tau_1, \nu_1)$ generated by this blocking consists of

$$\Big\langle (\{a_1\}, \{h_1\}), (\{a_3, a_4\}, \{h_2, h_4\}), \tau_1(h_1) = a_1, \tau_1(h_2) = a_3, \tau_1(h_4) = a_4, \Big\rangle$$

- The continuation economy $\mathcal{E}_{\geq t}(\{a_2, a_3\}, \tau_1, \nu_1)$ contains two allocations that are not blocked in any period $t' \geq 1$.
- a_1 gets $(h_1, 1)$, a_3 gets $(h_4, 2)$, and a_4 gets $(h_2, 2)$;
- a_1 gets $(h_4, 2)$, a_3 gets $(h_1, 2)$, and a_4 gets $(h_2, 2)$.
- Agent a_3 prefers what she gets in both allocations to consuming $(h_3, 1)$.

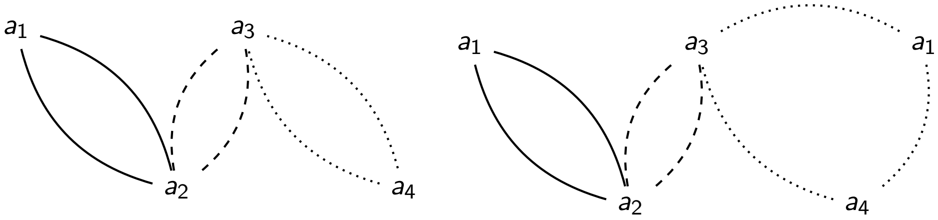

Figure: In bold the exchange between $\{a_1, a_2\}$ according to (σ, μ); in dashed the exchange performed by $\{a_2, a_3\}$ according to (τ_1, ν_1); in dotted the two unblocked exchanges that can take place in period 2.

An allocation in the Dynamic Core

- Consider the following allocation:

$$(\sigma', \mu') = \left(\begin{array}{cc} a_1 = \sigma'_1(h_1) & \mu'_1(a_1) = h_1 \\ a_2 = \sigma'_1(h_3) & \mu'_1(a_2) = h_3 \\ a_3 = \sigma'_1(h_2) & \mu'_1(a_3) = h_0 \\ \hline a_3 = \sigma'_1(h_4) & \mu'_1(a_3) = h_4 \\ a_4 = \sigma'_2(h_2) & \mu'_2(a_4) = h_2 \end{array} \right)$$

- In period 1 a_1 consumes her endowment, a_2 and a_3 exchange, a_2 consumes and a_3 remains in the market; in period 2 a_3 exchanges with a_4 and they consume.
- Every agent consumes her most preferred object with the exception of a_1.
- a_1 can block in period 1 by remaining in the market.

The continuation economy $\mathcal{E}_{\geq t} = (\{a_1\}, \tau_1', \nu_1')$ consists of

$$\Big\langle (\{a_2, a_3\}, \{h_2, h_3\}), (\{a_1, a_4\}, \{h_1, h_4\}), \\ \tau_1'(h_1) = a_1, \tau_1'(h_2) = a_2, \tau_3'(h_3) = a_3, \tau_1'(h_4) = a_4 \Big\rangle.$$

$\mathcal{E}_{\geq t}(\{a_1\}, \tau_1', \nu_1')$ has two unblocked allocations; in both a_2 gets $(h_3, 1)$, and a_4 gets $(h_2, 2)$:

1. a_1 gets $(h_4, 2)$ and a_3 gets $(h_1, 2)$.
2. a_1 gets $(h_1, 2)$ and a_3 gets $(h_4, 2)$

For agent a_1, $(h_1, 2)$ is a worst outcome than $(h_1, 1)$, therefore remaining in the market is not a period-1 blocking for the coalition $\{a_1\}$.

Rational Expectations

- Our solution has similarities with the perfect α-Core in Kotowski (2019) and dynamic stability in Doval (2022) defined for two-sided matching.
- An essential characteristic of our stability notion is that the continuation economy starts from the same period of the blocking and the deviating agents form expectations already starting over the same period they block.

