ORDER ROUTING AND MARKET QUALITY: WHO BENEFITS FROM INTERNALIZATION?

UMUT ÇETİN AND ALBINA DANILOVA

ABSTRACT. The practice of retail internalization has been a controversial topic since the late 1990s. The crux of this debate is whether this practice benefits, via the price improvement relative to exchange, or disadvantages, via the reduced liquidity on exchange, retail traders. To answer this question we set two models of market design that differ in their mode of liquidity provision: in the model capturing retail order internalization the liquidity is provided by market makers (representing wholesalers) competing for the retail order flow in a Bertrand fashion, whereas in the model characterizing the open exchange the price-taking competitive agents act as liquidity providers. We discover that, when liquidity providers in both market designs are risk averse, routing of the marketable orders to the wholesalers is preferred by *all* retail traders: informed, uninformed and noisy.

In addition to addressing optimal order routing problem, we identify a universal parameter that allows comparison of market liquidity, profit and value of information across different markets and demonstrate that the risk aversion of liquidity providers fundamentally changes market outcomes. In particular, we observe mean reverting inventories, price reversal, and lower market depth as the result of retail investor (informed or not) absorbing large shocks in their inventory to compensate for the unwillingness of liquidity providers to bear risk.

1. INTRODUCTION

The retail internalization, i.e. the practice of routing the marketable retail orders to wholesalers such as Citadel, Susquehanna, and Wolverine, has been a major focus for regulators since the late 1990s.[1] This regulators' attention is unsurprising given the size of the market and the proportion of retail orders affected by this practice.

---

[1]The regulatory debate on internalization is far from over: while UK Financial Conduct Authority (FCA) has banned payment for order flow (PFOF) since 2012 and the European Union is following the suit in their 2021 Capital Markets Union Package (see https://ec.europa.eu/commission/presscorner/detail/en/ip_21_6251), the US Securities and Exchange Commission (SEC) is still assessing the impact of such a ban on market quality (and in their earlier, 1997, assessment SEC did not find internalization harmful to market quality).

Indeed, Bryzgalova et al. (2022) report that the average share of internalized trades in the total weekly stock trading volume is 17%, and has an upward trend. Moreover, most of the retail orders received by brokers are routed to wholesalers[2].

The main argument for banning payment for order flow (PFOF) is that it reduces liquidity on exchanges since the internalization of *uninformed* retail orders increases the information asymmetry on the exchanges leading to higher spreads. This in turn increases retail orders' execution costs as the price improvement that the wholesalers must provide is based on the best quotes from the exchange[3]. Easley et al. (1996) and Bessembinder and Kaufman (1997) similarly argue that PFOF lowers the market quality since the wholesalers *cream skim* the uninformed traders.

This argument hinges on the assumption that the retail orders are uninformed, consistently with empirical studies at that time (see, e.g. Barber and Odean (2000, 2008)). However, more recent studies (Kaniel et al. (2008), Barber et al. (2008), Kaniel et al. (2012), Kelley and Tetlock (2013), Fong et al. (2014), Barrot et al. (2016), Boehmer et al. (2021)) show that the retail order flow can predict future returns, which suggests it can contain informed trading.

Our paper enters the debate on optimal routing of the retail order flow by comparing two market designs: in the first model, that extends Garleanu et al. (2008), the liquidity is supplied by perfectly competitive agents of unit mass (as a model for the lit exchange); in the second, that extends Kyle (1985), it is provided by a finite number of imperfectly competitive market makers, i.e. wholesalers. Considering both informed and uninformed types of strategic traders allows us to analyse the retail agents' preferences for trading venues whether the retail order is informed or uninformed. We find that, independently of the information content of the retail order, strategic traders prefer their orders to be routed to the wholesalers when the liquidity providers (market makers and competitive agents) are risk averse. Remarkably this profit gain of strategic traders doesn't come at the expense of noise traders who are indifferent between two venues. This result stems from the fact that (price taking)

---

[2]The US Securities and Exchange Commission (2010) Concept Release on Equity Market Structure states: "A review of the order routing disclosures required by Rule 606 of Regulation NMS of eight broker-dealers with significant retail customer accounts reveals that nearly 100% of their customer market orders are routed to OTC market makers." The concept release estimates the amount of payment for order flow is 0.1 cent per share or less. https://www.sec.gov/rules/concept/2010/34-61358.pdf (2010)

[3]See the CFA Institute study at https://www.cfainstitute.org/en/advocacy/policy-positions/payment-for-order-flow-in-the-united-kingdom.

competitive agents enjoy positive utility gain – the result remarkably similar to Biais et al. (2000) – whereas the market makers, who have the market power to set the prices, compete the utility gain away to zero. This difference of utility gain is then transferred to the strategic trader giving her higher profit in wholesalers' market.

These findings are consistent with the proponents of internalization who argue that it enhances market quality by increasing the number of agents competing to execute order flow. For example, Hansch et al. (1998) show that internalized trades pay lower spreads than non-internalized ones and that bid-ask spreads are not affected by the level of internalization. Battalio (1997) and Battalio et al. (1997) demonstrate, correspondingly, that the transaction costs are not increased by the introduction of PFOF and internalizing dealers.

This paper also contributes to theoretical market microstructure literature in two ways. First, the market structure with competitive agents extends Garleanu et al. (2008) to allow adverse selection. Second, the model with market makers considered in this paper extends the continuous time version of the model introduced in Kyle (1985) by allowing the market makers to be risk averse and by providing an explicit description of their competition. Endowing the liquidity providers with CARA utility is not only more in line with empirical research than standard risk neutrality assumption[4], but also allows to assess the impact of risk aversion on market liquidity.

The risk aversion of liquidity providers alters the equilibrium outcome drastically. One of the main differences is that, in contrast with the earlier extensions of the Kyle model, the insider's trades induce a reversal in the total demand (and hence in the price) in both models of liquidity provision process, i.e. insider trading ceases to be *inconspicuous*. This is a remarkable departure from the results of Kyle (1985) and related literature, where the distribution of the total demand is unaltered by the presence of private information, and is fully determined by the noise trading. Inventories of liquidity suppliers are mean reverting in equilibrium irrespectively of the nature of liquidity provision and independently of whether the strategic trader is informed or not. The driving force behind this result is that the risk aversion

---

[4]The literature indicates that the trading behaviour of the market makers is affected by their inventory positions even after isolating the adverse selection component of their quotes (see Huang and Stoll (1997) and Madhavan and Smidt (1993) for NYSE, Hansch et al. (1998) for LSE, Bjønnes and Rime (2005) for FX; for a survey of related literature and results, see Sections 1.2 and 1.3 in Biais et al. (2005)). This leads to the conclusion that the market makers are risk averse as inventory should play no role for a risk neutral market maker once the adverse selection is accounted for.

4 UMUT ÇETİN AND ALBINA DANILOVA

of liquidity providers makes them less willing to bear risk. Instead of paying the extra compensation for the inventory risk, the risk neutral strategic trader chooses to reduce it by absorbing (a part of) excess demand of noise traders thus lowering market makers' inventory (and associated risk). This causes the total demand to mean revert – a result unanimously supported by empirical studies (see, among others, Huang and Stoll (1997), Madhavan and Smidt (1993), Hansch et al. (1998), and Bjønnes and Rime (2005)). Moreover, the fact that the mean reversion is more pronounced in the presence of private information for reasonable market parameters suggests a new paradigm of empirical research which has so far attributed the mean reversion solely to the inventory costs.

In addition to studying equilibrium outcomes resulting from different market designs of liquidity provision and risk aversion of agents, we are also able to compare the market depth and value of information across different models and markets due to identification of a universal parameter, *the market adjusted risk aversion* ρ_M. Remarkably, the market equilibrium characteristics, once normalized with the same equilibrium characteristics in Kyle's model, turned out be fully determined by $\rho_M = \rho\sigma\gamma$ rather then by individual market characteristics such as liquidity provider' risk aversion ρ, volatility of the noisy trades, σ, and volatility of the asset, γ, in isolation.

Risk aversion considerably reduces the market depth as expected. Moreover, higher ρ_M leads to lower market depth since the liquidity providers require additional compensation for the risk that they bear. The same mechanism is responsible for the monotone relationship between normalized (by Kyle's market depth) market depth and market adjusted risk aversion.

A surprising outcome of the model, however, is that the value of private information is non-monotone in ρ_M in both markets. In particular we observe that it is decreasing for small market adjusted risk aversion and increasing for sufficiently large values. The reason for this apparently counter-intuitive behaviour is the fact that the value of information consists of two components. One component reflects insider's profits purely due to her private information, while the other is due to her participation in risk sharing. The equilibrium mean reverting demand causes the former to decrease in the market adjusted risk aversion, since this profit is collected by bringing the price to its fundamental value, and the stronger is the mean reversion, the more effort is needed to do so. On the other hand, the latter must increase in the market adjusted risk aversion as a result of risk sharing. These two components combined

result in a non-monotone dependency of the value of information on market adjusted risk aversion: for small ρ_M the value of information purely due to private information dominates, whereas for higher ρ_M this value is largely determined by profits due to participation in risk sharing.

These observations show that a mere introduction of risk averse market makers to the setting of Kyle (1985) changes the equilibrium outcome fundamentally. The previous attempt to investigate the effect of such an extension was carried out in Subrahmanyam (1991), who considered a one-period model. Due to the strategic behaviour of the insider, a direct extension of this one-period model to a multi-period, let alone a continuous-time one, is not possible, as discussed in Subrahmanyam (1991). As we demonstrate in this paper one can, however, derive the closed-form equilibrium in a continuous-time setting in the spirit of Back (1992).

Related work contains two streams of literature. In the first stream that stems from the extension of the Kyle's model liquidity providers are market makers who compete in a Bertrand fashion and set the quotes. Extensions of the Kyle's model include, but are not limited to, continuous-time formulation by Back (1992), risk-averse insider by Baruch (2002), long-lived private information by Back and Pedersen (1998), markets with default risk by Campi et al. (2013), competition among insiders by Back et al. (2000), and stochastic volatility of noise trades by Collin-Dufresne and Fos (2016). In all these models the optimal strategy of the insider is inconspicuous. In the second strand of literature that stems from Garleanu et al. (2008) the liquidity providers are price-taking dealers that correspond our competitive agents in Section 3. Back et al. (2020) is the work in this stream of literature that is closest to the model studied in Section 3. Back et al. (2020) recasts Garleanu et al. (2008) as the insider trading model with risk-averse dealers by employing optimal transport and formulating the problem under the risk-neutral measure rather than physical as we do. As their focus is primarily on the solvability of this extension, they do not consider the liquidity providers that set the quotes, i.e. market maker, precluding the analysis of the choice of the trading venue that is the main focus of our paper.

The rest of the paper is structured as follows. In Section 2 we describe the market structure. Section 3 establishes the existence of equilibrium when liquidity is provided by competitive agents and characterises it. Section 4 derives an equilibrium in the market makers' case. Section 5 discusses market parameters resulting from equilibria

derived in Sections 3 and 4 and their relationship with the Kyle model. Section 6 concludes.

2. Market structure

We consider a market in which a single risky asset is traded. The fundamental value of this asset, V – a normally distributed random variable with mean μ and variance γ^2 – becomes public knowledge at some future time $t = 1$. For simplicity of exposition, we assume that the risk free interest rate is 0.

There are three types of agents that interact in this market:

i) Noise traders, whose demands are random, price inelastic, and do not reveal any information about the value of V. In particular, we assume that their cumulative demand at time t is given by Z_t – a Brownian motion with mean 0 and variance σ^2, independent of V. That is, $Z_t = \sigma B_t$, where B is a standard Brownian motion independent of V.

ii) A single insider, who knows V from time $t = 0$ onwards, and is risk neutral. We will denote insider's cumulative demand at time t by X_t. The information of the insider at time t, $\mathcal{F}_t^I$, is generated by observing the price of the risky asset on $[0, t]$ and V.

iii) Liquidity suppliers observe only the net demand[5] of the risky asset, $Y = X + Z$, thus, their information at time t, $\mathcal{F}_t^M$, is generated by observing Y only on the interval $[0, t]$.

   We also assume that they have identical CARA utilities U with the common risk aversion parameter ρ. More precisely, $U(x) = 1 - e^{-\rho x}$. We consider two market designs where the liquidity suppliers are either

   a) *perfectly competitive agents* that form a continuum of mass one and take prices P as given, or

   b) *market makers* who compete in a Bertrand fashion for the net demand of the risky asset. The number of market makers is assumed to be finite.

Since the insider is given the realization of $V = v$, she assigns probability 1 to this event. On the other hand, since V is a continuous random variable and the liquidity suppliers do not observe V, they assign probability 0 to the event $V = v$. Thus, their probability measures are not only different but *singular* which implies that insider's

---

[5]That is, we assume that the traders cannot choose a particular liquidity supplier to trade with. Hence, the demand is aggregated before being sent to the liquidity suppliers.

probability measure is *not* a measure change from the liquidity suppliers' one. This is one of the main mathematical challenges of the insider trading models.

Nevertheless, it is possible to express liquidity suppliers' measure via the insiders' one. Indeed, since insider views V as deterministic and liquidity suppliers view V as random, the probability of a given event from the point of view of liquidity supplier should be an average – with respect to the distribution of V – of the probabilities of this event from the point of view of the insider for all possible realizations of V. Mathematically, this is expressed as

$$\mathbb{P}(E) = \int_{\mathbb{R}} \mathbb{P}^v(E)\mathbb{P}(V \in dv) \text{ for any } E \in \mathcal{F}$$

where $\mathbb{P}^v$ is the probability measure of the insider[6] who is given the information that $V = v$, $\mathbb{P}$ is the liquidity suppliers' probability measure and $\mathcal{F}$ is the set of all possible events in our model, i.e. a sigma-algebra generated by V and $(B_t)_{t=0}^1$.

To keep the exposition rigorous, we need to introduce an information flow that underpins the model as well as the probability measure on $\mathcal{F}$. They can be viewed as information flow and probability measure of a fictitious agent who observes the fundamental value V and the evolution of the noise trades, B. We denote by P the probability measure on $\mathcal{F}$ and by $\mathcal{F}_t$ the filtration[7] generated by V and $(B_s)_{s\in[0,t]}$.

3. Competitive agents equilibrium

Before defining equilibrium in this market we define the admissible strategies for the competitive agents and the insider. We look for an equilibrium where the price is of the following form:

$$dP_t^X = m_t dt + \lambda_t(dZ_t + dX_t), \tag{3.1}$$

for some stochastic processes m, λ and X such that[8] P^X is exists, well defined and adapted to $(\mathcal{F}_t^M)$. We denote by $\mathcal{L}(P^X)$ the set of portfolio strategies of competitive agents, i.e. $\theta \in \mathcal{L}(P^X)$ means that θ is adapted to $(\mathcal{F}_t^M)$ and associated portfolio wealth process, $(\int_0^t \theta_s dP_s^X)_{t\in[0,1]}$ is well-defined (see Section IV.2 of Protter (2005)).

---

[6]Note that $\mathbb{P}^v$ is the joint distribution of V and B evaluated at $V = v$.
[7]Note that the insider is assumed to observe the fundamental value and the price process but not the noisy demand process. Thus her filtration does not have to coincide with $\mathcal{F}_t$.
[8]We also assume that m_t and λ_t are measurable with respect to $\mathcal{F}_t^M$ and X_t – the insider strategy – is measurable with respect to $\mathcal{F}_t^I$.

Moreover, since equilibrium price processes should not allow for arbitrage opportunities, we will only consider process P^X admitting an equivalent local martingale measure; that is, there exists a measure $\mathbb{Q}$ (equivalent to measure $\mathbb{P}$) under which P^X is an $\mathcal{F}^M$-local martingale. The processes satisfying these conditions are said to be *compatible with no-arbitrage* and the set of such processes is denoted by $\mathcal{P}(X)$.

Definition 3.1. *Let m, σ and X be given and consider $P^X \in \mathcal{P}(X)$. For each competitive agent the trading strategy θ is admissible given P^X if $(\int_0^t \theta_s dP_s^X)_{t\in[0,1]}$ is a $(\mathbb{Q}, \mathcal{F}^M)$-martingale, where $\mathbb{Q}$ is the unique equivalent local martingale measure for P^X. The set of admissible strategies for competitive agents is denoted by $\mathcal{A}(m, \lambda, X)$.*

The set of admissible strategies for the insider is different as she takes into account the feedback effect of her trading.

