The Decision-Conflict Logit

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Outline

Baseline **Luce/logit** models:

$$\rho(a,A) = \frac{u(a)}{\sum_{b\in A} u(b)} \qquad \text{(no outside option)}$$

$$\rho(a,A) = \frac{u(a)}{\sum_{b\in A} u(b) + u(o)}, \quad o \notin A \qquad \text{(with outside option)}$$

This paper:

$$\rho(a,A) = \frac{u(a)}{\sum_{b\in A} u(b) + D(A)}$$

- Main focus: D depends on u
  - Active-choices & deferrals when decisions “easy” & “hard”
- Simple & applicable extension of Luce (1959)

Notation

X: finite set of active-choice alternatives

$o \notin X$: deferral/outside option

$\mathcal{M} := \{A \neq \emptyset : A \subseteq X\}$: collection of menus

$\rho(a, A)$: choice probability of $a \in A \in \mathcal{M}$

ρ is a **random non-forced-choice** model:

$$\rho(o, A) \equiv 1 - \sum_{a \in A} \rho(a, A) \geq 0 \quad \text{for all } A \in \mathcal{M}$$

For $A \subset B$: $\rho(A, B) := \sum_{a \in A} \rho(a, B)$

General Model

Definition

ρ is a generalized Luce/logit model with an outside option *if there are* $u : X \to \mathbb{R}_{++}$ *and* $D : \mathcal{M} \to \mathbb{R}_+$ *s.t.*

$$\rho(a, A) \quad = \quad \frac{u(a)}{\sum_{b \in A} u(b) + D(A)} \tag{1}$$

and (u, D) is unique up to a common positive linear transformation

A1 (Positivity)
For all $a \in A \in \mathcal{M}$: $\rho(a, A) > 0$

A2 (Active-Choice Luce Axiom)
For all $A, B \in \mathcal{M}$ and all $a, b \in A \cap B$: $\dfrac{\rho(a, A)}{\rho(b, A)} = \dfrac{\rho(a, B)}{\rho(b, B)}$ $\left(\textit{allows for } \frac{\rho(o,A)}{\rho(b,A)} \neq \frac{\rho(o,B)}{\rho(b,B)}\right)$

Proposition

Any ρ can be written as in (1) *iff it satisfies A1–A2.*

General Model

Special cases:

- $D(A) \equiv 0$: standard model *without* outside option
- $D(A) \equiv \overline{v} > 0$: standard model *with* outside option
- $D(A) = 0 \Leftrightarrow |A| = 1$: **decision-conflict logit (DCL)**
  - a problem has *some* degree of difficulty iff ≥ 2 alternatives

A3 (Desirability & Complexity)
For all $A \in \mathcal{M}: \quad \rho(A, A) = 1 \iff |A| = 1$

Corollary

ρ is a DCL iff it satisfies A1–A3.

General Model: Argument

A1, **A2** & basic adaptation of proof in Luce (1959) lead to $u : X \to \mathbb{R}_{++}$ s.t.

$$\rho(a, A) = \left(1 - \rho(o, A)\right) \frac{u(a)}{\sum_{b \in A} u(b)}, \quad (2)$$

where $u(a) := \dfrac{\rho(a, X)}{\rho(z, X)}$ for some fixed $z \in X$.

Now observe that (2) can be re-written as

$$\rho(a, A) = \frac{u(a)}{\sum_{b \in A} u(b) + D(A)} \quad (3)$$

for

$$D(A) := \frac{\rho(o, A)}{1 - \rho(o, A)} \sum_{b \in A} u(b),$$

with $\left(D(A) = 0 \Leftrightarrow |A| = 1\right)$ iff **A3** also holds. ■

First Special Case: Quadratic Logit

ρ is a **quadratic logit** if $\exists\ \widehat{u} : X \to \mathbb{R}_{++}$ s.t.

$$\rho(a, A) = \left(\frac{\widehat{u}(a)}{\sum_{b \in A} \widehat{u}(b)} \right)^2$$

An interpretation:

- Hesitant DM info-samples from a menu twice, with noise
- Independent draws
- More likely to choose $a \in A$ when **both** draws favourable

