Competing to Commit: Markets with Rational Inattention*

Carlo M. Cusumano† Francesco Fabbri‡ Ferdinand Pieroth§

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Abstract

Two homogeneous-good firms compete for a consumer's unitary demand. The consumer is rationally inattentive and pays entropy costs to process information about firms' offers. Compared to a collusion benchmark, competition produces two effects. As in standard models, competition puts downward pressure on prices. But, additionally, an *attention effect* arises: The consumer engages in trade more often. This alleviates the commitment problem that firms have when facing inattentive consumers and increases trade efficiency. For high enough attention costs, the attention effect dominates the effect on prices: Firms' profits are higher under competition than under collusion.

Keywords: Rational inattention; Competition; Endogenous demand
JEL codes: D21; D43; D83

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†Yale University, Department of Economics; `carlo.cusumano@yale.edu`
‡Princeton University, Department of Economics; `ffabbri@princeton.edu`
§Yale University, Department of Economics; `ferdinand.pieroth@yale.edu`

1 Introduction

Consumers' ability to process information about prices is naturally at the heart of the idea of competition. Standard models of Bertrand competition assume that consumers can perfectly spot different firms' offers and optimally choose the best one. However, in many situations, finding the best offer requires costly attention. Consider, for example, a consumer deciding which mortgage to apply for, where to purchase life insurance, or which food delivery service to use. Even if all the information needed for optimal decision-making is available, the consumer still has to process this information. Mortgage and life insurance contracts can be challenging to understand, and learning which delivery service offers the lowest fees or the best promotions requires time and cognitive effort.

If information processing is costly, rational consumers must not only decide which offer to accept but also how much attention to pay to each offer. Since firms' price-setting decisions depend on how strongly consumers react to price changes, understanding how consumers strategically allocate attention to offers is crucial. In particular, consumers' endogenous attention allocation can be a novel channel through which changes in the market structure shape economic outcomes.

This paper studies the impact of competition in markets with costly information processing. We show that increasing the level of competition has two effects. If the consumer's attention strategy remained unchanged, prices would be lower under competition than under collusion. This *pricing effect* is the same as in standard models with an exogenous downward-sloping demand curve. However, the consumer's optimal attention strategy is not fixed and depends on the prices she conjectures. Since the consumer correctly anticipates the pricing effect in equilibrium, increasing the level of competition also changes the consumer's optimal attention allocation. This new *attention effect* causes the consumer to engage in trade more often, effectively increasing her demand curve pointwise. For a parameter range of information costs, we show that the attention effect dominates the pricing effect: Firms' equilibrium profits are *higher* when they compete than when they collude.

We introduce rational inattention to an otherwise standard Bertrand duopoly setting. Our model consists of a representative consumer and two firms that sell a good of common quality.[1] The firms make take-it-or-leave-it offers to the consumer simultaneously. The consumer has unitary demand and chooses an information structure to learn about the good's quality and the firms' prices. Following an extensive literature building on Sims (2003), we take the information processing cost, or *attention cost*, to be proportional to the expected entropy reduction. After processing the information, the consumer decides whether and from which firm to buy the good.

Our model applies particularly well to markets where an offer is difficult to understand or compare across firms. Examples of this kind are complex loan contracts and insurance contracts, as described above. More generally, our model captures settings where consumers need to process information about offers, for example because they are comprised of several individual prices, fees, and discounts. Focusing on information processing allows us to model consumers' attention as a continuous variable.[2] In particular, rational inattention provides a tractable framework to analyze the resulting trade-off between optimal decision-making and costly information processing.

We characterize the firms' and consumer's behavior by using the solution concept of Bayes Nash Equilibrium. To avoid an infinite multiplicity of equilibrium outcomes, we impose a refinement that requires the consumer's strategy to be *robust to vanishing perturbations* (RVP). Because entropy costs ignore off-path events, rational inattention does not place any restriction on the consumer's behavior following a firm's deviation. RVP requires that for all possible deviations in the equilibrium price-setting behavior of the firms, the consumer strategy must be optimal against some vanishing belief perturbation consistent with the deviation.[3] This means that the consumer's off-path behavior can be rationalized by some arbitrarily small perturbations in the firms' strategies.

To isolate the effects of competition, we first establish a benchmark where the

[1]We extend our results to any number of firms in Section 4.3.

[2]Models with search costs (Stahl, 1989) or captive consumers (Varian, 1980) assume binary price information: Consumers either observe the price or not. In contrast, the rational inattentive consumer chooses information flexibly, allowing us to study the effects of subtle attention changes.

[3]RVP extends the notion of *credible best response* introduced by Ravid (2020) to a multi-firm setting, and is similar in spirit but weaker than Selten's (1975) trembling-hand perfection.

firms collude. In this collusive setting, the unique RVP trading equilibrium outcome is identical to the unique credible trading equilibrium outcome of Ravid's (2020) ultimatum bargaining game. Using this equivalence and Ravid's results, it follows that trade can be sustained under collusion if and only if the parameter k that governs the consumer's unit cost of information processing is below a threshold k^t. Moreover, trade under collusion is always inefficient as it occurs with a probability strictly lower than one. Efficient trade would require the consumer always to purchase the good, which implies never paying costly attention to prices, since any information is inconsequential to her purchase decision. However, colluding firms, lacking commitment power, would overcharge a completely inattentive consumer. This contradicts the optimality of the consumer's purchasing decision.

In our model of competition, we are interested in RVP equilibria in which both firms trade with positive probability, which we call *competitive trading equilibrium*.[4] We show that such an equilibrium exists if and only if trade can be sustained under collusion, i.e., if and only if k is below the trade-threshold k^t. Moreover, whenever a competitive trading equilibrium exists, it is unique.

The competitive trading equilibrium outcome depends on the unit cost of information processing k. We show that there exists an efficiency-threshold $k^e \in (0, k^t)$ such that if k is lower than k^e, then trade is efficient. This is in stark contrast with the collusive benchmark where trade is always inefficient. The consumer buys with probability one and disregards any information about the value of firms' offers relative to the no-purchase outside option. Since the consumer only processes information about the difference between the firms' prices, and not their absolute value, colluding firms would exploit this situation and coordinate on overcharging the consumer. In contrast, competing firms have incentives to undercut each other since the consumer still pays attention to how offers compare. If $k < k^e$, this attention strategy is enough to sustain the efficient trade outcome. As k increases, the level of detail with which the consumer processes information about price differences decreases and, consequently, the firms charge higher prices. If k is above

[4]Competitive trading equilibria are the only equilibria in which competitive forces are present. If one of the firms trades with probability zero, the active firm faces the same environment as a monopolist.

k^e, these prices become too large for the consumer to always buy, so she needs to make use of the no-trade outside option to discipline the firms further. In the region $k \in (k^e, k^t)$, trade is therefore inefficient. When k is above k^t, not even fully resorting to the no-trade outside option sufficiently disciplines the firms' pricing behavior. As a result, trade cannot be sustained in equilibrium, irrespective of the number of firms in the market and their incentives to compete.

Our analysis compares the competitive equilibrium outcome to the collusive one. We find that competition always increases trading efficiency when a competitive trading equilibrium exists. To see why, note that the consumer's attention strategy governs the demand the firms face. In particular, the attention strategy determines, for any pair of prices the firms offer, how likely the consumer is to buy from each of them. For a fixed attention strategy, the standard *pricing effect* of competition implies that competing firms charge lower prices than colluding ones. In turn, since the consumer correctly anticipates the firms' pricing strategies in equilibrium, and since her optimal attention strategy depends on these anticipated pricing strategies, competition also produces an *attention effect*. Due to the firms' better offers, the consumer wants to engage in trade more often when the firms compete, leading to a pointwise increase in the endogenous demand the firms face.[5]

Our main result shows that the increased trade efficiency of competitive markets can lead to a higher producer surplus. In particular, a parameter region of relatively high attention costs exists where firms achieve higher profits by competing than by colluding. Intuitively, if k is close to k^t, the consumer focuses on deciding *whether* to buy and pays less attention to comparing the firms' offers. This implies that competing firms' pricing behavior approximates that of colluding firms in this parameter region. Nevertheless, competing firms face a more favorable demand, leading to higher profits under competition than under collusion.[6]

A key insight behind our results is that competition alleviates the consequences of

[5] Demand changes also affect the firms' equilibrium pricing. In particular, for a fixed market structure, a firm's optimal price is higher the more the consumer engages in trade. As a result, the equilibrium effect of competition on prices is ambiguous: Prices decrease due to the pricing effect but increase due to the attention effect.

[6] While the consumer's trading probability goes to zero under both market structures, the ratio between the two is always bounded away from one.

firms' lack of commitment power in markets with inattentive consumers. If colluding firms could commit to charging a price equal to the good's quality, they would extract the full surplus. However, this pricing strategy does not align with firms' incentives *ex-post*, as they would find it optimal to exploit the consumer's inattention, ultimately leading to trade inefficiency. Under competition, firms have incentives to charge lower prices, alleviating the losses due to the lack of commitment power. The resulting higher trade efficiency yields our main result.

Our findings contrast with those of standard models that ignore information processing costs. Since the firms produce perfect substitutes, competition in frictionless models keeps demand unchanged and only puts downward pressure on prices, thus reducing firms' profits. Instead, when the consumer is rationally inattentive, we show that the attention effect of competition leads to an outward shift of demand, dominating the pricing effect for attention costs close enough to the trade-threshold k^t.

In addition to the producers, consumers can also benefit from competition: There exists a parameter range of attention costs, where competition simultaneously improves the welfare of *both* sides of the market. Obtaining general results about consumer surplus and total welfare is difficult as the consumer's payoff includes entropy-based information costs. However, we establish that the consumer benefits from competition whenever trade is efficient or the attention costs are relatively high. In the latter case, by our main result, producers may benefit as well.

Our analysis extends to competitive markets with any finite number of firms. The trade-threshold k^t is constant in the industry size. On the other hand, the parameter range where efficient trade can be supported in equilibrium expands with the number of firms as both the pricing and attention effect become stronger. Similarly, a region of sufficiently high attention costs always exists such that adding competing firms increases the producer surplus.

Related Literature. Our findings contribute to a recent literature (Ravid 2020; Denti, Marinacci, and Rustichini 2022; Wolitzky 2023) documenting inefficient trade in monopoly markets featuring consumers unable to observe offers perfectly. These works interpret inefficiency as a lack of monopolist's commitment power, which cannot avoid overcharging the inattentive consumer. We contribute to this literature by introducing the novel attention effect, which states that the demand increases pointwise due to increased competition. Our results show that competition alleviates the firms' commitment problem, allowing for efficient trade even when consumers are inattentive.[7]

Our modeling approach connects us to the literature studying rational inattention to uncertainty jointly controlled by nature and players, e.g., products' quality and prices. The issue of off-path unrestricted consumer behavior is central to this literature, which addresses it following different approaches.[8] We characterize the consumer's best response *everywhere* following Ravid (2020) by imposing robustness to firms' small mistakes. Instead, in a framework similar to ours, Matějka and McKay (2012) solves this issue by restricting the set of attention of strategies available to the consumer. In a monopoly model, Matějka (2015) constrains off-path behavior by subgame perfection, as the seller commits to a public price schedule.

We further relate to the literature on consumer search (Diamond 1971; Burdett and Judd 1983; Stahl 1989; Cachon, Terwiesch, and Xu 2008). Apart from delivering different equilibrium predictions, we differ substantially from this literature in how we regard consumers: Rational inattention models the consumer's information processing decision in a continuous manner, while this decision is binary in search models.

Our work also connects with the behavioral literature that justifies frictions in the observability of prices by bounded rationality (Varian 1980; Spiegler 2006; Gabaix and Laibson 2006; Hefti 2018; Heidhues, Johnen, and Koszegi 2021). De Clippel,

[7]Board and Pycia (2014) find a similar effect in a Coasian framework. They show that competition endogenously generates an outside option for consumers, allowing firms to avoid lowering their prices as it would occur under the Coase conjecture. Our work differs in two ways. First, the commitment problem stems from inattentive consumers rather than the possibility to make multiple offers over time. Second, while in our setting an increase in trade efficiency is responsible for the possibly larger profits, the mechanism works through increased prices in Board and Pycia (2014).

[8]This problem does not arise if attention is devoted to exogenous variables only, since the almost-sure solution provided by Matějka and McKay (2015) suffices in this case. This framework is used by a growing literature that investigates the market consequences of rational inattention to exogenous variables (Martin 2017; Boyacı and Akçay 2018; Yang 2019; Ravid, Roesler, and Szentes 2022; Mensch and Ravid 2022; Thereze 2022a; Thereze 2022b; Wu 2022; Albrecht and Whitmeyer 2023). Instead, we focus mainly on rational inattention to endogenous equilibrium objects.

Eliaz, and Rozen (2014) model attention as the number of markets consumers explore, finding that consumer welfare can be higher if the expected level of attention is lower. Armstrong and Vickers (2022) show that firms' entry may decrease consumer and increase producer surplus when consideration sets are *exogenous*. Our results are driven instead by *endogenous* attention allocation.[9] Attention shapes demand also in Bordalo, Gennaioli, and Shleifer (2016). The difference with our approach is that our rationally inattentive consumer allocates attention *ex-ante*, while their salient thinker allocates attention based on which attribute is salient *ex-post*.

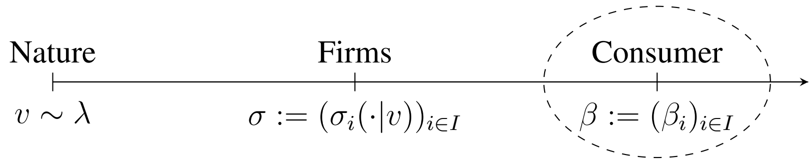
The literature on strategic price complexity provides a different rationale for consumers' mistakes by allowing firms to compete, besides prices, on the complexity level of their price structure (Carlin 2009; Piccione and Spiegler 2012; Chioveanu and Zhou 2013; Spiegler 2016). Instead, we take price complexity as given and compare the consumer's best response across different market structures.

2 Model

Two identical firms with zero marginal costs compete for a consumer's unitary demand. Product quality is common, stochastic, and perfectly observed by the firms. After observing the product quality, each firm makes a simultaneous offer to the consumer. The consumer does not directly observe the product quality or the firms' offers. Instead, she holds beliefs about their joint distribution and costly processes information to improve her purchasing decision. We interpret the consumer's information processing decision as an attention problem and use these terms interchangeably in our analysis. Following the literature on rational inattention initiated by Sims (2003), we call the consumer *rationally inattentive*, and we assume that the attention cost is entropy-based, as described below.

[9]Consumers in Armstrong and Vickers (2022) always trade whenever they observe at least one offer. Therefore, an increase in the producer surplus *implies* a decrease in the consumer surplus. This implication is not valid in our framework, as competition expands the trading surplus of the economy, allowing the coexistence of higher producer and consumer surplus. See Section 4.2 for more results on consumer surplus.

Game structure. The following timeline summarizes the game structure of our model. We formalize each element below.