Definition 3.2. *Let m and λ be given. The trading strategy X is admissible if it is absolutely continuous, the associated price process is compatible with no-arbitrage, that is,*

$$P_t := P_0 + \int_0^t m_u du + \int_0^t \lambda_u(dZ_u + dX_u) \in \mathcal{P}(X),$$

and no doubling strategies are allowed, i.e.

$$\mathbb{E}^v \int_0^1 P_t^2 dt < \infty.$$

The set of admissible strategies for the insider is denoted by $\mathcal{A}^I(m, \lambda)$.

In addition to the standard *no-doubling* condition as in Back (1992) we also require that the price process associated with the insider's trading strategy should not offer arbitrage opportunities to the competitive agents since, otherwise, the model admits no equilibrium.

Definition 3.3. *An equilibrium in this market is given by $((m^*, \lambda^*), \theta^* = (\theta^{*,a})_{a\in[0,1]}, X^*)$ such that*

$$P_u^* = P_0^* + \int_0^u m_t^* dt + \int_0^u \lambda_t^*(dZ_t + dX_t^*),$$

$P^* \in \mathcal{P}(X^*)$, $\theta^{*,a} \in \mathcal{A}(m^*, \lambda^*, X^*)$ for all $a \in [0,1]$, $X^* \in \mathcal{A}^I(m^*, \lambda^*)$ and the following are satisfied:

(1) **(Optimality for competitive agents)** *For all $a \in [0,1]$ $\theta^{*,a}$ solves*

$$\sup_{\theta\in\mathcal{A}(m^*,\lambda^*,X^*)} \mathbb{E}\left[1 - \exp\left\{-\rho\left(\int_0^1 \theta_t dP_t^* + \theta_1(V - P_1^*)\right)\right\}\right];$$

(2) **(Optimality for the insider)** X^* *solves*

$$\sup_{X \in \mathcal{A}^I(m^*, \lambda^*)} \mathbb{E}^v \left[\int_0^1 X_t dP_t + X_1(V - P_1) \right],$$

where

$$P_t = P_0^* + \int_0^t m_u^* du + \int_0^t \lambda_u^* (dZ_u + dX_u);$$

(3) **(Market clearing)** $\int_0^1 \theta_t^{*,a} da = -(X_t^* + Z_t)$ *for all* $t \in [0,1]$.

This definition of equilibrium formalizes a continuous-time market in which Kyle's insider trades with "dealers" of Garleanu et al. (2008). To be more precise this notion of equilibrium postulates that the demand pressure faced by the dealers comprises of the optimal strategy of the insider and the inelastic demand of the noise traders. By doing so it partially endogenizes the total order flow that is fully exogenous in Garleanu et al. (2008). As in Kyle, the insider takes into account the fact that her actions influence the prices whereas the liquidity suppliers are price takers as in Garleanu et al. (2008).

In case of risk-neutral competitive agents the price process is a martingale and, therefore, the equilibrium coincides with that of Kyle (1985) in which the price is an affine function of demand. The next theorem establishes the existence of an equilibrium where the price process is still an affine function of demand for an arbitrary risk aversion parameter.

Theorem 3.1. *There exists an equilibrium* $((m^*, \lambda^*), \theta^* = (\theta^{*,a})_{a \in [0,1]}, X^*)$, *where*

$$P_0^* = \mu, \quad m^* = 0, \quad \lambda^* = \frac{\rho \gamma^2}{2} + \sqrt{\frac{\rho^2 \gamma^4}{4} + \frac{\gamma^2}{\sigma^2}},$$

$$\theta^{*,a} = -(X_t^* + Z_t), \quad \forall a \in [0,1], \quad and \quad X_t^* = (1-t) \int_0^t \frac{\frac{V-\mu}{\lambda^*} - Z_s}{(1-s)^2} ds. \ (3.2)$$

Moreover, the equilibrium characteristics of the market are as follows:

(1) *From the point of view of the insider the equilibrium demand and price dynamics are given by*

$$dY_t^* = \sigma dB_t + \frac{\frac{V-\mu}{\lambda^*} - Y_t^*}{1-t} dt, \tag{3.3}$$

$$dP_t^* = \lambda^* \sigma dB_t + \frac{V - P_t^*}{1-t} dt, \tag{3.4}$$

and her ex-ante profit is given by

$$\mathbb{E}\left[\frac{(P_0 - V)}{2\lambda^*} + \frac{\sigma^2}{2}\lambda^*\right] = \frac{\gamma^2}{2\lambda^*} + \frac{\sigma^2}{2}\lambda^*. \tag{3.5}$$

(2) *From the point of view of the competitive agents the equilibrium demand and price dynamics are given by*

$$dY_t^* = \sigma d\beta_t^* - \sigma^2\frac{\lambda^*\rho}{1 + \lambda^*\rho\sigma^2(1-t)}Y_t^* dt, \tag{3.6}$$

$$dP_t^* = \lambda^*\sigma d\beta_t^* - \sigma^2\frac{\lambda^*\rho}{1 + \lambda^*\rho\sigma^2(1-t)}(P_t^* - \mu)dt, \tag{3.7}$$

where β^* is an $\mathcal{F}^M$-Brownian motion. Their expected utility equals

$$1 - \frac{\lambda^*\sigma}{\gamma}e^{-\frac{\lambda^*\rho\sigma^2}{2}}. \tag{3.8}$$

The proof of the above theorem is technical and is delegated to the appendix. However, the idea of the proof is easy to grasp once we observe that the equilibrium price $P_t^* = \lambda^* Y_t^* + \mu$ is a linear function of demand; thus, the optimality condition for the insider's trading strategy is the same as in Back (1992). That is, the insider drives the market price to the fundamental value V. Moreover, how she achieves that does not affect her utility and in particular the total demand does not have to be a Brownian motion as in Back (1992). Hence, to determine the optimal demand process that ensures convergence of the price to the fundamental value, one needs to solve the optimization problem of the competitive agents. This is achieved by equating the stochastic discount factor to their marginal utility, which allows us to identify the law of the equilibrium demand process from the point of view of the competitive agents. Finally, to identify the equilibrium demand from the point of view of the insider, we need to ensure that from her point of view the equilibrium terminal price coincides with the fundamental value and the law of the demand process in the smaller information set of the competitive agents remains the same. This is obtained via a Doob's h-transform technique that has been developed for precisely these types of problems.

Although the in-depth discussion of the results of this Theorem is postponed to Section 5, a few remarks on the main features of the obtained equilibrium are in order. First, observe that the competitive agents obtain strictly positive utility from trading as a compensation for risk they bear. Thus, given that they started with zero utility, they enjoy positive utility gain in this market design, as in Biais et al. (2000).

Furthermore, comparison of the optimal insider strategy X^* given by (3.2) with equation (13) in Back (1992) reveals that[9] the response of the insider to the choice of price as a function of total demand by the competitive agents is the same as in Back (1992). However, in contrast with Back (1992) where the law of equilibrium demand process and the noisy demand process coincided, the law of demand process given by (3.6) is different from the one of the noisy trade process (as former is Ornstein-Uhlenbeck process and latter is a Browninan motion). Thus, differently from Back (1992), in this setting the optimal strategy is not inconspicuous.

This difference arises from the fact that the inventory, due to the risk aversion, is a source of risk for liquidity suppliers (whereas it isn't, due to risk neutrality, in Back (1992)). Thus, for a given market depth the liquidity suppliers are willing to bear only limited fluctuations in their inventory. As the insider's strategy is optimal as long as the price converges to the fundamental value at the end, she opts for a strategy that keeps the inventory of competitive agents mean reverting around 0.

Keeping the liquidity suppliers' inventory mean reverting around 0 is still optimal even if the strategic trader is uninformed as is shown in the next theorem.

Theorem 3.2. *There exists an equilibrium* $((m^*, \lambda^*), \theta^* = (\theta^{*,a})_{a \in [0,1]}, X^*)$, *where*

$$\lambda^* = \frac{\rho \gamma^2}{2}, \quad P_t^* = \begin{cases} \lambda^* Y_t^* + \mu, & t < 1 \\ 2\lambda^* Y_1^* + \mu, & t = 1 \end{cases},$$

$$dX_t^* = -\frac{\lambda^* \sigma^2 \rho}{2 + \lambda^* \rho \sigma^2 (1-t)} (X_t^* + Z_t) dt, \quad t < 1,$$

$$\Delta X_1^* = -\frac{X_{1-}^* + Z_1}{2}, \quad and$$

$$\theta^{*,a} = -(X^* + Z), \quad \forall a \in [0,1].$$

Moreover, the ex-ante profit of the strategic trader is given by $\frac{\rho \gamma^2 \sigma^2}{4}$ *and the expected utility of the competitive agents equals*

$$1 - \sqrt{1 + \frac{\lambda^* \rho \sigma^2}{2}} e^{-\frac{\lambda^* \rho \sigma^2}{2}} \tag{3.9}$$

Comparing results of Theorem 3.1 and Theorem 3.2 shows that the speed of mean reversion in absence of private information, $\frac{\lambda^* \sigma^2 \rho}{2 + \lambda^* \rho \sigma^2 (1-t)}$, is smaller than the one with private information, $\frac{\lambda^* \sigma^2 \rho}{1 + \lambda^* \rho \sigma^2 (1-t)}$. This implies that although the mean reversion of

---

[9]Note that in our case $h(y) = \mu + \lambda^* y$.

demand is borne out of inventory management considerations, its speed is determined by the total amount of perceived risk, which is higher in the presence of asymmetric information. Smaller total perceived risk (and therefore lower required compensation) also underpins the lower utility gain of competitive agents in the absence of private information which is evident from comparison of (3.9) and (3.8).

4. Market makers equilibrium

Before determining the optimal behaviour of the agents in this market, we first need to understand how the orders are allocated among the market makers. Recall that the liquidity providers observe only total demand process which implies that the orders can not be split between different providers. Thus, orders are combined before arriving to the market makers and the Bertrand competition dictates that the total order is executed at the best available quote. To gain an intuition as to how the winning quote is determined, consider a small time interval $[s, s + \Delta)$ and observe that the number of shares to be allocated is $\Delta Y := Y_{s+\Delta} - Y_s$. Assume that Δ is such that ΔY is arbitrarily small. Each market maker provides a quote (λ^i, ϕ^i), where $\lambda^i > 0$. That is, the market maker i is willing to absorb this demand at the price $\lambda^i \Delta Y + \phi^i$ both at ask (if $\Delta Y > 0$) and at bid (if $\Delta Y < 0$).

Observe that if the market makers quote different ϕ^is, than the buy order ($\Delta Y > 0$) is priced at $\min_i \phi^i + \lambda^{\arg\min_i \phi^i} \Delta Y$ while the sell order is priced at $\max_i \phi^i + \lambda^{\arg\max_i \phi^i} \Delta Y$ leading to a negative bid-ask spread and profitable round-trip trades. As trading is continuous, those round-trip trades warrant infinite profits for any trader. Thus, it is optimal for the market makers to quote the same ϕ.

If $\phi^i = \phi$ for all i, than, independently from the sign of ΔY, the market maker quoting the smallest λ^i provides the best price for this order. That is, the market makers's competition reduces to the provision of the smallest Kyle's lambda.

The order is allocated according to the *price-time priority*. That is, the market maker quoting the best price of the given order size gets the whole order. If there are several market makers quoting the best price, the order is allocated to the one who has submitted this quote first. Finally, as convention, the order is allocated to a single market maker at random if several market makers submit the best price at the

same time. We also rule out *shuffling*: the winning market maker continues to receive the total market order as long as his quotes are: a) the best and b) don't change[10] .

4.1. **Agents' objectives.** As we are searching for a linear equilibrium, we consider the quotes of the form (λ^i, ϕ^i) where both λ^i and ϕ^i are piecewise constant and right-continuous – that is, the price of execution is known at the time of execution – functions of time. Moreover, as follows from the above discussion the market makers must quote the same ϕ at all times, i.e. all ϕ^i should be the same. The following definition summarizes these observations:

Definition 4.1. *The vector of quotes* $(\Lambda, \Phi) := \left[(\lambda^1, \phi^1) \ldots (\lambda^N, \phi^N)\right]$ *submitted by the market makers is admissible if* $\lambda^i > 0$ *and* ϕ^i *are piecewise constant and right-continuous functions of time with finite number of discontinuities such that* $\phi^i = \phi$ *for all* $i = 1, \ldots, N$*. The set of admissible vectors is denoted by* $\mathcal{H}^M$*.*

Next, we formalize the above discussion on determining the winning quote (and therefore the market quote). Let $\mathcal{S}$ be the set of right-continuous piecewise constant functions on $[0, 1]$. Given an admissible vector of quotes, (Λ, Φ) we may define the function $H : \mathcal{H}^M \mapsto \mathcal{S} \times \mathcal{S}$ by

$$H((\Lambda, \Phi)) := (\lambda, \phi), \text{ where } \lambda(t) = \min_i \lambda^i(t), \; \forall t \in [0, 1] \text{ and } \phi = \phi^1. \qquad (4.10)$$

Note that the function H maps the quotes submitted by the market makers into the market quote at which the retail orders are executed by selecting the quote with the smallest Kyle's lambda. We will denote by $\mathcal{H}$ the set of market quotes (λ, ϕ) that result from admissible vector of quotes submitted by the market makers. That is,

$$\mathcal{H} = \{(\lambda, \phi) : (\lambda, \phi) = H((\Lambda, \Phi)), \; (\Lambda, \Phi) \in \mathcal{H}^M\}.$$

Note that for the given market quote $(\lambda, \phi) \in \mathcal{H}$ the market price P evolves as

$$P_t = \phi(t) + \int_0^t \lambda(s)dY_s. \qquad (4.11)$$

Since the insider observes the market price, she observes the market quote (λ, ϕ) as well as the demand process Y as $\lambda > 0$ for $(\lambda, \phi) \in \mathcal{H}$. Therefore, the insider can perfectly infer the demand of the liquidity traders since she knows her own demand. This implies that her information is $\mathcal{F}_t^I = \sigma(V, Z_s; s \leq t)$.

---

[10]That is, if several market makers give the same (best) quote at the same time, then randomly selected market maker absorbs the total order until either the best or his quote quote changes.

Definition 4.2. *Given* $(\lambda, \phi) \in \mathcal{H}$, *X is an insider's admissible strategy for this quote (abbreviated as $X \in \mathcal{A}(\lambda, \phi)$) if it satisfies the following.*

(1) *X is adapted to $\mathcal{F}^I$.*

(2) *No doubling strategies are allowed:*

$$\mathbb{E}^v \int_0^1 P_t^2 \, dt = \mathbb{E}^v \int_0^1 \left(\int_0^t \lambda(s) \left\{ dX_s + dZ_s \right\} + \phi(t) \right)^2 dt < \infty, \qquad (4.12)$$

where $\mathbb{E}^v$ is the expectation taken with respect to $\mathbb{P}^v$.

(3) *X has finite variation.*

Condition 1) is the minimal requirement that the strategy is implementable. Condition 2) prevents the insider from following doubling strategies as in Back (1992). Condition 3) relaxes the standard, due to arguments of Back (1992), assumption of absolutely continuous strategies by allowing the insider's strategy X to have finitely many jumps (and thus X of finite variation). The jumps can not be excluded ex ante in this model since, as arguments of Back (1992) show sub-optimality of insider's bulk order (i.e. jumps in X) only when λ is continuous, the jumps in X might be optimal at (finitely many) points of discontinuity of λ.

Observe that if $X \in \mathcal{A}(\lambda, \phi)$, then the terminal wealth of the insider is given by[11]

$$W_1^X := \int_0^1 X_{s-} \, dP_s + X_1(V - P_1) = \int_0^1 (V - P_s) \, dX_s. \qquad (4.13)$$

The first term in the terminal wealth corresponds to continuous trading in the risky asset, while the second term exists due to potential discontinuity in the asset price when the value becomes public knowledge at time $t = 1$. The second expression for the wealth follows from integration by parts and X having finite variation.

As we assume that the insider is risk neutral, her optimisation problem consists of finding the strategy that maximises her expected profit for the given market quote.

We search for symmetric equilibria that exists since all market makers are identical. To analyse the deviations from a symmetric vector of quotes consider $(\Lambda, \Phi) \in \mathcal{H}^M$ where $(\lambda^i, \phi^i) = (\lambda, \phi)$ for $i \neq j$ and the quote of the deviating market maker given by $(\lambda^j, \phi^j) = (\tilde{\lambda}, \phi)$. Before quantifying the profitability of deviation we need to formalise the order allocation process for given total demand process Y.

