Asset Pricing in a Low Rate Environment

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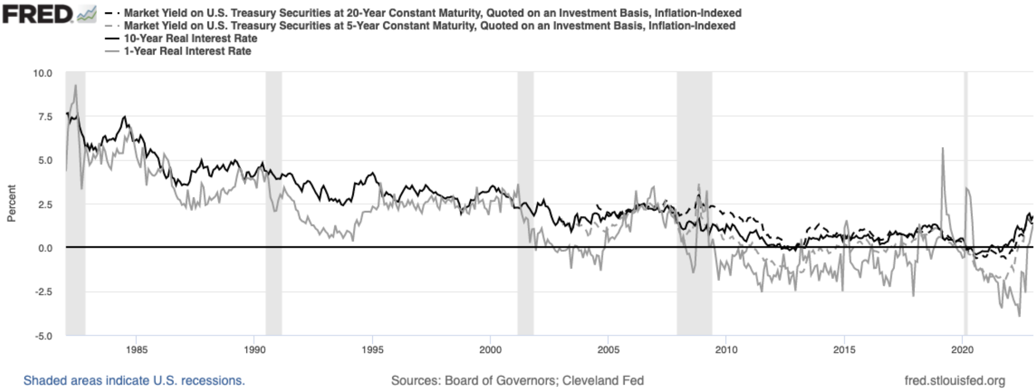
Real interest rates are often low and below the growth rate of GDP



This paper

1. Develop a tractable model of a production economy that consistent with
   1.1 business cycle statistics
      1.1.1 standard deviation of consumption growth
      1.1.2 standard deviation of output growth
   1.2 asset prices
      1.2.1 low and smooth risk free rate, with $r < g$
      1.2.2 a high market price of risk
2. Investigate conditions for the possibility of infinite debt roll over by the government
   → possibility of infinite debt roll over depends on aggregate risk
   → impossible in our baseline calibrations, even with $r - g = -3.5\%$
3. Provide a tractable and efficient way to solve the model with a simulated path version of Krusell and Smith (1998)

✗ Not in this paper: Welfare. For welfare, see *e.g.* Brumm et al. (2021); Aguiar et al. (2021); Amol and Luttmer (2022)

Model ingredients

- Our model combines
  - Firm entry and exit
  - Idiosyncratic income risk along the lines of Imrohoroğlu (1989); Aiyagari (1994); Huggett (1993); Bewley (1983)
  - Limited stock market participation as Chien et al. (2011)

X Our model does not need
  - habits or non-standard preferences (we have time separable expected utility preferences with CRRA utility)
  - huge levels of risk aversion (we have relative risk aversion $\gamma = 5.5$)

Literature

- Limited stock market participation
  e.g. Vissing-Jørgensen (2002); Vissing-Jørgensen and Attanasio (2003); Guvenen (2009); Chien et al. (2011)
- Production based asset pricing with idiosyncratic risk
  e.g. Storesletten et al. (2007), Favilukis (2013)
- Infinite debt roll over with $r < g$
  e.g. Blanchard (2019); Mian et al. (2021); Aguiar et al. (2021); Kocherlakota (2022); Bloise and Reichlin (2022)

Model

Firms entry and exit

- A fraction Γ of initial firm survive and the remainder exit losing their capital
- Surviving firms produce today and choose their capital for tomorrow
- New capital is subject to adjustment costs, so price of capital > 1
- Exiting firms are replaced by new startups who start producing next period
- Startup enter smaller; fraction $s < 1$ of average firm size next period
- New capital in startups is not subject to adjustment costs
- There are rents from creating capital, especially for startups
- measure Γ of incumbents and a measure $1 - \Gamma$ of startups
- Γ and s will determine growth rate of surviving firms

Firms technology

- continuum of firms produces single consumption good with identical production technology

$$y_t^i = \xi(z_t)(k_t^i)^\alpha (A_t l_t^t)^{1-\alpha} \tag{1}$$

  - z_t: aggregate TFP shock. Discreticed first order Markov process
  - $\xi(z_t)$: aggregate TFP (deviation from trend)
  - k_t^i: capital input
  - l_t^i: labor input
  - $A_t = (1+g)^t$: deterministic trend growth in labor augmenting technology
- chooses investment subject to adjustment costs

$$k_{t+1}^{i*} = k_t^i(1-\delta) + i_t^i \tag{2}$$

$$d_t^i = y_t^i - \omega_t l_t^i - i_t^i - \psi(k_{t+1}^{i*}, k_t^i) \tag{3}$$

$$\psi(k', k) := \xi^{\text{adj}} k \left(\frac{k'}{k} - (1 - \delta + x^{\text{target}})\right)^2 \tag{4}$$