1. Product quality v is drawn according to a probability measure $\lambda \in \Delta(\mathbb{R})$. We assume that λ has strictly positive finite support, i.e., $\operatorname{supp}\lambda =: V \subseteq (0,\infty)$ is finite.
2. After observing the product quality realization v, each firm $i \in I := \{1,2\}$ makes a simultaneous offer to the consumer. We denote by $\sigma_i : V \to \Delta(\mathbb{R}_+)$ firm i's strategy and by $x_i \in \mathbb{R}_+$ the price firm i charges.
3. We refer to each profile of the exogenous product quality and the endogenous firms' prices (v, x_1, x_2) as a *state*. The consumer does not directly observe the realized state. Rather, she holds beliefs and pays attention by selecting an *information structure* to learn about it. The attention cost is proportional to the mutual information between states and signals, which equals the expected entropy reduction between the consumer's prior belief over states and her posterior beliefs obtained via Bayesian updating. After a signal realizes, the consumer makes a purchasing decision, and the game ends.

   Without loss of generality, we restrict the consumer's strategy space to *recommendation strategies*, which we refer to as *attention strategies*.[10] A recommendation strategy β is a profile (β_1, β_2) such that $\beta_i : V \times \mathbb{R}^2_+ \to [0,1]$ denotes the *conditional* probability of accepting the offer of firm $i \in I$. That is, $\beta_i(v, x_1, x_2)$ is the probability of receiving the recommendation "accept i's offer" given the realized state (v, x_1, x_2). Naturally, for every $(v, x_1, x_2) \in V \times \mathbb{R}^2_+$, it holds that $\sum_{i\in I} \beta_i(v, x_1, x_2) \le 1$; with the remaining probability, the consumer accepts no offer.

   Denote the consumer's prior belief over states by $\mu \in \Delta(V \times \mathbb{R}^2_+)$.[11] Exploiting

[10] See Matějka and McKay (2015) for the optimality of recommendation strategies when the consumer is restricted to pure strategies and Ravid (2019) for an extension to mixed strategies.

[11] We endow $V \times \mathbb{R}^2_+$ with the product σ-algebra between the discrete σ-algebra on V and the

the restriction to recommendation strategies, we write mutual information as

$$I(\mu, \beta) := H(\mathbb{E}_\mu[\beta]) - \mathbb{E}_\mu[H(\beta)], \tag{1}$$

where $H(p) = -p_1 \log(p_1) - p_2 \log(p_2) - (1 - p_1 - p_2)\log(1 - p_1 - p_2)$ is the Shannon entropy associated with the probability measure $(p_1, p_2, 1 - p_1 - p_2)$ consistent with $p = (p_1, p_2)$. Mutual information as formalized in equation (1) asserts that attention costs are proportional to the difference in entropy between the conditional and the unconditional distribution of playing each action. In other words, $I(\mu, \beta)$ captures how much the uncertainty about a specified plan of action decreases with information.[12]

Payoffs. Once firms make offers and the consumer selects a recommendation strategy, payoffs are obtained. The consumer's utility given product quality $v \in V$ is

$$U := \sum_{i \in I} (v - x_i)\beta_i(v, x_1, x_2) - k \cdot I(\beta, \mu).$$

That is, the consumer's utility equals her gains from trade net of the costs of processing information. The parameter $k > 0$ is the consumer's unit cost of information processing and represents the cost assigned to each bit of processed information.

The payoff obtained by each firm $i \in I$ equals

$$\Pi_i^C := \beta_i(v, x_1, x_2) \cdot x_i,$$

where the superscript C stands for "Competition." The competing firms in our model adopt the standard profit-maximizing behavior. Since the consumer uses a recommendation strategy, $\beta_i(v, x_1, x_2)$ represents the endogenous demand firm i faces.

standard Borel σ-algebra on $\mathbb{R}^2_+$. We also endow both $\Delta(\mathbb{R}^2_+)$ and $\Delta(V \times \mathbb{R}^2_+)$ with the topology of strong convergence.

[12]Entropy-based information costs are tractable and allow us to make sharp economic predictions. However, they are not necessary for our results. See Cusumano, Fabbri, and Pieroth (2022) for further discussion.

Discussion. In our framework, prices are not directly observable by the consumer. Instead, she chooses an information structure that (stochastically) determines what she learns about offers. As motivated in the Introduction and discussed also in Ravid (2020), this assumption captures situations where offers entail complex contracts or include multiple prices. Additionally, this assumption can be viewed as modeling a consumer at a very early stage of the purchasing decision, when she does not yet have access to prices. Consider, for example, the decision of where to have dinner. While restaurant prices are perfectly observable, one has to physically go to the restaurant or visit their website to actually access the menu. Either of these processes involves costs in time and cognitive effort. If ex-post switching costs are high,[13] this leads to the same trade-off between optimal decision-making and costly information processing studied in our model.

2.1 Equilibrium Refinement

We adopt Bayes Nash Equilibrium (BNE) as the solution concept for our duopoly model with rational inattention. The *assessment* (μ, σ, β) is a BNE if (i) μ is *consistent* with σ,[14] (ii) β is a best response to μ, and (iii) for every $i \in I$, σ_i is a best response to σ_{-i} given β.

As discussed by Ravid (2020), standard BNE is too permissive to make sharp predictions about equilibrium outcomes in games with rational inattention directed toward endogenous equilibrium quantities. As the consumer's attention cost is prior-dependent, it is unaffected by off-path contingencies. Therefore, despite the firms' optimal behavior depending on the consumer's reaction on and off-path, BNE does not require the consumer's off-path threats to be credible. As Ravid (2020) shows, one can avoid this problem by allowing firms to make arbitrarily small mistakes on-path. We follow this approach and refine BNE by imposing an additional property on the consumer's best response which we call *robustness to vanishing perturbations* (RVP). RVP requires the consumer's strategy to be justified under some arbitrarily small belief perturbations both on and off the conjectured path of play. It

[13]This must include costs for switching to the outside option, which in this example could be a delayed dinner at home. We thank an anonymous referee for pointing this out.

[14]The belief $\mu \in \Delta(V \times \mathbb{R}^2_+)$ is *consistent* with the profile σ if for every $v \in V$, and for every Borel measurable set $E \subseteq \mathbb{R}^2_+$, we have $\mu(v, E) = \lambda(v) \cdot \int_E 1 \, \mathrm{d}\sigma_1(\cdot|v) \otimes \sigma_2(\cdot|v)$.

implies that the consumer no longer considers perfectly informative off-path signals to be costless.

Definition 1. *Let μ be consistent with profile σ and let β be a best response to μ. We say that β is robust to vanishing perturbations (RVP) if for every $v^* \in V$ and $x_1^*, x_2^* \geq 0$, there exists a sequence $(\mu^n, \tilde{\sigma}^n)$ such that for all $v \in V$ and $n \in \mathbb{N}$*

- $\tilde{\sigma}^n(\cdot|v) \in \Delta(\mathbb{R}_+^2)$ *is a possibly correlated probability measure on $\mathbb{R}_+^2$,*
- $\tilde{\sigma}^n(x_1^*, x_2^*|v^*) > 0$,
- $\tilde{\sigma}^n(\cdot|v) \to \sigma_1(\cdot|v) \otimes \sigma_2(\cdot|v)$ *strongly,*
- μ^n *is consistent with $\tilde{\sigma}^n$,*
- β *is a best reply to μ^n.*

Definition 2. (μ, σ, β) *is an RVP equilibrium if it is a BNE and β is RVP.*

RVP extends the notion of *credible best response* introduced by Ravid (2020) to a multi-firm setting, and is similar in spirit, albeit considerably weaker than, Selten's (1975) trembling-hand perfection.[15] Like Ravid, we allow belief perturbations to vary with off-path deviations while trembling hand perfection does not. Moreover, we allow for correlated off-path belief perturbations about the firms' offers.[16]

We solve our duopoly model using RVP equilibrium (hereafter, just *equilibrium*). Despite its weakness, this refinement is strong enough to obtain sharp predictions regarding the offers accepted by the consumer on-path and the overall trade probability. These variables are sufficient to characterize the most important economic statistics of our model: Trade efficiency, producers' profits, and consumer surplus.

2.2 Consumer's Best Response

In this subsection, we describe the consumer's RVP best response. While it is possible to characterize the consumer's optimal attention strategy against arbitrary beliefs, to analyze the equilibrium effect of competition, it is sufficient to restrict

[15] Formally, RVP requires that for every state, a sequence of vanishing belief perturbations exists such that (i) the sequence puts a positive probability on that state, and (ii) β is a best reply to every element of the sequence. Instead, trembling hand perfection requires that a sequence of vanishing *full-support* belief perturbations exists such that β is a best reply to every element of the sequence.

[16] Allowing for off-path belief correlation has two advantages: It makes our refinement particularly weak, and allows for cleaner proofs. Off-path belief correlation, however, is not necessary for our analysis: Our results still hold if we impose independent beliefs instead.

attention to symmetric assessments. We provide an informal explanation of this fact after Lemma 1.

Definition 3. *We say that: (i) β is symmetric if $\beta_i(v, x_1, x_2) = \beta_{-i}(v, x_2, x_1)$ for all $v \in V$, $x_1, x_2 \geq 0$; (ii) $\sigma = (\sigma_1, \sigma_2)$ is symmetric if $\sigma_1 = \sigma_2$; (iii) the assessment (μ, σ, β) is symmetric if μ is consistent with σ, and both σ and β are symmetric.*

Lemma 1. *Let (μ, σ, β) be a symmetric assessment. Then, β is a RVP best response to μ if and only if, for every $v \in V$ and $x_1, x_2 \geq 0$, we have*

$$\beta_i(v, x_1, x_2) = \frac{\pi_i \cdot e^{\frac{v - x_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v - x_j}{k}} + 1 - \pi_1 - \pi_2} \quad (i \in I) \tag{2}$$

where

(i) $\pi_i = \mathbb{E}_\mu [\beta_i]$ *for every* $i \in I$,

(ii) there exists $\pi \in [0, 1/2]$ *such that* $\pi_1 = \pi_2 = \pi$.

Moreover, exactly one of the following statements is true:

(iii) $\pi = 0$ *and* $\mathbb{E}_\mu \left[e^{\frac{v - x_i}{k}} \right] \leq 1$ *for every* $i \in I$,

(iv) $\pi = 1/2$ *and* $\mathbb{E}_\mu \left[\left(e^{\frac{v - x_1}{k}} + e^{\frac{v - x_2}{k}} \right)^{-1} \right] \leq 1/2$,

(v) $\pi \in (0, 1/2)$, $\mathbb{E}_\mu \left[e^{\frac{v - x_i}{k}} \right] \geq 1 \ \forall i \in I$, *and* $\mathbb{E}_\mu \left[\left(e^{\frac{v - x_1}{k}} + e^{\frac{v - x_2}{k}} \right)^{-1} \right] \geq 1/2$.

To understand Lemma 1, recall that in a standard rational inattention framework, a decision maker's optimal behavior is characterized by a multinomial logit formula μ-almost surely.[17] Our refinement extends this feature: Lemma 1 shows that β is a consumer's symmetric RVP best response to μ if and only if it displays a multinomial logit formula adjusted for the consumer's prior beliefs *everywhere*.

More specifically, equation (2) identifies the entire class of RVP best responses to μ. Point (i) is the standard optimality condition of rational inattention problems: Each π_i must be equal to the average probability of trade with that firm. Accordingly, each π_i represents the consumer's *trade engagement level* with firm $i \in I$. Since β is symmetric if and only if $\pi_1 = \pi_2 = \pi$, point (ii) imposes symmetry on β. Finally, points (iii), (iv), and (v) characterize when the symmetric trade engagement level π is optimal given that μ is consistent with a symmetric strategy profile

[17] See Csiszár (1974), Matějka and McKay (2015), and Denti, Marinacci, and Montrucchio (2020).

of the firms.

To see why we can restrict the equilibrium analysis of competition to symmetric assessments, consider the following argument.[18] First, competition only plays a role in equilibrium when both firms actively trade with the consumer. Otherwise, the unique active firm behaves like a monopolist, making de facto inconsequential the presence of the competitor. Furthermore, firms are ex-ante identical, so the consumer cannot trade with them asymmetrically. Intuitively, if the consumer traded with firm 1 *more often*, i.e., $\pi_1 > \pi_2 > 0$, firm 1 would charge higher equilibrium prices. However, this would induce the consumer to trade *less often* with firm 1, a contradiction. Thus, any equilibrium where firms actively compete must feature a consumer's symmetric recommendation strategy. In turn, this implies symmetric equilibrium play from the firms.

We call $(\psi, \xi) \in [0, 1] \times \mathbb{R}_+^V$ an *equilibrium outcome* whenever $\psi = \pi_1 + \pi_2$ is the consumer's overall trade engagement level, and $(\xi(v))_{v \in V}$ are the symmetric equilibrium offers accepted by the consumer on-path. We say that two assessments are *outcome equivalent* if they imply the same equilibrium outcome. Abusing notation, the equilibrium outcome associated with a no-trade equilibrium is $(0, \emptyset)$.

3 Benchmark: Collusion

To understand the equilibrium effects of competition, we formulate a benchmark case where the firms collude. Under collusion, the firms perfectly internalize each others' profits and set prices jointly. For this reason, the producer surplus describes the preferences of each colluding firm:

$$\Pi^M := \sum_{j \in I} \beta_j(v, x_1, x_2) \cdot x_j,$$

where M stands for "Monopoly." All other aspects of the model remain unchanged.

Our model with collusion is outcome equivalent to the ultimatum bargaining model of Ravid (2020), which we henceforth refer to as the *monopoly model* for

[18] See Cusumano, Fabbri, and Pieroth (2022) for a formal proof.

simplicity of exposition.[19] The intuition is as follows. If the consumer trades with only one of the two firms in equilibrium, collusion is *de facto* equivalent to a monopoly. On the other hand, if both firms are active, the consumer's attention strategy satisfies the multinomial logit of equation (2) adjusted for some $\pi_1, \pi_2 > 0$. In the appendix, we show that in equilibrium, sellers' offers under collusion are symmetric and, more importantly, equal to the monopolist's offer of Ravid's (2020) model when facing the *aggregate demand*. Intuitively, when firms perfectly internalize each other's profits, they have no incentive to charge different prices.[20] Moreover, if they charge the same price, they face the same aggregate demand as a monopolist. As a result, they act as if they were serving the consumer in a monopoly market, which implies that the analysis of Ravid (2020) applies verbatim to our collusion benchmark.

Let $k^t > 0$ be defined by

$$\mathbb{E}_\lambda \left[e^{v/k^t - 1} \right] = 1. \tag{3}$$

The following result characterizes the main equilibrium predictions of the collusive trading equilibrium, i.e., the equilibrium under collusion in which the consumer trades with positive probability.

Proposition 1. *[Theorem 1 and Corollary 1 of Ravid (2020)] A collusive trading equilibrium outcome exists if and only if $k < k^t$. If a collusive trading equilibrium outcome exists, it is unique and equilibrium trade is inefficient. That is, $\pi_1 + \pi_2 < 1$.*

Proposition 1 emphasizes two main features of the analysis of Ravid (2020), which extend to our collusion benchmark and will subsequently be used as a comparison point for our findings under competition. First, trade cannot be sustained as an equilibrium outcome if attention costs are too high. Second, equilibrium trade is never efficient: The consumer's probability of buying the product is always smaller than 1. The main force at play for both results is that the consumer does not process

[19]Since products are perfectly homogeneous, the equivalence between collusion and monopoly is straightforward without rational inattention. However, in the presence of information processing costs, colluding firms may use different prices to influence the consumer's attention. Nevertheless, our analysis shows that this strategic manipulation of the consumer's attention does not bite in equilibrium.