As was discussed in the beginning of this section, the deviating market maker j gets the infinitesimal change in the demand at time $t > 0$ if his Kyle's lambda is the

---

[11]For a derivation and motivation of this equation see Back (1992).

smallest, i.e. $\tilde{\lambda}(t) < \lambda(t)$. Let's denote by τ_j the first time that the market maker j is not allocated the order. If $\tilde{\lambda}(0) > \lambda(0)$, τ_j is obviously 0. If they are equal, τ_j is still 0 if the market maker was not allocated the order at time 0. In case he wins the allocation at time 0, the time τ^j is given by

$$\tau^j = \begin{cases} \inf\{t \geq 0 : \tilde{\lambda}(t) > \lambda(t)\}, & \text{if } \tilde{\lambda}(0) = \lambda(0); \\ \inf\{t \geq 0 : \tilde{\lambda}(t) \geq \lambda(t)\}, & \text{if } \tilde{\lambda}(0) < \lambda(0), \end{cases}$$

where by convention the infimum of an empty set is ∞. Then, according to our time-priority and no shuffling conventions, his holdings, Y^j, evolve as

$$dY_t^j = \left(\mathbf{1}_{[\tau^j > t]} + \mathbf{1}_{[\tau^j \leq t]} \mathbf{1}_{[\tilde{\lambda}(t) < \lambda(t)]} \right) dY_t. \tag{4.14}$$

Note that the time-priority and no-shuffling imply that after losing the order allocation for the first time, the only way to receive new market orders is via undercutting the market.

We now turn to the characterisation of an equilibrium $((\Lambda, \Phi), X)$, where all market makers submit the same quote (λ, ϕ) and X is an optimal strategy for (λ, ϕ).

Recall that the market makers do not compete on ϕ and consider a situation where market maker j deviates by submitting the quote (λ^j, ϕ). Let $\tilde{\lambda} := \min\{\lambda, \lambda^j\}$ and note that the market quote corresponding to this deviation is $(\tilde{\lambda}, \phi)$. First consider the case in which the deviation doesn't alter the market quote, i.e. $\tilde{\lambda} = \lambda$. Since the insider bases her decisions on the market quotes, such a deviation will not change her trading strategy. Thus, the total demand process Y remain unchanged.

Then the terminal wealth of the deviating market maker is given by

$$G_1^j(X) := - \int_0^1 Y_{s-}^j \, dP_s + Y_1^j(P_1 - V), \tag{4.15}$$

where P is the price process corresponding to the market quote (λ, ϕ) and Y^j is as in (4.14) with $Y = X + Z$. Thus, such a deviation is not profitable if $\mathbb{E}\left(U\left(G_1^j(X) \right) \right)$ is less than the expected utility had the market maker j quoted (λ, ϕ).

If $\tilde{\lambda} \neq \lambda$, i.e. the market quote changes, the insider's optimal strategy may change. Denote by $\mathcal{X}(\tilde{\lambda}, \phi)$ the set of optimal strategies of the insider given this market quote and fix $\tilde{X} \in \mathcal{X}(\tilde{\lambda}, \phi)$. Then, the terminal wealth G_1^j of the deviating market maker is given by (4.15), where P is the price process corresponding to the market quote $(\tilde{\lambda}, \phi)$ and Y^j is as in (4.14) with $Y = \tilde{X} + Z$. We will say that this deviation is not

profitable if the expected utility of the deviating market maker from terminal wealth is lower for all $\tilde{X} \in \mathcal{X}(\tilde{\lambda}, \phi)$.

Note that in our definition of equilibrium, the market makers take into account the response of the insider when considering deviations. That is, the equilibrium that we consider is of multi-leader Stackelberg-Nash-Bertrand type in which the group of market makers is the leader and the insider is the follower. The market makers compete on price in a non-cooperative Nash game while taking into account the optimal response of the insider.

4.2. **Equilibrium.** The feedback from the insider's strategy makes the analysis of the actions of the deviating market maker quite challenging. Nevertheless, the nature of the optimal response of the insider as given by Proposition C.1 implies that deviations in market maker's quotes should not decrease the market quote λ on $[0, 1)$ as this will bring an infinite loss to the deviating market maker. Moreover, the Proposition C.1 also implies that, although market quote λ can decrease at time 1, there is a minimal value for λ at time 1 undercutting which will result in infinite loss for the deviating market maker.

Therefore, as soon as the market λ is a constant λ^* on $[0, 1)$ and the final market λ is set at this minimal value, the deviating market maker, once quit making the market, can not win order allocation at any future time as undercutting the market quote either on $[0, 1)$ or at 1 will result in infinite loss. Thus, he has only two options: 1) quote λ^* at the beginning and quit making the market at some point, and 2) undercut at time 0 and possibly leave market at some future time.

Note that the market quote (and thus the insider's strategy) remains the same in the first case. Consequently, if the market maker's utility of leaving market is a martingale, he will be indifferent between quitting and continuing making the market.

Although the market quotes and the corresponding insider strategies change in the second case, any optimal response of the insider leads to the full revelation of the fundamental value at the time the deviating market maker stops making the market. As the market maker's utility increase in λ, the deviating market maker's utility is negative as soon as the equilibrium one is zero.

These two observations lead to the following theorem, proof of which is delegated to the appendix.

Theorem 4.1. *There exist two equilibria* $((\Lambda^*, \Phi_i^*, X_i^*))_{i=1,2}$, *where*

$$\Lambda^*(t) = \left(\lambda^* \mathbf{1}_{[t<1]} + \frac{\lambda^*}{2} \mathbf{1}_{[t=1]}\right) \mathbf{1}, \ where \ \lambda^* = \frac{\rho\gamma^2}{2} + \sqrt{\frac{\rho^2\gamma^4}{4} + \frac{\gamma^2}{\sigma^2}},$$

$$\Phi_i^* = \phi_i^* \mathbf{1}, \ where \ \phi_i^* = \mu - \frac{(-1)^i}{\rho}\sqrt{\rho^2\gamma^2 + 2\frac{\rho\gamma^2}{\lambda^*\sigma^2}\log\left(\frac{\gamma}{\lambda^*\sigma}\right)},$$

$$dX_{i,t}^* = \sigma^2\left(a(t)(X_{i,t}^* + Z_t) + b_i(t) + \frac{p_x^{(i)}}{p^{(i)}}\left(t, X_{i,t}^* + Z_t; 1, \frac{V - \phi_i^*}{\lambda^*}\right)\right)dt, \ where$$

$$a(t) = -\frac{\lambda^*\rho}{1 + \lambda^*\rho\sigma^2(1-t)},$$

$$b_i(t) = (-1)^i\frac{1}{2}\log(1 + \lambda^*\rho\sigma^2(1-t))\sqrt{\frac{a(t)}{-\lambda^*\rho\sigma^2(1-t) + \log(1 + \lambda^*\rho\sigma^2(1-t))}},$$

and $p^{(i)}$ *is the transition density of the Ornstein-Uhlenbeck process*

$$dR_t^{(i)} = \sigma dB_t + \sigma^2(a(t)R_t^{(i)} + b_i(t))dt. \tag{4.16}$$

(1) *From the point of view of the insider the equilibrium demand and price dynamics are given by*

$$dY_t^{*,i} = \sigma dB_t + \frac{1}{1-t}\left(\frac{V - \phi_i^*}{\lambda^*} - \frac{\alpha_{1,i}(t)}{\rho\lambda^*} + b_i(t)\sigma^2(1-t) - Y_t^{*,i}\right)dt, \tag{4.17}$$

$$dP_t^{*,i} = \lambda^*\sigma dB_t + \frac{1}{1-t}\left(V - \frac{\alpha_{1,i}(t)}{\rho} + b_i(t)\lambda^*\sigma^2(1-t) - P_t^{*,i}\right)dt, \tag{4.18}$$

where

$$\alpha_{1,i}(t) = (-1)^i\sqrt{-a(t)\left(\lambda^*\rho\sigma^2(1-t) - \log(1 + \lambda^*\rho\sigma^2(1-t))\right)}.$$

The ex-ante profit of the insider is given by

$$\mathbb{E}\left[\frac{1}{2\lambda^*}(P_0 - V)^2 + \frac{\lambda^*\sigma^2}{2}\right] = \frac{(\phi_i^* - \mu)^2}{2\lambda^*} + \frac{\gamma^2}{2\lambda^*} + \frac{\lambda^*\sigma^2}{2}. \tag{4.19}$$

(2) *From the point of view of the market makers the equilibrium demand and price dynamics are given by*

$$dY_t^{*,i} = \sigma d\beta_t + \sigma^2(a(t)Y_t^{*,i} + b_i(t))dt, \tag{4.20}$$

$$dP_t^{*,i} = \lambda^*\sigma d\beta_t + \sigma^2(a(t)(P_t^{*,i} - \phi_i^*) + \lambda^*b_i(t))dt, \tag{4.21}$$

where β *is a* $\mathcal{F}^M$-*Brownian motion. The market makers' expected utility is* 0.

Although in-depth analysis and discussion of the results of the above theorem is postponed until next section, some key observations can be made by direct comparison of the above theorem with Theorem 3.1. First of all, observe that the ex-ante profit of the insider is lower in the open market (3.5) then in wholesale market (4.19) – that is, the insider prefers her order routed to wholesaler. Also note that market makers have zero utility whereas competitive agents have positive utility (3.8) which allows to conclude that the additional profit of the insider comes from market makers "competing away" their profits.

Furthermore, the equilibrium market parameters are the same apart from the mean reversion level of total demand in liquidity suppliers filtration, as evident from equations (4.20) and (3.6). As the target level of inventory for competitive agents is 0, we conclude that competition among makers results in them taking additional inventory risk. The inventory risk depends on the magnitude of inventory and not its sign giving raise to two equilibria, with the magnitude of the target inventory determined by the zero-utility condition. This phenomenon is not an artefact of asymmetric information as it appears in the equilibrium with uninformed strategic stated below. As in the case of competitive agents, the speed of mean reversion is smaller in absence of private information due to smaller perceived risk by the liquidity suppliers.

Theorem 4.2. *There exist equilibria* $(\Lambda^*, \Phi_i^*, X_i^*)_{i=1,2}$, *where*

$$\Lambda^* = \lambda^* \mathbf{1}, \ with \ \lambda^*(t) = \frac{\rho \gamma^2}{2} \mathbf{1}_{[t<1]} + \frac{\rho \gamma^2}{4} \mathbf{1}_{[t=1]},$$

$$\Phi_i^* = \phi_i^* \mathbf{1}, \ with \ \phi^* = (-1)^i \frac{\rho \sigma \gamma^2}{2\sqrt{3}} + \mu,$$

$$dX_{i,t}^* = \mathbf{1}_{[t<1]} \sigma^2 \left(a(t)(X_{i,t}^* + Z_t) + b_i(t) \right) dt + \mathbf{1}_{[t=1]} \frac{\mu - \phi_i^* - \lambda^*(1-)(X_{i,1-}^* + Z_1)}{\lambda^*(1-)}, \ where$$

$$a(t) = -\frac{\rho^2 \gamma^2}{4},$$

$$b_i(t) = -\frac{\rho(\mu - \phi_i^*)}{2}.$$

Moreover, the equilibrium dynamics of the demand is given by

$$dY_{i,t}^* = \sigma dB_t - \frac{\rho \sigma^2}{2} \left(\frac{\rho \gamma^2}{2} Y_{i,t}^* + \mu - \phi_i^* \right) dt, \quad t < 1;$$

$$\Delta Y_{i,1}^* = 2 \frac{\mu - \phi_i^*}{\rho \gamma^2} - Y_{i,1-}^*,$$

and the expected profit of the strategic trader in equilibrium equals $\frac{\rho\sigma^2\gamma^2}{3}$.

5. COMPARATIVE DYNAMICS

In this section we consider the case $V \sim N(0, \gamma)$, i.e. $\mu = 0$. Our aim is to analyse the impact of risk aversion on the market liquidity, in particular *depth* and *resilience*, *price efficiency*, *reversal* of the equilibrium prices, *insider's profit*, and the value of private information, i.e. the difference between the profits of the insider and a strategic trader with no information. Recall that depth corresponds to the order size necessary to move the price by one unit; the resilience is the speed with which the prices converge to the fundamental value; price efficiency is a measure of remaining uncertainty in the prices; and, finally, the reversal reflects the autocorrelation of returns.

Our comparative analysis will show that there is single quantity that parametrises the above measures of liquidity across markets that differ on the risk aversion of liquidity suppliers, inventory and adverse selection risks. More precisely, all measures of liquidity when normalised by their Kyle (risk-neutral) counterparts will only depend on *the market adjusted risk aversion* parameter $\rho_M := \rho\gamma\sigma$. One way to understand the universality of this parameter is to observe that the utility of liquidity providers from wealth W is $1 - \exp(-\rho_M \frac{W}{\gamma\sigma})$. Note that $\frac{W}{\gamma\sigma}$ is dimensionless since the unit of W is time × currency whereas γ^2 is in units of time × currency^2 and σ^2 is in units of time. By making the risk dimensionless we can shift our attention from idiosyncratic features of each market towards the general mechanism underpinning the market liquidity. Although in what follows we shall consider all values of ρ_M, the interval $[0, 1)$ is of particular interest since it encompasses all reasonable values observed in markets[12].

For the equilibria considered in Sections 3 and 4 we will have two benchmark models to compare to. The first model is the one with risk-neutral market makers as in Kyle (1985), in which case the depth of the market is constant and equals $\frac{1}{\lambda_K} := \frac{\sigma}{\gamma}$, resilience is a function of time and is given by $R(t) = \frac{1}{1-t}$, efficiency equals $\Sigma_K(t) := \gamma^2(1 - t)$, and the ex-ante profit of the insider is $\pi_K := \gamma\sigma$. We will also

---

[12]As dividends are less volatile than asset prices, if we consider risk aversion parameter less than 10 and yearly volatility less than 0.3, we will be in that range.

consider[13]

$$M(s) := \lim_{\substack{t-s=\varepsilon \\ u-s=\varepsilon \\ \varepsilon \to 0}} \frac{\text{Cov}_s\left(P_t - P_s, P_u - P_t\right)}{\sqrt{\text{Var}_s(P_t - P_s)\text{Var}_s(P_u - P_t)}\varepsilon}, \tag{5.22}$$

where Cov_s and Var_s are covariance and variance given market's information at time s. If M is positive, it indicates a price momentum whereas a negative value implies price reversal. In Kyle's model M is equal to 0 at all times.

We will also compare our models to the benchmark models with strategic trader described in the previous section. As the strategic trader does not possess any private information, the equilibrium prices do not carry any information about the fundamental value, and therefore the resilience is undefined and efficiency is constant, γ^2. However, we can calculate the market depth and the profit of the strategic trader. According to Theorems 3.2 and 4.2 the market depth is $\frac{1}{\lambda_0} := \frac{2}{\rho\gamma^2}$ in both models. While the profit of the strategic trader equals $\frac{\rho\gamma^2\sigma^2}{4}$ in the market with competitive agents, her profit becomes $\frac{\rho\gamma^2\sigma^2}{3}$ when she trades against market makers. Thus, it is evident that given a choice the strategic trader would choose market makers to execute their orders.

To analyse the relative impact of inventory and private information on price formation, we need to measure the change in price impact as a result of private information. Since the equilibrium market depth does not depend on the market type, we shall use the results of Theorems 4.1 and 4.2 to conduct our analysis. We denote by λ and λ_0 the inverse market depths in case of insider and strategic (and uninformed) trader correspondingly. It is evident that $\lambda > \lambda_0$. Thus, the market depth decreases when the strategic trader gains access to private information. This decrease is due to asymmetric information and we will use it as a measure of the component of the equilibrium price affected by adverse selection. Direct calculations show that the adverse selection component, $\lambda - \lambda_0 = \sqrt{\frac{\rho^2\gamma^4}{4} + \frac{\gamma^2}{\sigma^2}}$ is increasing in risk aversion and decreasing in noise volatility. This is intuitive since the liquidity providers require higher compensation for facing adverse selection as their risk aversion increases. Moreover, higher noise volatility makes total demand less informative to the liquidity providers which decreases the informational component of the price adjustment as well as its proportion. Naturally, one expects that the cost of holding an inventory increases

---

[13]The additional normalisation by ε is an artefact of continuous time models. For t and u close to s, the covariance is of order $(t-s)^2$, thus, a further normalisation is needed since the variances are also of order $t - s$.