Firms problem

- A firm's Bellman equation is given by

$$v_t^i = \max_{k_{t+1}^{i*}} d_t^i + \Gamma \sum_{z_{t+1}} p_t^{z_{t+1}} v_{t+1}^{i*}, \tag{5}$$

- $p_t^{z_{t+1}}$:= price of aggregate state-contingent security
- FOC

$$1 + \frac{\partial \psi(k_{t+1}^{i*}, k_t^i)}{\partial k_{t+1}^{i*}} = \Gamma \sum_{z_{t+1}} p_{z_{t+1}}^t \left(r_{t+1}^K + 1 - \delta - \frac{\partial \psi(k_{t+2}^{i**}, k_{t+1}^{i*})}{\partial k_{t+2}^{i**}} \right) \tag{6}$$

Firm Aggregation

- value of producing firms

$$V_t = D_t + \sum_{z_{t+1}} p_t^{z_{t+1}} \left(V_{t+1} \underbrace{\frac{\Gamma}{\Gamma + (1-\Gamma)s}}_{\text{disc. from entry and exit}} \right).$$

- value of time t startups

$$V_t^{\text{startup}} := \sum_{z_{t+1}} p_t^{z_{t+1}} V_{t+1} \frac{s(1-\Gamma)}{\Gamma + (1-\Gamma)s},$$

- aggregate dividends

$$D_t = Y_t - \omega_t L_t - I_t^{\text{incumbent}} - \Xi_t$$

- aggregate adjustment costs

$$\Xi_t = \Gamma K_t \xi^{\text{adj}} (g_k - (1 - \delta + x^{\text{target}}))^2$$

- aggregate firm Euler equation

$$1 + 2\xi^{\text{adj}} \left(g_t^k - (1 - \delta + x^{\text{target}}) \right) = \Gamma \sum_{z_{t+1}} p_t^{z_{t+1}} \left(r_{t+1}^K + 1 - \delta + \xi^{\text{adj}} \left(\left(g_{t+1}^k \right)^2 - (1 - \delta + x^{\text{target}})^2 \right) \right)$$

- growth rate of capital in firms

$$g^k = \frac{1}{\Gamma + (1-\Gamma)s} \frac{K_{t+1}}{K_t}$$

- aggregate rents from startup creation

$$\Omega_t = \frac{s(1-\Gamma)}{\Gamma + (1-\Gamma)s} \left(\sum_{z_{t+1}} p_t^{z_{t+1}} V_{t+1} - K_{t+1} \right) \tag{7}$$

Households

- identical, times-separable, vNM utility, where instantaneous utility is CRRA

$$U((c_t)_{t=0}^{\infty}) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{8}$$

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \tag{9}$$

- households face idiosyncratic income risk. Idiosyncratic shock: η_t, with transition probability $\pi_\eta(\eta_{t+1}|\eta_t)$ determines labor endowment $l(\eta_t)$
- two types of traders as in Chien et al. (2011)
  - measure $1-\mu$ of non-participants (aka bond traders), can only trade a one period bond with borrowing constraint $b_t \geq 0$, and budget constraint

$$c_t = \underbrace{l(\eta_t)\omega(z^t)}_{\text{labor inc.}} + \underbrace{b_{t-1}}_{\text{fin. wealth}} - \underbrace{b_t p^b(z^t)}_{\text{savings}} \tag{10}$$

  - measure μ of advanced traders (aka Arrow traders), can trade a full set of aggregate state-contingent securities, receive rents from startup creation, face borrowing constraint $a_{t-1}^{z_t} \geq 0$ and budget constraint

$$c_t = \underbrace{l^A(\eta_t)\omega(z^t)}_{\text{labor inc.}} + \underbrace{\frac{1-\Gamma}{\Gamma\mu}\zeta(z^t)}_{\text{startup inc.}} + \underbrace{a_{t-1}^{z_t}}_{\text{fin. wealth}} - \underbrace{\sum_{z_{t+1}\in\mathcal{Z}} a_t^{z_{t+1}} p^{z_{t+1}}(z^t)}_{\text{savings}} \tag{11}$$

- the prices of the aggregate state-contingent securities, allow us to price all aggregate assets!