Observation

A quadratic-logit $\rho = (\widehat{u})^2$ is the particular DCL (u, D) where:

$$u(a) := \widehat{u}(a)^2$$

$$D(\{a, b\}) := 2\widehat{u}(a)\widehat{u}(b)$$

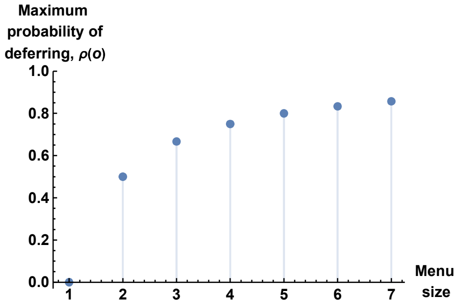
$$D(A) \equiv \sum_{\substack{a,b \in A, \\ a \neq b}} D(\{a, b\}) \qquad \text{(additivity)}$$

Quadratic Logit: Deferral Upper Bounds

Prediction

For every menu A:

$$\rho(o,A) \leq 1 - \frac{1}{|A|},$$
$$\rho(o,A) = 1 - \frac{1}{|A|} \iff \widehat{u}(a) = \widehat{u}(b) \quad \forall a,b \in A$$



Quadratic Logit: Dominance, Overload & No Overload

Prediction

Let $\rho = (\hat{u})^2 = (u, D)$ & $U(A) := \sum_{a \in A} u(a) \equiv \sum_{a \in A} \hat{u}(a)^2$.

Then, for menus $A \supset B$:

$$\rho(o, A) \leq \rho(o, B) \iff \underbrace{\frac{D(A) - D(B)}{D(B)}}_{\substack{\textit{marginal cost} \\ \textit{from menu expansion}}} \leq \underbrace{\frac{U(A) - U(B)}{U(B)}}_{\substack{\textit{marginal benefit} \\ \textit{from menu expansion}}}$$

$\longrightarrow$ **The quadratic logit *is not* a Random Utility Model**

Example:

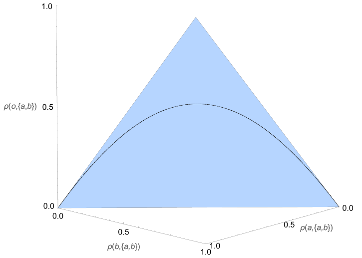
Option	$\hat{u}$	$\rho(\cdot, \{a, b\})$	$\rho(\cdot, \{a, b, c\})$	$\rho(\cdot, \{a, b, c, d\})$
a	10	0.980	0.250	0.007
b	0.1	0.001	0.000	0.000
c	9.9	–	0.245	0.007
d	100	–	–	0.694
o	–	**0.019**	$\nearrow$ **0.505** $\searrow$	**0.292**

$\longrightarrow$ Such **"roller-coasting"** of o in line with Chernev et al (2015) meta-analysis

Quadratic Logit: Binary-Menu Characterization

A4 (Symmetric Deferral Odds)
For all $a, b \in X$:

$$\frac{1}{2} \cdot \frac{\rho(o, \{a, b\})}{\rho(a, \{a, b\})} = \left(\frac{1}{2} \cdot \frac{\rho(o, \{a, b\})}{\rho(b, \{a, b\})} \right)^{-1}$$



Theorem

ρ on X is a quadratic logit on the binary menus of *X iff it satisfies A1–A4.*

Quadratic Logit: Argument

Step 1: Start with: $\rho = (u, D)$, $X := \{a_1, \dots, a_k\}$ & $u(a_i) := \frac{\rho(a_i,X)}{\rho(a_1,X)}$.