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$t=1$		$t=2$
a_1	a_2	a_3
$h_3, 2$	$h_2, 1$	$h_2, 2$
$h_1, 1$	$h_3, 2$	$h_1, 2$
$\vdots$	$h_2, 2$	$h_3, 2$

Relation with the Core and Strong Core

- Assume a pre-stage game in which all the agents can make binding agreements (analogous to dynamic matching with contracts)
- Suppose that (σ, μ) is an allocation in the pre-stage game. Then, (σ, μ) is **blocked** if there is a coalition $S \subseteq A$ and an allocation (τ, ν) such that all agents in S strictly prefers (τ, ν) over (σ, μ).
- The **Core** of an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^{n}, \omega \rangle$ consists of the set of all allocations that cannot be blocked.
- The Dynamic Core and the Core are unrelated.

Relation with the Core (I)

$t=1$		$t=2$	
a_1	a_2	a_3	a_4
$h_3, 2$	$h_3, 2$	$h_4, 2$	$h_1, 2$
$h_2, 1$	$h_1, 1$	$h_3, 2$	$h_2, 2$
$\vdots$	$\vdots$	$\vdots$	$h_4, 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$(\sigma, \mu) = \left(\begin{array}{cc} a_1 = \sigma_1(h_2) & \mu_1(a_1) = h_2 \\ a_2 = \sigma_1(h_1) & \mu_1(a_2) = h_1 \\ \hline a_3 = \sigma_2(h_3) & \mu_2(a_3) = h_3 \\ a_4 = \sigma_2(h_4) & \mu_2(a_4) = h_4 \end{array} \right)$$

An allocation in the Dynamic Core that is not in the Core.

Relation with the Core (II)

$t=1$	$t=2$	
a_1	a_2	a_3
$h_3, 2$	$h_1, 2$	$h_2, 2$
$h_1, 1$	$h_3, 2$	$h_1, 2$
$h_2, 2$	$h_2, 2$	$h_3, 2$
$h_1, 2$	$\vdots$	$\vdots$

$$(\sigma, \mu) = \left(\begin{array}{cc} a_1 = \sigma_1(h_1) & \mu_1(a_1) = h_1 \\ \hline a_2 = \sigma_2(h_3) & \mu_2(a_2) = h_3 \\ a_3 = \sigma_2(h_2) & \mu_2(a_3) = h_2 \end{array} \right)$$

An allocation in the Core that is not in the Dynamic Core

- In private economies in any Strong Core allocation the consumption choice is the same for every agents.

Theorem

Let $\mathcal{E} = \langle (A_t, H_t)_{t=1}^{n}, \omega \rangle$ be a private economy. Then, the Strong Core of $\mathcal{E}$ is essentially unique.

Theorem

In private economies, there exists a Strong Core allocation which is in the Dynamic Core.

The Dynamic Core and the Strong Core

$t=1$			$t=2$
a_1	a_2	a_3	a_4
$h_3, 1$	$h_3, 1$	$h_4, 2$	$h_1, 2$
$h_1, 1$	$h_2, 1$	$\vdots$	$h_2, 2$
$\vdots$	$\vdots$	$\vdots$	$h_3, 2$
$\vdots$	$\vdots$	$\vdots$	$h_4, 2$

$$\left(\begin{array}{ll} a_1 = \sigma_1(h_3) & \mu_1(a_1) = h_3 \\ a_2 = \sigma_1(h_2) & \mu_1(a_2) = h_2 \\ a_3 = \sigma_1(h_1) & \mu_1(a_3) = h_0 \\ \hline a_3 = \sigma_2(h_4) & \mu_2(a_3) = h_4 \\ a_4 = \sigma_2(h_1) & \mu_2(a_4) = h_1 \end{array}\right) \qquad \left(\begin{array}{ll} a_1 = \sigma'_1(h_1) & \mu'_1(a_1) = h_1 \\ a_2 = \sigma'_1(h_3) & \mu'_1(a_2) = h_3 \\ a_3 = \sigma'_1(h_2) & \mu'_1(a_3) = h_0 \\ \hline a_3 = \sigma'_2(h_4) & \mu'_2(a_3) = h_4 \\ a_4 = \sigma'_2(h_2) & \mu'_2(a_4) = h_2 \end{array}\right)$$

Both allocations are in the Dynamic Core, but only the allocation on the left is in the Strong Core.