Asset Market Clearing

- Total HH assets must equal value of existing firms for each shock

$$w_t^{\text{households}} := w_t^A + w_t^B = V_t^{\text{firm}}$$

Intuition for the mechanism

Calibration

Exogenous parameters

- choose survival rate $\Gamma = 96.5\%$ and startups size $s = 0.75$ to obtain an exit rate of 3.5% and employment growth of roughly 0.7%
- $\mu = 10\%$ of advanced traders, as in Chien et al. (2011)
- relative risk aversion $\gamma = 5.5$
- capital share $\alpha = 0.33$
- depreciation of capital $\delta = 0.1$
- deterministic trend growth rate $g = 2\%$
- AR(1) proces for TFP: $\rho^{\text{tfp}} = 0.8145$, following Guvenen (2009), standard deviation of innovations $\sigma^{\text{tfp}} = 2.47\%$ to match a standard deviation of output growth of 2.6% ▸ Why no random walk component in GDP?
- idiosyncratic shock process: estimate a two-state Markov process to the non-permanent idiosyncratic component in Storesletten et al. (2004), abstracting from CCV ▸ Details on the idiosyncratic shock process

$$l(\eta_t) = \begin{pmatrix} 0.463 \\ 1.537 \end{pmatrix} \qquad \pi_\eta = \begin{pmatrix} 0.892 & 0.108 \\ 0.108 & 0.892 \end{pmatrix} \tag{12}$$

Endogenous parameters

We use the two remaining free parameters, time preference parameter β and adjustment cost parameter ξ^{adj} to jointly match the average interest rate

$$r := \mathrm{E}\left[\frac{1}{p^b(z^t)}\right]$$

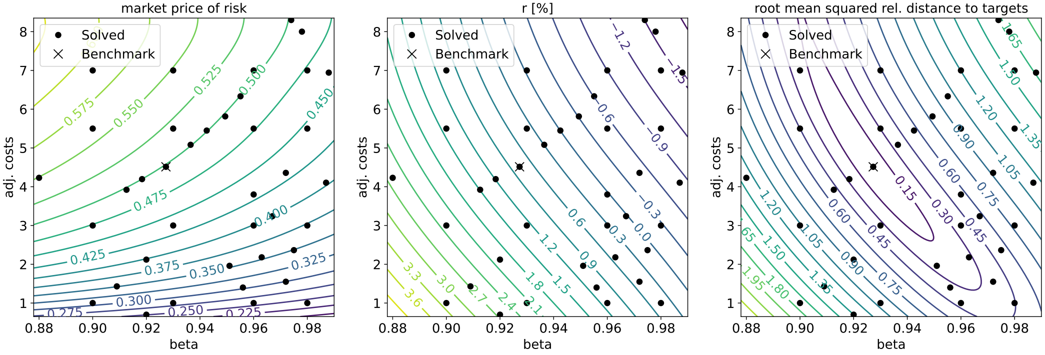
and the average market price of risk

$$mpr := \mathrm{E}\left[\frac{\sqrt{\sum_{z_{t+1}\in\mathcal{Z}} \pi(z_{t+1}|z_t)\left(\frac{p^{z_{t+1}}}{\pi(z_{t+1}|z_t)} - p^b(z^t)\right)^2}}{p^b(z^t)}\right]$$

Matching the two targets

In the benchmark calibration we target a interest rate of $r = 0.8\%$ and a market price of risk of $mpr = 50\%$.

We use Gaussian Processes (see Scheidegger and Bilionis (2019)) to approximate the mapping from parameters to moments of interest.

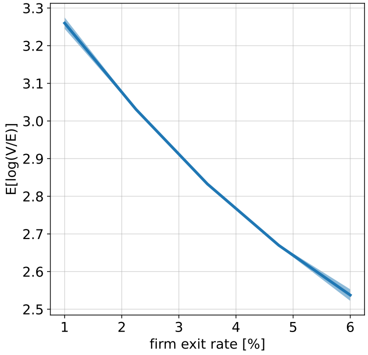


Match and other moments

	Model	Data	Rep. agent
r^{bond}	0.8 %	0.8 %	19 %
MPR	50 %	50 %	12 %
std output growth	2.6 %	2.6 %	2.6 %
std r^{bond}	2.0 %	1.8-2.9 %	3.1 %
std agg. consumption growth	2.0 %	1.4-2.0 %	2.2 %
log V/E	2.8	2.8-3.0	1.8

Role of firm exit

We vary the death rate and recalibrate the patience and adjustment costs to hold the average interest rate and market price of risk constant



$\Rightarrow$ Firm exit is needed for a reasonable price earnings ratio

Very low interest rates

Lowering the interest rate

We vary the time preference parameter and the adjustment costs parameter to compare different interest rate levels, keeping the market price of risk at 50%.