Step 2: By (u, D) & **A2**: ρ is quadratic-logit on binary menus $\Leftrightarrow \exists \left(\widehat{u}(a_1), \dots, \widehat{u}(a_k)\right)$ s.t.

$$\begin{pmatrix} \widehat{u}(a_1)^2 \\ \widehat{u}(a_2)^2 \\ \vdots \\ \widehat{u}(a_k)^2 \\ 2\widehat{u}(a_1)\widehat{u}(a_2) \\ 2\widehat{u}(a_1)\widehat{u}(a_3) \\ \vdots \\ 2\widehat{u}(a_1)\widehat{u}(a_k) \\ 2\widehat{u}(a_2)\widehat{u}(a_3) \\ \vdots \\ 2\widehat{u}(a_{k-1})\widehat{u}(a_k) \end{pmatrix} = \begin{pmatrix} u(a_1) \\ u(a_2) \\ \vdots \\ u(a_k) \\ D(\{a_1, a_2\}) \\ D(\{a_1, a_3\}) \\ \vdots \\ D(\{a_1, a_k\}) \\ D(\{a_2, a_3\}) \\ \vdots \\ D(\{a_{k-1}, a_k\}) \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \frac{\rho(a_2,X)}{\rho(a_1,X)} \\ \vdots \\ \frac{\rho(a_k,X)}{\rho(a_1,X)} \\ \frac{\rho(o,\{a_1,a_2\})}{\rho(a_1,\{a_1,a_2\})} \\ \frac{\rho(o,\{a_1,a_3\})}{\rho(a_1,\{a_1,a_3\})} \\ \vdots \\ \frac{\rho(o,\{a_1,a_3\})}{\rho(a_1,\{a_1,a_k\})} \\ \frac{\rho(o,\{a_2,a_3\})}{\rho(a_2,\{a_2,a_3\})} \frac{\rho(a_2,X)}{\rho(a_1,X)} \\ \vdots \\ \frac{\rho(o,\{a_{k-1},a_k\})}{\rho(a_{k-1},\{a_{k-1},a_k\})} \frac{\rho(a_{k-1},X)}{\rho(a_1,X)} \end{pmatrix} \quad (4)$$

Step 3:
Reduce the dimensionality of (4) by noting that $\widehat{u}(a_i) = \frac{1}{2}D(\{a_1, a_i\})$ for $1 \neq i \leq k$.

Step 4: Show that the reduced system is solved iff **A4** holds. ■

Quadratic Logit: Duopolistic-Game Application

Two firms, $i = 1, 2$, sell to **one consumer/homogeneous unit mass**

Single product differentiated in **quality**, q_i, and **price**, p_i

Cost: $c(q_i) = q_i$

Utility: $u(q_i, p_i) = \dfrac{q_i}{p_i}$ **Income:** $I > 0$

Simultaneous competition with complete information:

$$\max_{0 \le q_i \le p_i \le I} \left(\frac{\frac{q_i}{p_i}}{\frac{q_i}{p_i} + \frac{q_j}{p_j}} \right)^s \cdot (p_i - q_i),$$

where $s = 1$ for **logit** market shares & $s = 2$ for **quadratic-logit** ones

Utilitarian consumer welfare:

$$W\big((q_1, p_1), (q_2, p_2)\big) = \rho((q_1, p_1)) \cdot u(q_1, p_1) + \rho((q_2, p_2)) \cdot u(q_2, p_2)$$

Quadratic Logit: Duopolistic-Game Application

Proposition

Logit equilibrium:

$$(q_1^*, p_1^*) = (q_2^*, p_2^*) = \left(\frac{I}{3}, I\right)$$

Quadratic-logit equilibrium:

$$(q_1^{**}, p_1^{**}) = (q_2^{**}, p_2^{**}) = \left(\frac{I}{2}, I\right)$$

Equilibrium outcomes:

$$\rho^*(q_i, p_i) = \frac{1}{2}, \quad \rho^*(o) = 0, \quad \pi^* = \frac{I}{3}, \quad W^* = \frac{1}{3} \textit{ with certainty}$$

$$\rho^{**}(q_i, p_i) = \frac{1}{4}, \quad \rho^{**}(o) = \frac{1}{2}, \quad \pi^{**} = \frac{I}{8}, \quad W^{**} = \frac{1}{4} \textit{ in expectation}$$

Quadratic Logit: Discrete-Choice Estimation

Let $A = \{a_1, \ldots, a_k\}$ & modify the **Holman-Marley-McFadden** setup:

(1) Assume 2 rounds of sampling, with $u^l(a_i) = \beta \cdot x_i + \epsilon_i^l$ for $l = 1, 2$

