Racing with a rearview mirror: innovation lag and investment dynamics*

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February 27, 2023

Abstract

We analyze a dynamic investment model in which short-lived agents sequentially decide how much to invest in a project of uncertain feasibility. The outcome of the project (success/failure) is observed after a fixed lag. We characterize the unique equilibrium and show that, in contrast with the case without lag, the unique equilibrium dynamics is not in thresholds. If the initial belief is relatively high, investment decreases monotonically as agents become more pessimistic about the feasibility of the innovation. Otherwise, investment is not monotonic in the public belief: players alternate periods of no investment and periods of positive, decreasing investment. The reason is that the outcome lag creates competition between a player and her immediate predecessors. A player whose predecessors did not invest may find investment attractive even if she is more pessimistic about the technology than her predecessors. We compare the total investment obtained in this equilibrium with that obtained with an alternative reward scheme where a mediator collects all the information about the players' experiences until some deadline, and splits the payoff between all the players who obtained a success before the deadline.

Keywords: Risky investment; Outcome lag; Payoff rivalry.
JEL codes: C73, D83

1 Introduction

When it comes to research or innovation, the return on investment is usually not immediate. In physics, large-scale experiments such as those carried out to discover exo-planets, or to prove the existence of the Higgs bozon, require the development of specific measuring instruments, so that it takes several years to observe the results of these experiments. In biology, the development of a new drug requires several validation steps that often take several years. In the start-up economy, it can take several years before a new application is

*We are grateful to Nenad Kos and the seminar participants to the One World Mathematical Game Theory seminar. This research was supported by the French National Research Agency (ANR-17-CE38-0005-01) and by the Labex MME-DII (ANR11-LBX-0023-01).

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known and massively used. Timing would not be an issue in a world without competition. However, as illustrated in the COVID-19 vaccine race, for investors it is often crucial to complete their project before their competitors.

The aim of this paper is to investigate how investment decisions are affected by the time it takes before observing the outcome of one's investment when investors are rewarded only if they are the first to obtain a success. To do so, we analyze a stylized model of dynamic investment in which short-lived agents sequentially choose how much to invest in a project of uncertain feasibility. If the project is not feasible, investment is lost and yields no payoff. If it is feasible, investment can generate a positive payoff after a fixed time lag Δ, with a probability that increases with the invested amount. In the main model, the game ends the first time a player receives a payoff, even though some players may have invested after this player. We call this reward scheme the *winner-takes-all* mechanism. Because of the outcome lag, a player who must decide how much to invest at time t knows that players who invested before time $t-\Delta$ have invested at a loss, but does not know whether investments made by players between $t-\Delta$ and t will be successful or not. Moreover, because players learn from predecessors' experiences in a Bayesian fashion, they become more pessimistic about the feasibility of the project as time passes. Therefore, this model features both a negative payoff externality and a positive informational externality. What is the tradeoff faced by the player investing at time t? Investment will yield the player a positive payoff if 1) her investment is successful, which can happen only if the project is feasible and 2) she is not preempted by a player investing between $t-\Delta$ and t. As time passes, players can only become more pessimistic about the feasibility of the project, which decreases the desirability of investment. However, the effect of competition may not be monotonic in time. Indeed, the innovation lag creates competition between the player investing at time t and all the players investing between $t-\Delta$ and t. Therefore, even if player t is necessarily more pessimistic than her predecessors, she may face a smaller risk of preemption and thus find investment more desirable than her predecessors.