[20]More concretely, if firm i charges a higher price, the consumer optimally shifts her demand away from i, which makes it optimal for the cartel to lower i's price, a contradiction.

enough information to sustain efficient trade. If information costs are too high, the consumer does not pay enough attention to prevent firms from overcharging, making it sub-optimal for her to trade with positive probability. For lower information costs, trade can be sustained in equilibrium, but it is always inefficiently low. To understand why, suppose that the consumer trades with certainty. Always accepting either offer is equivalent to never using the no-trade *outside option*, which, in an RVP equilibrium, implies that the consumer disregards learning about prices in absolute terms. As a result, colluding firms could coordinate on a simultaneous price increase, making equilibrium offers too unappealing to sustain positive trade.

4 Competition

To study the impact of competition, we first identify equilibria in which competitive forces are present. Robustness to vanishing perturbations implies that competition delivers a finite multiplicity of equilibrium outcomes. First, there always exists a trivial no-trade equilibrium outcome, where firms overcharge the consumer, who consequently does not trade. Second, there is a class of equilibria where the consumer only trades with one firm. This class is outcome equivalent to all collusive trading equilibria. Finally, there is an equilibrium where the consumer trades with both firms. Since this is the unique equilibrium class where firms actively compete on-path, we call this equilibrium the *competitive trading equilibrium*. In this section, we characterize its properties and describe the equilibrium effects of competition.

4.1 The Competitive Trading Equilibrium

As argued in Section 2.2, every competitive trading equilibrium must be symmetric. Given the consumer's best reply of Lemma 1, firms behave as if they are facing a symmetric downward-sloping multinomial logit demand. The following lemma characterizes the firms' equilibrium strategies.

Lemma 2. *Suppose (μ, σ, β) is a competitive trading equilibrium. For every $v \in V$,* *each seller $i \in I$ plays a symmetric pure strategy $\sigma_i(\cdot|v) = \delta_{x^C(v)}$ given by*

$$x^C(v) = k \cdot (1 + \phi(v)) \tag{4}$$

where $\phi = \phi(v)$ is the unique solution to

$$\left(1 + e^{\phi} \cdot \frac{1 - 2\pi}{\pi e^{\frac{v-k}{k}}}\right) \phi = 1. \tag{5}$$

Lemma 2 follows from the fact that firms are not facing a perfectly elastic demand, even though their products are homogeneous. Notice that the consumer behaves as if products were differentiated because the information she gets is noisy. It is too costly to process information about ex-post gains from trade perfectly and, as a result, the consumer is not certain about the best offer and makes mistakes. This mechanism explains why, according to equation (4), the firms can charge prices above marginal costs in equilibrium.

The relationship between the consumer's trade engagement level and firms' offers is central to our analysis. The function ϕ captures the price-setting incentives of the firms. Note that for fixed product quality v and information cost k, equation (5) shows that $\phi(v)$ is increasing in the consumer's trade engagement level π. If firms submit appealing offers to the consumer, the consumer chooses a high trade engagement level, in line with Lemma 1. At the same time, if the consumer engages more in trade, demand expands and elasticity declines. As a result, the firms submit worse offers.

The following theorem characterizes the existence and uniqueness of the competitive trading equilibrium by identifying the region of attention costs where competition sustains trade in equilibrium. Recall from equation (3) that k^t is defined as the unique solution to $\mathbb{E}_\lambda \left[e^{v/k^t - 1}\right] = 1$.

Theorem 1. *A competitive trading equilibrium exists if and only if $k < k^t$. If a competitive trading equilibrium exists, it is unique.*

Competition cannot sustain trade when attention costs are too high. The consumer is unwilling to process any information, implying that demand does not change with firms' offers. As a result, firms overcharge the consumer, leading to a breakdown of trade. Conversely, a competitive trading equilibrium exists if attention costs are moderately low. The consumer is willing to process some information to find the best offer. Consequently, the firms face downward-sloping demand curves and make appealing offers to the consumer.

The theorem also shows that whenever a competitive trading equilibrium exists, it is unique. For instance, suppose there is a second competitive trading equilibrium in which the consumer's overall trade engagement level is higher. Due to this expansion in demand, firms' marginal revenue is higher everywhere, prompting firms to make less appealing offers compared to the original equilibrium. This induces the consumer to *reduce* the overall trade engagement level, a contradiction.

An immediate implication of Theorem 1 is that competition cannot sustain trade when collusion could not: In both cases, a trading equilibrium exists if and only if $k < k^t$. When the attention costs exceed k^t, the consumer does not process any information. Since this includes information about which offer is better, the downward pressure on prices induced by competition vanishes when $k > k^t$, implying that competition cannot prevent a breakdown in trade.

4.2 Equilibrium Effects of Competition

We investigate the impact of competition in markets with rational inattention by comparing the competitive and the collusive trading equilibrium outcomes.

The pricing effect. Lemma 1 implies that any symmetric RVP best response follows equation (2) and is thus pinned down by π, the trade engagement level with each firm. The following lemma describes the pricing effect of competition: If the consumer's trade engagement level, and therefore attention strategy, were fixed across market structures, competing firms would charge lower prices than colluding firms.

Denote by $x^C(v;\pi)$ a competitive firm's optimal price when the consumer's symmetric trade engagement level with each firm equals π. Similarly, let $x^M(v;2\pi)$ be the optimal price of colluding firms facing an overall trade engagement level of 2π.

Lemma 3. *For any fixed symmetric RVP consumer's best response, competing firms charge lower prices than colluding firms. Formally, for all $\pi \in (0, 1/2)$ and $v \in V$,* *we have* $x^C(v;\pi) < x^M(v;2\pi)$.[21]

Recall that the trade engagement level determines the downward-sloping demand the firms face. The pricing effect follows since, for any given demand, competing firms may benefit from undercutting the competitor, while collusive firms do not.

The attention effect. Proposition 2 describes a novel effect of competition that we name the *attention effect*: The consumer's endogenous demand expands when firms compete. In other words, for any level of attention cost, the overall trade engagement level under competition is strictly higher than under collusion. Competition thus alleviates the efficiency losses that occur under collusion, where trade is always inefficient due to costly information processing. Formally, denote by $2\pi^C := \pi_1^C + \pi_2^C$ and $\pi^M := \pi_1^M + \pi_2^M$ the overall trade engagement level under competition and collusion, respectively.

Proposition 2. *The consumer engages in trade more often under competition than under collusion: For any* $k \in (0, k^t)$, $0 < \pi^M < 2\pi^C \leq 1$.

The intuition is as follows. Suppose both competing firms face half the demand faced by a monopolist. The resulting offers would be more favorable to the consumer due to the *pricing effect*, i.e., the fact that competing firms have stronger incentives to charge low prices since they do not internalize each other's profits. However, the consumer would trade more often at these lower prices. Therefore, the resulting equilibrium trade probabilities have to satisfy $\pi^M < 2\pi^C$.

As a consequence of the attention effect, we show in Proposition 3 that an equilibrium with efficient trade, which we call an *efficient equilibrium*, exists when the consumer's unit attention cost is relatively low. To see why, suppose trade is efficient, i.e., the consumer trades with each firm with equal probability $\pi = 0.5$. By Lemma 2, this implies that firms charge a price of $x(v) = 2k$ for all $v \in V$. From Lemma 1, we know that, under this configuration of prices, the consumer wants to trade with certainty if and only if $\mathbb{E}_\lambda \left[e^{2-v/k}\right] \leq 1$. Thus, let $k^e > 0$ be the unique

[21]For $\pi = 0$, all prices constitute a best response for the firms regardless of the market structure. For $\pi = 1/2$, the colluding firms' optimal pricing strategy is not well defined, since they would always prefer to charge a higher price, while the competing firms optimally charge a finite price.

solution to

$$\mathbb{E}_\lambda\left[e^{2-v/k^e}\right] = 1.$$

Notice that k^e is lower than the threshold characterizing the existence of a competitive trading equilibrium, i.e., $k^e < k^t$.[22]

Proposition 3. *Under competition, an efficient equilibrium exists if and only if* $k \leq k^e$.

When trade is efficient, the consumer demand does not react to offers in absolute terms but only to price differences.[23] Under collusion, this attention strategy does not prevent firms from submitting unreasonable offers. In contrast, as competing firms have incentives to undercut the competitor's offer, this strategy is effective in disciplining prices under competition. Moreover, equilibrium prices strictly increase with k. Intuitively, the higher the unit attention cost k, the less the consumer reacts to price changes. This effect leads firms to charge a higher equilibrium price, which explains the existence of the threshold $k^e > 0$ that characterizes equilibrium trade efficiency. If the consumer's attention costs exceed k^e, the constant equilibrium price becomes too high relative to the expected quality, and buying with certainty is not optimal for the consumer.

Observe that in an efficient equilibrium attention costs are equal to zero and, therefore, the economy achieves first-best social welfare. When $k \leq k^e$, the consumer uniformly randomizes between the firms' offers and does not pay information processing costs on path. As a result, the (ex-post) social welfare of the economy is maximal and equals v. This prediction contrasts with the collusion benchmark: When the firms collude, the consumer must resort to the no-trade outside option to discipline firms' pricing strategies, which results in welfare losses.

Figure 1 illustrates Propositions 2 and 3. It displays the equilibrium trade engagement level under competition and collusion as a function of $k \in (0, k^t)$. For values of $k \leq k^e$, the unique competitive trading equilibrium features efficient trade, i.e., $2\pi^C = 1$. For $k > k^e$, $2\pi^C$ is decreasing in k, but as Proposition 2 shows, it is

[22] $\mathbb{E}_\lambda[e^{2-v/k^t}] > \mathbb{E}_\lambda[e^{1-v/k^t}] = \mathbb{E}_\lambda[\frac{1}{e^{v/k^t-1}}] \geq \frac{1}{\mathbb{E}_\lambda\left[e^{v/k^t-1}\right]} = 1$. Therefore, $k^e < k^t$.

[23] From equation (2), if $\pi = 1/2$, the consumer's symmetric best response becomes $\beta_i(v, x_1, x_2) = 1/(1 + e^{\frac{x_i - x_j}{k}})$ for every $i \in I$.

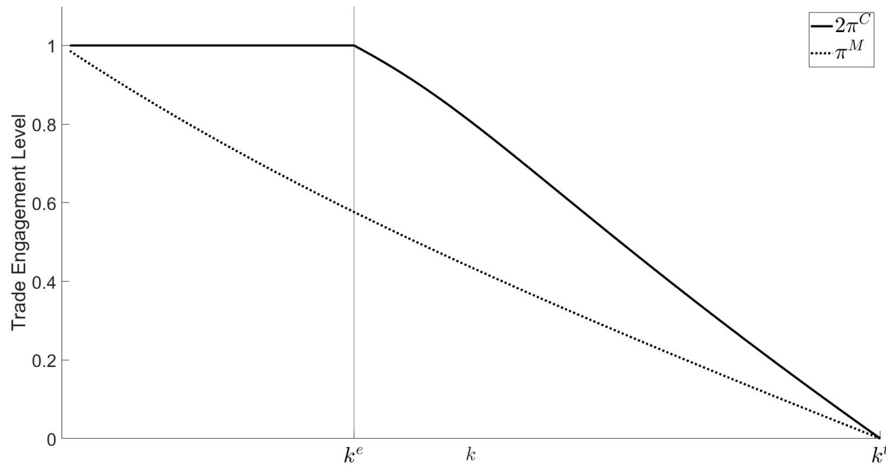


Figure 1: Overall trade engagement level in the competitive and collusive trading equilibrium with a binary quality distribution.

always strictly above π^M.

The following corollary describes the consumer's observable behavior under competition.

Corollary 1. *In a competitive trading equilibrium, the consumer's attention strategy satisfies*

$$\beta_i(v, x^C(v), x^C(v)) = \begin{cases} 1/2 & \textit{if } k \leq k^e \\ 1 - k/x^C(v) & \textit{if } k \in (k^e, k^t) \end{cases} \quad \textit{for all } v \in V.$$

When the cost of information processing is small, the equilibrium resembles the standard Bertrand competition outcome. Figure 1 shows that the consumer buys with certainty when $k \in (0, k^e)$ and Figure 2 shows that she does so at a price that does not vary with quality. However, as the consumer's information processing costs grant the industry positive market power, firms successfully submit offers above marginal cost. When k is above k^e, we observe an equilibrium outcome resembling Ravid's (2020) analysis: Trade is inefficient and firms' offers depend on the product's quality (see Figure 2). In the limit as $k \uparrow k^t$, the firms' behavior is identical in both settings because the consumer's information processing about rel-

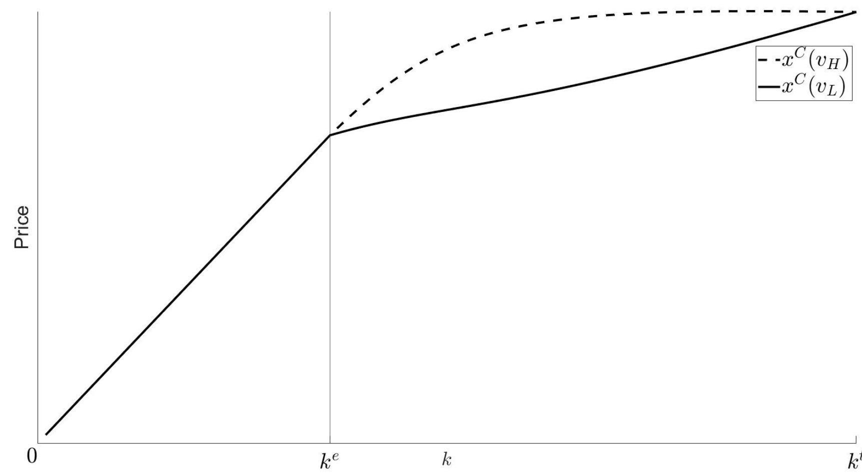


Figure 2: Equilibrium prices in the competitive trading equilibrium with a binary quality distribution.

ative prices vanishes.[24] This explains why the range of attention costs that support trade is the same under competition and collusion, as shown by Theorem 1.

Producer surplus. Propositions 2 and 3 show that trade efficiency increases with competition, which implies that the sum of producer surplus and consumer trade surplus, i.e., the consumer surplus without considering attention costs, increases. The remainder of this section studies which side benefits from competition. Without information processing costs, i.e., at $k = 0$, competition benefits the consumer at the expense of the firms. Since the equilibrium outcomes are continuous in k, this also holds for small information processing costs.

In contrast, Theorem 2 shows that when attention costs are high enough within the range $(0, k^t)$, the aggregate producer surplus under competition is *higher* than under collusion. For such costs, the result says that the positive effect that competition has on demand *dominates* the negative pricing effect: The attention effect prevails over the pricing effect.

[24] As the overall trade engagement level decreases, the likelihood that the consumer considers the competitor's offer vanishes, and the firms focus on prevailing over the no-trade outside option: Each competing firm's objective approximates the one of a colluding firm.

