The Falsification Adaptive Set in Linear Models with Instrumental Variables that Violate the Exogeneity or Exclusion Restriction

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Introduction

- Classical linear model $Y = X\beta + \mathcal{E}$ with an endogenous variable, estimated by method of instrumental variables (IVs), multiple correlated instruments $\mathbf{Z}$.
- Standard assumption $\text{cov}(\mathbf{Z}, \mathcal{E}) = 0$.
- When model is falsified, $\text{cov}(\mathbf{Z}, \mathcal{E}) \neq 0$. Need to specify the invalidity of the instruments, *exclusion* or *exogeneity* assumption.
- Masten and Poirier (2021) (MP) introduced the *Falsification Adaptive Set* (*FAS*) to report an estimate of when the baseline model is falsified.
- It is the set of just-identified estimands of the models where each relevant instrument in turn is used as the just-identifying instrument and the other instruments are included as controls.
- Reflects the model uncertainty that arises from falsification of the baseline model.

Introduction

- $\text{cov}(Z, \mathcal{E}) \neq 0$, let $\mathcal{E} = A + \widetilde{\mathcal{E}}$, where A is the unobserved confounder, $\text{cov}(X, A) \neq 0$.
- Violation of *Exclusion* restriction. $\mathbf{Z}_{\text{dir}}$ has direct effect, $\widetilde{\mathcal{E}} = \mathbf{Z}_{\text{dir}}\gamma_{\text{dir}} + \dot{\mathcal{E}}$, $\mathbf{Z}_{\text{indir}}$ has indirect effect, $A = \mathbf{Z}_{\text{indir}}\gamma_{\text{indir}} + \dot{A}$, on Y.
- An invalid instrument can have both a direct and indirect effect.
- Then $Y = X\beta + \mathbf{Z}_{\text{dir}}\gamma_{\text{dir}} + \mathbf{Z}_{\text{indir}}\gamma_{\text{indir}} + U$, $U = \dot{A} + \dot{\mathcal{E}}$.
- Then instruments that do not have a direct and/or indirect effect, $\gamma_{\text{dir}} = \gamma_{\text{indir}} = \mathbf{0}$, are valid if they satisfy the conditional *Exogeneity* assumption $\text{cov}(\mathbf{Z}_{\text{val}}, U) = \mathbf{0}$, and are invalid otherwise, $\text{cov}(\mathbf{Z}_{\text{inval}}, U) = \boldsymbol{\alpha}$.

Introduction

- *FAS* of MP derived under violation of the exclusion restriction only.
- We derive a different *FAS* for violation of the exogeneity assumption only.
- Natural extension is then a generalized *FAS* that considers violations of both assumptions.
- This is the set of all possible just-identified estimands where the just-identifying instrument is relevant, there are a maximum of $k_z 2^{k_z - 1}$ of these, where k_z is the number of instruments.
- If there is at least one valid and relevant instrument, then our *FAS* will contain β.

Model, Assumptions and Definitions

- The general model specification is given by

$$Y = X\beta + \mathbf{Z}^T\boldsymbol{\gamma} + U,$$

$$\text{cov}\left(\mathbf{Z}, U\right) = \boldsymbol{\alpha}.$$

- *Relevance*: The k_z-vector $\text{cov}\left(\mathbf{Z}, X\right) \neq \mathbf{0}$.
- *Sufficient variation*: The $k_z \times k_z$ matrix $\boldsymbol{\Sigma}_z := \text{var}\left(\mathbf{Z}\right)$ is invertible.

The baseline model assumes all instruments to be valid, satisfying the exclusion and exogeneity assumptions:

- *Exclusion*: $\gamma_\ell = 0$ for all $\ell \in \{1, \ldots, k_z\}$.
- *Exogeneity*: $\alpha_\ell = 0$ for all $\ell \in \{1, \ldots, k_z\}$.

Model, Assumptions and Definitions

- *Valid Instrument*: An instrument Z_ℓ is a valid instrument if both the exogeneity and the exclusion assumption hold, $\alpha_\ell = \gamma_\ell = 0$.
- *Invalid Instrument*: An invalid instrument violates either the exogeneity assumption, $\alpha_\ell \neq 0$, or the exclusion assumption, $\gamma_\ell \neq 0$, but not both, $\gamma_\ell \alpha_\ell = 0$. (Otherwise an instrument is an endogenous variable itself).
- *Our objective*: If there is a valid and relevant instrument, then the *FAS* contains β.