β	ξ^{adj}	MPR	mean r^{bond}	std r^{bond}	mean $r^{\text{Firm, }K}$	std $r^{\text{Firm, }K}$	std Y_{t+1}/Y_t	std C_{t+1}/C_t
0.9740	8.300	49.9 %	-1.55 %	2.6 %	1.5 %	7.2 %	2.6 %	2.1 %
0.9600	7.000	50.4 %	-0.89 %	2.5 %	1.9 %	6.7 %	2.6 %	2.1 %
0.9551	6.333	49.9 %	-0.60 %	2.4 %	2.0 %	6.5 %	2.6 %	2.1 %
0.9494	5.814	49.8 %	-0.37 %	2.3 %	2.2 %	6.0 %	2.6 %	2.0 %
0.9425	5.449	50.1 %	-0.06 %	2.2 %	2.4 %	5.7 %	2.6 %	2.0 %
0.9365	5.082	50.1 %	0.33 %	2.2 %	2.6 %	5.5 %	2.6 %	2.0 %
0.9273	4.514	49.8 %	0.79 %	2.1 %	2.9 %	5.2 %	2.6 %	2.0 %
0.9126	3.922	49.8 %	1.47 %	1.9 %	3.4 %	4.8 %	2.6 %	2.0 %

Low interest rates and asset prices

- p^{console}: price of a console with a deterministic payout stream $A_t = (1+g)^t$

$\mathrm{E}\left[r^{\text{bond}}\right]$	$\mathrm{E}\left[\log(V/E)\right]$	$\mathrm{E}\left[p^{\text{console}}\right]$	$\mathrm{E}\left[\text{pdv}(Lw)\right]$	$\mathrm{E}\left[\text{pdv}(V)\right]$	$\mathrm{E}\left[\text{pdv}(V^{\text{start-ups}})\right]$
-1.55 %	3.16	524	455	3.83	47.2
-0.89 %	3.05	195	171	3.62	19.3
-0.60 %	3.02	162	140	3.52	16.1
-0.37 %	2.99	133	115	3.45	13.6
-0.06 %	2.94	106	90.8	3.34	11.2
0.33 %	2.89	90	77.3	3.27	9.89
0.79 %	2.84	73	61.8	3.15	8.31
1.47 %	2.75	55	46.3	2.99	6.72

▸ low interest rates and the wealth distribution

Infinite debt rollover

Simple example

- With certainty: if and only if $r \leq g$, is infinite debt rollover possible
- with uncertainty: $\mathrm{E}[r] \leq \mathrm{E}[g]$ is neither necessary nor sufficient (see Kocherlakota (2022), Bloise and Reichlin (2022))

Simple example

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- with uncertainty: $\mathrm{E}[r] \leq \mathrm{E}[g]$ is neither necessary nor sufficient (see Kocherlakota (2022), Bloise and Reichlin (2022))
- consider an S-state Markov chain
- let Q denote the $S \times S$ matrix of Arrow prices
- value of an infinite payout stream of 1

$$\lim_{n \to \infty} (I + Q + \ldots + Q^n) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \tag{13}$$

converges $\Leftrightarrow$ value of largest eigenvalue of Q is $< 1 \Leftrightarrow$ infinite debt rollover impossible

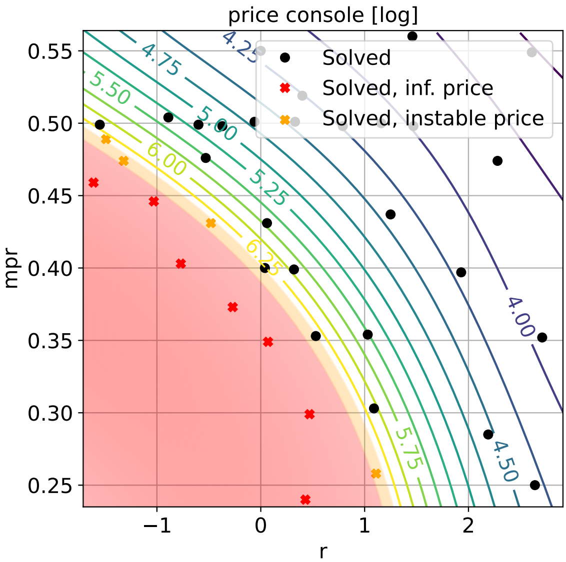
Simple example continued

- Consider

$$Q_\epsilon = \begin{pmatrix} \frac{1}{R_1} & \epsilon & \dots & \epsilon \\ \frac{1}{R_2} & \epsilon & \dots & \epsilon \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{R_S} & \epsilon & \dots & \epsilon \end{pmatrix}$$