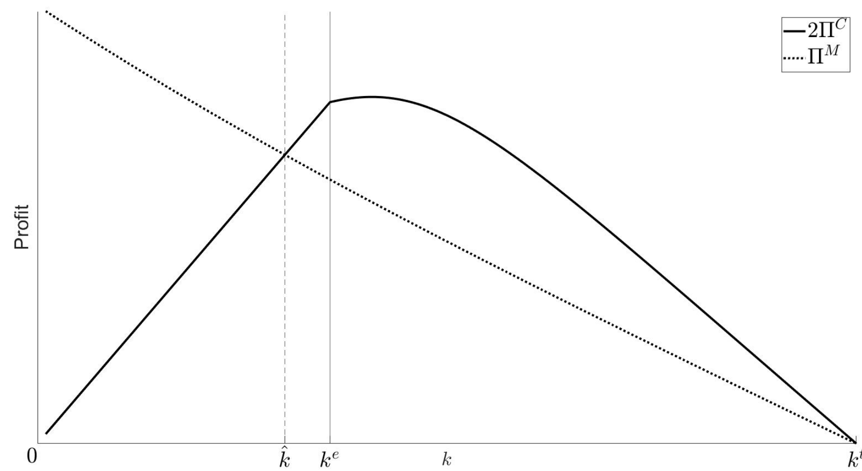


Figure 3: producer surplus in the competitive and collusive trading equilibrium with a binary quality distribution.

For every attention cost $k \in (0, k^t)$, let $\Pi^M(k)$ and $2\Pi^C(k)$ be the aggregate producer surplus in the collusive and the competitive trading equilibrium, respectively.

Theorem 2. *There exists $\hat{k} \in (0, k^t)$ such that the producer surplus is higher under competition than under collusion for all k between $\hat{k}$ and k^t, i.e., $2\Pi^C(k) > \Pi^M(k) > 0$ for all $k \in (\hat{k}, k^t)$.*

Figure 3 illustrates the content of the theorem for a specific distribution of v. The sum of the competitors' profits is larger than the collusive profits for information costs in the interval $(\hat{k}, k^t)$. Theorem 2 states that such a region exists for any distribution of v.

The proof of Theorem 2 revolves around the use of *L'Hopital's rule* to prove that

$$\lim_{k \uparrow k^t} \frac{\pi^C(k)}{\pi^M(k)} > \frac{1}{2},$$

even though trading probabilities under competition and collusion are both converging to zero as $k \uparrow k^t$. This fact implies that, as k grows large, (i) the behavior of each firm in the competitive trading equilibrium approximates the equilibrium behavior of the colluding firms and, at the same time, (ii) each firm faces strictly more than half the aggregate equilibrium demand under collusion. As a consequence of (i) and (ii), we conclude that for k close to k^t, the firms' total profits are strictly higher under competition than under collusion, proving the statement.

Theorem 2 implies that competition can alleviate the commitment problem faced by colluding firms. Colluding firms cannot extract the full surplus since they cannot commit to a pricing strategy, and instead price according to their ex-post incentives. When firms compete, these incentives change, which ultimately leads the consumer to expand her demand due to the attention effect. The takeaway is that firms can benefit from pricing competitively as this reduces the efficiency loss due to their lack of commitment.

Consumer surplus. Under collusion, rational inattention allows the consumer to obtain a positive utility in situations where costless information does not. Competition reverses this logic: When multiple firms are active, the consumer's surplus is smaller than the total surplus for any $k > 0$, which implies that it is smaller than the surplus obtained in the frictionless benchmark ($k = 0$).

In general, characterizing how consumer surplus reacts to competition for a fixed unit cost of information processing is difficult.[25] Competition puts downward pressure on prices for any given trade engagement level by the consumer, making her better off. However, at the same time, the attention effect leads the consumer to engage in trade more often, which increases the price the firms charge. The net effect is therefore ambiguous. Moreover, whether competition makes information processing less or more costly is ambiguous in equilibrium unless $k \leq k^e$.

Lemma 4 shows that if the consumer is uncertain about the quality of the product, she always benefits from competition whenever expected market offers decrease. Denote by U^m the *consumer surplus*, i.e., the consumer's expected gains from trade minus her information processing costs, and by x^m the firms' offers when the mar-

[25]Consumer surplus is easy to characterize only when quality is commonly known, i.e., V is a singleton. Suppose $V = \{v_o\}$ with $v_o > 0$. Under collusion, offers equal v_o, implying that the consumer has no trading surplus. Under competition, Lemma 2 implies that offers equal v_o if $k > k^e = v_o/2$, and $2k \leq v_o$ otherwise. Additionally, since firms use pure strategies and quality is known, the consumer does not incur any information processing costs on path in both cases. Therefore, the consumer benefits strictly from competition when $k < v_o/2$ and receives the same surplus otherwise.

ket structure is $m \in \{C, M\}$.

Lemma 4. *If quality is uncertain, the consumer surplus is strictly higher under competition whenever, in expectation, competitive prices are lower than under collusion: If* $|V| > 1$, *then* $\mathbb{E}_\lambda[x^C(v)] \leq \mathbb{E}_\lambda[x^M(v)]$ *implies* $\mathbb{E}[U^C] > \mathbb{E}[U^M]$.

To prove Lemma 4, we show that competition implies a more dispersed distribution of the consumer's gains from trade than collusion. Since rationally inattentive agents enjoy risk in (ex-post) utils, we conclude the argument by invoking well-established results in risk theory (Meyer, 1977).

We use Lemma 4 to identify two sufficient conditions under which competition unambiguously helps the consumer: Efficient trade, $k \leq k^e$, and high information processing costs, $k \uparrow k^t$. In light of Theorem 2, this last result implies that when information processing costs are high enough, competition can (Pareto) improve the economic situation of both sides of the market: Producers and consumer altogether.

Proposition 4. *There exists* $\bar{k} \in [k^e, k^t)$ *such that the expected consumer surplus weakly increases with competition whenever* $k \in (0, k^e] \cup (\bar{k}, k^t)$. *Moreover, if* $\max\{\bar{k}, \hat{k}\} < k < k^t$, *competition increases surplus for both sides of the market.*

4.3 More than Two Firms

In this subsection, we extend the analysis of Section 4.2 to the presence of more than two firms.[26] When firms perfectly internalize each other profits, they jointly maximize the producer surplus, implying that each firm effectively behaves as if facing the consumer in a monopoly market. As a result, the equilibrium predictions under collusion are independent of the number of firms $N \geq 2$.

The number of active firms matters under competition since it impacts the strength of both the pricing and attention effect. We proceed to show that equilibrium trade efficiency expands, implying that the region of attention costs where competition sustains efficient trade becomes larger as more firms compete. Furthermore, the attention effect still dominates the pricing effect when k grows large, ensuring that Theorem 2 extends to an arbitrary number of firms.

[26] Since the results of Section 2.2 naturally extend to $N \geq 2$ firms, we omit to discuss the consumer's optimal behavior. In particular, RVP characterizes the consumer's best response *everywhere* in the state space $V \times \mathbb{R}_+^N$ by a multinomial logit formula adjusted for the parameters $(\pi_1, ..., \pi_N)$.

We say that a trading equilibrium is *competitive* if *all* firms are active. First, we characterize the firms' optimal behavior in equilibrium.

Lemma 5. *Suppose* (μ, σ, β) *is a competitive trading equilibrium with* $N \geq 2$ *firms. For every* $v \in V$*, each seller* $i \in I$ *plays a symmetric pure strategy* $\sigma_i(\cdot|v) = \delta_{x^C(v,N)}$ *given by*

$$x^C(v, N) = k \cdot (1 + \phi(v, N))$$

where $\phi = \phi(v, N)$ *is the unique solution to*

$$\left((N-1) + e^{\phi} \cdot \frac{1 - N\pi^N}{\pi^N e^{\frac{v-k}{k}}}\right)\phi = 1. \tag{6}$$

Equation (6) is the counterpart to equation (5) with $N \geq 2$ firms, where $\pi^N \in (0, 1/N]$ represents the symmetric trade engagement level of the consumer with each firm in equilibrium.[27] It shows that for any fixed overall trade engagement level $\bar{\pi}$ of the consumer, the firm's undercutting incentives intensify as the number of active firms increases: The pricing effect becomes stronger as N gets large.[28]

The following result describes the properties of the competitive trading equilibria with more than two firms. Let $k^e(N) > 0$ be defined by $\mathbb{E}\left[e^{\frac{N}{N-1} - v/k^e(N)}\right] = 1$, and denote by $\Pi^C(N)$ the expected profit of each firm in the competitive trading equilibrium with N active firms.

Proposition 5. *Suppose there are* $N \geq 2$ *firms.*

- *(i) A competitive trading equilibrium exists if and only if* $k < k^t$*. If a competitive trading equilibrium exists, it is unique.*
- *(ii) The consumer's overall trade engagement level in the competitive trading equilibrium increases with* N*.*
- *(iii) Under competition, an efficient equilibrium exists if and only if* $k \leq k^e(N)$*. Furthermore,* $k^e(N)$ *strictly increases in N.*
- *(iv) For any market size, a region of attention costs exists such that adding competing firms increases producer surplus. Formally, there exists* $\hat{k} \in (k^e(N), k^t)$ *such that* $(N+1) \cdot \Pi^C(N+1) > N \cdot \Pi^C(N)$ *for all* $k \in (\hat{k}, k^t)$*.*

[27] As in the duopoly case, a competitive trading equilibrium with $N \geq 2$ firms must be symmetric.

[28] To see why, fix $N_1, N_2 \geq 2$ and suppose that $N_1\pi^{N_1} = N_2\pi^{N_2} = \bar{\pi} \in (0, 1]$. Equation (6) implies that $\phi(v, N_1) < \phi(v, N_2)$ if and only if $N_1 > N_2$.

The existence and uniqueness properties of the competitive trading equilibria do not change with the number of active firms. For every N, by Lemma 5, the function $\phi(v, N)$ strictly increases with π, which implies that at most one competitive trading equilibrium exists.[29] Moreover, as $\pi \downarrow 0$, the function $\phi(v, N)$ converges to 0, irrespective of the value of N. This implies that the pricing effect vanishes when it becomes prohibitively costly for the consumer to process information about offers. As a result, competition with $N \geq 2$ firms supports equilibrium trade only when $k < k^t$, like under collusion.

Although the number of $N \geq 2$ firms is inconsequential for the existence and uniqueness of competitive trading equilibria, varying the number of active firms affects trade efficiency. The intuition behind parts (ii) and (iii) of Proposition 5 follows the one of Propositions 2 and 3. As N grows, firms have stronger incentives to undercut their competitors' offers. Anticipating this, the consumer engages in trade more often, implying that the overall trade engagement level increases with N. In other words, the attention effect becomes stronger as the number of active firms increases. For analogous reasons, $k^e(N)$ strictly increases with N, and efficient trade becomes easier to sustain when N grows. This feature implies that introducing an additional competing firm restores trade efficiency for a range of attention costs.

Nevertheless, as the number of firms grows, i.e., $N \uparrow \infty$, competition does not guarantee efficient trade if the quality of the product is uncertain. If prices exceed average quality, the consumer is not willing to buy with certainty. When quality is uncertain, this would be the case in an efficient equilibrium for k close to k^t, regardless of the number of competitors. Thus, there always remains a parameter region where trade is inefficient. The following corollary summarizes this point.

Corollary 2. *If quality is stochastic, a region with inefficient trade always exists, i.e., if* $|V| > 1$, $\lim_{N\uparrow\infty} k^e(N) := k^e(\infty) < k^t$.

In Section 4.2, we showed that the attention effect of competition can be large enough to increase the producer surplus as we move from collusion to competition. Part (iv) of Proposition 5 adds to this by showing that for any number $N \geq 2$ of firms, there exists a parameter region of information costs, such that adding another

[29] Otherwise, the equilibrium where the consumer engages in trade more often is the one where, on-path, the terms of trade are worse for the consumer, a contradiction.

firm increases the producers' combined profits. Adding a firm strengthens both the pricing and the attention effect. For information cost k close to k^t, the intuition of our main result extends: The additional attention effect dominates the additional pricing effect and total profits increase in the number of firms.

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Appendix – For Online Publication

Proof of Lemma 1

Proof. ("Only if" direction.) As a first step, we show that any RVP best response of the consumer must display an adjusted multinomial logit formulation *everywhere*.

Claim 1. *Suppose β is a RVP best response to μ. Then, for every $w \in V$ and $y_1, y_2 \geq 0$, it holds that*

$$\beta_i(w, y_1, y_2) = \frac{\pi_i \cdot e^{\frac{w-y_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{w-y_j}{k}} + 1 - \pi_1 - \pi_2}, \ (i \in I) \tag{7}$$

where (π_1, π_2) is a solution to problem

$$\max_{\pi_1', \pi_2' \geq 0} \mathbb{E}_\mu \left[\log \left(\pi_1' \cdot e^{\frac{v-x_1}{k}} + \pi_2' \cdot e^{\frac{v-x_2}{k}} + (1 - \pi_1' - \pi_2') \right) \right] \ \textit{s.t.}\ \pi_1' + \pi_2' \leq 1. \tag{8}$$

In particular, for every $i \in I$, it holds that

$$\pi_i = \mathbb{E}_\mu[\beta_i]. \tag{9}$$

Proof of Claim 1. Let μ be consistent with a strategy profile σ of the sellers. Suppose that β is a best response to μ and that it is RVP. Since β is a best response to μ, known results show that[30] (7) holds μ-almost surely, where (π_1, π_2) is a solution to problem (8). Now, fix $v \in V$ and $x_1, x_2 \geq 0$ arbitrarily. Since β is RVP, there exists a sequence $(\mu^n, \tilde{\sigma}^n)$ with the desired properties such that β is a best response to μ^n for every $n \in \mathbb{N}$. Once again, we know that β must take the following logit functional form in (7) μ^n-a.s. for every $n \in \mathbb{N}$, where each (π_1, π_2) is replaced by (π_1^n, π_2^n), which is a solution to

$$\max_{\pi_1', \pi_2' \geq 0} \mathbb{E}_{\mu^n} \left[\log \left(\pi_1' \cdot e^{\frac{v-x_1}{k}} + \pi_2' \cdot e^{\frac{v-x_2}{k}} + (1 - \pi_1' - \pi_2') \right) \right] \ \text{subject to}\ \pi_1' + \pi_2' \leq 1.$$

Since β is a best reply to all μ^n, and $\mu^n(v, x_1, x_2) > 0$ for all n by assumption, it

[30] See Matějka and McKay (2015), and Denti, Marinacci, and Montrucchio (2020).

must be that $(\pi_1^n, \pi_2^n) = (\bar{\pi}_1, \bar{\pi}_2)$ for some $\bar{\pi}_1, \bar{\pi}_2 \in [0,1]$.