The Intertemporal Top-Trading Cycle

- We provide an extension of the Top-Trading Cycle, the **Intertemporal Top Trading Cycle** (ITTC) that identifies an allocation in the Dynamic Core.
- To provide an intuition on how the ITTC algorithm works consider a 2-period private economy.

Informal description of the ITTC (in private economies)

- **Pointing:** Each agent points to the object belonging to her preferred feasible pair and each object points to its owner.
- **Clearing:** Consider any cycle composed only by agents and objects that are present at time $t = 1$, and perform the exchanges accordingly; every agent consumes at time $t = 1$ the object she points. Remove agents and objects.
- **Trimming and Clearing:** Consider any cycle involving some agents (and objects) that enter in period 1 and some that enter in period 2. Identify each chain starting from an object entered in period 1 that ends with an agent who points to an object that enters in period 2. Perform the exchanges along the chain accordingly, and assign the first object of the chain to the last agent in the chain. All agents in the chain except the last one consume at $t = 1$ the object they point. Remove all agents who consume and objects that are consumed.

Informal description of the ITTC

- Repeat this procedure until it exhausts all exchanges performed in period 1, that is until every agent entered in period 1 either consumes at $t = 1$ or points to an object that enters in period 2.
- Every agent who is still present in period 2 points to her preferred remaining object, and each object points to its current owner. Any time a cycle is formed, an agent consumes at time $t = 2$ the object is pointing.
- The outcome of the ITTC algorithm is an allocation in the Dynamic Core.

Theorem

Let $\mathcal{E} = \langle (A_t, H_t)_{t=1}^{n}, \omega \rangle$ be an economy. Then, the output of any ITTC is an allocation in the Dynamic Core of the economy $\mathcal{E}$.

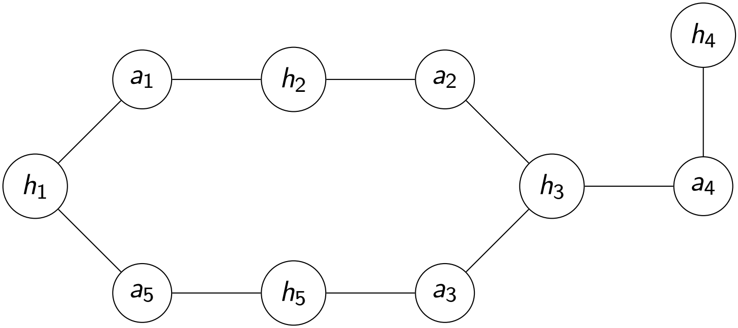
Example

Consider the following private 2-period economy with 5 agents and 5 objects.

$t = 1$			$t = 2$	
a_1	a_2	a_3	a_4	a_5
$h_2, 1$	$h_3, 1$	$h_5, 2$	$h_3, 2$	$h_1, 2$
$h_4, 2$	$h_1, 1$	$h_2, 1$	$h_4, 2$	$h_5, 2$
$h_1, 1$	$h_2, 1$	$h_3, 1$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

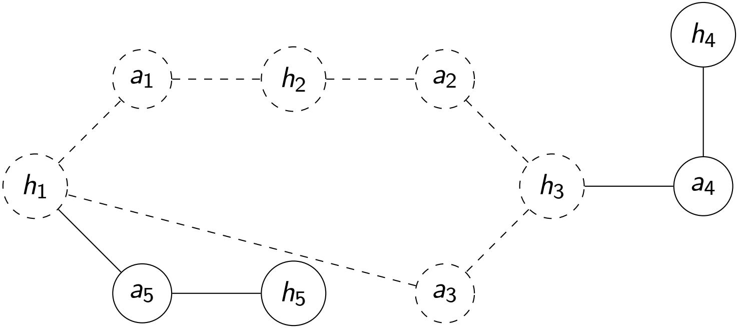
ITTC

- Each object points to its owner and each agents points her most preferred object.
- Notice that there is an inter-temporal cycle $(h_1, a_1, h_2, a_2, h_3, a_3, h_5, a_5)$