  - consider $\epsilon \to 0$, largest eigenvalue $< 1 \Leftrightarrow R_1 > 1$.
  - this poses basically no restriction on the average interest rate.
  - all we can say is
    i) for debt rollover to be possible, there must be one state s for which $R_s < 1$
    ii) for debt rollover to be impossible there must be a state s' for which $R_{s'} > 1$
- a model to investigate the possibility of infinite debt rollover requires a theory for state prices, allowing for $\mathrm{E}[r] < g$ $\Rightarrow$ our model is ideally suited for this

Infinite debt rollover in our model



⇒ infinite debt rollover impossible for our benchmark calibration ▸ Computational Method

Conclusion

- we provide a tractable model of a production economy with heterogeneous agents, which
  1. is consistent with business cycles statistics
  2. is consistent with a low interest rate and a high market price of risk
  3. allows for $\mathrm{E}[r] < \mathrm{E}[g]$
  4. provides a theory for state-prices
- we use this model to investigate the possibility of infinite debt roll-over $\rightarrow$ benchmark calibration does not allow for infinite debt rollover
- More in the paper:
  - Computational method for a simulated-path Krusell and Smith (1998) algorithm
  - More on the wealth distribution and low interest rates

Thank you!

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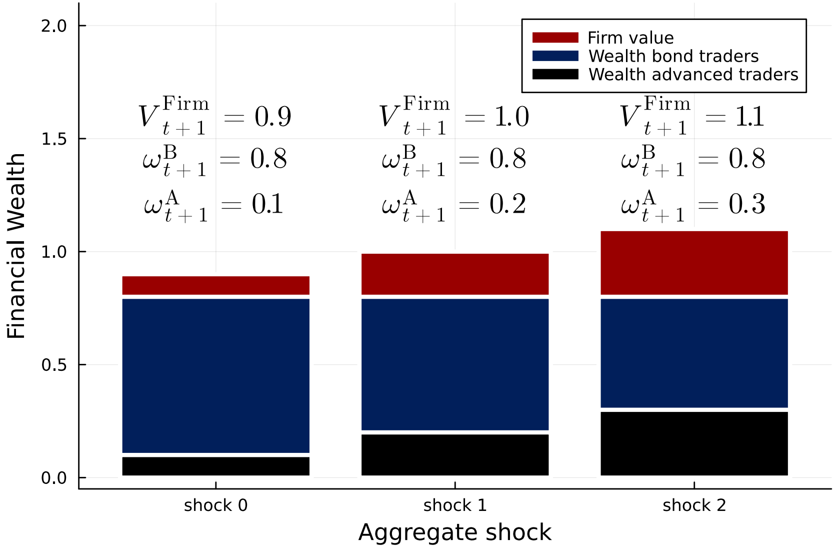
Intuition for the mechanism in Chien et al. (2011)



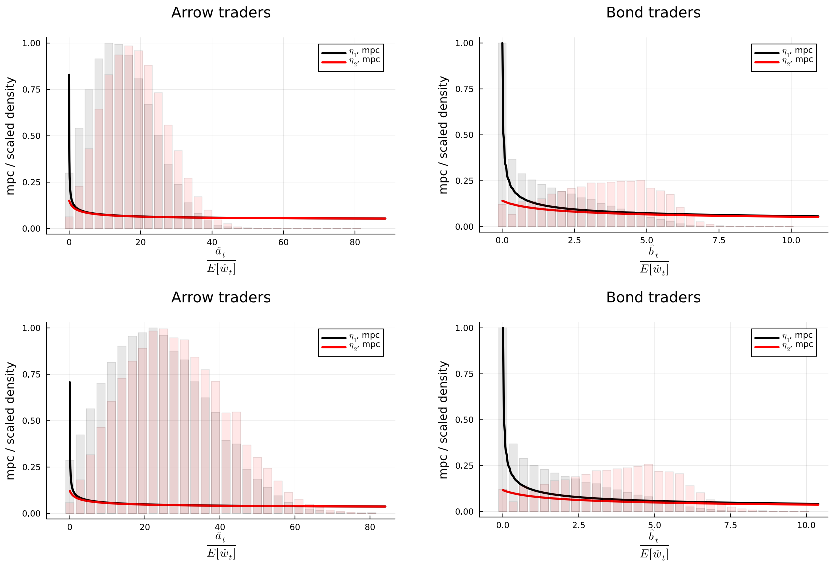
Asset Markt Clearing

Low interest rates and the wealth distribution

- λ^A: wealth share held by advanced traders
- $C^{A/B}$: aggregate consumption by each group of traders
- $\mu(\widehat{a/b}_t = 0)$: fraction of constrained households