TABLE 1. Market depth

	Competitive Agents	Market Makers
Insider trader	$\frac{1}{\lambda_K}\frac{1}{\frac{\rho_M}{2}+\sqrt{\frac{\rho_M^2}{4}+1}}$	$\frac{1}{\lambda_K}\frac{1}{\frac{\rho_M}{2}+\sqrt{\frac{\rho_M^2}{4}+1}}$
Strategic trader	$\frac{2}{\rho\gamma^2}=\frac{1}{\lambda_K}\frac{2}{\rho_M}$	$\frac{2}{\rho\gamma^2}=\frac{1}{\lambda_K}\frac{2}{\rho_M}$

in risk aversion, even in the absence of private information, thus the component of the price determined by purely inventory considerations is increasing in risk aversion, too. In Figure 1 we plot $(\lambda - \lambda_0)/\lambda$ (proportion of price due to adverse selection) as a function of market adjusted risk aversion. As one can observe, the increase in the cost of inventory holding is strong enough to reduce the proportion of price affected by private information. However, this inventory component never dominates, and accounts for a half of the price differential at most.
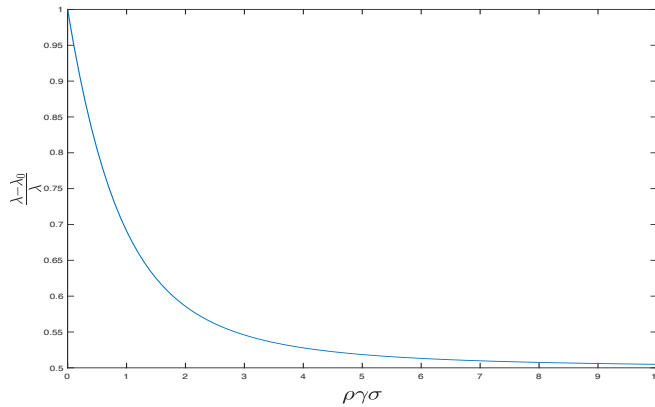

FIGURE 1. The figure shows the dependency of the adverse selection proportion, which is defined to be $(\lambda - \lambda_0)/\lambda$, of the equilibrium price on the market adjusted risk aversion.

As observed above, in both equilibria with an insider the market depth is the same (for convenience we report the market depth in all four equilibria in Table 1) and smaller than $\frac{1}{\lambda_K}$. This implies that the depth of the market with risk averse liquidity suppliers is smaller than the depth observed in Kyle (1985). The reason for the observed decrease in the market depth is the risk sharing between the liquidity suppliers and the insider as the risk aversion makes the liquidity providers require additional

compensation for large quantities of risk. This effect gets more pronounced as the risk aversion coefficeint ρ increases and the market approaches to complete illiquidity when the risk aversion rate gets high. One can also show that the market depth increases with the volatility of noise trading since the total order process contains less information and therefore the price is less sensitive to it.

On the other hand, if we compare the loss in liquidity due to risk aversion, that is, the depth with risk averse liquidity providers normalised by the one in Kyle (1985), we observe in Figure 2 that the loss in liquidity become larger as the market adjusted risk aversion parameter increases, i.e. the market conditions get riskier from the point of view of liquidity providers. The Figure 2 emphasises that to study the market liquidity behaviour we must focus on the collective impact of market parameters via ρ_M rather than their effect in isolation. For instance, by keeping the risk aversion ρ and $\frac{\gamma}{\sigma}$ constant, we can obtain strikingly different market behaviour from complete illiquidity when γ and σ are large to liquidity levels of risk-neutral liquidity suppliers for small γ and σ.
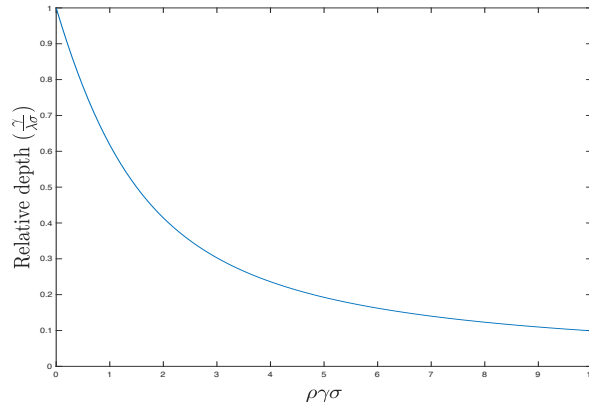

FIGURE 2. The figure depicts how the relative depth, i.e. $\frac{\gamma}{\lambda\sigma}$ depends on ρ_M.

It is evident from price dynamics (3.4) and (4.18) that in both equilibria convergence to the fundamental value occurs at rate $\frac{1}{1-t}$, which is the same level of resilience as Kyle (1985). Thus, we can conclude that the resilience is unaffected by risk aversion or market design.

Efficiency is a measurement of how informative the market prices are. Using Lemma B.2, we get

$$\Sigma(t) = \mathrm{Var}(P_1|\mathcal{F}_t^M) = \mathrm{Var}(\lambda Y_1|Y_t) = \frac{\lambda^2\sigma^2(1-t)}{1+\lambda\rho\sigma^2(1-t)}$$

$$= \Sigma_K(t)\frac{2+\rho_M(1+\sqrt{\rho_M^2+4})}{2+\rho_M(1+\sqrt{\rho_M^2+4})(1-t)}$$

in both market makers and competitive agents equilibrium.

Direct differentiation show that Σ is decreasing in t, which reflects the fact that prices become more informative as time progresses as a result of the insider's effort to drive prices to the fundamental value. Moreover, it is clear that $\Sigma(t) \geq \Sigma_K(t)$, i.e. the prices become less efficient when the liquidity providers become risk averse. This behaviour is not completely driven by ρ: as Figure 3 illustrates the loss of efficiency is monotone in ρ_M. In particular, as $\rho_M \to \infty$, Σ behaves like γ^2, independent of t, indicating a complete loss of efficiency.

The monotonicity in ρ_M is to be expected as the insider has to participate in risk sharing to make the equilibrium possible as the market gets riskier for the liquidity providers. She does so by forcing the total demand to mean revert around a time varying level which provides sufficient level of risk sharing. More precisely, the total demand dynamics in both equilibria is given by

$$dY_t = \sigma d\beta_t + \sigma^2 a(t)\left(Y_t + \frac{b(t)}{a(t)}\right)dt, \tag{5.23}$$

where

$$\sigma^2 a(s) = -\frac{2}{1+\sqrt{1+\frac{4}{\rho_M^2}} - 2s},$$

$b = 0$ for the competitive agents equilibrium or $b_i(t)$ in the market makers one.

It is evident that the level of mean reversion is determined by the liquidity suppliers, and thus carry no information about the fundamental value. This results in a slower convergence to the fundamental value, which in turn implies lower efficiency. When the market adjusted risk aversion parameter increases, liquidity suppliers demand more risk sharing from the insider and, therefore, stronger mean reversion, which manifests itself in $|a|$ being increasing in ρ_M. As a result, the total order stays closer to the long run mean, which makes the speed of convergence to the fundamental value

decrease with ρ_M, i.e. the market becomes less efficient when market becomes riskier for the liquidity suppliers.

This loss of efficiency and the associated off-load of the risk by the liquidity providers becomes particularly striking as ρ_M increases to infinity. In this case the scaled equilibrium demand, $\tilde{Y} := \frac{Y}{\sigma}$, converges to the solution of

$$d\tilde{Y}_t = dB_t - \frac{\tilde{Y}_t}{1-t}dt,$$

since σb_i vanishes in the limit. As the solution to the above equation is a standard Brownian bridge with $\tilde{Y}_1 = 0$, this implies that, after we control for the noise volatility, *all* inventory risk is shifted from the liquidity providers to the insider and *all* asymmetric information risk is shifted to the noise traders.

This off-load of liquidity risk by liquidity suppliers also features in both strategic trader equilibria. Indeed, in the model with market makers simple calculations reveal that

$$d\tilde{Y}_t = dB_t - \frac{\rho_M^2}{4}(\tilde{Y}_t \pm \frac{1}{\sqrt{3}})dt,$$

the solution of which is $\tilde{Y}_1 = \pm\frac{1}{\sqrt{3}} + e^{-\frac{\rho_M^2}{4}}\left(\pm\frac{1}{\sqrt{3}} + \int_0^1 e^{\frac{\rho_M^2 s}{4}}dB_s\right)$. As the market adjusted risk aversion parameter gets larger $\tilde{Y}_1 \to \pm\frac{1}{\sqrt{3}}$ – a level of inventory that provides zero expected utility.

On the other hand, if the liquidity is supplied by competitive agents, the limiting dynamics of $\tilde{Y}$ as $\rho_M \to \infty$ is given by

$$d\tilde{Y}_t = dB_t - \frac{\tilde{Y}_t}{1-t}dt$$

when the strategic trader is uninformed. Thus, once again, we get $\tilde{Y}_1 = 0$ and the competitive agents carry no liquidity risk in the limit.

As higher volatility of the noise trades implies that liquidity suppliers face higher inventory risk and therefore require more risk sharing which, in turn, slows down price discovery, one would expect that the market efficiency is affected by σ. The above discussion indeed supports this observation as ρ_M is proportional to σ. This is in stark contrast with the risk neutral case, where the efficiency is independent of the volatility of the noise trading as the inventory risk is not priced in.
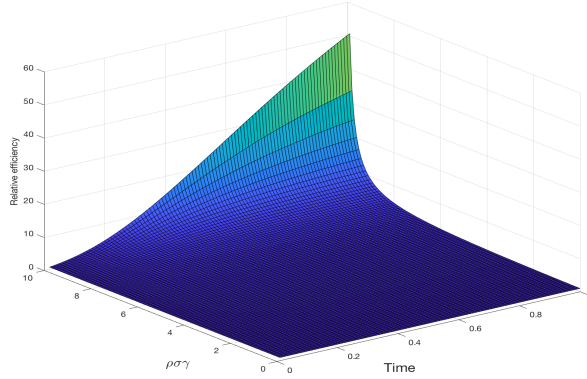
FIGURE 3. Relative efficiency is a measurement for residual uncertainty. The above shows it as a function of time and market adjusted risk aversion ρ_M.

When the liquidity suppliers become risk averse, the prices exhibit reversal. That is, the large buy orders are usually followed by sell orders and vice versa. This is apparent from the evolution of equilibrium demand given in (5.23). To compute the intensity of reversal, observe that in all equilibria P is of the form $P = \lambda Y + \phi$ and therefore

$$
\begin{aligned}
M(s) \quad &:= \quad \lim_{\substack{t-s=\varepsilon \\ u-s=\varepsilon \\ \varepsilon \to 0}} \frac{\mathrm{Cov}_s\left(Y_t - Y_s, Y_u - Y_t\right)}{\sqrt{\mathrm{Var}_s(Y_t - Y_s)\mathrm{Var}_s(Y_u - Y_t)}\varepsilon} \\
&= \quad \lim_{\substack{t-s=\varepsilon \\ u-s=\varepsilon \\ \varepsilon \to 0}} \frac{\mathbb{E}_s\left((Z_t - Z_s)(Z_u - Z_t)\right)}{\sqrt{\mathrm{Var}_s(Z_t - Z_s)\mathrm{Var}_s(Z_u - Z_t)}\varepsilon},
\end{aligned}
$$

where $Z_t = Y_t - \mathbb{E}_s[Y_t]$. Direct calculations yield

$$
M(s) = \sigma^2 a(s) = -\frac{1}{\frac{1+\sqrt{1+\frac{4}{\rho_M^2}}}{2} - s}
$$

in both markets when an insider is present. Similar calculations show that $M(s) = -\frac{\rho_M^2}{4}$, thus, the reversal is constant in time in the model with market makers and uninformed strategic trader. Finally, when the uninformed strategic trader trades against the competitive agent, we have

$$
M(s) = -\frac{\rho_M^2}{4 + \rho_M^2(1 - s)},
$$

UMUT ÇETİN AND ALBINA DANILOVA

Table 2. Intensity of price reversal

	Competitive Agents	Market Makers
Insider trader	$\dfrac{2}{1+\sqrt{1+\frac{4}{\rho_M^2}}-2s}$	$\dfrac{2}{1+\sqrt{1+\frac{4}{\rho_M^2}}-2s}$
Strategic trader	$\dfrac{\rho_M^2}{4+\rho_M^2(1-s)}$	$\dfrac{\rho_M^2}{4}$

i.e. prices exhibit reversal in all models that we consider. For the convenience of the reader the intensity of price reversal, i.e. $-M(s)$, as a function of ρ_M and time for all four models is summarised in Table 2 and is plotted in Figures 4 and 5.

It is evident from Figures 4 and 5 that the intensity of price reversal is an increasing function of market adjusted risk aversion parameter ρ_M. The above discussion of market efficiency implies that this monotonicity is due to the risk sharing. Indeed, as observed above, risk sharing is responsible for stronger mean reversion in the demand as perceived risk faced by the liquidity providers grows, which in turn implies that reversal should be stronger as market adjusted risk aversion increases.

Moreover, the intensity of price reversal is non-decreasing in time. This seems to be in conflict with the risk-sharing idea between the insider and the liquidity suppliers since the remaining risk at time 0 is higher. However, taking into account that the liquidity suppliers derive their utility only from their consumption at time 1, it becomes apparent that they place more emphasis on their inventory being close to the target at time 1. Thus, risk-sharing implies stronger reversal at time 1 as we observe from all models.

Although the price reversal seems to be affected by the nature of liquidity provision when the strategic trader is uninformed, if we focus on the dynamics close to market termination, we see that it does not depend on the type of the provision. For both types of liquidity suppliers the reversal is stronger in the presence of insider due to the additional adverse selection risk and the resulting higher demand for risk sharing.

The last market characteristic we consider is the ex-ante profit of the insider. The expected profits of the insider, conditional on the value of V, and the ex-ante profits of uninformed strategic trader are summarised in Table 3. This table also reports
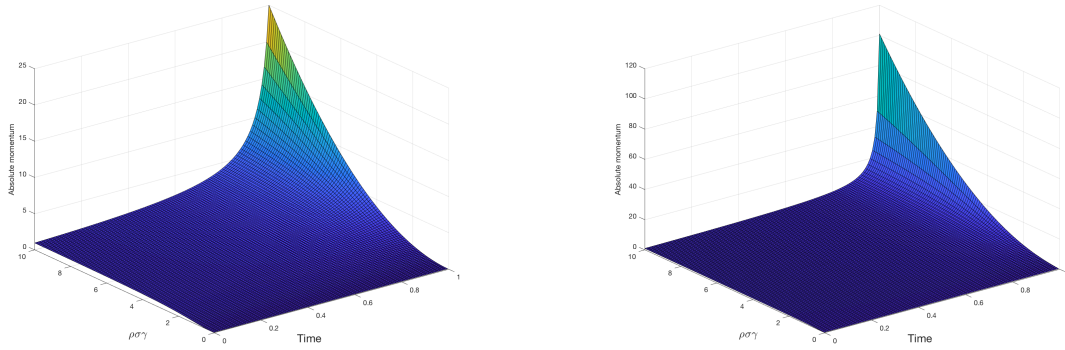
FIGURE 4. Intensity of price reversal in the market with competitive agents as a function of time and ρ_M. The left figure depicts price reversal in the presence of strategic trader and the right one in the presence of insider.
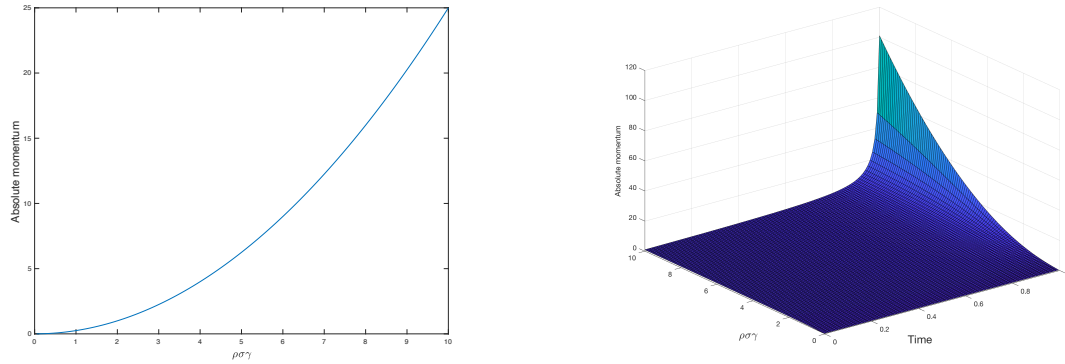

FIGURE 5. Intensity of price reversal in the market with market makers as a function of time and ρ_M. The left figure depicts price reversal in the presence of strategic trader and the left one in the presence of insider.

the value of information in both models of liquidity provision normalized by the corresponding value in the Kyle model.