(2) **Dominance:** $\rho(a_i, A) = Pr\big(u^l(a_i) \geq u^l(a_j)$ for all $j \leq k$ **&** $l \leq 2\big)$

(3) IID standard Gumbel noise across j ***and*** l

Observation

(1) − (3) *yield the* ***conditional quadratic logit***:

$$\rho(a_i, A) = \left(\frac{e^{\beta \cdot x_i}}{\sum_{j=1}^{k} e^{\beta \cdot x_j}} \right)^2 \qquad (4)$$

Estimation & evaluation:

1. Estimate β after dropping all o observations
2. Square all estimated choice probabilities & map residual to o
3. Find MSE of quadratic-logit estimates & compare with o-logit & o-nested-logit

Quadratic Logit: General Characterization

A5 (Balancing Odds).
For all $A, B \in \mathcal{M}$ with $B \supset A$, and for all $a, b \in A$:

$$\frac{\rho(o,A)}{1-\rho(o,A)} = \sum_{\substack{a,b\in A,\\ a\neq b}} \left(\frac{\rho(o,\{a,b\})}{1-\rho(o,\{a,b\})} \cdot \frac{\rho(\{a,b\},B)}{\rho(A,B)} \right)$$

Corollary

ρ on X is a quadratic logit iff it satisfies A1–A5.

Second Special Case: Reciprocal Utility Differences

Call any ρ *asymmetric* if $\rho(a, A) \neq \rho(b, A)$ for all $a, b \in A \subseteq X$

An asymmetric DCL ρ is a **reciprocal utility differences** model if

$$D(\{a,b\}) \quad = \quad \frac{\lambda}{|u(a) - u(b)|}$$

for some $\lambda > 0$ and all a, b

- Decision harder when alternatives similarly appealing, other things equal
- λ captures sensitivity to such decision difficulty
- Inspired by the ***Drift Diffusion Model***
  - Indeed, in the DDM we have $\mathbb{E}[DT_{a,b}] = \frac{\lambda}{|u(a)-u(b)|} \cdot \left(\frac{1-e^{\frac{\lambda}{\sigma^2}|u(a)-u(b)|}}{1+e^{\frac{\lambda}{\sigma^2}|u(a)-u(b)|}} \right)$
- However:
  - explicit outside option here
  - no response times or diffusion sampling
- u & λ pin down D, so can write $\rho = (u, D, \lambda) \equiv (u, \lambda)$

3. Reciprocal Utility Differences: Characterization & Uniqueness

A6 (Odds-Ratio Proportionality)
If $\rho(a,\{a,b\}) > \rho(b,\{a,b\})$, $\rho(b,\{b,c\}) > \rho(c,\{b,c\})$ & $\rho(a,\{a,c\}) > \rho(c,\{a,c\})$*:*

$$\frac{\frac{\rho(o,\{a,b\})}{\rho(a,\{a,b\})}}{\frac{\rho(o,\{b,c\})}{\rho(b,\{b,c\})}} = \frac{\frac{\rho(o,\{a,b\})}{\rho(a,\{a,b\})} - \frac{\rho(o,\{a,c\})}{\rho(a,\{a,c\})}}{\frac{\rho(o,\{a,c\})}{\rho(a,\{a,c\})}} \qquad \text{and} \qquad \frac{\frac{\rho(o,\{a,b\})}{\rho(a,\{a,b\})}}{\frac{\rho(o,\{a,c\})}{\rho(a,\{a,c\})}} = \frac{\frac{\rho(a,\{a,c\})-\rho(c,\{a,c\})}{\rho(a,\{a,c\})}}{\frac{\rho(a,\{a,b\})-\rho(b,\{a,b\})}{\rho(a,\{a,b\})}}$$

Theorem

An asymmetric DCL ρ *on* X *is a* (u,λ)*-model* on the binary menus *of* X *iff it satisfies A6.*