$E\left[r^{\text{bond}}\right]$	$E\left[\lambda^A\right]$	$\text{Std}\left(\lambda^A\right)$	$\frac{E\left[\lambda^A \mid z_t=\text{good}\right]}{E\left[\lambda^A \mid z_t=\text{bad}\right]}$	$\text{Std}\left(\frac{C^A_{t+1}}{C^A_t}\right)$	$\text{Std}\left(\frac{C^B_{t+1}}{C^B_t}\right)$	$E\left[\mu(\hat{a}_t = 0)\right]$	$E\left[\mu(\hat{b}_t = 0)\right]$
-1.55 %	35.5 %	12.9 %	2.8	8.5 %	1.0 %	1.1 %	10.7 %
-0.89 %	33.4 %	12.1 %	2.8	8.6 %	1.0 %	1.3 %	10.8 %
-0.60 %	32.0 %	11.6 %	2.8	8.7 %	1.0 %	1.4 %	10.7 %
-0.37 %	31.0 %	11.2 %	2.7	8.5 %	1.0 %	1.5 %	10.8 %
-0.06 %	30.5 %	10.6 %	2.7	8.3 %	1.0 %	1.7 %	10.9 %
0.33 %	29.0 %	10.6 %	2.8	8.3 %	1.0 %	1.8 %	10.9 %
0.79 %	27.6 %	9.9 %	2.7	8.3 %	1.0 %	2.0 %	11.0 %
1.47 %	26.2 %	9.0 %	2.6	8.4 %	1.0 %	2.1 %	11.3 %

Low interest rates and the wealth distribution



Back to low interest rates and asset prices

Computational Method

Challenges I

- rich wealth distribution, within and across types

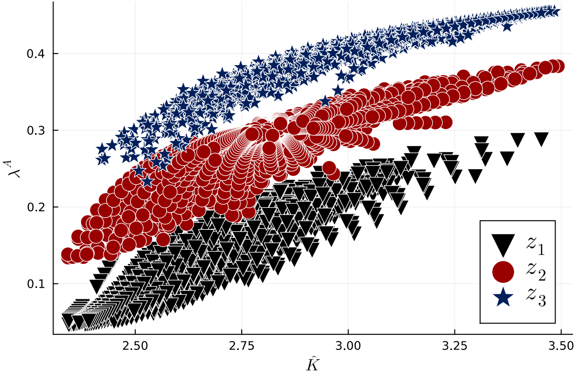
⇒ summarize the wealth distribution in the spirit of Krusell and Smith (1998) and Kubler and Scheidegger (2019), with (detrended) aggregate capital and the wealth share of advanced traders, $(\hat{K}_t, \lambda_t^A)$

- wealth distribution is moving a lot, affecting prices in a non-linear way

⇒ allow for non-linear dependence of policy functions on $(\hat{K}_t, \lambda_t^A)$

Challenges II

- summarizing endogenous aggregate state variables, $\hat{K}_t$ and λ_t^A, are heavily correlated



$\Rightarrow$ solve everything on simulated paths of aggregate variables

$\Rightarrow$ iterate between simulation and updating of all approximating functions

Challenges III

- need to have some extrapolation in the $\hat{K}_t$, λ_t^A, space
  - easier for aggregates
  - harder for households' consumption functions, which also depend on the asset holdings $c^{z_t,\eta_t}(\hat{a}_t, \hat{K}_t, \lambda_t^A)$

$\Rightarrow$ fit different simple functions for different wealth levels, which then only depend on $\hat{K}_t, \lambda_t^A$: $c^{z_t,\eta_t,\hat{a}_t}(\hat{K}_t, \lambda_t^A)$, for $\hat{a}_t$ on a grid

- naturally combines with the endogenous grid method by Carroll (2006)
- captures wealth dependent relative importance of $(\hat{K}_t, \lambda_t^A)$

Functions to approximate

We denote our approximations with $g_x(\cdot)$, where the index x denotes what is approximated. The approximations we need are