Observe that in both markets the profit of the insider can be expressed as

$$\frac{1}{2\lambda}\left(P_0 - V\right)^2 + \frac{\sigma^2}{2}\lambda = \frac{1}{2\lambda}\left(\Delta^{P_0} - \Delta^V\right)^2 + \frac{\sigma^2}{2}\lambda, \tag{5.24}$$

TABLE 3. Expected profit of insider/strategic trader

	Competitive Agents	Market Makers	Additional value
Insider trader	$\gamma\sigma\sqrt{1 + \frac{\rho_M^2}{4}}$	$\gamma\sigma\pi(\rho_M)$	$\gamma\sigma\Delta(\rho_M)$
Strategic trader	$\gamma\sigma\frac{\rho_M}{4}$	$\gamma\sigma\frac{\rho_M}{3}$	$\gamma\sigma\frac{\rho_M}{12}$
Normalized value of information	$\sqrt{1 + \frac{\rho_M^2}{4}} - \frac{\rho_M}{4}$	$v(\rho_M)$	

In above $\Delta(x) := \frac{\Delta_0(u(x))}{2x}$, $\Delta_0(x) := x\log x - x + 1$, $u(x) := 1 - \frac{2}{1+\sqrt{1+\frac{4}{x^2}}}$,

$v(x) := \frac{u(x)\log u(x) + \frac{1+7u(x)-8u^2(x)}{3u(x)}}{2x}$, and $\pi(x) := \frac{x}{3} + v(x)$.

where $\Delta^{P_0} = P_0 - \mu$ and $\Delta^V = V - \mu$. Since $\mathbb{E}[\Delta^V] = 0$, the ex-ante profit of the insider is given by

$$\frac{1}{2\lambda}(P_0 - \mu)^2 + \frac{\gamma^2}{2\lambda} + \frac{\sigma^2}{2}\lambda. \tag{5.25}$$

This profit consists of three parts, first of which disappears when there is no mispricing by the liquidity providers, i.e. when $P_0 = \mathbb{E}[V]$. We will identify this part of the profit as the gains due to the initial mispricing by the market makers resulting from their competition. To understand the role of the second component, i.e. $\frac{\gamma^2}{2\lambda}$, let's first consider the market with competitive agents. In this case, there is no mispricing and the anticipated profit (5.24) upon receiving the signal that corresponds to this part of the ex-ante profit is given by $\frac{(\Delta^V)^2}{2\lambda}$. When $V = 0$, i.e. the time-0 expected value of the asset by the competitive agents and the insider coincide, this part of the anticipated profit vanishes, justifying the identification of the expected value of this part as *purely informational profit*. Similarly, in the market with market makers the term $\frac{\gamma^2}{2\lambda}$ is identified as purely informational profit, too.

The profit of the strategic trader in market makers' equilibrium,

$$\frac{1}{2\lambda_0}(P_0 - \mu)^2 + \frac{\lambda_0\sigma^2}{2}, \tag{5.26}$$

has two components with the first one being due to the mispricing only and the second one due to providing insurance to the market makers. Thus, comparison of the latter to the third component of the insider's ex-ante profit reveals that the third part of the insider's profit is generated by providing insurance to the risk averse liquidity suppliers. The difference between the ex-ante profit of the insider and that of the

strategic trader is the value of information. This value of information normalized by its Kyle counterpart, i.e. $\gamma\sigma$, is reported in Table 3 and depicted in Figure 6.

It is apparent from Figure 6 that for reasonable values of market adjusted risk aversion parameter (i.e. $\rho_M \in [0,1)$), the value of information is smaller when the liquidity providers are risk averse. On the other hand, for large values of ρ_M the value of information in a market with risk averse liquidity providers becomes much higher then that in the Kyle's model, with the ratio going to infinity in the limit. Moreover, further examination of the Figure 6 and Table 3 reveals[14] that there is a value of the risk aversion parameter ρ that minimizes[15] the value of information and, therefore, reduces the incentive for the insider to participate in the market.

Figure 6 also reveals that although the normalized value of information is decreasing within the reasonable range of market adjusted risk aversion, it is non-monotone in general . The reason for that is the fact that the purely informational profit and the excess insurance profit react in opposite directions to changes in ρ_M. Indeed, as the risk aversion increases the market makers are willing to pay more for an insurance against large fluctuations in the demand. Moreover, in the presence of adverse selection this effect is magnified, which entails that λ is increasing faster than λ_0 as is evident in Table 1. On the other hand, the purely informational component is decreasing with an increase in the market adjusted risk aversion since, in the presence of strong mean reversion, it becomes more costly for the insider to drive the price to its fundamental value.

The difference in the normalized value of information between the markets with market makers and competitive agents displays a similar non-monotone pattern as illustrated by Figure 7. As one can observe, the information is less valuable when the liquidity is supplied by competitive agents in the reasonable range of market adjusted risk aversion . This implies that strategic traders in markets with market makers have more incentives to acquire private information under typical market conditions. Despite this nonlinear pattern, an insider who already possesses private information chooses to trade with market makers as direct calculations demonstrate that the function Δ from Table 3 is positive. The same table also shows that strategic traders

---

[14]Note that normalization doesn't involve ρ and therefore doesn't change the shape of the value of information as a function of ρ.

[15]This value will be outside the reasonable range for risk aversion as it is larger than 10 (20) for competitive agents (market makers) equilibrium.
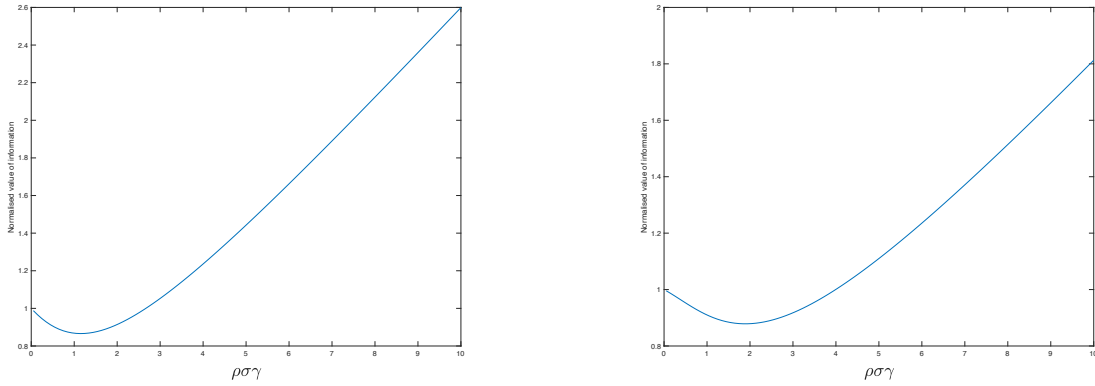
FIGURE 6. Normalized value of information in competitive agents equilibrium is reported in the left pane and the right plot illustrates corresponding value for the market makers equilibrium.
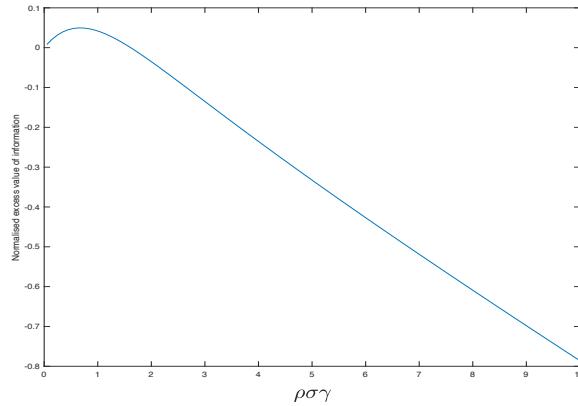

FIGURE 7. The difference between the normalized value of information in the market makers and competitive agents equilibrium as a function of ρ_M.

prefer to trade with market makers as well. The noise traders are indifferent between the two types of liquidity provision. Indeed, their expected profit is given by

$$\mathbb{E}[\sigma \int_0^T B_t dP_t] = \sigma \mathbb{E}[B_1 V - \int_0^1 P_t dB_t - \lambda \sigma] = -\lambda \sigma^2,$$

which is identical across the markets. This in particular implies that the profits of the competitive agents above the zero-utility level are passed to the strategic traders in market makers equilibrium. Thus, both informed and uninformed strategic traders

enjoy larger profits with market makers. It is easy to see from Table 3 that the uninformed strategic trader's profit increases by $\frac{1}{3}$ when trading with market makers. Moreover, direct calculations show that

$$\frac{1}{3} > \frac{\Delta(\rho_M)}{\sqrt{1 + \frac{\rho_M^2}{4}}}$$

for all $\rho_M > 0$, which implies that the increase in profit is higher for uninformed strategic traders. Finally, since all the traders prefer liquidity provided by the market makers, the trading desks executing client orders should optimally choose market makers over competitive agents.

6. Conclusion

We have investigated the role of market mechanism in liquidity and trader profits. Identification of a universal parameter, ρ_M, allowed us to compare equilibrium outcomes across different markets. The resulting equilibria let us conclude that the retail agents (informed, uninformed and noisy) would prefer their order routed to a market maker.

The risk aversion of liquidity providers alters the equilibria not only quantitatively but also qualitatively. The most significant distinction from the earlier models is that, as a result of risk sharing between the insider and the market makers, the demand process becomes mean reverting although the cumulative noise demand is postulated to be a Brownian motion.

This mean reversion is responsible for the other departures from the previous literature that we highlighted in this paper. In particular, it entails conspicuous insider trading, and price reversal.

We also studied the sensitivity of market parameters to risk aversion. We confirm the intuition that the market depth decreases, prices exhibit stronger reversal, become less efficient and converge to the fundamental value at a lower rate as the liquidity providers become more risk averse. An unforeseen consequence of our models is the non-monotone relationship between the market adjusted risk aversion and the value of private information. In particular, we observe that there exists a critical risk aversion level at which the amount a strategic trader is willing to pay for the private information is minimal.

References

Kerry Back. Insider trading in continuous time. *The Review of Financial Studies*, 5 (3):387–409, 1992.

Kerry Back and Hal Pedersen. Long-lived information and intraday patterns. *Journal of financial markets*, 1(3-4):385–402, 1998.

Kerry Back, C Henry Cao, and Gregory A Willard. Imperfect competition among informed traders. *The journal of finance*, 55(5):2117–2155, 2000.

Kerry Back, Francois Cocquemas, Ibrahim Ekren, and Abraham Lioui. Optimal transport and risk aversion in kyle's model of informed trading. *arXiv preprint arXiv:2006.09518*, 2020.

Brad M Barber and Terrance Odean. Trading is hazardous to your wealth: The common stock investment performance of individual investors. *The journal of Finance*, 55(2):773–806, 2000.

Brad M Barber and Terrance Odean. All that glitters: The effect of attention and news on the buying behavior of individual and institutional investors. *The review of financial studies*, 21(2):785–818, 2008.

Brad M Barber, Terrance Odean, and Ning Zhu. Do retail trades move markets? *The Review of Financial Studies*, 22(1):151–186, 2008.

Jean-Noel Barrot, Ron Kaniel, and David Sraer. Are retail traders compensated for providing liquidity? *Journal of Financial Economics*, 120(1):146–168, 2016.

Shmuel Baruch. Insider trading and risk aversion. *Journal of Financial Markets*, 5 (4):451–464, 2002.

Robert Battalio, Jason Greene, and Robert Jennings. Do competing specialists and preferencing dealers affect market quality? *The Review of Financial Studies*, 10 (4):969–993, 1997.

Robert H Battalio. Third market broker-dealers: Cost competitors or cream skimmers? *The Journal of Finance*, 52(1):341–352, 1997.

Hendrik Bessembinder and Herbert M Kaufman. A cross-exchange comparison of execution costs and information flow for nyse-listed stocks. *Journal of Financial Economics*, 46(3):293–319, 1997.

B. Biais, D Martimort, and J.C. Rochet. Competing mechanisms in a common value environment competing mechanisms in a common value environment. *Econometica*, 68(4):799–837, July 2000.

Bruno Biais, Larry Glosten, and Chester Spatt. Market microstructure: A survey of microfoundations, empirical results, and policy implications. *Journal of Financial Markets*, 8(2):217–264, 2005.

Geir Høidal Bjønnes and Dagfinn Rime. Dealer behavior and trading systems in foreign exchange markets. *Journal of Financial Economics*, 75(3):571–605, 2005.

Ekkehart Boehmer, Charles M Jones, Xiaoyan Zhang, and Xinran Zhang. Tracking retail investor activity. *The Journal of Finance*, 76(5):2249–2305, 2021.

Svetlana Bryzgalova, Anna Pavlova, and Taisiya Sikorskaya. Retail trading in options and the rise of the big three wholesalers. *Available at SSRN*, 2022.

Luciano Campi, Umut Çetin, and Albina Danilova. Equilibrium model with default and dynamic insider information. *Finance and Stochastics*, 17(3):565–585, 2013.

Umut Çetin and Albina Danilova. Markov bridges: Sde representation. *Stochastic Processes and their Applications*, 126(3):651–679, 2016.

Pierre Collin-Dufresne and Vyacheslav Fos. Insider trading, stochastic liquidity, and equilibrium prices. *Econometrica*, 84(4):1441–1475, 2016.

Freddy Delbaen, Peter Grandits, Thorsten Rheinländer, Dominick Samperi, Martin Schweizer, and Christophe Stricker. Exponential hedging and entropic penalties. *Mathematical finance*, 12(2):99–123, 2002.

David Easley, Nicholas M Kiefer, and Maureen O'Hara. Cream-skimming or profit-sharing? the curious role of purchased order flow. *the Journal of Finance*, 51(3):811–833, 1996.

Kingsley YL Fong, David R Gallagher, and Adrian D Lee. Individual investors and broker types. *Journal of Financial and Quantitative Analysis*, 49(2):431–451, 2014.

Nicolae Garleanu, Lasse Heje Pedersen, and Allen M Poteshman. Demand-based option pricing. *The Review of Financial Studies*, 22(10):4259–4299, 2008.

Oliver Hansch, Narayan Y Naik, and S Viswanathan. Do inventories matter in dealership markets? evidence from the london stock exchange. *The Journal of Finance*, 53(5):1623–1656, 1998.

Roger D Huang and Hans R Stoll. The components of the bid-ask spread: A general approach. *The Review of Financial Studies*, 10(4):995–1034, 1997.

Ron Kaniel, Gideon Saar, and Sheridan Titman. Individual investor trading and stock returns. *The Journal of finance*, 63(1):273–310, 2008.

Ron Kaniel, Shuming Liu, Gideon Saar, and Sheridan Titman. Individual investor trading and return patterns around earnings announcements. *The Journal of Finance*, 67(2):639–680, 2012.

Eric K Kelley and Paul C Tetlock. How wise are crowds? insights from retail orders and stock returns. *The Journal of Finance*, 68(3):1229–1265, 2013.

Albert S Kyle. Continuous auctions and insider trading. *Econometrica: Journal of the Econometric Society*, pages 1315–1335, 1985.

Ananth Madhavan and Seymour Smidt. An analysis of changes in specialist inventories and quotations. *The Journal of Finance*, 48(5):1595–1628, 1993.

Philip E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2005. ISBN 3-540-00313-4. doi: 10.1007/978-3-662-10061-5. URL `http://dx.doi.org/10.1007/978-3-662-10061-5`. Second edition. Version 2.1, Corrected third printing.

Avanidhar Subrahmanyam. Risk aversion, market liquidity, and price efficiency. *The Review of Financial Studies*, 4(3):417–441, 1991.