Now, let (v', x_1', x_2') be a generic element in the support of μ. Since $\tilde{\sigma}^n \to \sigma$ implies $\mu^n \to \mu$ in the topology of strong convergence, we have that $\mu^n(\text{Supp}(\mu)) > 0$ for large n. Because β has to be a best response at all n, we know that

$$\beta_i(v', x_1', x_2') = \frac{\bar{\pi}_i \cdot e^{\frac{v'-x_i'}{k}}}{\sum_{j=1,2} \bar{\pi}_j \cdot e^{\frac{v'-x_j'}{k}} + 1 - \bar{\pi}_1 - \bar{\pi}_2} = \frac{\pi_i \cdot e^{\frac{v'-x_i'}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v'-x_j'}{k}} + 1 - \pi_1 - \pi_2},$$

for all $i \in I$. Therefore, $\bar{\pi}_i = \pi_i$ for all $i \in I$. Lastly, that equation (9) holds follows from standard results. See, e.g., Matějka and McKay (2015), Corollary 2. □

Note that Claim 1 did not use symmetry. We now impose symmetry to prove points (ii), ..., (v) of Lemma 1, thus concluding the "only if" direction.[31] Suppose σ is symmetric, and let μ be consistent with σ. It is easy to see that μ is *symmetric*, i.e., $\mu(A) = \mu(A^{\text{sym}})$ for every measurable $A \subseteq V \times \mathbb{R}_+^2$, where $A^{\text{sym}} := \bigcup\{(v, x_1, x_2) : (v, x_2, x_1) \in A\}$ is the symmetric conjugate of A. Given the concavity of the objective function, it is without loss to focus on solutions (π_1, π_2) to (8) such that $\pi_1 = \pi_2$.[32] This proves point (ii) of Lemma 1. Points (iii), (iv) and (v) now follow from a standard analysis of problem (8), once the constraint $\pi_1' = \pi_2'$ is imposed. We omit the details.

("If" direction.) Let $\beta = (\beta_1, \beta_2)$ be given by (2) and satisfy points (i), ..., (v). Clearly, β is symmetric. Furthermore, given the symmetry of μ, we know that β is *a* best response to μ. Fix $v \in V$ and $x_1, x_2 \geq 0$ arbitrarily. To prove that β is indeed RVP, we distinguish three cases. For each case, we define a symmetric perturbation $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}_+^2)^V$ such that $\tilde{\sigma}'(x_1, x_2|v) > 0$. Then, for each $n \in \mathbb{N}$, we let $\tilde{\sigma}^n = \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$. By construction, $\tilde{\sigma}^n \to \sigma$ strongly and $\tilde{\sigma}^n(x_1, x_2|v) > 0$ for every $n \in \mathbb{N}$. Let μ^n be consistent with $\tilde{\sigma}^n$. It remains to define $\tilde{\sigma}'(\cdot|\cdot)$ and to show that β is indeed a best response to μ^n for each $n \in \mathbb{N}$ in each case.

Case 1. Suppose $\pi = \mathbb{E}_\mu[\beta_i] \in (0, 1/2)$. At the end of this proof, Lemma 6 shows that the other conditions displayed in point (v) of Lemma 1 are redundant.

[31] That equation (2) and point (i) of Lemma 1 hold follows already from Claim 1.

[32] If (π_1, π_2) solves (8) with $\pi_1 \neq \pi_2$, symmetry of μ implies (π_2, π_1) is a distinct solution. Because of concavity, $\frac{1}{2}(\pi_1, \pi_2) + \frac{1}{2}(\pi_2, \pi_1)$ is then a symmetric solution, proving the assertion.

Let $A := \frac{1}{2}\beta_1(v, x_1, x_2) + \frac{1}{2}\beta_1(v, x_2, x_1) > 0$. Note that $A = \frac{1}{2}\beta_2(v, x_1, x_2) + \frac{1}{2}\beta_2(v, x_2, x_1)$ due to symmetry. Also, $\beta_i(v, x, x)$ is strictly decreasing in $x \geq 0$, and that $\beta_i(v, v, v) = \pi$. There are two possibilities:

1) Suppose $A < \pi$. Then, there exists $\alpha \in (0, 1)$ and $\varepsilon > 0$ such that (i) $v - \varepsilon \geq 0$ and (ii) $\alpha A + (1-\alpha)\beta_i(v, v-\varepsilon, v-\varepsilon) = \pi$ for each $i \in I$. Let $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v') = \delta_{(v',v')}$ if $v' \neq v$, and $\tilde{\sigma}'(\cdot|v') = \alpha\left(\frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(x_2,x_1)}\right) + (1-\alpha)\delta_{(v-\varepsilon,v-\varepsilon)}$ when $v' = v$. Since $\mathbb{E}_{\mu^n}[\beta_i] = \pi$ for each $i \in I$, β is a best response to μ^n for each $n \in \mathbb{N}$ by Lemma 6 below and Corollary 2 of Matějka and McKay (2015).

2) Suppose $A \geq \pi$. Then, there exists $\alpha \in (0, 1)$ and $\varepsilon \geq 0$ such that $\alpha A + (1-\alpha)\beta_i(v, v+\varepsilon, v+\varepsilon) = \pi$ for each $i \in I$. Let $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v') = \delta_{(v',v')}$ if $v' \neq v$, and $\tilde{\sigma}'(\cdot|v') = \alpha\left(\frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(x_2,x_1)}\right) + (1-\alpha)\delta_{(v+\varepsilon,v+\varepsilon)}$ when $v' = v$. Like before, $\mathbb{E}_{\mu^n}[\beta_i] = \pi$ for each $i \in I$ implies that β is a best response to μ^n for each $n \in \mathbb{N}$ as required.

Case 2. Suppose $\pi = 0$, so that $\beta_1 = \beta_2 = 0$. Let $A := \frac{1}{2}e^{\frac{v-x_1}{k}} + \frac{1}{2}e^{\frac{v-x_2}{k}} > 0$. There are two possibilities:

1) If $A \leq 1$, let $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v') = \delta_{(v',v')}$ if $v' \neq v$, and $\tilde{\sigma}'(\cdot|v') = \frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(x_2,x_1)}$ when $v' = v$. By construction, for every $n \in \mathbb{N}$, we have $\mathbb{E}_{\mu^n}\left[e^{\frac{v-x_i}{k}}\right] \leq 1$ for each $i \in I$. This implies that β is a best reply to μ^n for all $n \in \mathbb{N}$ as required.

2) If $A > 1$, there exists $\alpha \in (0, 1)$ and $\varepsilon > 0$ such that $\alpha A + (1-\alpha)e^{-\varepsilon/k} \leq 1$. Let $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v') = \delta_{(v',v')}$ if $v' \neq v$, and $\tilde{\sigma}'(\cdot|v') = \alpha\left(\frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(x_2,x_1)}\right) + (1-\alpha)\delta_{(v+\varepsilon,v+\varepsilon)}$ when $v' = v$. For all $n \in \mathbb{N}$, we have $\mathbb{E}_{\mu^n}\left[e^{\frac{v-x_i}{k}}\right] \leq 1$ for all $i \in I$. This implies that β is a best reply to μ^n for all $n \in \mathbb{N}$ as required.

Case 3. The proof for the case $\pi = 1/2$ is similar to the one for $\pi = 0$ (see Case 2). We omit the details. □

Lemma 6. *Let μ be symmetric, and β be given by (2). If $\pi = \mathbb{E}_\mu[\beta_i] \in (0, 1/2)$ for all $i \in I$, then $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \geq 1$ for all $i \in I$, and $\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right] \geq 1/2$.*

Proof. For every $y > 0$ and $\gamma \in (0, 1/2)$, let

$$g(y,\gamma) = \frac{1-2\gamma}{\gamma y + (1-2\gamma)} \quad \text{and} \quad h(y,\gamma) = \frac{\gamma}{\gamma + (1-2\gamma)y}.$$

Note that g and h are strictly decreasing and convex in $y > 0$ for every $\gamma \in (0, 1/2)$. Moreover, $g(y,\gamma) = 1 - 2\gamma$ iff $y = 2$, and $h(y,\gamma) = 2\gamma$ iff $y = 1/2$.

From Jensen's inequality, we have

$$2\pi = \mathbb{E}_\mu[\beta_1+\beta_2] = \mathbb{E}_\mu\left[h\left(\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}, \pi\right)\right] \geq h\left(\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right], \pi\right),$$

which implies that $\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right] \geq 1/2$. Similarly,

$$1-2\pi = 1-\mathbb{E}_\mu[\beta_1+\beta_2] = \mathbb{E}_\mu\left[g\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}, \pi\right)\right] \geq g\left(\mathbb{E}_\mu\left[e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right], \pi\right),$$

which implies that $\mathbb{E}_\mu\left[e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right] \geq 2$. Since μ is symmetric, it holds that $\mathbb{E}_\mu\left[e^{\frac{v-x_1}{k}}\right] = \mathbb{E}_\mu\left[e^{\frac{v-x_2}{k}}\right]$. Therefore, $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \geq 1$ for every $i \in I$. □

Proof of Proposition 1

Proof. We show that colluding firms offer the same price as a monopolist facing the aggregate trade engagement level. Since the functional form of the monopolist's best response is enough to characterize the unique trading equilibrium in Ravid (2020), this proves that monopoly and collusion are equilibrium outcome-equivalent. Given this fact, the proposition follows from Theorem 1 and Corollary 1 in Ravid (2020).

Fix $v \in V$. Suppose firms face individual demands given by

$$Q^i(x_1, x_2) := \frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2} \quad (i \in I).$$

Colluding firms solve the problem (P): $\max_{x_1,x_2 \geq 0} \Pi^M(x_1, x_2)$, where $\Pi^M(x_1, x_2) := \sum_{i=1,2} Q^i(x_1, x_2) \cdot x_i$. If $0 = \pi_i < \pi_j$ for some $i \in I$, the problem (P) is identical to the problem solved by the monopolist in Ravid (2020). The equilibrium outcome equivalence between monopoly and collusion is, therefore, immediate. Thus, suppose $\pi_i > 0$ for all $i \in I$, and let $D := \sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2 > 0$. The FOCs associated to problem (P):

$$D = \frac{\pi_j e^{\frac{v-x_j}{k}}}{k} \cdot (x_i - x_j) + \frac{x_i}{k}(1 - \pi_i - \pi_j), \quad \forall i \in I. \tag{10}$$

An interior solution exists,[33] and is characterized by the FOCs. Combing equations (10) across $i \in I$ yields $\frac{x_1 - x_2}{k} \cdot D = 0$, which is true iff $x_1 = x_2$. Let $x_1 = x_2 = x$. Equation (10) becomes $D = \frac{x}{k}(1 - \pi_1 - \pi_2)$, and admits unique solution x^* given by

$$x^* = k \left(1 + W\left(\frac{\pi_1 + \pi_2}{1 - \pi_1 - \pi_2} e^{\frac{v-k}{k}}\right)\right), \tag{11}$$

where $y \mapsto W(y)$ is the Lambert function.[34] Equation (11) is identical to equation (6) in Ravid (2020), characterizing the monopolist optimal equilibrium pricing when the consumer's overall trade engagement level is $\pi_1 + \pi_2 = \pi^M$. Thus, monopoly and collusion models are equilibrium outcome-equivalent, as required. □

Proof of Lemma 2

Proof. A profile (μ, β, σ) is a competitive trading equilibrium if and only if it is symmetric trading equilibrium. That is, (a) μ is consistent with σ, (b) β is given by (2) with $\pi > 0$, and (c) σ is symmetric and σ_i is a best response to σ_{-i} given β.

We focus on the equilibrium behavior of the firms. Fix $v \in V$ arbitrarily. From Milgrom and Roberts (1990), for fixed symmetric logit demand β of the consumer, the unique NE of the pricing game played by the firms is pure and symmetric. To characterize it, suppose (b) holds and let $\sigma_i(\cdot|v) = \delta_{x_i(v)}$ for all $i \in I$. Then, taking $\pi \in (0, 1/2]$ and $-i$'s offer $x_{-i}(v) = x_{-i}$ as given, firm i solves:

[33] For all $x_j \geq 0$, if $x_i = 0$, the LHS of (10) is strictly greater than the RHS. This implies that any solution to (P) (if it exists) must be interior. Conversely, there exists a $\bar{x}_i > 0$ such that, for all $x_i \geq \bar{x}_i$, the RHS of (10) is strictly greater than the LHS for all $x_j \geq 0$. This means that $\Pi^M(x_i, x_j)$ is eventually decreasing in x_i for all $x_j \geq 0$, implying that a bounded solution exists.

[34] The Lambert function is defined as the inverse of $z \mapsto ze^z$.

$$\max_{x_i \geq 0} \frac{x_i \cdot \pi e^{\frac{v-x_i}{k}}}{\pi \cdot \left(e^{\frac{v-x_i}{k}} + e^{\frac{v-x_{-i}}{k}}\right) + 1 - 2\pi}$$

The first-order condition can be written as

$$\frac{d\Pi_i^C(v, x_1, x_2)}{dx_i} = \beta_i(v, x_1, x_2) \left(1 - \frac{x_i}{k} \cdot [1 - \beta_i(v, x_1, x_2)]\right) = 0. \qquad (12)$$

Note that $\beta_i > 0$ for all $x_i \geq 0$ and $\lim_{x_i \to \infty} \beta_i = 0$. Therefore, $x_i \mapsto \frac{d\Pi_i^C(v,x_1,x_2)}{dx_i}$ crosses zero exactly once from above. It follows that $x_i \mapsto \Pi_i^C(v, x_i, x_{-i})$ admits a unique (interior) global maximum characterized by the FOC. We rearrange (12) and use symmetry to see that, in equilibrium, $x_i(v) = x_{-i}(v) =: x^C(v)$ satisfies:

$$x^C(v; \pi) = k \cdot \left[1 + \frac{\pi e^{\frac{v - x^C(v;\pi)}{k}}}{\pi e^{\frac{v - x^C(v;\pi)}{k}} + 1 - 2\pi}\right].$$

Define $\phi(v; \pi) := \pi e^{\frac{v - x^C(v;\pi)}{k}} / \left(\pi e^{\frac{v - x^C(v;\pi)}{k}} + 1 - 2\pi\right)$. Then, the equilibrium firm behavior is given by $x^C(v; \pi) = k \cdot [1 + \phi(v; \pi)]$, where optimality requires that

$$\left(1 + e^{\phi(v;\pi)} \frac{1 - 2\pi}{\pi e^{\frac{v-k}{k}}}\right) \phi(v; \pi) = 1.$$

The above equation uniquely pins down $\phi(v; \pi) > 0$ for fixed $\pi \in (0, 1/2]$: The LHS is continuously increasing in ϕ, it goes to 0 as $\phi \downarrow 0$ and goes to ∞ as $\phi \uparrow \infty$. This concludes the proof of the lemma. □

Proof of Theorem 1

Proof. We start with the proof of the if and only if statement. The necessity proof is as follows. As we argued in Lemma 2, in any competitive trading equilibrium, the sellers charge a price $x_i(v) = x^C(v)$ strictly above k for each $v \in V$. Now, suppose by way of contradiction that a trading equilibrium exists but $k \geq k^t$, i.e., $\mathbb{E}_\lambda \left[e^{v/k-1}\right] \leq 1$. In equilibrium, we would have $\mathbb{E}_\mu \left[e^{\frac{v-x(v)}{k}}\right] < \mathbb{E}_\lambda \left[e^{v/k-1}\right] \leq 1$.

This is in contradiction with our hypothesis of on-path equilibrium trade.[35] Thus, $k < k^t$ is necessary for the existence of a competitive trading equilibrium.