$$g_K\left(z_t, \hat{K}_t, \lambda_t^A\right) \approx \hat{K}_{t+1} \tag{14}$$

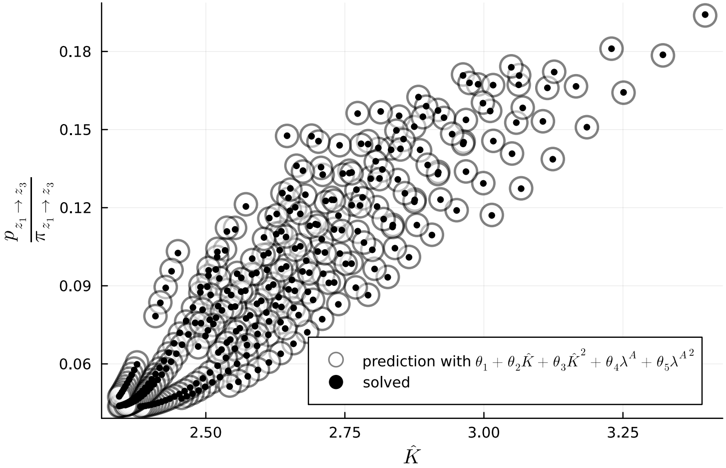
$$g_{V^{\text{firm}}}\left(z_t, \hat{K}_t, \lambda_t^A\right) \approx \hat{V}_t^{\text{firm}} \tag{15}$$

$$g_{\lambda^A}\left(z_t, \hat{K}_t, \lambda_t^A, z_{t+1}\right) \approx \lambda_{t+1|z_{t+1}}^A \tag{16}$$

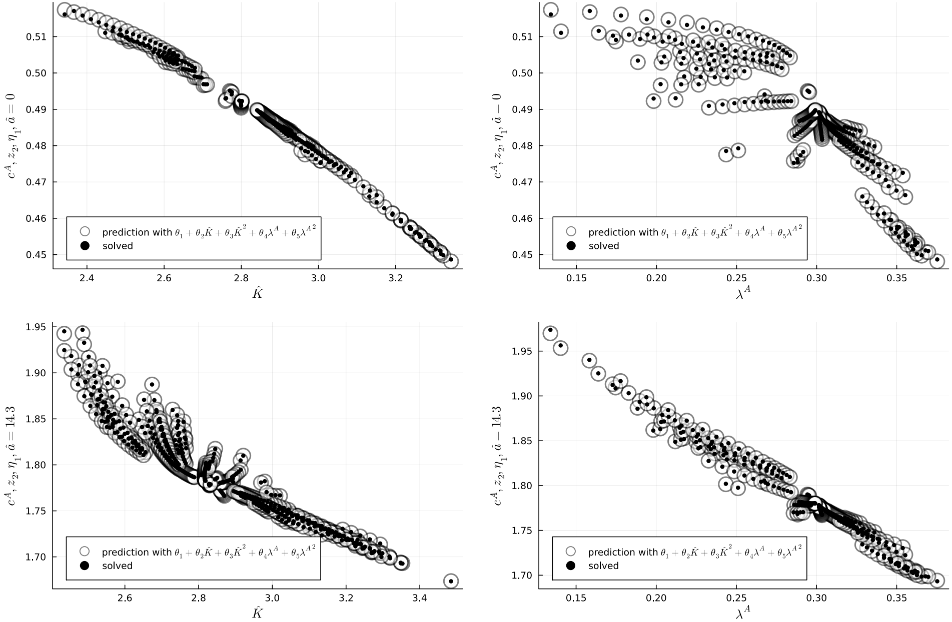
$$g_{c^A}\left(z_t, \hat{K}_t, \lambda_t^A, \eta_t, \frac{\hat{a}_{t-1}^{z_t}}{1+g}\right) \approx \hat{c}_t^A \tag{17}$$

$$g_{c^B}\left(z_t, \hat{K}_t, \lambda_t^A, \eta_t, \frac{\hat{b}_{t-1}}{1+g}\right) \approx \hat{c}_t^B. \tag{18}$$

Necessity of the wealth share of advanced traders



Relative importance of $(\hat{K}_t, \lambda_t^A)$ by wealth



Accuracy of policies

	$\hat{c}^A$ [%]	$\hat{c}^B$ [%]	$\hat{c}^A$ [%] η_1	$\hat{c}^B$ [%] η_1	$\hat{c}^A$ [%] η_2	$\hat{c}^B$ [%] η_2	$\hat{V}^{\text{firm}}$ [%]
Mean	0.01	0.00	0.01	0.00	0.01	0.00	0.01
90th percentile	0.02	0.00	0.02	0.00	0.02	0.00	0.01
99th percentile	0.06	0.01	0.07	0.01	0.04	0.01	0.02