Identified Set, Falsification Frontier and FAS$_{excl}$

MP maintains the conditional exogeneity assumption $\boldsymbol{\alpha} = \mathbf{0}$, but relax the exclusion assumption:

- *Partial Exclusion*: There are known constants $\delta_\ell \geqslant 0$ such that $|\gamma_l| \leqslant \delta_\ell$ for $\ell = 1, \ldots, k_z$.

Define

$$\boldsymbol{\pi} := \boldsymbol{\Sigma}_z^{-1} \text{cov}(\mathbf{Z}, X)\,; \;\; \boldsymbol{\psi} := \boldsymbol{\Sigma}_z^{-1} \text{cov}(\mathbf{Z}, Y)\,.$$

As $\boldsymbol{\alpha} = \mathbf{0}$ by assumption, we have that $\boldsymbol{\psi} = \boldsymbol{\pi}\beta + \boldsymbol{\gamma}$.

The identified set for β is then given by

$$\mathcal{B}(\boldsymbol{\delta}) = \{b \in \mathbb{R} : -\boldsymbol{\delta} \leqslant (\boldsymbol{\psi} - \boldsymbol{\pi} b) \leqslant \boldsymbol{\delta}\}\,.$$

Identified Set, Falsification Frontier and FAS_{excl}

Let $\mathcal{L}_{rel}$ denote the set of relevant instruments

$$\mathcal{L}_{rel} = \{\ell \in \{1, \ldots, k_z\} : \pi_\ell \neq 0\}.$$

The falsification adaptive set is then given in Theorem 2 of MP as

$$FAS_{excl} = \left[\min_{\ell \in \mathcal{L}_{rel}} \frac{\psi_\ell}{\pi_\ell}, \max_{\ell \in \mathcal{L}_{rel}} \frac{\psi_\ell}{\pi_\ell}\right].$$

As MP point out in their Lemma 1, for $\ell \in \mathcal{L}_{rel}$, $\frac{\psi_\ell}{\pi_\ell} \left(= \beta + \frac{\gamma_\ell}{\pi_\ell}\right)$ are the IV/2sls estimands in the just identified model specification

$$Y = X\beta_\ell + \boldsymbol{Z}_{\{-\ell\}}^T \boldsymbol{\gamma}_{\{-\ell\}} + U_\ell \tag{1}$$

where $\boldsymbol{Z}_{\{-\ell\}} = \boldsymbol{Z} \setminus \{Z_\ell\}$, and using Z_ℓ as the excluded just-identifying instrument, see also Windmeijer et al. (2021).

Identified Set, Falsification Frontier and FAS$_{excl}$

- FAS_{excl} contains β if $0 \in \left[\min_{\ell \in \mathcal{L}_{rel}} \frac{\gamma_\ell}{\pi_\ell}, \max_{\ell \in \mathcal{L}_{rel}} \frac{\gamma_\ell}{\pi_\ell}\right]$
- *Sufficient:* When $\boldsymbol{\alpha} = \mathbf{0}$, $\beta \in FAS_{excl}$ if there is a relevant and valid instrument with $\gamma_\ell = 0$.

Estimation

MP proposes to estimate the relevant set by

$$\widehat{\mathcal{L}}_{rel} = \{\ell \in \{1, \ldots, k_z\} : F_\ell \geqslant C_n\},$$

where F_ℓ is the first-stage F-statistic for $H_0 : \pi_\ell = 0$. They set as default value $C_n = 10$. This is same as first-stage hard thresholding procedure of Guo et al. (2018),

$$\left| \frac{\widehat{\pi}_\ell}{\text{se}\left(\widehat{\pi}_\ell\right)} \right| \geqslant \sqrt{C_n} = 3.16.$$

Let $\widehat{\beta}_\ell = \frac{\widehat{\psi}_\ell}{\widehat{\pi}_\ell}$ be the IV estimator of β_ℓ in the just-identified model specification with $\boldsymbol{Z}_{\{-\ell\}}$ included as controls. Then FAS_{excl} is estimated by

$$\widehat{FAS}_{excl} = \left[\min_{\ell \in \widehat{\mathcal{L}}_{rel}} \widehat{\beta}_\ell, \max_{\ell \in \widehat{\mathcal{L}}_{rel}} \widehat{\beta}_\ell \right].$$

A maximum of k_z IV, or two OLS regressions.