We now turn to the sufficiency direction. We split the proof in two parts. First, we restrict attention to values of k for which trade occurs with probability 1. We then consider the remaining parameter values. To this end, we need to introduce some further notation. Let k^e be the unique solution to $\mathbb{E}_\lambda\left[e^{2-v/k^e}\right] = 1$. Notice that

$$\mathbb{E}_\lambda\left[e^{2-v/k^t}\right] > \mathbb{E}_\lambda\left[e^{1-v/k^t}\right] = \mathbb{E}_\lambda\left[\frac{1}{e^{v/k^t-1}}\right] \geq \frac{1}{\mathbb{E}_\lambda\left[e^{v/k^t-1}\right]} = 1.$$

Thus, $0 < k^e < k^t$.

Suppose first that $k \leq k^e$. Take $x^C(v) = 2k$ for all $v \in V$. Observe that this configuration of prices is an equilibrium of the pricing game played by the firms when they face a symmetric logit demand with $\pi = 1/2$. At the same time, from Lemma 1, a symmetric trade engagement level $\pi = 1/2$ is consistent with this configuration of prices if and only if $\mathbb{E}_\lambda\left[e^{2-v/k}\right] \leq 1$, or equivalently, $k \leq k^e$. Therefore, a symmetric efficient equilibrium exists. Since a symmetric efficient equilibrium is indeed a competitive trading equilibrium, we are done.

Now, consider the case where $k \in (k^e, k^t)$, or equivalently, $\mathbb{E}_\lambda\left[e^{2-v/k}\right] > 1$ and $\mathbb{E}_\lambda\left[e^{v/k-1}\right] > 1$. Define the functions $\phi = \phi(v; p)$ and $x = x^C(v; p)$ as in Lemma 2 with π replaced by p. Let $F = F(p)$ be defined as

$$F(p) := \mathbb{E}_\lambda\left[\frac{e^{\frac{v-k}{k}} \cdot e^{-\phi}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p)}\right]. \tag{13}$$

Since the function $F(\cdot)$ satisfies $F(p) = \frac{1}{p}\mathbb{E}[\beta_i(v, x^C(v;p), x^C(v;p))]$ for every $i \in I$, a symmetric RVP trading equilibrium where trade occurs with probability strictly between 0 and 1 exists if $F(p^*) = 1$ for some $p^* \in (0, 1/2)$.[36] We prove this by relying on the Intermediate Value Theorem, hence exploiting the continuity of $F(\cdot)$ in $p \in (0, 1/2]$. In particular, we show that there exists $0 < p_0 < p_1 < 1/2$ such that for all $p \in (0, p_0)$, we have $F(p) > 1$, and for all $p \in (p_1, 1/2)$, we have $F(p) < 1$.

[35] From Lemma 1, the consumer's trade engagement level with each firm would equal zero.

[36] See Lemma 6 above.

Existence of $0 < p_1 < 1/2$*:* We exploit the fact that $F(\cdot)$ is continuously differentiable. This follows from the Implicit Function Theorem that guarantees that $\phi(v;p)$ is continuously differentiable in $p \in (0, 1/2]$ for all $v \in V$.[37] Given that V is finite and $\phi(v;p) \uparrow 1$ as $p \uparrow 1/2$, for every $\varepsilon > 0$ there exists a $\bar{p}_1 \in (0, 1/2)$ such that $\phi(v;p) > 1 - \varepsilon$ for all $v \in V$ and $p \in (\bar{p}_1, 1/2)$. Fix $\varepsilon > 0$ and $\delta > 0$ small enough so that $\mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k} \right] - \delta > 1$, and let $\bar{p}_1$ be the p-threshold corresponding to ε.[38] For every $v \in V$, define

$$A(v) := \max_{p \in [\bar{p}_1, 1/2]} e^{1-v/k+\phi(v;p)} \cdot \frac{\partial}{\partial p} \phi(v;p) \cdot \frac{D_{\max}(p)}{D_{\min}(p)}$$

where

$$D_{\max}(p) := \max_{v \in V} \left(2p + (1-2p) \cdot e^{\phi(v;p)+1-v/k} \right)^2 > 0$$

$$D_{\min}(p) := \min_{v \in V} \left(2p + (1-2p) \cdot e^{\phi(v;p)+1-v/k} \right)^2 > 0.$$

We make two observations.

Obs. 1: Each $A(v)$ is a well-defined real number since it is the maximum value of a continuous function on a compact support. Again by the finiteness of V, there exists $\bar{p}_2 \in (0, 1/2)$ such that $(1-2p) \cdot A(v) \leq \delta$ for all $v \in V$ and $p \in (\bar{p}_2, 1/2)$.

Obs. 2: Since $D_{\max}(p), D_{\min}(p) \to 1$ as $p \uparrow 1/2$, we have that $D_{\max}(p)/D_{min}(p) \to 1$ as $p \uparrow 1/2$. Therefore, there exists $\bar{p}_3 \in (0, 1/2)$ such that $D_{\max}(p)/D_{min}(p) \leq 1 + \delta/2$ for all $p \in (\bar{p}_3, 1/2)$.

Now, let $\bar{p} = \max\{\bar{p}_1, \bar{p}_2, \bar{p}_3\} < 1/2$. For all $p \in (\bar{p}, 1/2)$, we have:

$$F'(p) = \mathbb{E}_\lambda \left[\frac{2 \cdot \left(e^{1+\phi(v;p)-v/k} - 1 \right) - (1-2p) \cdot e^{\phi(v;p)+1-v/k} \cdot \frac{\partial}{\partial p} \phi(v;p)}{\left(2p + (1-2p) \cdot e^{\phi(v;p)+1-v/k} \right)^2} \right]$$

[37] More formally, for $\phi \in (0, \infty)$, $v \in V$, and $p \in (0, 1/2 + \tau)$, let

$$G(p, \phi, v) := \phi \cdot \left(1 + e^{\phi} \cdot (1-2p) / \left[p \cdot e^{\frac{v-k}{k}} \right] \right) - 1.$$

For $\tau > 0$ small enough, the assumptions of the Implicit Function Theorem are satisfied by G. Thus, there exists continuously differentiable $\bar{\phi}(v;p)$ on $(0, 1/2 + \tau) \times V$ such that $G(p, \bar{\phi}(v;p), v) = 0$ for all $v \in V$ and $p \in (0, 1/2 + \tau)$. Let $\bar{\phi}(v;p) = \phi(v;p)$ on $V \times (0, 1/2]$.

[38] Such $\varepsilon, \delta > 0$ exist because $\mathbb{E}_\lambda \left[e^{2-v/k} \right] > 1$ by assumption.

$$
\begin{aligned}
&\geq \mathbb{E}_\lambda\left[\frac{2}{D_{\max}(p)}\cdot e^{1+\phi(v;p)-v/k}-\frac{2}{D_{\min}(p)}-(1-2p)\cdot\frac{e^{\phi(v;p)+1-v/k}}{D_{\min}(p)}\cdot\frac{\partial}{\partial p}\phi(v;p)\right]\\
&=\frac{1}{D_{\max}(p)}\cdot\mathbb{E}_\lambda\left[2\cdot e^{1+\phi(v;p)-v/k}-\frac{D_{\max}(p)}{D_{\min}(p)}\left(2+(1-2p)\cdot e^{\phi(v;p)+1-v/k}\cdot\frac{\partial}{\partial p}\phi(v;p)\right)\right]\\
&\geq \frac{1}{D_{\max}(p)}\cdot\mathbb{E}_\lambda\left[2\cdot e^{1+\phi(v;p)-v/k}-2\cdot\frac{D_{\max}(p)}{D_{\min}(p)}-(1-2p)\cdot A(v)\right]\\
&\geq \frac{2}{D_{\max}(p)}\cdot\mathbb{E}_\lambda\left[e^{1+\phi(v;p)-v/k}-1-\delta\right]\geq\frac{2}{D_{\max}(p)}\cdot\mathbb{E}_\lambda\left[e^{2-\varepsilon-v/k}-1-\delta\right]>0,
\end{aligned}
$$

where the first inequality comes from the fact that $\frac{\partial}{\partial p}\phi(v;p)\geq 0$.[39] Since $F(1/2)=1$ and $F'(p)>0$ for all $p\in(0,1/2)$ close to 1/2, the existence of p_1 follows.

Existence of $0<p_0<p_1<1/2$*:* Given that V is finite, $\phi(v;p)\downarrow 0$ and $2p\cdot e^{\frac{v-k}{k}}\cdot e^{-\phi}+(1-2p)\to 1$ as $p\downarrow 0$, for every $\varepsilon>0$ there exists a $\underline{p}\in(0,1/2)$ such that $\phi(v;p)<\varepsilon$ and $2p\cdot e^{\frac{v-k}{k}}\cdot e^{-\phi}+(1-2p)<1+\varepsilon$ for all $v\in V$ and $p\in(0,\underline{p})$. Let $\varepsilon>0$ be small enough so that $\mathbb{E}_\lambda\left[e^{v/k-1-\varepsilon}\right]/(1+\varepsilon)>1$. Such an $\varepsilon>0$ exists because $\mathbb{E}_\lambda\left[e^{v/k-1}\right]>1$. For all $p\in(0,\underline{p})$, we have:

$$
F(p)=\mathbb{E}_\lambda\left[\frac{e^{\frac{v-k}{k}}\cdot e^{-\phi}}{2p\cdot e^{\frac{v-k}{k}}\cdot e^{-\phi}+(1-2p)}\right]\geq\frac{\mathbb{E}_\lambda\left[e^{\frac{v-k}{k}}\cdot e^{-\phi}\right]}{1+\varepsilon}\geq\frac{\mathbb{E}_\lambda\left[e^{v/k-1-\varepsilon}\right]}{1+\varepsilon}>1.
$$

Thus, a $p_0\in(0,p_1)$ with the desired properties exists. This concludes the proof of existence of a competitive equilibrium.

Uniqueness: Once again, we distinguish between two cases. First, suppose $k\leq k^e$, or equivalently, $\mathbb{E}\left[e^{2-v/k}\right]\leq 1$. From the proof of existence, we know that a competitive efficient equilibrium exists. We want to show that no other symmetric trading equilibrium can exist. For each $p\in(0,1/2]$ and $v\in V$, let $\phi=\phi(v;p)$, $x=x^C(v;p)$, and $F=F(p)$ be defined as above. Note that $F(1/2)=1$. To prove that no other symmetric trading equilibrium exists, it is sufficient to show that $F(p)\neq 1$ for all $p\in(0,1/2)$. With this goal in mind, first note that $\phi(v;p)$ is strictly increasing in $p\in(0,1/2]$ for every $v\in V$, and that $\phi(v;1/2)=1$. Thus, given that V is finite, when p is strictly below 1/2, there exists $\varepsilon>0$ small enough such that $\phi(v;p)<1-\varepsilon$ for all $v\in V$. Second, observe that $\mathbb{E}\left[e^{2-c-v/k}\right]<1$ for

[39]See the proof of Lemma 8.

any constant $c > 0$. Now, fix $p \in (0, 1/2)$ and its corresponding $\varepsilon > 0$. We have

$$\begin{aligned} F(p) &= \mathbb{E}_\lambda \left[\frac{e^{\frac{v-k}{k}} \cdot e^{-\phi}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p)} \right] = \mathbb{E}_\lambda \left[\frac{1}{2p + (1-2p) \cdot e^{\phi + 1 - v/k}} \right] \\ &> \mathbb{E}_\lambda \left[\frac{1}{2p + (1-2p) \cdot e^{2-\varepsilon - v/k}} \right] \geq \frac{1}{2p + (1-2p) \cdot \mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k} \right]} > 1. \end{aligned}$$

where the first strict inequality comes from $\phi = \phi(v; p) < 1 - \varepsilon$ for all $v \in V$, the weak inequality is an application of Jensen's inequality, and the last inequality is implied by $\mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k} \right] < 1$. Hence, $F(p) \neq 1$ for all $p < 1/2$ as required.

Now, consider the case where $k \in (k^e, k^t)$. Suppose towards a contradiction that there exist $0 < p^* < p^{**} < 1/2$ such that $F(p^*) = F(p^{**}) = 1$. Define $\gamma \in (0, 1)$ implicitly by $p^{**} = \gamma p^* + (1-\gamma)1/2$. We have

$$\begin{aligned} F(p^{**}) &= \mathbb{E}_\lambda \left[\frac{1}{2p^{**} + (1-2p^{**}) \cdot e^{\phi(v;p^{**}) + 1 - v/k}} \right] \\ &= \mathbb{E}_\lambda \left[\frac{1}{2\gamma p^* + 1 - \gamma + \gamma(1-2p^*) \cdot e^{\phi(v;p^{**}) + 1 - v/k}} \right] \\ &< \mathbb{E}_\lambda \left[\frac{1}{1 - \gamma + \gamma \left(2p^* + (1-2p^*) \cdot e^{\phi(v;p^*) + 1 - v/k}\right)} \right] \leq 1. \end{aligned}$$

The first inequality follows from the fact that $p^{**} > p^*$ and that $\phi(v; p)$ is strictly increasing in $p \in (0, 1/2)$ for all $v \in V$. In order to prove the second inequality, we define $g(\gamma) := \mathbb{E}_\lambda \left[\frac{1}{1-\gamma+\gamma\left(2p^*+(1-2p^*)\cdot e^{\phi(v;p^*)+1-v/k}\right)} \right]$. Note that $g(0) = 1$ and $g(1) = F(p^*) = 1$. It remains to show that $g(\gamma)$ is convex for all $\gamma \in [0, 1]$. Taking the second derivative, we get

$$g''(\gamma) = \mathbb{E}_\lambda \left[\frac{2\left(2p^* + (1-2p^*) \cdot e^{\phi(v;p^*) + 1 - v/k} - 1\right)^2}{\left(1 - \gamma + \gamma\left(2p^* + (1-2p^*) \cdot e^{\phi(v;p^*) + 1 - v/k}\right)\right)^3} \right] \geq 0.$$

Thus, we reached the contradiction that $F(p^{**}) < 1$. We conclude that there is at most one $p \in (0, 1/2)$ such that $F(p) = 1$, and the proof of uniqueness. □

Proof of Lemma 3

Proof. From Lemma 2, we know that the firms price according to $x^C(v;\pi^C) = k \cdot (1 + \phi(v;\pi^C))$. Under collusion,[40] for every $v \in V$, each active firm plays a strategy $\sigma^M(\cdot|v) = \delta_{x^M(v)}$ such that $x^M(v;\pi^M) = k \cdot \left(1 + W\left(\frac{\pi^M}{1-\pi^M} e^{v/k-1}\right)\right)$, and $W(\cdot)$ is the Lambert function. Compared to the equilibrium price formula of the competition model, we note that the only difference is that $W\left(\frac{\pi^M}{1-\pi^M} e^{v/k-1}\right)$ is replaced by $\phi(v;\pi^C)$. Fix $p \in (0, 1/2)$ arbitrarily. According to Lemma 2, $\phi(v;p)$ is the unique solution to equation (5), where π is replaced by p. Note that (5) is equivalent to $\frac{p}{1-2p} e^{v/k-1} = \phi \cdot \frac{p}{1-2p} e^{v/k-1} + \phi e^{\phi}$. Therefore

$$\frac{2p}{1-2p} e^{v/k-1} > \frac{p}{1-2p} e^{v/k-1} = \phi(v;p) \left(\frac{p}{1-2p} e^{v/k-1} + e^{\phi(v;p)} \right) > \phi(v;p) e^{\phi(v;p)}.$$

Applying the Lambert function on both sides yields $\phi(v;p) < W\left(\frac{2p}{1-2p} e^{v/k-1}\right)$, which implies the result. □

Proof of Proposition 2

Proof. Fix $k \in (0, k^t)$, and let $(\mu^M, \sigma^M, \beta^M)$ and $(\mu^C, \beta^C, \sigma^C)$ be the unique symmetric equilibrium under collusion and competition respectively associated with the cost parameter k. Set $\pi^M = \mathbb{E}_{\mu^M}[\beta_1^M + \beta_2^M]$ and $\pi^C = \mathbb{E}_{\mu^C}[\beta_i^C]$ for each $i \in I$.