Appendix A. Proofs of main theorems

Proof of Theorem 3.1. We will start with proving that $P^* \in \mathcal{P}(X^*)$. Note that

$$dX_t^* = \left\{ -\sigma^2 \frac{\lambda^* \rho}{1 + \lambda \rho \sigma^2 (1-t)} (X_t^* + Z_t) + \sigma^2 \frac{p_x(t, X_t^* + Z_t; 1, \frac{V-\mu}{\lambda^*})}{p(t, X_t^* + Z_t; 1, \frac{V-\mu}{\lambda^*})} \right\} dt, \quad \text{(A.27)}$$

where p is the transition density of the Ornstein-Uhlenbeck process R defined by

$$dR_t = \sigma d\beta_t - \sigma^2 \frac{\lambda^* \rho}{1 + \lambda^* \rho \sigma^2 (1-t)} R_t dt, \quad \text{(A.28)}$$

where β is a Brownian motion. Thus, it follows from Theorem 2.2 in Çetin and Danilova (2016) that $X^* + Z$, in its own filtration, is a weak solution of (A.28) since R_1 has the same distribution as $\frac{V-\mu}{\lambda}$. Therefore,

$$dP_t^* = \lambda^* \left(\sigma d\beta_t^* - \sigma^2 \frac{\rho(P_t^* - \mu)}{1 + \lambda^* \rho \sigma^2 (1-t)} dt \right), \quad \text{(A.29)}$$

and β^* is an $\mathcal{F}^M$-Brownian motion, which clearly admits a unique martingale measure.

Moreover, Theorem 2.2 in Çetin and Danilova (2016) also yields that $X^* + Z$ converges a.s. to $\frac{V-\mu}{\lambda^*}$. Thus, in view of Proposition C.1, X^* is optimal provided it is admissible. Indeed, using the transition density of the bridge process provided by

Theorem 2.2 in Çetin and Danilova (2016), it can be directly verified that

$$\mathbb{E}^v \int_0^1 (\lambda^*(X_t^* + Z_t))^2 dt < \infty.$$

Thus, X^* is admissible, and, therefore optimal. Moreover, the value function reads

$$\psi(0,0) = \frac{1}{2\lambda^*}(P_0 - V)^2 + \frac{\lambda^*\sigma^2}{2}.$$

Taking expectation yields ex-ante profit as claimed.

We next turn to the optimisation problem for competitive agents, which is given by

$$\sup_{\theta \in \mathcal{A}(0,\lambda^*,X^*)} \mathbb{E}\left[1 - \exp\left\{-\rho\left(\int_0^1 \theta_t dP_t^*\right)\right\}\right],$$

where P^* is given by (A.29) and $P_0^* = \mu$.

Let us introduce $Y^* := X^* + Z$ and observe that

$$dY_t^* = \sigma d\beta_t^* - \sigma^2 \frac{\lambda^* \rho Y_t^*}{1 + \lambda^* \rho \sigma^2 (1-t)} dt.$$

In view of Theorem 2.1 in Delbaen et al. (2002) the optimal final wealth of the competitive agent is given by

$$\frac{1}{\rho}\left(\mathbb{E}^{\mathbb{Q}}[\log L_1] - \log L_1\right),$$

where

$$L_1 = \exp\left(\int_0^1 \frac{\rho\lambda^* Y_t^*}{1 + \lambda^* \rho \sigma^2 (1-t)} dY_t^* + \frac{\sigma^2 \rho^2 (\lambda^*)^2}{2} \int_0^1 \left(\frac{Y_t^*}{1 + \lambda^* \rho \sigma^2 (1-t)}\right)^2 dt\right)$$

and $\mathbb{Q}$ is the equivalent measure defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = L_1$. Note that Y^* is a $\mathbb{Q}$-martingale with quadratic variation $d[Y_t^*, Y_t^*] = \sigma^2 dt$. Moreover, integration by parts applied to

$$\frac{(Y_t^*)^2}{1 + \lambda^* \rho \sigma^2 (1-t)}$$

yields

$$\log L_1 = \frac{\lambda^* \rho}{2}\left((Y_1^*)^2 - \log(1 + \lambda^* \rho \sigma^2))\right).$$

This in turn implies

$$\mathbb{E}^{\mathbb{Q}}[\log L_1] = \frac{\sigma^2}{2}\int_0^1 \frac{\rho^2(\lambda^*)^2\sigma^2 t}{(1 + \lambda^* \rho \sigma^2 (1-t))^2} dt = \frac{1}{2}\left(\lambda^* \rho \sigma^2 - \log(1 + \lambda^* \rho \sigma^2)\right).$$

Therefore, the final optimal wealth of the competitive agent is given by

$$\frac{\lambda^*}{2}(\sigma^2 - (Y_1^*)^2).$$

This proves the optimality of θ^* since $\int_0^1 \theta_t^* dP_t^* = \frac{\lambda^*}{2}(\sigma^2 - (Y_1^*)^2)$.

Moreover, the expected utility is given by

$$\mathbb{E}\left[1 - \exp\left(-\mathbb{E}^{\mathbb{Q}}[\log L_1] + \log L_1\right)\right] = 1 - e^{-\mathbb{E}^{\mathbb{Q}}[\log L_1]} = 1 - \sqrt{1 + \lambda^*\rho\sigma^2}e^{-\frac{\lambda^*\rho\sigma^2}{2}}.$$

Using the identity $\lambda^*\rho\gamma^2 = (\lambda^*)^2 - \frac{\gamma^2}{\sigma^2}$ we arrive at (3.8).

Since it is obvious that the market clears, $((m^*, \lambda^*), \theta^* = (\theta^{*,a})_{a\in[0,1]}, X^*)$ is an equilibrium. Finally, the equilibrium price process follows (3.4) due to (A.27) and the explicit form of p.

$\square$

Proof of Theorem 3.2. Observe that the pricing rule in terms of the notation of Proposition C.1 can be expressed as $\lambda(t) = \lambda^*$, $t < 1$, $\lambda(1) = 2\lambda^*$, $\phi = \mu$, $m_0 = \mu$ and $c_0 = 2\lambda^*$. Thus, Condition C2 is satisfied with $2\lambda(1)\lambda(1-) = c_0^2$. Moreover, since ΔY^* satisfies (C.38), it suffices to show the admissibility to prove the optimality of the stated strategy. This, however, follows from the fact that Y^* is Gaussian and P^* is an affine function of Y^*.

We next turn to the optimisation problem for competitive agents, which is given by

$$\sup_{\theta\in\mathcal{A}(0,\lambda^*,X^*)} \mathbb{E}\left[1 - \exp\left\{-\rho\left(\int_0^1 \theta_{t-} dP_t^* + \theta_1(V - P_1^*)\right)\right\}\right].$$

Since all stochastic processes involved in the above expression are independent of V, the above reduces to

$$\sup_{\theta\in\mathcal{A}(0,\lambda^*,X^*)} \mathbb{E}\left[1 - \exp\left\{-\rho\left(\int_0^1 \theta_{t-} dP_t^* - 2\lambda^*Y_1^*\theta_1 - \frac{\rho\gamma^2}{2}\theta_1^2\right)\right\}\right].$$

Maximising over θ_1 yields that $\theta_1^* = -Y_1^*$. Thus, the optimisation problem becomes

$$\sup_{\theta\in\mathcal{A}(0,\lambda^*,X^*)} \mathbb{E}\left[1 - \exp\left\{-\rho\left(\int_0^1 \theta_{t-} dP_t^* + \lambda^*(Y_1^*)^2\right)\right\}\right]$$

$$= \sup_{\theta\in\mathcal{A}(0,\lambda^*,X^*)} \mathbb{E}\left[1 - \exp\left\{-\rho\left(\int_0^1 \theta_{t-} dP_t^* + \frac{\lambda^*}{4}(Y_{1-}^*)^2\right)\right\}\right]$$

Next, consider the process

$$N_t := \exp(-\alpha_2(t)(Y_t^*)^2 - \alpha_0(t)),$$

where

$$\alpha_2(t) := -\frac{a(t)}{2} - \frac{\lambda^*\rho}{4 + 2\lambda^*\rho\sigma^2(1-t)}$$

$$a(t) := -\frac{\lambda^*\rho}{1 + \lambda^*\rho\sigma^2(1-t)}$$

$$\alpha_0(t) = \sigma^2 \int_t^1 \alpha_2(s)ds.$$

Direct calculations show that

$$dN_t = -2\alpha_2(t)Y_t\sigma dB_t.$$

Thus, under $\tilde{\mathbb{P}}$ defined by $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \sqrt{\frac{1+\lambda^*\rho\sigma^2}{1+\frac{1}{2}\lambda^*\rho\sigma^2}}N_1$, Y^* follows

$$dY_t^* = \sigma d\tilde{B}_t + \sigma^2 a(t)Y_t^* dt, \quad t < 1,$$

where $\tilde{B}$ is a $\tilde{\mathbb{P}}$-Brownian motion. Consequently, the optimisation problem can be cast under $\tilde{\mathbb{P}}$ as follows

$$\sup_{\theta\in\mathcal{A}(0,\lambda^*,X^*)} \tilde{\mathbb{E}}\left[1 - \sqrt{\frac{1 + \frac{1}{2}\lambda^*\rho\sigma^2}{1 + \lambda^*\rho\sigma^2}}\exp\left\{-\rho\left(\int_0^{1-}\theta_t dP_t^*\right)\right\}\right]$$

In view of Theorem 2.1 in Delbaen et al. (2002) the optimal final wealth of the competitive agent is given by

$$\frac{1}{\rho}\left(\mathbb{E}^{\mathbb{Q}}[\log L_1] - \log L_1\right),$$

where

$$L_1 = \exp\left(\int_0^1 \frac{\rho\lambda^* Y_t^*}{1 + \lambda^*\rho\sigma^2(1-t)}\sigma d\tilde{B}_t - \frac{\sigma^2\rho^2(\lambda^*)^2}{2}\int_0^1\left(\frac{Y_t^*}{1 + \lambda^*\rho\sigma^2(1-t)}\right)^2 dt\right)$$

and $\mathbb{Q}$ is the equivalent measure defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = L_1$. Note that Y^* is a $\mathbb{Q}$-martingale with quadratic variation $d[Y_t^*, Y_t^*] = \sigma^2 dt$. Moreover, integration by parts applied to

$$\frac{(Y_t^*)^2}{1 + \lambda^*\rho\sigma^2(1-t)}$$

yields

$$\log L_1 = \frac{\lambda^* \rho}{2} \left((Y_{1-}^*)^2 - \log(1 + \lambda^* \rho \sigma^2))) \right).$$

This in turn implies

$$\mathbb{E}^{\mathbb{Q}}[\log L_1] = \frac{\sigma^2}{2} \int_0^1 \frac{\rho^2 (\lambda^*)^2 \sigma^2 t}{(1 + \lambda^* \rho \sigma^2 (1 - t))^2} dt = \frac{1}{2} \left(\lambda^* \rho \sigma^2 - \log(1 + \lambda^* \rho \sigma^2) \right).$$

Therefore, the final optimal wealth of the competitive agent is given by

$$\frac{\lambda^*}{2}(\sigma^2 - (Y_{1-}^*)^2).$$

This proves the optimality of θ^* since $\int_0^{1-} \theta_t^* dP_t^* = \frac{\lambda^*}{2}(\sigma^2 - (Y_{1-}^*)^2)$.

Moreover, in view of $\tilde{E}[L_1] = 1$, the expected utility is therefore given by

$$\tilde{\mathbb{E}} \left[1 - \sqrt{\frac{1 + \frac{1}{2}\lambda^* \rho \sigma^2}{1 + \lambda^* \rho \sigma^2}} \exp \left(-\mathbb{E}^{\mathbb{Q}}[\log L_1] + \log L_1 \right) \right] = 1 - \sqrt{1 + \frac{1}{2}\lambda^* \rho \sigma^2} e^{-\frac{\lambda^* \rho \sigma^2}{2}},$$

which proves (3.9).

Since the market clears, $((m^*, \lambda^*), \theta^* = (\theta^{*,a})_{a \in [0,1]}, X^*)$ is an equilibrium. □

Proof of Theorem 4.1. It follows from Theorem 2.2 in Çetin and Danilova (2016) that $X^* + Z$, in its own filtration, is a weak solution of (4.16) since $R_1^{(i)}$ has the same distribution as $\frac{V - \phi_i^*}{\lambda^*}$. In particular,

$$dY_t^* = \sigma^2(a(t)Y_t^* + b(t))dt + \sigma d\beta_t^* \qquad (\text{A.30})$$

for some Brownian motion β^*. This readily proves (4.20) and (4.21).

Moreover, Theorem 2.2 in Çetin and Danilova (2016) also yields that $X^* + Z$ converges a.s. to $\frac{V - \phi_i^*}{\lambda^*}$. Thus, in view of Proposition C.1, X^* is optimal provided it is admissible. Indeed, using the transition density of the bridge process provided by Theorem 2.2 in Çetin and Danilova (2016), it can be directly verified that

$$\mathbb{E}^v \int_0^1 (\lambda^*(X_t^* + Z_t))^2 dt < \infty.$$

Thus, X^* is admissible, and, therefore optimal. Moreover, the value function given by Proposition C.1 reads

$$\psi(0,0) = \frac{1}{2\lambda^*}(P_0 - V)^2 + \frac{\lambda^* \sigma^2}{2}.$$

Taking expectation yields ex-ante profit as claimed.

Next define

$$G_t := - \int_0^t Y_s^* \, dP_s^* + Y_t^* (P_t^* - V)$$

and note that for $t < 1$, G_t is the final wealth of an agent who got the allocation at time 0 and decided to quit market making at time t. Moreover, since $\Delta Y_1^* = 0$, G_1 is the wealth of a market maker who follows the stated strategy until time 1 and wins the allocation at time 0 when all market makers quote λ^*.

Note that the above in conjunction with (A.30) yield that if all market makers quote λ^*, their utility is 0. Indeed, since $P_t^* = \lambda^* Y_t^* + \phi^*$ and $P_1^* = V$,

$$\begin{aligned}
M_t := 1 - \mathbb{E}_t[U(G_t)] &= \mathbb{E}_t \left[\exp \left(\rho \lambda^* \int_0^t Y_s^* \, dY_s^* - \rho \lambda^* Y_t^* (Y_t^* - Y_1^*) \right) \right] \\
&= \exp \left(-\frac{\rho \lambda^*}{2} ((Y_t^*)^2 + \sigma^2 t) \right) \mathbb{E}_t \left[\exp \left(\rho \lambda^* Y_t^* Y_1^* \right) \right] \\
&= \exp \left(-\frac{\rho \lambda^* \sigma^2 t}{2} + \alpha_1(t) Y_t + \alpha_2(t) Y_t^2 \right),
\end{aligned}$$

where

$$\alpha_1(t) := \lambda^* \rho \sigma^2 e^{\sigma^2 A(1)} \int_t^1 e^{-\sigma^2 A(s)} b(s) ds, \text{ and}$$

$$\alpha_2(t) := \lambda^* \rho e^{\sigma^2 (A(1) - A(t))} + \frac{1}{2} (\lambda^*)^2 \rho^2 \sigma^2 \int_t^1 e^{2\sigma^2 (A(1) - A(s))} ds - \frac{\lambda^* \rho}{2},$$

and $A(t) := \int_0^t a(s) ds$ using the moment generating function of the normal distribution associated with the conditional distribution of Y_1^* given Y_t^*. By Lemma B.2 M is a martingale. Therefore, $\mathbb{E}[U(G_1)] = 1 - M_0 = 0$. Since $U(0) = 0$, the ex-ante utility of an arbitrary market maker is given by $\frac{1}{N} \mathbb{E}[U(G_1)] = 0$ when all market makers quote λ^*.

Observe that Proposition C.1 implies that the market λ must be an increasing function of time on $[0, 1)$. Moreover, quoting a λ below $\frac{\lambda^*}{2}$ at time 1 leads to infinite loss and, therefore, is suboptimal. In conjunction with our conventions on time priority this yields only two options for the deviating market maker: i) quote a higher λ starting from time $s \geq 0$, which amounts to stopping making the market on $[s, 1]$, or ii) quote a smaller λ at time 0 with the possibility of increasing it at later dates and opportunity to decrease at time 1. Let (λ, ϕ) denote the market quote resulting from the deviation.

Consider Case i). In this case we can assume without loss of generality that deviating market maker j quotes

$$\lambda^j(t) = \lambda^* \mathbf{1}_{[t<\tau^j]} + (\lambda^* + 1)\mathbf{1}_{[\tau^j \leq t \leq 1]}.$$

Observe that the market quote resulting from this deviation is still the equilibrium market quote (λ^*, ϕ^*). Thus, by our convention, the insider follows the same equilibrium strategy in this setting as well. In particular $\Delta Y_1 = 0$ and $P_{1-} = V$. Thus, the final wealth of the market maker is given by

$$G_1^j = -\int_0^{\tau^j} Y_t^* dP_t^* + Y_{\tau^j}^*(P_{\tau^j}^* - V) = G_{\tau^j}.$$

Since M is a martingale, it follows that $\mathbb{E}[U(G_1^j)] = 0$.