ITTC

- Trim and clear the cycle $(a_1, h_2, a_2.h_3, a_3, a_5, h_1)$

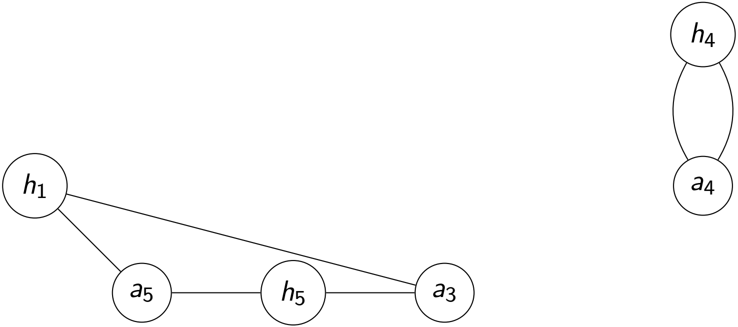


- This results in the following "partial assignment" describing the exchanges and consumption choices in period 1:

$$(\sigma_1, \mu_1) = \left(\begin{array}{ll} a_1 = \sigma_1(h_2) & \mu_1(a_1) = h_2 \\ a_2 = \sigma_1(h_3) & \mu_1(a_2) = h_3 \\ a_3 = \sigma_1(h_1) & \mu_1(a_3) = h_0 \end{array} \right)$$

ITTC

The following two cycles are formed in the following steps of the algorithm.



They illustrate exchanges and consumption choices in period 2:

$$(\sigma_2, \mu_2) = \left(\begin{array}{ll} a_3 = \sigma_2(h_5) & \mu_2(a_3) = h_5 \\ a_4 = \sigma_2(h_4) & \mu_2(a_4) = h_4 \\ a_5 = \sigma_2(h_1) & \mu_2(a_5) = h_1 \end{array} \right)$$

Properties of the ITTC

Theorem

The ITTC mechanism is Pareto efficient.

Theorem

The ITTC mechanism is group strategy-proof.

Properties of the ITTC

Let $p \in \mathbb{R}_+^{|H|}$ denote a price vector and p_h the price of the object h with $p_\emptyset = 0$.

Definition

Given a private economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ an allocation (σ, μ) can be supported as dynamic competitive equilibrium for a profile $\succ \in \mathcal{L}$ if there exists a price vector p such that for all $a \in A$ the following conditions hold:

1. $p_{\sigma_t^{-1}(a)} \leq p_{\sigma_{t-1}^{-1}(a)}$ for all $t \in \{1, ..., n\}$
2. if $(\tau, \nu) \succ_a (\sigma, \mu)$ then $p_{\tau_t^{-1}(a)} > p_{\tau_{t-1}^{-1}(a)}$ for some $t \in \{1, ..., n\}$

Theorem

In private economies, the output of the ITTC can be supported as a dynamic competitive equilibrium.

Conclusions

- We provide a novel solution concept, the Dynamic Core, for dynamic one-sided matching models in which agents and object arrive over time, and objects are either privately or collectively owned.
- The Dynamic Core and the Core are unrelated solution concepts
- In private economies the Dynamic Core and the Strong Core has a non empty intersection.
- We present a dynamic version of the Gale's TTC mechanism, named the Intertemporal Top-Trading Cycle (ITTC). The ITTC identifies an allocation in the Dynamic Core at every preference profile, it is Pareto efficient, and group strategy proof. For private economies, its outcome can be supported as a dynamic competitive equilibrium.

Conclusions

- A real-life problem in which our model can be applied is kidney transplantation. The problem has an inherently dynamic structure and the ownership structure resembles the one of the model.
- New patients and organs continuously arrive over time, and often patients waiting for a transplant face intertemporal tradeoffs having to decide whether to accept the kidney for transplantation, or decline the offer and rejoin the candidate pool for a future reassignment.
- The ITTC algorithm can be used to incorporate recent proposals to merge allocation programs that allocate deceased donor organs with kidney exchange programs (Sönmez, Ünver, and Yenmez, 2020).