Failure of Exogeneity Assumption

- We now assume that $\boldsymbol{\gamma} = \mathbf{0}$, but invalid instruments violate the exogeneity assumption, $\text{cov}\,(\mathbf{Z}, U) = \boldsymbol{\alpha}$.
- MP argue that, mathematically, the same technical analysis can be used as for the violation of the exclusion restriction, as from linear projection,

$$Y = X\beta + \mathbf{Z}^T\boldsymbol{\eta} + \dot{U}.$$

- However, as

$$\boldsymbol{\eta} = \boldsymbol{\Sigma}_z^{-1}\text{cov}\,(\mathbf{Z}, U) = \boldsymbol{\Sigma}_z^{-1}\boldsymbol{\alpha},$$

  and with correlated instruments, it could be the case that all $\eta_\ell \neq 0$ even when there are valid instruments present.
- Therefore FAS_{excl} is not guaranteed to contain β if there is a valid and relevant instrument .

Failure of Exogeneity Assumption

- Define the k_z-vectors $\boldsymbol{\pi}^*$ and $\boldsymbol{\psi}^*$ with ℓ-th elements given by

$$\pi_\ell^* := \left(\text{var}\left(Z_\ell\right)\right)^{-1} \text{cov}\left(Z_\ell, X\right); \;\; \psi_\ell^* := \left(\text{var}\left(Z_\ell\right)\right)^{-1} \text{cov}\left(Z_\ell, Y\right),$$

  for $\ell = 1, \ldots, k_z$.

- By same arguments as for the FAS_{excl}, we obtain the FAS_{exo} as

$$FAS_{exo} = \left[\min_{\ell \in \mathcal{L}_{rel}^*} \frac{\psi_\ell^*}{\pi_\ell^*}, \max_{\ell \in \mathcal{L}_{rel}^*} \frac{\psi_\ell^*}{\pi_\ell^*} \right],$$

  where $\mathcal{L}_{rel}^* = \left\{ \ell \in \{1, \ldots, k_z\} : \pi_\ell^* \neq 0 \right\}$.

Failure of Exogeneity or Exclusion Assumption

- We now consider the full specification

$$Y = X\beta + \boldsymbol{Z}^T \boldsymbol{\gamma} + U,$$

$$\text{cov}\left(\boldsymbol{Z}, U\right) = \boldsymbol{\alpha}.$$

- Together with the assumption that an invalid instrument can violate either the exclusion or the conditional exogeneity assumption.
- We therefore consider all possible just-identified model specifications.
- For $k_z = 2$, there are 4 of these, and the resulting FAS is simply $FAS_{excl} \cup FAS_{exo}$.
- If there is a relevant and valid instrument, then clearly $\beta \in FAS$.

General Results

- For general k_z there are $s_{k_z} = k_z s^{k_z - 1}$ just identified model specifications, or just identifying transformed instruments
- For example, for $k_z = 3$ we have

$$\widetilde{\boldsymbol{Z}} = \left(Z_1, Z_{1|2}, Z_{1|3}, Z_{1|23}, Z_2, Z_{2|1}, Z_{2|3}, Z_{2|13}, Z_3, Z_{3|1}, Z_{3|2}, Z_{3|12} \right)^T ,$$

with e.g.

$$Z_{1|2} = Z_1 - Z_2 \left(\mathrm{var}\left(Z_2\right)\right)^{-1} \mathrm{cov}\left(Z_2, Z_1\right).$$

- Then define the s_{k_z}-vectors $\widetilde{\boldsymbol{\pi}}$ and $\widetilde{\boldsymbol{\psi}}$ with j-th elements

$$\widetilde{\pi}_j := \left(\mathrm{var}\left(\widetilde{Z}_j\right)\right)^{-1} \mathrm{cov}\left(\widetilde{Z}_j, X\right); \; \widetilde{\psi}_j := \left(\mathrm{var}\left(\widetilde{Z}_j\right)\right)^{-1} \mathrm{cov}\left(\widetilde{Z}_j, Y\right).$$

General Results

- The set of relevant instruments is specified as

$$\widetilde{\mathcal{L}}_{rel} = \left\{ j \in \left\{ 1, \ldots, s_{k_z} \right\} : \widetilde{\pi}_j \neq 0 \right\}$$