If the deviating market maker chooses to undercut at time 0, then he has the option to increase λ on $[0,1)$, can quit the market at $t \in (0,1)$ by quoting λ^* or above, and can decide to meet the bulk order at $t = 1$. Let (λ, ϕ) denote the market quote resulting from such a deviation and $\hat{G}_1$ be the gains of a fictitious market maker as defined in Remark 2. Suppose that the market maker quits making the market at $t^* \in (0,1]$ with $t^* = 1$ and $\Delta Y_1^j \neq 0$ meaning that he makes the market until the end. Denoting his gains process by G^j we have

$$\begin{aligned}
G_1^j =& \hat{G}_1 - \frac{\lambda(1-)\sigma^2}{2}(1-t^*) - \frac{(P_{1-} - V)^2}{2\lambda(1-)}\mathbf{1}_{[t^*<1]} + \Delta Y_1(V - P_1)\mathbf{1}_{[\Delta Y_1^j = 0]} \\
=& -\frac{(P_0 - V)^2}{2\lambda(0)} + \frac{\sigma^2}{2}\int_0^{t^*}\lambda(s)ds - \Delta Y_1(V - P_1)\mathbf{1}_{[\Delta Y_1^j \neq 0]} + \frac{(P_{1-} - V)^2}{2\lambda(1-)}\mathbf{1}_{[t^*=1]} \\
\leq& -\frac{(P_0 - V)^2}{2\lambda(0)} + \frac{\sigma^2}{2}\lambda^*,
\end{aligned}$$

where the second equality follows from (C.46) and the last one follows from Remark 1 when $t^* = 1$ and $\Delta Y_1^j \neq 0$. In case $t^* = 1$ and $\Delta Y_1^j = 0$, $\lambda(1) > \lambda(1-)/2$ and, therefore $P_{1-} = V$ in view of Proposition C.1. Observe that the price priority rule implies that $\Delta Y_1^j = 0$ if $t^* < 1$. This shows $\mathbb{E}[U(G_1^j)] < 0$.

Hence, $((\Lambda^*, \Phi_i^*), X^*)$ is an equilibrium.

To conclude, observe that the dynamics given by (4.17) and (4.18) follow from the fact that $Y^* = X^* + \sigma B$ and the explicit form of the transition density of R. □

Proof of Theorem 4.2. In view of Proposition C.1, X^* is optimal provided it is admissible. Since the price process is Gaussian, (4.12) clearly holds. Moreover, the value

function reads

$$\psi(0,0) = \frac{1}{2\lambda^*(0)}(P_0 - \mu)^2 + \frac{\lambda^*(0)\sigma^2}{2} = \frac{\rho\gamma^2\sigma^2}{4}.$$

Next recall that for $t < 1$

$$G_t = -\int_0^t Y_s^* \, dP_s^* + Y_t^*(P_t^* - V)$$

is the final wealth of a market maker who got the allocation at time 0 and decided to quit market making at time t as in the proof of Theorem 4.1. Note that since in equilibrium Y has a jump at 1, the final wealth of a market maker who follows the stated strategy until time 1 and wins the allocation at time 0 would be different. To be more precise

$$G_1 = -\int_0^{1-} Y_s^* \, dP_s^* + Y_{1-}^*(P_{1-}^* - V) + \Delta Y_1^*(P_1^* - V).$$

Note that the above yields that if all market makers quote λ^*, their utility is 0. Indeed, since $P_t^* = \lambda^*(0)Y_t^* + \phi^*$ for $t < 1$, we can define M via

$$\begin{aligned} M_t := 1 - \mathbb{E}_t[U(G_t)] &= \mathbb{E}_t\left[\exp\left(\rho\lambda^*(0)\int_0^t Y_s^* \, dY_s^* - \rho Y_t^*(\lambda^*(0)Y_t^* + \phi^* - V)\right)\right] \\ &= \exp\left(-\frac{\rho^2\gamma^2\sigma^2 t}{4} + \rho(\mu - \phi^*)Y_t^* + \frac{\rho^2\gamma^2}{4}(Y_t^*)^2\right), \end{aligned}$$

where the last equality follows from the independence of V and Y^* and the expression for λ^*. Direct application of Ito's formula yields that M is a martingale on $[0,1)$. Furthermore, defining

$$M_1 := 1 - \mathbb{E}_1[U(G_1)]$$

one can directly verify that

$$M_1 = \exp\left(V\rho\frac{\mu - \phi^*}{\lambda^*(0)} - \frac{\sigma^2\rho\lambda^*(0)}{2} - \rho\frac{\mu^2 - (\phi^*)^2}{2\lambda^*(0)}\right) = 1 - U\left(-\frac{(\mu - \phi^*)^2}{2\lambda^*(0)} + \frac{\sigma^2\lambda^*(0)}{2} + \frac{\mu - \phi^*}{\lambda^*(0)}(\mu - V)\right).$$

(A.31)

Thus, $\mathbb{E}[M_1] = 1$. Since $U(0) = 0$, utility of any market maker is given by $\frac{1}{N}\mathbb{E}[U(G_1)] = 0$.

Observe that Proposition C.1 implies that the market λ must be an increasing function of time on $[0,1)$ and

$$2\lambda(1)\lambda(1-) \geq 1.$$

This implies that a deviating market maker who doesn't make the market on $[0,1)$ cannot submit a winning quote at time 1 as undercutting at time 1 would cause

infinite loss according to Proposition C.1. In conjunction with our conventions on time priority this yields only two options for the deviating market maker: i) quote a higher λ starting from time $s \geq 0$, which amounts to stopping making the market on $[s, 1]$, or ii) quote a smaller λ at time 0 with the possibility of increasing it at later dates and an opportunity to decrease at time 1.

In the first case, if the market maker j quotes a higher λ starting at $s < 1$, he will get no order on $[s, 1]$ due to the no-shuffling rule and the fact that by undercutting market lambda at time 1 will result in infinite loss. Note that for this deviating strategy the resulting market quote does not change and therefore the equilibrium strategy of the strategic trader stated in the theorem is still optimal. Since M is a martingale and the utility of the deviating market maker is given by $1 - \mathbb{E}[M_s]$, such a deviation is suboptimal.

If the deviating market maker chooses to undercut at time 0, then he has two options: i) quit making the market at $t^* \in (0, 1]$ by quoting the equilibrium λ or above, and ii) make the market on $[0, 1]$ by undercutting throughout.

Let (λ, ϕ) denote the market quote resulting from such a deviation and $\hat{G}_1$ be the gains of a fictitious market maker as defined in Remark 2. Suppose that the market maker quits making the market at $t^* \in (0, 1]$. Denoting his gains process by G^j, we have via Remark 2 that

$$
\begin{aligned}
G_1^j &= \hat{G}_1 - \frac{\lambda(1-)\sigma^2}{2}(1 - t^*) + \frac{(\mu - V)^2 - (P_{1-} - V)^2}{2\lambda(1-)} + \Delta Y_1(V - P_1) \\
&= -\frac{(P_0 - \mu)^2}{2\lambda(0)} + \frac{\sigma^2}{2}\int_0^{t^*} \lambda(s)ds + Y_{t^*}(\mu - V) \\
&= -\frac{(\phi - \mu)^2}{2\lambda(0)} + \frac{\sigma^2}{2}\int_0^{t^*} \lambda(s)ds + \frac{\mu - \phi}{\lambda(0)}(\mu - V),
\end{aligned}
$$

where we use the fact that $P_{t^*} = \mu$ in view of Proposition C.1 since the strategic trader constructs a bridge at t^*. Thus, direct comparison with (A.31) and the fact that $\lambda < \lambda^*$ on $[0, t^*)$ reveals that deviation is not profitable.

Moreover, if the deviating market maker improves the equilibrium quotes on $[0, 1]$, his total gains are given by $\hat{G}_1$. If $2\lambda(1) = \lambda(1-)$, Proposition C.1 implies $Y_1 = \frac{\mu - \phi}{\lambda(1-)}$. Otherwise, $Y_1 = \frac{\mu - \phi}{\lambda(0)}$. Also note that Remark 1 and (C.47) yield that

$$
\hat{G}_1 \leq -\frac{(\phi - \mu)^2}{2\lambda(0)} + \frac{\sigma^2}{2}\int_0^1 \lambda(s)ds + Y_1(\mu - V).
$$

Therefore, another comparison with (A.31) and the fact that $\lambda < \lambda^*$ on $[0, 1]$ leads to the claimed suboptimality.

Finally the dynamics of Y^* follows from that $Y_i^* = \sigma B + X_i^*$. $\qquad\qquad\qquad\square$

Appendix B. Exponential martingales of Ornstein-Uhlenbeck processes

Lemma B.1. *Suppose Y solves*

$$dY_t = \sigma^2 \left(a(t)Y_t + b(t)\right) dt + \sigma dB_t. \quad Y_0 = 0,$$

for some continuously differentiable functions a and b. Then,

$$M_t := \exp \left(\alpha_0(t) + \alpha_1(t)Y_t + \alpha_2(t)Y_t^2\right)$$

is a martingale such that

$$
\begin{aligned}
\alpha_0(1) &= -\frac{\lambda\rho\sigma^2}{2}, \\
\alpha_1(1) &= 0, \\
\alpha_2(1) &= \frac{\lambda\rho}{2}
\end{aligned}
$$

and MY is a martingale if and only if

$$
\begin{aligned}
-2\alpha_2(t) &= a(t) = -\frac{\lambda\rho}{1 + \lambda\rho\sigma^2(1-t)}, & \text{(B.32)} \\
b(t) &= \alpha_1(t) = 0, & \text{(B.33)} \\
\alpha_0(t) &= -\frac{\lambda\rho\sigma^2}{2} + \frac{1}{2}\log(1 + \lambda\rho\sigma^2(1-t)). & \text{(B.34)}
\end{aligned}
$$

Proof. Application of Ito's formula yields the following necessary and sufficient conditions on the coefficients α_i for M to be a local martingale:

$$
\begin{aligned}
2\alpha_2(t)a(t) + 2\alpha_2^2(t) + \frac{\alpha_2'(t)}{\sigma^2} &= 0; & \text{(B.35)} \\
\frac{\alpha_1'(t)}{\sigma^2} + \alpha_1(t)a(t) + 2\alpha_2(t)\alpha_1(t) + 2\alpha_2(t)b(t) &= 0; & \text{(B.36)} \\
\frac{\alpha_0'(t)}{\sigma^2} + \alpha_1(t)b(t) + \alpha_2(t) + \frac{\alpha_1^2(t)}{2} &= 0. & \text{(B.37)}
\end{aligned}
$$

As a result

$$dM_t = M_t(\alpha_1(t) + 2\alpha_2(t)Y_t)\sigma dB_t.$$

Thus, in order for MY to be a local martingale we must have

$$\alpha_1(t) = -b(t) \text{ and } \alpha_2(t) = -\frac{a(t)}{2}.$$

Plugging above into (B.36) yields $b' = -\sigma^2 ab$. Thus, $b \equiv 0$ since $b(1) = 0$. Moreover, the remaining ODEs can be rewritten as

$$\begin{aligned} a'(t) &= -\sigma^2 a^2(t); \\ \alpha_0'(t) &= \frac{\sigma^2}{2}a(t), \end{aligned}$$

which yields the claim in view of the boundary of conditions. Indeed, Y is an Ornstein-Uhlenbeck process, which in turn implies that M is a martingale by direct calculations. Moreover, Girsanov's theorem yields that MY is a martingale as well.   □

Lemma B.2. *Suppose Y solves*

$$dY_t = \sigma^2 \left(a(t)Y_t + b(t) \right) dt + \sigma dB_t, \quad Y_0 = 0,$$

for some continuously differentiable functions a and b. Then, Y_t is normally distributed given Y_s with mean

$$Y_s e^{\sigma^2(A(t)-A(s))} + \int_s^t e^{\sigma^2(A(t)-A(r))}\sigma^2 b(r) dr,$$

and variance

$$v^2(s,t) := \int_s^t e^{2\sigma^2(A(t)-A(r))}\sigma^2 dr, \text{ where } A(t) := \int_0^t a(s) ds.$$

Moreover,

$$M_t := \exp\left(\alpha_0(t) + \alpha_1(t)Y_t + \alpha_2(t)Y_t^2 \right)$$

is a martingale, where

$$\begin{aligned} \alpha_0(t) &:= -\frac{\lambda\rho\sigma^2}{2}t, \\ \alpha_1(t) &:= \lambda\rho\sigma^2 e^{\sigma^2 A(1)} \int_t^1 e^{-\sigma^2 A(s)} b(s) ds, \text{ and} \\ \alpha_2(t) &:= \lambda\rho e^{\sigma^2(A(1)-A(t))} + \frac{1}{2}\lambda^2\rho^2\sigma^2 \int_t^1 e^{2\sigma^2(A(1)-A(s))} ds - \frac{\lambda\rho}{2}, \end{aligned}$$

and $A(t) := \int_0^t a(s)ds$, if and only if the following are satisfied:

$$
a(t) = -\frac{\lambda\rho}{1 + \lambda\rho\sigma^2(1-t)} \quad and
$$

$$
b(t) = \pm\frac{1}{2}\log(1 + \lambda\rho\sigma^2(1-t))\sqrt{\frac{a(t)}{-\lambda\rho\sigma^2(1-t) + \log(1 + \lambda\rho\sigma^2(1-t))}}.
$$

Proof. The first claim is well-known and follows from direct calculations.

We can identify α_2 by using the moment generating function of a chi-squared random variable as follows: Note that

$$
M_t := \exp\left(\alpha_0(t) + \alpha_2(t)\left(Y_t + \frac{\alpha_1(t)}{2\alpha_2(t)}\right)^2 - \frac{\alpha_1^2(t)}{4\alpha_2(t)}\right).
$$

By using the moment generating function of chi-squared r.v. and equating the coefficients of Y_s^2 in $M_s = \mathbb{E}[M_t|\mathcal{F}_s]$, we find

$$
\alpha_2(t)e^{2\sigma^2(A(t)-A(s))} = \alpha_2(s)(1 - 2\alpha_2(t)v^2(s,t)).
$$

Next take $t = 1$ in above and use the definition for α_2 to arrive at

$$
\phi^2(t) = \left[2\phi(t) + \lambda\rho\sigma^2\int_t^1 \phi^2(u)du - 1\right]\left(1 - \lambda\rho\sigma^2\int_t^1 \phi^2(u)du\right),
$$

where

$$
\phi(t) = e^{\sigma^2(A(1)-A(t))}.
$$

The above is equivalent to

$$
\left(1 - \phi(t) - \lambda\rho\sigma^2\int_t^1 \phi^2(u)du\right)^2 = 0.
$$

Thus, using the fact that $\phi(1) = 1$, we obtain

$$
\phi(t) = \frac{1}{1 + \lambda\rho\sigma^2(1-t)},
$$

which implies

$$
a(t) = -\frac{\lambda\rho}{1 + \lambda\rho\sigma^2(1-t)}.
$$

Also observe that $\alpha_i s$ must also satisfy (B.35)-(B.37) since M is a local martingale. Note that (B.36) is redundant since $\alpha_2 = -a/2$ by direct manipulation. Thus, (B.37) simplifies to

$$
-\frac{\rho\lambda}{2} + \alpha_1(t)b(t) - \frac{a(t)}{2} + \frac{\alpha_1^2(t)}{2} = 0.
$$

By substituting $b = \frac{\alpha_1'}{\sigma^2 a}$ and $v = \alpha_1^2$, we get the ODE

$$\frac{v'}{\sigma^2 a} + v = \lambda\rho + a,$$

which has the following solution:

$$v(t) = -a(t)\left(\lambda\rho\sigma^2(1-t) - \log(1 + \lambda\rho\sigma^2(1-t))\right).$$

Note that v is always positive since the term in the parentheses is decreasing to 0 at t=1.

Calculating b via $b = \frac{\alpha_1'}{\sigma^2 a}$ yields the claimed form of b.