If $k \leq k^e$, the result follows from Proposition 3 and Corollary 1 of Ravid (2020). In words, while an efficient equilibrium cannot exist under collusion, it is the only competitive trading equilibrium outcome. Hence, $0 < \pi^M < 1 = 2\pi^C$, as required.

Now assume that $k \in (k^e, k^t)$. Let $W(v; 2p) := W\left(\frac{2p}{1-2p} e^{v/k-1}\right)$ for all $v \in V$ and $p \in (0, 1/2)$. From Lemma 3, we know $\phi(v;p) < W(v;2p)$. Following the proof of Theorem 1 in Ravid (2020), the overall equilibrium engagement level in the collusion benchmark is given by $\pi^M = 2p^M$, where p^M the unique solution in $(0, 1/2)$ to the equation:

$$G(2p) := \mathbb{E}_\lambda \left[\frac{e^{\frac{v-k}{k}} \cdot e^{-W(v;2p)}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-W(v;2p)} + (1-2p)} \right] = 1. \tag{14}$$

[40] See the proof of Proposition 1, and Proposition 2 in Ravid (2020).

Let F be defined as in the proof of Theorem 1. We have

$$1 = G(2p^M) = \mathbb{E}_\lambda \left[\frac{1}{2p^M + (1-2p^M) \cdot e^{W(v;2p^M)+1-v/k}} \right]$$
$$< \mathbb{E}_\lambda \left[\frac{1}{2p^M + (1-2p^M) \cdot e^{\phi(v;p^M)+1-v/k}} \right] = F(p^M),$$

where the strict inequality follows from Lemma 3. From the proof of Theorem 1, we conclude that $p^M < \pi^C$. This is equivalent to $\pi^M < 2\pi^C$. □

Proof of Proposition 3

Proof. Follows directly from the proof of Theorem 1. □

Proof of Corollary 1

Proof. Follows directly from the proof of Lemma 2. □

Proof of Theorem 2

Preliminary analysis for the collusion benchmark. For each $k \in (0, k^t]$, let $F_k^M : [0,1) \to \mathbb{R}_+$ be defined as $F_k^M(p) := \mathbb{E}_\lambda \left[\frac{1}{p+(1-p)\cdot e^{W(p,v,k)+1-v/k}} \right]$. Again, we abuse notation and write $W(p,v,k)$ for $W\left(\frac{p}{1-p} e^{v/k-1}\right)$, where $W(\cdot)$ is the Lambert function. We are interested in the solution $p^M(k)$ to $F_k^M(p) = 1$. By the Implicit Function Theorem,[41] we know that whenever this solution exists, it is continuously differentiable. In his Theorem 1, Ravid (2020) shows that $p^M(k)$ exists uniquely in $(0,1)$ whenever $k \in (0, k^t)$. The following Lemma characterizes additional properties that $p^M(k)$ satisfies as k ranges in $(0, k^t)$.

Lemma 7. *We have:*

(i) $\lim_{k \uparrow k^t} p^M(k) = 0$.

(ii) $\lim_{k \uparrow k^t} \frac{\partial}{\partial k} p^M(k) = -\mathbb{E}_\lambda \left[\frac{v}{(k^t)^2} \cdot e^{v/k^t - 1} \right] / \mathbb{E}_\lambda \left[\frac{2 - e^{1-v/k^t}}{e^{2\cdot(1-v/k^t)}} \right]$.

[41] More precisely, one can show that there exists $\tau_k, \tau_p > 0$ small enough so that the assumptions of the Implicit Function Theorem are satisfied once the domain of $F_k^M(\cdot)$ is extended to let k range in $(0, k^t + \tau_k)$ and p range in $(-\tau_p, 1)$.

Proof. (i): Recall from Ravid (2020) that $F_k^M(\cdot)$ crosses the line $y = 1$ only once from above.[42] Therefore, it is sufficient to show that (#): for every $p \in (0,1)$, there exists $k_p \in (0, k^t)$ such that for all k strictly between k_p and k^t, $F_k^M(p) < 1$.

Since the Lambert function $W(\cdot)$ is strictly increasing, $W(p, v, k)$ is strictly decreasing in k for every $p \in (0,1)$ and $v \in V$. It further satisfies $W(p, v, k) > 0$ for all $p \in (0,1)$, $v \in V$ and $k > 0$. Fix $p \in (0,1)$ arbitrarily. Given the finiteness of V, there exists $c_p > 0$ such that $W(p, v, k) > c_p$ for all $v \in V$ and $k \in (0, k^t)$. Since $\mathbb{E}_\lambda\left[e^{v/k^t-1}\right] = 1$, we have $\mathbb{E}_\lambda\left[e^{v/k^t-1-c_p}\right] < 1$. Therefore, continuity implies that there exists k_p strictly between 0 and k^t so that $\mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1$ for all $k \in (k_p, k^t)$. Fix any such k. We have:

$$F_k^M(p) = \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{W(p,v,k)+1-v/k}}\right] \leq \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{c_p+1-v/k}}\right]$$
$$\leq 2p + (1-2p)\cdot \mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1.$$

Thus, (#) holds.

(ii): For each $k \in (0, k^t)$, we totally differentiate the equation $F_k^M(p(k)) = 1$ to obtain:[43]

$$\frac{\partial}{\partial k} p^M(k) = -\frac{A_M}{B_M} \tag{15}$$

where

$$A_M = \mathbb{E}_\lambda\left[\frac{v\cdot(1-p^M(k))\cdot e^{W(p^M(k),v,k)+1-v/k}}{k^2\cdot D_M^2\cdot(1+W(p^M(k),v,k))}\right],$$

$$B_M = \mathbb{E}_\lambda\left[\frac{1}{D_M^2}\cdot\left(1 - e^{W(p^M(k),v,k)+1-v/k} + (1-p^M(k))\cdot\frac{e^{W(p^M(k),v,k)}\cdot W'\left(\frac{p^M(k)}{1-p^M(k)}e^{v/k-1}\right)}{(1-p^M(k))^2}\right)\right],$$

and

$$D_M = p^M(k) + (1-p^M(k))\cdot e^{W(p^M(k),v,k)+1-v/k}.$$

As $k \uparrow k^t$, we know from (i) that $p^M(k) \to 0$. Therefore, $A_M \to \mathbb{E}_\lambda\left[\frac{v}{(k^t)^2}\cdot e^{v/k^t-1}\right]$ and $B_M \to \mathbb{E}_\lambda\left[\frac{2-e^{1-v/k^t}}{e^{2\cdot(1-v/k^t)}}\right]$.[44] This concludes the proof of Lemma 7. □

[42]This is shown by Ravid (2020) in the proof of Theorem 1.

[43]To derive equation (18), we used the fact that $W'(x) = \frac{W(x)}{x\cdot(1+W(x))}$ for all $x > 0$.

[44]Here, we used the fact that $W'(x) = 1$ as $x \downarrow 0$.

Preliminary analysis for the competition model. For each $k \in (k^e, k^t]$, we define $F_k^C : [0, 1/2) \to \mathbb{R}_+$ as $F_k^C(p) := \mathbb{E}_\lambda \left[\frac{1}{2p+(1-2p)\cdot e^{\phi(v;p,k)+1-v/k}} \right]$, where for $p > 0$, we let $\phi(v; p, k)$ be defined as the unique solution to equation (5), and we set $\phi(v; 0, k) := 0$ for all $v \in V$ and $k \in (k^e, k^t]$. Let $p^C(k)$ be a solution to $F_k^C(p) = 1$. From Theorem 1, we know that $p^C(k)$ exists and is unique for all $k \in (k^e, k^t)$. Again, by the Implicit Function theorem we know that $p^C(k)$ is continuously differentiable on (k^e, k^t). The next Lemma provides additional properties that $p^C(k)$ satisfies.

Lemma 8. *We have:*

(i) $\lim_{k \uparrow k^t} p^C(k) = 0$.

(ii) $\lim_{k \uparrow k^t} \frac{\partial}{\partial k} p^C(k) = -\mathbb{E}_\lambda \left[\frac{v}{(k^t)^2} \cdot e^{v/k^t - 1} \right] / \mathbb{E}_\lambda \left[\frac{2(1-e^{1-v/k^t})+1}{e^{2\cdot(1-v/k^t)}} \right]$.

Proof. (i): We show that (#): for every $p \in (0, 1/2)$, there exists $k_p \in (k^e, k^t)$ such that for all k strictly between k_p and k^t, $F_k^C(p) < 1$. Given our proof of Theorem 1, (#) implies that for all k sufficiently close to k^t, $p^C(k) < p$, proving the statement.

From equation (5), $\phi(v; p, k)$ is strictly decreasing in k for every $p \in (0, 1/2)$ and $v \in V$, and satisfies $\phi(v; p, k) > 0$ for all $p \in (0, 1/2)$, $v \in V$ and $k > 0$. Fix $p \in (0, 1/2)$ arbitrarily. Given the finiteness of V, there exists $c_p > 0$ such that $\phi(v; p, k) > c_p$ for all $v \in V$ and $k \in (k^e, k^t]$. Since $\mathbb{E}_\lambda \left[e^{v/k^t - 1} \right] = 1$, we have $\mathbb{E}_\lambda \left[e^{v/k^t - 1 - c_p} \right] < 1$. Therefore, continuity implies that there exists k_p strictly between k^e and k^t so that $\mathbb{E}_\lambda \left[e^{v/k - 1 - c_p} \right] < 1$ for all $k \in (k_p, k^t)$. Fix any such k. We have:

$$
\begin{aligned}
F_k^C(p) = \mathbb{E}_\lambda \left[\frac{1}{2p + (1-2p) \cdot e^{\phi(v;p,k)+1-v/k}} \right] &\leq \mathbb{E}_\lambda \left[\frac{1}{2p + (1-2p) \cdot e^{c_p + 1 - v/k}} \right] \\
&\leq 2p + (1-2p) \cdot \mathbb{E}_\lambda \left[e^{v/k - 1 - c_p} \right] < 1.
\end{aligned}
$$

Thus, (#) holds.

(ii): We first totally differentiate equation (5) to find the partial derivatives of ϕ with respect to p and k. That is, $\phi_p(v; p, k) := \frac{\partial}{\partial p} \phi(v; p, k)$ and $\phi_k(v; p, k) :=$

$\frac{\partial}{\partial k}\phi(v;p,k)$. After some algebra, one can show that

$$\phi_p(v;p,k) = \frac{1-\phi(v;p,k)}{(1-2p)\cdot\left(p + e^{\phi(v;p,k)}\cdot(1+\phi(v;p,k))\frac{1-2p}{e^{v/k-1}}\right)} \geq 0, \tag{16}$$

$$\phi_k(v;p,k) = -\frac{v}{k^2}\cdot\frac{\phi(v;p,k)e^{\phi(v;p,k)}}{\frac{p}{1-2p}e^{v/k-1} + e^{\phi(v;p,k)}(1+\phi(v;p,k))} \leq 0. \tag{17}$$

Note that, as $k \uparrow k^t$ and, therefore, $p \to 0$, we have $\phi \to 0$. Therefore, $\phi_p \to e^{v/k^t-1}$ and $\phi_k \to 0$ as $k \uparrow k^t$.

Next, we totally differentiate the equation $F_k^C(p^B(k)) = 1$ with respect to $k > k^e$. One can show that

$$\frac{\partial}{\partial k}p^C(k) = -\frac{A_C}{B_C} \tag{18}$$

where

$$A_C = \mathbb{E}_\lambda\left[\frac{1}{D_C^2}\cdot\left((1-2p^C(k))e^{\phi(v;p^C(k),k)+1-v/k}\cdot\left(\frac{v}{k^2}+\phi_k(v;p^C(k),k)\right)\right)\right],$$

$$B_C = \mathbb{E}_\lambda\left[\frac{1}{D_C^2}\cdot\left(2(1-e^{\phi(v;p^C(k),k)+1-v/k}) + (1-2p^C(k))\cdot e^{\phi(v;p^C(k),k)+1-v/k}\cdot\phi_p(v;p^C(k),k)\right)\right],$$

and

$$D_C = 2p^C(k) + (1-2p^C(k))\cdot e^{\phi(v;p^C(k),k)+1-v/k}.$$

Letting $k \uparrow k^t$, we conclude that

$$\frac{\partial}{\partial k}p^C(k) \to -\mathbb{E}_\lambda\left[\frac{v}{(k^t)^2}\cdot e^{v/k^t-1}\right]/\mathbb{E}_\lambda\left[\frac{2(1-e^{1-v/k^t})+1}{e^{2\cdot(1-v/k^t)}}\right]$$

as required. □

Concluding the proof of Theorem 2. We use *L'Hopital's rule* to show that as $k \uparrow k^t$, the ratio $p^B(k)/p^M(k)$ is bounded above 1/2 *strictly*. Formally:

Lemma 9. *There exists* $\Theta > 0$ *such that*

$$\lim_{k\uparrow k^t} \frac{p^C(k)}{p^M(k)} > \frac{1}{2} + \Theta.$$

Proof. Note that $\lim_{k\uparrow k^t} \frac{\partial}{\partial k} p^M(k)$ exists and is different from 0. Therefore, by *L'Hopital's rule*

$$\lim_{k\uparrow k^t} \frac{p^C(k)}{p^M(k)} = \lim_{k\uparrow k^t} \frac{\frac{\partial}{\partial k} p^C(k)}{\frac{\partial}{\partial k} p^M(k)} = \frac{\mathbb{E}_\lambda \left[\frac{2-e^{1-v/k^t}}{e^{2\cdot(1-v/k^t)}}\right]}{\mathbb{E}_\lambda \left[\frac{2(1-e^{1-v/k^t})+1}{e^{2\cdot(1-v/k^t)}}\right]} = \frac{1}{2 - \frac{\mathbb{E}_\lambda\left[e^{2(v/k^t-1)}\right]}{2\mathbb{E}_\lambda\left[e^{2(v/k^t-1)}\right]-1}}.$$

Since

$$2\mathbb{E}_\lambda \left[e^{2(v/k^t-1)}\right] - 1 > \mathbb{E}_\lambda \left[e^{2(v/k^t-1)}\right] - 1 \geq 0$$

because of Jensen inequality, the conclusion of the lemma follows. □

As the last step, note that as $k \uparrow k^t$, $p^M(k), p^C(k) \to 0$. It follows that $x_k^M(v), x_k^C(v) \to k^t$ for all $v \in V$. Now, fix $\varepsilon > 0$ so small that $1 + 2(\Theta - \varepsilon) > (k^t + \varepsilon)/(k^t - \varepsilon)$, and let $\hat{k} \in (k^e, k^t)$ be such that $p^C(k)/p^M(k) > 1/2 + \Theta - \varepsilon$ and $x_k^m(v) \in (k^t - \varepsilon, k^t + \varepsilon)$ for all $k > \hat{k}$, $v \in V$, and $m \in \{C, M\}$. For all $k > 0$, we have that $2p^C(k)(k^t - \varepsilon) > p^M(k)(k^t + \varepsilon)$ if and only if

$$2 \cdot \frac{p^C(k)}{p^M(k)} > \frac{k^t + \varepsilon}{k^t - \varepsilon}. \tag{19}$$

Notice that (19) holds by assumption as long as $k \in (\hat{k}, k^t)$. Since by construction we have $\Pi^C(k) \geq p^C(k)(k^t - \varepsilon)$ and $p^M(k)(k + \varepsilon) \geq \Pi^M(k)$, we conclude that $2\Pi^C(k) > \Pi^M(k)$ for all $k \in (\hat{k}, k^t)$ as required. *Q.E.D.*

Proof of Lemma 4

Proof. The proof of Lemma 4 relies on the following lemma.