- The generalized falsification adaptive set is

$$FAS = \left[\min_{j \in \widetilde{\mathcal{L}}_{rel}} \frac{\widetilde{\psi}_j}{\widetilde{\pi}_j}, \max_{j \in \widetilde{\mathcal{L}}_{rel}} \frac{\widetilde{\psi}_j}{\widetilde{\pi}_j} \right] = \left[\min_{j \in \widetilde{\mathcal{L}}_{rel}} \widetilde{\beta}_j, \max_{j \in \widetilde{\mathcal{L}}_{rel}} \widetilde{\beta}_j \right],$$

  where, for $j \in \widetilde{\mathcal{L}}_{rel}$, the IV estimands are given by

$$\widetilde{\beta}_j = \frac{\widetilde{\psi}_j}{\widetilde{\pi}_j} = \frac{\operatorname{cov}\left(\widetilde{Z}_j, Y\right)}{\operatorname{cov}\left(\widetilde{Z}_j, X\right)}.$$

- $\beta \in FAS$ if there is a relevant and valid instrument.

Empirical Example, $k_z = 3$

- Empirical analysis of roads and trade by Duranton et al. (2014).
- The outcome variable is a measure of how much a city exports. The one considered here is called the "propensity to export weight".
- The treatment variable is the log number of kilometers of interstate highway within a city in 2007.
- There are three potential instruments
  - $Z_1 = \textit{Plan}$ is the log number of kilometers of highway in the city according to a planned highway construction map, approved by the federal government in 1947
  - $Z_2 = \textit{Railroads}$ is the log number of kilometers of railroads in the city in 1898
  - $Z_3 = \textit{Exploration}$ is a measure of the quantity of historical exploration routes that passed through the city.
- Partial sample correlations are $\widehat{\rho}_{12} = 0.57$, $\widehat{\rho}_{13} = 0.34$ and $\widehat{\rho}_{23} = 0.11$.
- If for example $\boldsymbol{\gamma} = (0, 0, \gamma_3)^T$ and $\boldsymbol{\alpha} = (\alpha_1, 0, 0)^T$ with $\gamma_3 \neq 0$ and $\alpha_1 \neq 0$, then $Z_{2|3}$ is a valid instrument. $Z_{2|3}$ is the just-identifying instrument Z_2 when Z_3 is included as a control and Z_1 omitted from the instrument set.

Empirical Example, $k_z = 3$

Table 1: IV estimation results, Duranton et al. (2014, Table 5, column 2)

	Instruments						
	Z_1, Z_2, Z_3	$Z_{1\|2,3}$	$Z_{2\|1,3}$	$Z_{3\|1,2}$	Z_1	Z_2	Z_3
hway	0.57	0.28	3.16	−0.32	0.55	1.09	0.13
	(0.16)	(0.25)	(1.39)	(0.86)	(0.17)	(0.26)	(0.38)
F	90.30	58.13	6.97	20.00	154.5	35.84	15.97
J p	0.043						
		$Z_{1\|2}$	$Z_{1\|3}$	$Z_{2\|1}$	$Z_{2\|3}$	$Z_{3\|1}$	$Z_{3\|2}$
hway		0.22	0.40	3.74	1.18	−0.61	−0.02
		(0.21)	(0.16)	(1.90)	(0.26)	(1.11)	(0.38)
F		81.14	122.45	5.29	34.31	14.27	31.07
$\widehat{FAS}_{excl}$	$[-0.32, 0.28]$						
$\widehat{FAS}_{exo}$	$[0.13, 1.09]$						
$\widehat{FAS}$	$[-0.61, 1.18]$						

Notes: Outcome variable "propensity to export weight", $n = 66$. Additional controls "log employment" and "Market access (export)". Heteroskedasticity robust test statistics and (standard errors). Z_1 is instrument "Plan", Z_2 is "Railroads", Z_3 is "Exploration".

Conclusions

- We have generalized the FAS_{excl} of MP to properly take into account possibe violations of both the exclusion and exogeneity assumptions.
- Report the *FAS* when the model has been falsified
- Alternatively, use valid/invalid instrument selection methods, as in Kang et al. (2016), Windmeijer et al. (2019), Guo et al. (2018) and Windmeijer et al. (2021).