When there is no outcome lag, the equilibrium strategy is of the cutoff type: players make the maximal investment effort if their belief that the project is feasible is above some threshold and invest nothing otherwise. In contrast, with a positive lag, the equilibrium strategy does not have a cutoff structure. When players are initially very pessimistic, none of them ever invests in the project. Otherwise, first players make the maximal investment until some cutoff time that depends on the prior belief. Afterwards, they invest only a fraction of their resource in the project, according to a pattern that depends on the prior belief. When they are initially very optimistic, investment decreases with time and belief in a discontinuous way, jumping downward at regular intervals (as a consequence, some investment levels are never realized in equilibrium). Surprisingly, for intermediate prior beliefs, investment is non-monotonic in time or belief: the investment dynamics feature periods of no investment and periods of decreasing, but positive investment, even though players are continuously more pessimistic about the project as time passes. To understand this result, imagine three generations of researchers: Gen X, Gen Y and Gen Z. Each generation chooses in turn how much to invest, and the outcome of a given generation is realized only two generations after. Suppose Gen X invests in the project. Gen Y does not know yet the outcome of Gen X, thus they have the same belief about the feasibility of the project. However, because Gen X has already invested, there is a positive probability for Gen Y to be preempted by Gen X, which makes investment riskier and less desirable for them than for Gen X. Suppose Gen Y decides not to invest and that Gen X does not complete the project. Because of this failure, Gen Z is more pessimistic than both their predecessors. But because Gen Y did not invest, Gen Z does not face any risk of preemption, and may thus find investment optimal even while being more pessimistic than Gen Y.

We compare our equilibrium investment dynamics with that obtained with an alternative reward scheme called "*hidden-equal-sharing mechanism*, which works as follows: outcomes are observed only by a principal who keeps them secret until some deadline T. At time T, outcomes are revealed to all and the benefit of the project is shared among all the players who obtained a success before $T - \Delta$. The principal chooses T in such a way that the total amount of investment is maximal. We find that the total investment may but must not be larger with the hidden-equal-sharing mechanism. Let us illustrate why in the three generations example. Suppose now that $T = 4$. On the one hand, investment may be more attractive for Gen Y in the hidden-equal-sharing, because it may yield a positive payoff even if Gen X is successful. Also, Gen Z is not informed of Gen X's outcome and is therefore more optimistic about the feasibility of the project. On the other hand, conditional on the innovation being feasible, the expected profit is smaller for Gen Z, since they may have to share the benefits with Gen X or Gen Y. We find a necessary and sufficient condition on the initial belief under which the "hidden equal-sharing" entails a larger total investment.

Related literature. The framework we use to model learning is borrowed from the exponential-bandit literature, in which long-lived players trade off exploration vs. exploitation (Keller, Rady and Cripps (2005), Keller and Rady (2010), Rady and Klein (2011), etc). Some authors have analyzed how imperfect observation of players' actions or outcomes can increase the exploration efforts in equilibrium (Bonatti and Hörner (2011), Heidhues, Rady and Strack (2015), Marlats and Ménager (2021)). Bimpikis and Drokopoulos (2014) and Che and Hörner (2015) study how a principal should disclose information to improve aggregate learning. In all these papers, externalities are only informational. There is a strand of literature mixing learning and competition issues. For instance, Choi (1991), Malueg and Tsutsui (1997) or Moscarini and Squintani (2010) analyze R&D competition under a winner-takes-all mechanism. Das and Klein (2021.a, 2021.b) analyze a patent race in a two-arm exponential bandit framework. Bimpikis, Ehsani and Mostagir (2014) and Halac, Kartik and Liu (2017) also address the issue of contest design in a dynamic environment. The paper by Halac, Kartik and Liu (2017) is the most closely related to our paper. They characterize the optimal contest in a class of mechanisms in which the principal chooses both a prize-sharing policy and a disclosure policy. They show that the optimal contest of this class is either a winner-takes-all with immediate disclosure of the outcomes or a hidden-equal-sharing mechanism. Only few papers in game and decision theory address the effects of delayed feedbacks. Gordon, Marlats and Ménager (2021) analyze a team problem where partners work together to achieve a project which is commonly known to be feasible. Players learn immediately whether they succeed but observe their partners' outcome only after a fixed lag. The observation lag has an effect that is similar to that in our paper, as, in equilibrium, players alternate between periods in which they exert the maximal effort and periods in which they make no effort at all.

The remainder of this paper is organized as follows. Section 2 sets up the model. Section 3 characterizes the unique equilibrium in the winner-takes-all framework. In Section 4, we compare our results to that obtained with the hidden-equal-sharing mechanism.