Lemma 10. *There exists threshold* $v^* > 0$ *such that* $x^M(v) > x^C(v)$ *iff* $v \geq v^*$.

Proof of Lemma 10. Let π^M be the overall equilibrium engagement level of the consumer when the firms collude, and $2\pi^C$ be the overall engagement level of the consumer in the competitive trading equilibrium. For every $v \in V$, let $W^M(v) = W\left(\frac{\pi^M}{1-\pi^M}e^{v/k-1}\right)$ and $\phi^C(v) = \phi(v;\pi^C)$ solving (5). Since $x^M(v) = k(1+W^M(v))$ and $x^C(v) = k(1+\phi^C(v))$, it follows that $x^M(v) > x^C(v)$ if and only if $W^M(v) > \phi^C(v)$. By the definition of the Lambert function, $W^M(v)e^{W^M(v)} = \frac{\pi^M}{1-\pi^M}e^{v/k-1}$. Moreover, $x \mapsto xe^x$ is a strictly increasing function of $x > 0$. Therefore, $W^M(v) > \phi^C(v)$ if and only if

$$\frac{\pi^M}{1-\pi^M}e^{v/k-1} > \phi^C(v)e^{\phi^C(v)}. \tag{20}$$

From equation (5), we know that $\phi^C(v)e^{\phi^C(v)} = \frac{1-\phi^C(v)}{2} \cdot \frac{2\pi^C}{1-2\pi^C}e^{v/k-1}$. Therefore, (20) is equivalent to

$$\frac{1-\phi^C(v)}{2} < \frac{\pi^M}{1-\pi^M} \cdot \frac{1-2\pi^C}{2\pi^C}. \tag{21}$$

Because $\phi^C(v)$ is strictly increasing in v, the conclusion of the lemma follows. $\square$

Known results in rational inattention show that, in equilibrium,[45]

$$\mathbb{E}[U^M] = \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^M}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right]$$

$$\mathbb{E}[U^C] = \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right].$$

Consider the random variables Y^C and Y^M defined by $Y^C(v) := v - x^C(v)$ and $Y^M(v) = v - x^M(v)$. Let G^C and G^M be the CDF of Y^C and Y^M respectively, and define $\omega := \mathbb{E}_\lambda[x^M(v)] - \mathbb{E}_\lambda[x^C(v)]$. By assumption, $\omega \geq 0$. Finally, denote with u_1 and u_0 the maximal and minimal element in the support of Y^C respectively. From Lemma 10, we know that $u_0 \leq Y^M \leq u_1$ with probability 1. Furthermore, one can verify that $\omega \geq 0$ together with Lemma 10 imply

$$\int_u^{\bar{u}} G^C(y)dy \leq \int_u^{\bar{u}} G^M(y)dy \;\text{ for all } u \in [u_0, u_1].$$

This means that any expected utility maximizer with an increasing and convex

[45] See, e.g., Lemma 2 in Matějka and McKay (2015). See also Denti, Marinacci, and Montrucchio (2020).

Bernoulli utility function $w : [u_0, u_1] \to \mathbb{R}$ would prefer the lottery Y^C over Y^M (see Theorem 4 in Meyer (1977)). Observe that for every $\pi \in [0, 1/2]$, we have

$$k \cdot \mathbb{E}_{\mu^M}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] = k \cdot \mathbb{E}\left[\ln\left(2\pi \cdot e^{\frac{Y^M}{k}} + 1 - 2\pi\right)\right],$$

$$k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] = k \cdot \mathbb{E}\left[\ln\left(2\pi \cdot e^{\frac{Y^C}{k}} + 1 - 2\pi\right)\right].$$

Furthermore, the function $y \in (0, +\infty) \mapsto \ln\left(2\pi \cdot e^{y/k} + 1 - 2\pi\right)$ is strictly increasing and strictly convex in $y > 0$ whenever $\pi \in (0, 1/2)$. Therefore,

$$\begin{aligned}
\mathbb{E}[U^M] &= \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^M}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] \\
&= k \cdot \mathbb{E}_{\mu^M}\left[\ln\left(\pi^M \cdot e^{\frac{v-x}{k}} + 1 - \pi^M\right)\right] = k \cdot \mathbb{E}\left[\ln\left(\pi^M \cdot e^{\frac{Y^M}{k}} + 1 - \pi^M\right)\right] \\
&\leq k \cdot \mathbb{E}\left[\ln\left(\pi^M \cdot e^{\frac{Y^C}{k}} + 1 - \pi^M\right)\right] = k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(\pi^M \cdot e^{\frac{v-x}{k}} + 1 - \pi^M\right)\right] \\
&< \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] = \mathbb{E}[U^C].
\end{aligned}$$

where the first inequality is implied by $\pi^M \in (0, 1)$, while the last strict inequality is implied by the fact that the overall engagement level π^M is not a best response to μ^C. We conclude that $\mathbb{E}[U^C] > \mathbb{E}[U^M]$ as required. □

Proof of Proposition 4

Proof. First, we show that the threshold $\bar{k} \geq k^e$ exists. To this goal, we use equation (21) introduced earlier. Specifically, the proof of Theorem 2 shows that while both π^C and π^M converge to 0 as $k \uparrow k^t$, we have $\lim_{k \uparrow k^t} \frac{\pi^C(k)}{\pi^M(k)} = \frac{1}{2 - \frac{\mathbb{E}_\lambda\left[e^{2(v/k^t - 1)}\right]}{2\mathbb{E}_\lambda\left[e^{2(v/k^t-1)}\right] - 1}}$. Such limit is strictly less than 1 because $\mathbb{E}_\lambda\left[e^{2(v/k^t-1)}\right] > \left(\mathbb{E}_\lambda\left[e^{v/k^t-1}\right]\right)^2$ due to Jensen inequality. (Recall that $\mathbb{E}_\lambda\left[e^{v/k^t-1}\right] = 1$ by definition.) Therefore, while the LHS of equation (21) converges to $\frac{1}{2}$ because $\phi^C(v) \downarrow 0$ as $k \uparrow k^t$, the RHS of (21) is converging to a limit strictly greater than 1/2. As a result, equation (21) is satisfied eventually (i.e., as k approaches k^t from below) for all $v \in V$. The existence of the threshold $\bar{k}$ follows immediately from this observation.

We now show that $\mathbb{E}_\lambda[x^M(v)] \geq \mathbb{E}_\lambda[x^C(v)]$ for all $k \in (0, k^e]$. To this goal, we begin by showing that the aggregate engagement level under collusion is weakly larger than the engagement level each competitive firm experience in the efficient equilibrium. Formally:

Lemma 11. *Suppose $k \leq k^e$. Then, $\pi^M \geq 1/2$.*

Proof. Ravid (2020) shows that the function $F^M(\cdot)$ defined in the proof of Theorem 2 is strictly convex when $k < k^t$, and satisfies $F^M(0) > 1 > F^M(1^-)$. This implies that, if $\pi^M \in (0,1)$ is the unique solution to $F^M(\pi^M) = 1$, we have $F^M(\pi) \geq 1$ if and only if $\pi^M \geq \pi$. Therefore, it is sufficient to show that:[46]

$$\frac{1}{2}F^M(1/2) = \mathbb{E}_\lambda\left[\frac{W(1/2,v)}{W(1/2,v)+1}\right] \geq 1/2 \tag{22}$$

whenever

$$\mathbb{E}_\lambda\left[e^{1-v/k}\right] \leq 1/e, \tag{23}$$

where $W(\pi, v) = W\left(\frac{\pi}{1-\pi}e^{v/k-1}\right)$. Observe that we can interpret (22) as an objective function and (23) as a constraint set on the distribution over quality levels $\lambda \in \Delta(\mathbb{R}_+)$. To simplify the problem, change variable from v to $y = e^{1-v/k}$. That is, let $\mathcal{F} := \{F \in \Delta(\mathbb{R}_+) : F \text{ is finitely supported}\}$. We need to show that $V^* \geq 1/2$, where

$$V^* := \inf_{F \in \mathcal{F}} \mathbb{E}_F\left[\frac{W(1/y)}{W(1/y)+1}\right] \quad \text{subject to} \quad \mathbb{E}_F[y] \leq 1/e.$$

The function $y \mapsto H(y) := \frac{W(1/y)}{W(1/y)+1}$ is strictly decreasing and strictly convex in $y \geq 0$. Therefore, V^* is achieved by the degenerate distribution $F = \delta_{1/e}$. Plugging in $y = 1/e$ in $H(\cdot)$, we get $H(1/e) = 1/2$, that is (22) holds. This shows that $\pi^M \geq 1/2$, as required. □

Given equations (4), the monopolist's equilibrium pricing strategy, and the fact that $\phi^C(v) = 1$ for all $v \in V$ when competitive trade is efficient, to complete the proof of Proposition 4, it is sufficient to show that $\mathbb{E}_\lambda[W(\pi^M, v)] \geq 1$, for all $k \leq$

[46] Observe that $k \leq k^e$ if and only if (23) holds.

k^e. To this goal, we use the optimization approach introduced earlier once again. Formally, define $\mathcal{F} := \{F \in \Delta(\mathbb{R}_+) : F \text{ is finitely supported}\}$. Since $\pi^M \geq 1/2$ (Lemma 11), it is enough to argue that $V^{**} \geq 1$, where

$$V^{**} := \inf_{F \in \mathcal{F}} \mathbb{E}_F[W(1/y)] \quad \text{subject to} \quad \mathbb{E}_F[y] \leq 1/e.$$

The function $y \mapsto G(y) := W(1/y)$ is strictly decreasing and strictly convex. Therefore, V^{**} is achieved at $F = \delta_{1/e}$. Since, $G(1/e) = 1$, we are done. □

Proof of Lemma 5 (sketch)

Proof. Suppose (μ, σ, β) is a competitive trading equilibrium with $N \geq 2$ firms. That such equilibrium assessment must be symmetric follows from the same arguments used for the duopoly model. From Milgrom and Roberts (1990), we know that for every $v \in V$, each firm $i \in I$ uses a pure strategy $\sigma_i(\cdot|v) = \delta_{x(v,N)}$ that solves

$$\max_{x_i \geq 0} \frac{\pi e^{\frac{v-x_i}{k}}}{\pi\left(e^{\frac{v-x_i}{k}} + \sum_{j \neq i} e^{\frac{v-x(v,N)}{k}}\right) + 1 - N\pi} \cdot x_i$$

The result follows from a re-arranging of the FOCs. See the proof of Lemma 2. □

Proof of Proposition 5 (sketch)

Proof. **Part** (i)**:** Follows from a simple extension of the proof of Theorem 1.

Part (ii)**:** Follows from a straightforward extension of the proof of Proposition 2. Here, we show how the instrumental first step is extended.

Lemma 12 (Pricing effect for an arbitrary number of firms)**.** *Fix $N_1 > N_2 \geq 2$ arbitrarily, and for every $j \in \{1, 2\}$, let $\phi_j > 0$ be the unique solution to*

$$\phi_j \left(N_j - 1 + e^{\phi_j} \frac{1 - N_j \pi^j}{\pi^j e^{v/k-1}}\right), \tag{24}$$

where $\pi^j \in (0, 1/N_j]$. If $N_1\pi^1 = N_2\pi^2$, then $\phi_2 > \phi_1$.

Proof. Equation (24) can be equivalently re-written as

$$\pi^j e^{v/k-1} = \phi_j(N_j - 1)\pi^j e^{v/k-1} + (1 - N_j\pi^j)\phi_j e^{\phi_j}.$$

Suppose by way of contradiction that $\phi_2 \leq \phi_1$. Then, $\phi_2 e^{\phi_2} \leq \phi_1 e^{\phi_1}$ which, given our assumption $N_1\pi^1 = N_2\pi^2$, implies $\pi_2(1 - \phi_2(N_2 - 1)) \leq \pi^1(1 - \phi_1(N_1 - 1))$. This is equivalent to $\frac{N_1}{N_2}(1 - \phi_2(N_2 - 1)) \leq 1 - \phi_1(N_1 - 1)$ which, in turn, implies $\phi_2 \geq \frac{N_1-1}{N_2-1}\phi_1 + (\frac{N_1}{N_2} - 1) > \phi_2$, a contradiction. □

Part (iii)**:** With $N \geq 2$ firms, the maximal price that can be sustained in a symmetric equilibrium is $x(v, N) = k \cdot \frac{N}{N-1}$.[47] Therefore, an efficient equilibrium exists if and only if $k \leq k^e(N)$, where $k^e(N)$ is the unique solution to $\mathbb{E}_\lambda\left[e^{\frac{N}{N-1} - v/k}\right] = 1$.

Part (iv)**:** Fix $N \geq 2$ arbitrarily and let $k \in (k^e(N), k^t)$. For each $p \in [0, 1/N)$, define $F_k^{C,N}(p)$ as $F_k^{C,N}(p) := \mathbb{E}_\lambda\left[\frac{1}{Np + (1-Np)\cdot e^{\phi(p,v,k,N)+1-v/k}}\right]$ where, for $p > 0$, we let $\phi(p, v, k, N)$ be defined as the unique solution to equation (6), and we set $\phi(0, v, k, N) := 0$ for all $v \in V$ and $k \in (k^e(N), k^t)$. For fixed k, the consumer trade engagement level with each firm in the competitive trading equilibrium is given by the unique solution to $F_k^{C,N}(p) = 1$. Denote with $p^{C,N}(k)$ such solution. Because as $k \uparrow k^t$, offers are converging to k^t irrespective of N, the crucial step to prove (iv) is to show that, for $N_1 > N_2 \geq 2$,

$$\lim_{k \uparrow k^t} \frac{p^{C,N_1}(k)}{p^{C,N_2}(k)} > \frac{N_2}{N_1} + \Theta, \tag{25}$$

for some $\Theta > 0$. Using the same arguments as in the proof of Theorem 2, one obtains

$$\lim_{k \uparrow k^t} \frac{p^{C,N_1}(k)}{p^{C,N_2}(k)} = \frac{\mathbb{E}_\lambda\left[\frac{N_2\left(1-e^{1-v/k^t}\right)+1}{e^{2(1-v/k^t)}}\right]}{\mathbb{E}_\lambda\left[\frac{N_1\left(1-e^{1-v/k^t}\right)+1}{e^{2(1-v/k^t)}}\right]},$$

which implies that (25) is satisfied. □

[47] This price corresponds to $\pi = 1/N$, which generalizes the case $\pi = 1/2$ of the duopoly setting.