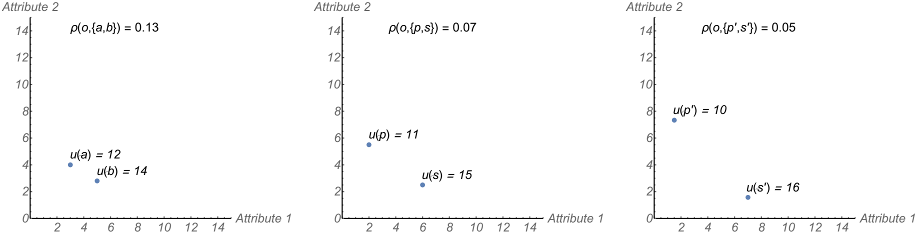
Moreover, any (u,λ) *representation of such* ρ *is unique up to a* ***positive linear-quadratic transformation****:*

$$(u,\lambda) \approx (u',\lambda') \quad \Longleftrightarrow \quad u' = \alpha u \quad \& \quad \lambda' = \alpha^2 \lambda \quad \textit{for } \alpha > 0$$

Reciprocal Utility Differences: Absolute & Relative Attractiveness

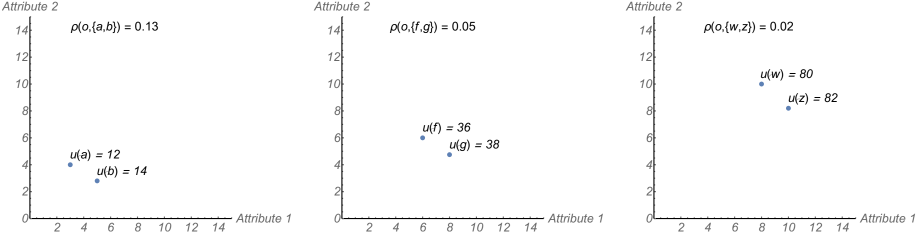
A “relative attractiveness” effect

(Dhar, 1997; Bhatia & Mullett, 2016)



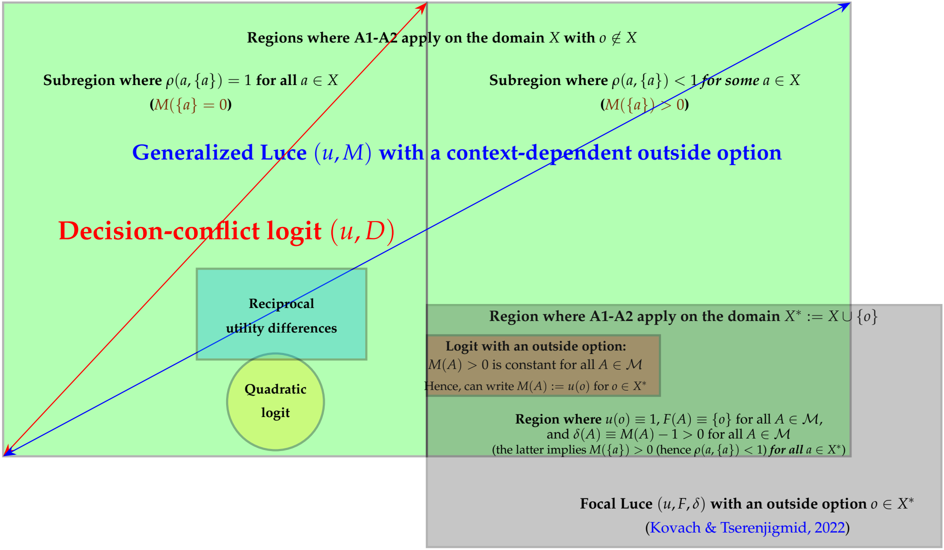
An “absolute attractiveness” effect

(Tversky & Shafir, 1992; White et al., 2015; Bhatia & Mullett, 2016)



Note: both effects generated with $\lambda = 8$ & symmetric Cobb-Douglas u

Logical Connections



Summary

General contribution:

- Extension of Luce with menu-dependent utility of outside option
  - Central idea: deferring is more attractive in harder decisions
- Helps explain deferral & overload-related phenomena
- Applicable

Disciplined special cases:

1. Quadratic logit
2. Reciprocal utility differences

- Common prediction: deferring at $\{a, b\}$ more likely as $u(a) \to u(b)$