Accuracy of forecasting functions

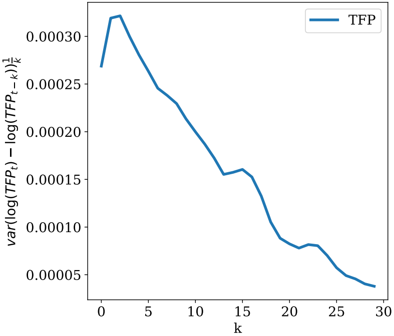
	$\hat{K}^A$ [%]	λ^A [%]	λ^A [%] $z_1 \to z_1$ $z_1 \to z_2$ $z_1 \to z_3$	λ^A [%] $z_2 \to z_1$ $z_2 \to z_2$ $z_2 \to z_3$	λ^A [%] $z_3 \to z_1$ $z_3 \to z_2$ $z_3 \to z_3$
Mean	0.00	0.01	0.01 0.01 0.01	0.01 0.01 0.01	0.01 0.01 0.01
90th percentile	0.00	0.02	0.02 0.02 0.02	0.02 0.02 0.02	0.02 0.02 0.01
99th percentile	0.00	0.03	0.03 0.03 0.03	0.04 0.04 0.04	0.03 0.03 0.03

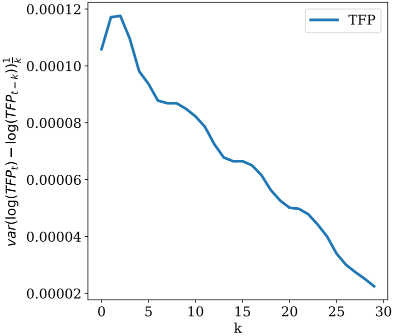
back to infinite debt rollover

Why no random walk component for TFP?

Following Cochrane (1988), we look at

$$\frac{1}{k}\text{var}(\log(\text{TFP}_t) - \log(\text{TFP}_{t-k})) \overbrace{\rightarrow}^{k \text{ increasing}} \begin{cases} 0 & \text{if: TFP is trend-stationary.} \\ \text{const.} & \text{if: TFP follows a random walk.} \end{cases}$$





Left: for "Business Sector TFP" data from Fernald (2014), 1960 - 2018.
Right: for "Total Factor Productivity at Constant National Prices" data from FRED, 1960 - 2018.

▸ Exogenous Parameters

Idiosyncratic shock process I

Storesletten et al. (2004) estimate a process for log earnings of the form

$$y_{it} = g(x_{it}^h, Y_t) + u_{it}^h \tag{19}$$

$$u_{it} = \alpha_i + z_{it}^h + \epsilon_{it} \tag{20}$$

$$z_{it}^h = \rho z_{i,t-1}^{h-1} + \eta_{it} \tag{21}$$

$$\alpha_i \sim \text{iid } N(0, \sigma_\alpha^2) \tag{22}$$

$$\epsilon_{it} \sim \text{iid } N(0, \sigma_\epsilon^2) \tag{23}$$

$$\eta_{it} \sim \text{iid } N(0, \sigma_t^2) \tag{24}$$

$$\sigma_t^2 = \begin{cases} \sigma_E^2 & \text{in agg. expansions} \\ \sigma_C^2 & \text{in agg. contractions} \end{cases}. \tag{25}$$

Storesletten et al. (2004) estimate $\sigma_\epsilon = 0.25$, $\rho = 0.95$ and frequency weighted average of $\mathrm{E}[\sigma_t] = 0.17$.

Idiosyncratic shock process II

We abstract from age and the CCV mechanism and focus on the non-permanent idiosyncratic component of log earnings.

$$x_{it} = \epsilon_{it} + z_{it} \tag{26}$$

$$z_{it} = \rho z_{it-1} + \eta_{it} \tag{27}$$

$$\epsilon_{it} \sim \text{iid } N(0, \sigma_\epsilon^2) \tag{28}$$

$$\eta_{it} \sim \text{iid } N(0, \sigma_\eta^2), \tag{29}$$

where we take $\rho = 0.95$, $\sigma_\epsilon = 0.25$, and $\sigma_\eta = 0.17$ from Storesletten et al. (2004).

Idiosyncratic shock process III

Fit an AR(1) process of the form

$$x_{it} = \bar{x} + \rho_x x_{i,t-1} + \sigma_x \epsilon_{it} \tag{30}$$

$$\epsilon_t \sim \text{iid } N(0,1), \tag{31}$$

Discretize the AR(1) process into a two-state Markov chain using Rouwenhorst (1995) algorithm. Exponentiate the resulting state values and normalize them. We obtain

$$X = \begin{pmatrix} 0.463 \\ 1.537 \end{pmatrix} \tag{32}$$

$$\pi^X = \begin{pmatrix} 0.892 & 0.108 \\ 0.108 & 0.892 \end{pmatrix} \tag{33}$$

The resulting cross-sectional standard deviation of log earnings is 0.60 and matches the standard deviation of the process simulated in equations (19) - (25) and is in the ballpark of values typically used in the heterogeneous agents literature (see, *e.g.* Auclert et al., 2021). ▸ Exogenous Parameters