Finally, since Y is an Ornstein-Uhlenbeck process, that M is a martingale follows from direct calculations via the transition function.   □

APPENDIX C. STRATEGIC TRADER'S OPTIMALITY

Proposition C.1. *Let λ be a right-continuous and piecewise constant function on $[0,1]$ and $\phi(t) = \phi + \mathbf{1}_{[t=1]}\Delta\phi$ with $\Delta\phi = c_1 P_{1-} + c_0$ for some constant c_1, c_0. Consider the price process defined by*

$$P_t = \phi(t) + \int_0^t \lambda(s)\{dZ_s + dX_s\}.$$

Denote the trader's valuation of V by $\tilde{V}$, i.e $\tilde{V} = V$ for the insider and $\tilde{V} = \mu$ for the uninformed strategic trader.

(1) *The trader has finite profit if and only if the following conditions are satisfied:*
   C1 *λ is non-decreasing on $[0,1)$, and $\frac{2\lambda(1)}{\lambda(1-)} \geq (1+c_1)^2$.*
   C2 *If $\frac{2\lambda(1)}{\lambda(1-)} = (1+c_1)^2$, then $c_0 = -\mu c_1$ when $\tilde{V} = \mu$, and $c_0 = c_1 = 0$, when $\tilde{V} = V$.*

(2) *If the trader has finite profit, any admissible strategy is optimal if and only if*

$$\Delta Y_1 = \frac{\tilde{V} - c_0 - (1+c_1)P_{1-}}{2\lambda(1)}, \quad a.s., \tag{C.38}$$

$$P_{1-} = \frac{\tilde{V}((1+c_1)\lambda(1-) - 2\lambda(1)) - c_0(1+c_1)\lambda(1-)}{\lambda(1-)(1+c_1)^2 - 2\lambda(1)}, \quad a.s., \ if \ \frac{2\lambda(1)}{\lambda(1-)} > (1+c_1)^2, \tag{C.39}$$

and for any point t_i of discontinuity of λ before 1,

$$P_{t_i-} = \tilde{V}, \quad a.s.. \tag{C.40}$$

Moreover, (C.39) and (C.40) are equivalent to

$$Y_{1-} = \frac{-\tilde{V}(1+c_1)\lambda(1-)c_1 - c_0(1+c_1)\lambda(1-)}{\lambda(1-)(\lambda(1-)(1+c_1)^2 - 2\lambda(1))} + \frac{\tilde{V}-\phi}{\lambda_0}, \ a.s., \ if \ \frac{2\lambda(1)}{\lambda(1-)} > (1+c_1)^2,$$

$$(\text{C.41})$$

$$Y_{t_i} = \frac{\tilde{V}-\phi}{\lambda(0)}, \ a.s.. \tag{C.42}$$

Furthermore, the value function of the trader is given by

$$\psi(0,0) + \sum_{i=1}^{n} \frac{\sigma^2}{2}(1-t_i)(\lambda(t_i) - \lambda(t_{i-1})) + y^* \frac{(c_1\tilde{V}+c_0)^2}{4\lambda_1 - 2\lambda_{1-}(1+c_1)^2},$$

$$\text{where } y^* = \begin{cases} 1, \ \text{in the case } \frac{2\lambda_1}{\lambda_{1-}} > (1+c_1)^2 \\ 0, \ \text{otherwise.} \end{cases}$$

Proof. We will adapt Back's arguments in Back (1992) to our case. Let $(t_i)_{i=1}^n$ be the times at which λ jumps before time 1. The arguments of Back can be directly used to show that it is suboptimal for the insider's strategy to jump at times other than t_is. Thus, without loss of generality, X can jump only at t_is or at 1.

With the convention that $t_{n+1} = 1$ and $t_0 = 0$ define the function

$$\psi(t,y) = \sum_{i=0}^{n} \psi^i(t,y)\mathbf{1}_{[t_i,t_{i+1})}(t),$$

where

$$\psi^i(t,y) = \frac{1}{2\lambda_i}(\lambda_i(y-Y_{t_i}) + P_{t_i} - \tilde{V})^2 + \frac{\sigma^2}{2}\lambda_i(1-t),$$

with $\lambda_i = \lambda(t_i)$. Direct calculations yield

$$\psi(1-,Y_{1-}) = \psi(0,0) + \int_0^{1-}(P_t - \tilde{V})dY_t$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\left[\frac{1}{\lambda_i}(P_{t_i}-\tilde{V})^2 - \frac{1}{\lambda_{i-1}}(P_{t_i-}-\tilde{V})^2 - 2(P_{t_i}-\tilde{V})\Delta Y_{t_i} + \sigma^2(1-t_i)(\lambda_i - \lambda_{i-1})\right].$$

Using this and admissibility properties of X, in particular $d[X,X]_t = (\Delta X_t)^2$, the insider's optimization problem becomes

$$
\begin{aligned}
\sup_{X \in \mathcal{A}} \mathbb{E}^{\tilde{v}}[W_1^X] &= \sup_{X \in \mathcal{A}} \mathbb{E}^{\tilde{v}}\left[\int_0^1 \left(\tilde{V} - P_t\right) dX_t\right] \\
&= \mathbb{E}^{\tilde{v}}[\psi(0,0)] + \frac{1}{2}\sum_{i=1}^n \sigma^2(1-t_i)(\lambda_i - \lambda_{i-1}) - \inf_{X \in \mathcal{A}} \mathbb{E}^{\tilde{v}}\left[\psi(1-, Y_{1-}) - (\tilde{V} - P_1)\Delta Y_1\right. \\
&\qquad \left. -\frac{1}{2}\sum_{i=1}^n \left(\frac{1}{\lambda_i}(P_{t_i-} - \tilde{V})^2 - \lambda_i(\Delta Y_{t_i})^2 - \frac{1}{\lambda_{i-1}}(P_{t_i-} - \tilde{V})^2\right)\right]
\end{aligned}
$$

where $\mathbb{E}^{\tilde{v}} = \mathbb{E}^v$ if $\tilde{V} = V$ and $\mathbb{E}$ otherwise. In above the last equality is due to (4.12). Clearly, $\Delta Y_{t_i} = 0$ is optimal for $i \leq n$. Note, however, that the same cannot be said for ΔY_1. Indeed, since

$$P_1 = P_{1-} + \Delta\phi + \lambda_{n+1}\Delta Y_1 = (1 + c_1)P_{1-} + c_0 + \lambda_{n+1}\Delta Y_1,$$

the term $-(\tilde{V} - P_1)\Delta Y_1$ is minimised at

$$\Delta Y_1 = \frac{\tilde{V} - c_0 - (1 + c_1)P_{1-}}{2\lambda_{n+1}}.$$

Therefore, with this choice of jumps, the optimisation problem of the insider becomes

$$
\begin{aligned}
\sup_{X \in \mathcal{A}} \mathbb{E}^{\tilde{v}}[W_1^X] &= \mathbb{E}^{\tilde{v}}[\psi(0,0)] + \frac{1}{2}\sum_{i=1}^n \sigma^2(1-t_i)(\lambda_i - \lambda_{i-1}) \\
&\quad - \inf_{X \in \mathcal{A}} \mathbb{E}^{\tilde{v}}\left[\frac{1}{2\lambda_n}(P_{1-} - \tilde{V})^2 - \frac{1}{4\lambda_{n+1}}((1+c_1)P_{1-} + c_0 - \tilde{V})^2\right. \quad \text{(C.43)} \\
&\qquad \left. -\frac{1}{2}\sum_{i=1}^n \left(\frac{1}{\lambda_i}(P_{t_i} - \tilde{V})^2 - \frac{1}{\lambda_{i-1}}(P_{t_i} - \tilde{V})^2\right)\right]
\end{aligned}
$$

Suppose we have $\lambda_i < \lambda_{i-1}$ for some $i \leq n$. Then by choosing a strategy to make Y_{t_i} large, the strategic trader will drive the price P_{t_i} arbitrarily far away from $\tilde{V}$, thus, achieving infinite profits. Consequently, λ_i are increasing for $i \leq n$ and, therefore, it is optimal to achieve $P_{t_i} = \tilde{V}$ for all $i \leq n$.

Using the same arguments, we can deduce that $\frac{2\lambda_{n+1}}{\lambda_n} = \frac{2\lambda(1)}{\lambda(1-)} \geq (1+c_1)^2$ and that C2 holds whenever $\frac{2\lambda(1)}{\lambda(1-)} = (1+c_1)^2$.

On the other hand, if $\frac{2\lambda_{n+1}}{\lambda_n} > (1+c_1)^2$, the maximizer of $\frac{1}{4\lambda_{n+1}}((1+c_1)P_{1-} + c_0 - \tilde{V})^2 - \frac{1}{2\lambda_n}(P_{1-} - \tilde{V})^2$ is given by

$$P_{1-} = \frac{\tilde{V}((1+c_1)\lambda_n - 2\lambda_{n+1}) - c_0(1+c_1)\lambda_n}{\lambda_n(1+c_1)^2 - 2\lambda_{n+1}} = \frac{\tilde{V}((1+c_1)\lambda_{1-} - 2\lambda_1) - c_0(1+c_1)\lambda_{1-}}{\lambda_{1-}(1+c_1)^2 - 2\lambda_1}.$$
(C.44)

This yields that the upper bound for the value function of the insider is

$$\psi(0,0) + \sum_{i=1}^{n} \frac{\sigma^2}{2}(1-t_i)(\lambda_i - \lambda_{i-1}) + y^* \frac{(c_1\tilde{V} + c_0)^2}{4\lambda_{n+1} - 2\lambda_n(1+c_1)^2}.$$

In above,

$$y^* = \begin{cases} 1, & \text{in the case } \frac{2\lambda_{n+1}}{\lambda_n} > (1+c_1)^2 \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $t_n > 0$, $P_{1-} - \tilde{V} = \lambda(1-)(Y_{1-} - Y_{t_n}) = \lambda(1-)(Y_{1-} - \frac{\tilde{V}-\phi}{\lambda(0)})$. Moreover, if $t_n = 0$, we still have $P_{1-} - \tilde{V} = \lambda(1-)(Y_{1-} - \frac{\tilde{V}-\phi}{\lambda(0)})$ since $\lambda(1-) = \lambda(0)$. Thus, the Y_{1-} corresponding to the equality (C.44) in case $\frac{2\lambda_1}{\lambda_{1-}} > (1+c_1)^2$ is given by

$$\lambda(1-)\left(Y_{1-} - \frac{\tilde{V}-\phi}{\lambda(0)}\right) = \frac{-\tilde{V}(1+c_1)\lambda_{1-}c_1 - c_0(1+c_1)\lambda_{1-}}{\lambda_{1-}(1+c_1)^2 - 2\lambda_1}.$$

Hence, to conclude the proof, it is enough to show the existence of an admissible strategy for the insider achieving this upper bound. Observe that Y and P are related by

$$Y_{t_i-} - Y_{t_{i-1}} = \frac{1}{\lambda_{i-1}}\left[P_{t_i-} - P_{t_{i-1}-}\right], \qquad i = 1,\ldots,n+1,$$

with the convention that $P_{0-} = P_0, \phi(0-) = \phi(0)$. Thus, the above conditions on P are equivalent to

$$Y_{t_1} = c_1 \quad := \quad \frac{\tilde{V}-\phi}{\lambda_0}$$

$$Y_{t_i} - Y_{t_{i-1}} = c_i \quad := \quad 0, \qquad i = 2,\ldots,n,$$

$$Y_{1-} - Y_{t_n} = c_{n+1} \quad := \quad \frac{-\tilde{V}(1+c_1)\lambda_{1-}c_1 - c_0(1+c_1)\lambda_{1-}}{\lambda_{1-}(\lambda_{1-}(1+c_1)^2 - 2\lambda_1)}.$$

Direct calculations yield (C.41) and (C.42).

A strategy that drives Y to the above levels is given by

$$\alpha_t = \sigma^2 \frac{Y_t - Y_{t_{i-1}} - c_i}{t_i - t}, \qquad t \in [t_{i-1}, t_i), \ i = 1,\ldots,n+1.$$

Indeed, this strategy makes Y_- a Brownian bridge over the interval $[t_{i-1}, t_i]$ from $Y_{t_{i-1}}$ to $Y_{t_{i-1}} + c_i$. In particular, Y_- is Gaussian conditioned on $\tilde{V} = v$. Since λ as well as ϕ is deterministic on $[0, 1)$, P is a Gaussian process on $[0, 1)$, too. This implies $\mathbb{E}^{\tilde{v}}[(P_t - \phi(t))^2]$ is a finite continuous function of t on $[0, 1)$ and, therefore, the above strategy is admissible.   □

Remark 1. *In the setting of Proposition C.1, if the strategic (informed or otherwise) trader acts optimally then*

$$\frac{1}{2\lambda(1-)}(P_{1-} - \tilde{V})^2 - (\tilde{V} - P_1)\Delta Y_1 \le 0. \tag{C.45}$$

Indeed, (C.38) implies $\tilde{V} - P_1 = \lambda(1)\Delta Y_1$. Therefore,

$$\frac{1}{2\lambda(1-)}(P_{1-} - \tilde{V})^2 - (\tilde{V} - P_1)\Delta Y_1 = \frac{1}{2\lambda(1-)}(P_{1-} - \tilde{V})^2 - \frac{((1+c_1)P_{1-} + c_0 - \tilde{V})^2}{4\lambda(1)}.$$

However, the right hand side above is minimised by the strategic trader in view of (C.43). Note that the minimum is at most 0 since she can trade so that $P_{1-} = \tilde{V}$.

Remark 2. *Consider the gains process $\hat{G}$ associated to a fictitious market maker that holds all the orders submitted by the strategic trader and the noise traders and trades at the market quotes. This gains process corresponds to the sum of all the gains of the market makers. Observe that*

$$\hat{G}_1 = -\int_0^{1-} Y_s dP_s + Y_{1-}(P_{1-} - V) + \Delta Y_1(P_1 - V) = -\int_0^1 (V - P_t)dY_t + \sigma^2 \int_0^1 \lambda(t)dt.$$

Thus, when the strategic trader is informed,

$$\hat{G}_1 = \sigma^2 \int_0^1 \lambda(t)dt + \psi(1-, Y_{1-}) - \psi(0,0) - (V - P_1)\Delta Y_1$$

$$- \frac{1}{2}\sum_{i=1}^n \left[\frac{1}{\lambda_i}(P_{t_i} - V)^2 - \frac{1}{\lambda_{i-1}}(P_{t_i-} - V)^2 - 2(P_{t_i} - V)\Delta Y_{t_i}\right] - \frac{1}{2}\sigma^2 \int_0^{1-}(1-t)d\lambda(t)$$

$$= \frac{1}{2\lambda(1-)}(P_{1-} - V)^2 - (V - P_1)\Delta Y_1 - \frac{1}{2\lambda(0)}(P_0 - V)^2 + \frac{1}{2}\sigma^2 \int_0^1 \lambda(t)dt$$

$$- \frac{1}{2}\sum_{i=1}^n \left[\frac{1}{\lambda_i}(P_{t_i} - V)^2 - \frac{1}{\lambda_{i-1}}(P_{t_i-} - V)^2 - 2(P_{t_i} - V)\Delta Y_{t_i}\right]$$

Since the insider's optimal strategy has no jumps before time 1 and $P_{t_i} = V$ by optimality considerations, the last term vanishes. Thus,

$$\hat{G}_1 = \frac{1}{2\lambda(1-)}(P_{1-} - V)^2 - (V - P_1)\Delta Y_1 - \frac{1}{2\lambda(0)}(P_0 - V)^2 + \frac{1}{2}\sigma^2 \int_0^1 \lambda(t)dt. \quad \text{(C.46)}$$

If the strategic trader is uninformed,

$$\hat{G}_1 = -\int_0^{1-} Y_s dP_s + Y_{1-}(P_{1-} - \mu) + \Delta Y_1(P_1 - \mu) + Y_1(\mu - V).$$

Thus, similar considerations yield

$$\hat{G}_1 = \frac{(P_{1-} - \mu)^2}{2\lambda(1-)} - (\mu - P_1)\Delta Y_1 - \frac{(P_0 - \mu)^2}{2\lambda(0)} + Y_1(\mu - V) + \frac{1}{2}\sigma^2 \int_0^1 \lambda(t)dt. \quad \text{(C.47)}$$

Department of Statistics, London School of Economics and Political Science, 10 Houghton St., London, WC2A 2AE

E-mail address: `u.cetin@lse.ac.uk`

Department of Mathematics, London School of Economics and Political Science, 10 Houghton St., London, WC2A 2AE

E-mail address: `a.danilova@lse.ac.uk`