2 The set-up

Time is continuous and there is a continuum of players indexed by $t \in [0, +\infty)$. Each player t chooses at time t what fraction $k_t \in [0, 1]$ to invest in a risky technology at unit cost $\alpha > 0$. The technology can be good or bad. If it is bad, the technology never yields any payoff. If it is good, it yields a payoff of 1 at time $t + \Delta$ at the first jump of a time-inhomogeneous Poisson process with rate λk_t, with $\lambda > \alpha$. The first time a player receives a positive payoff is called a *breakthrough*, and the game stops after a breakthrough.

As player t plays only at time t, a pure strategy for player t is $k_t \in [0, 1]$ and a strategy profile is a function $\mathbf{k} : \mathbb{R}_+ \to [0, 1]$. To guarantee that players can always update their beliefs on the observation of past investments, we restrict the analysis to *admissible* strategy profiles, defined as profiles $\mathbf{k}$ such that $\int_{\underline{t}}^{\bar{t}} k_t dt$ is well-defined for every $\underline{t} \leq \bar{t}$.

Players observe the whole history of actions and outcomes but do not know the type of the technology, absent a breakthrough. The public belief at time t that the technology is good is denoted p_t, with $p_0 \in (0, 1)$ the initial common belief. The public belief is continuously updated on the basis of the observation of past investments. As a player's investment operates on the Poisson process with a time lag Δ, players have no feedback at all before Δ, hence $p_t = p_0$ for all $t \leq \Delta$. For $t \geq \Delta$, the public belief follows the law of motion

$$\dot{p}_t = -p_t(1 - p_t)\lambda k_{t-\Delta}. \tag{1}$$

Therefore, for every admissible strategy profile $\mathbf{k}$, the public belief at time t is defined as follows:[1]

$$p_t = \frac{p_0}{p_0 + (1 - p_0)e^{\lambda \int_0^{(t-\Delta)\mathbb{1}_{t \geq \Delta}} k_s ds}}. \tag{2}$$

[1]This is obtained by integrating (1) between Δ and $t > \Delta$ and using the initial condition $p_\Delta = p_0$.

Player t will have the opportunity to invest only if no breakthrough occurred before time t, that is if no player in $[0,(t-\Delta)\mathbb{1}_{t\geq\Delta}]$ made a successful investment.[2] Therefore, player t competes only with players in $[(t-\Delta)\mathbb{1}_{t\geq\Delta},t)$. With k_{-t} standing for the investment profile of players in $[(t-\Delta)\mathbb{1}_{t\geq\Delta},t)$, the expected payoff to player t is

$$u(k_t;k_{-t}) = \underbrace{-\alpha k_t}_{\text{instantaneous cost}} + \underbrace{p_t e^{-\lambda\int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds}\lambda k_t}_{\text{expected benefit}} \tag{3}$$

Let us explain why with a heuristic argument. Suppose that player t invests k_t during the interval $[t,t+dt)$ with $dt>0$, in the sense that she is the only player to invest during the interval and chooses $k_{t'}=k_t$ for every $t'\in[t,t+dt)$. This costs her $\alpha k_t dt$ and yields a success with probability $p_t\lambda k_t dt$. This success will give her a payoff of 1 in $t+\Delta$ if and only if she has not been preempted by a player in $[(t-\Delta)\mathbb{1}_{t\geq\Delta},t)$, which happens with probability $e^{-\lambda\int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds}$.

3 Equilibrium analysis

Defining $\mu_t := p_t e^{-\lambda\int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds}$ and $\underline{p} := \frac{\alpha}{\lambda}$, the payoff of player t can be expressed as $u(k_t,k_{-t}) = \lambda k_t\left(\mu_t-\underline{p}\right)$. The best response of player t to k_{-t} is thus to invest if and only if her confidence in the technology, p_t, and the probability of not being preempted by a competitor, $e^{-\lambda\int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds}$, are large enough:

$$k_t \begin{cases} =1 & \text{if } \mu_t>\underline{p}, \\ \in[0,1] & \text{if } \mu_t=\underline{p}, \\ =0 & \text{if } \mu_t<\underline{p}. \end{cases} \tag{4}$$

To interpret this, imagine that player t faces no competition, i.e., $k_s=0$ for every $s\in[(t-\Delta))\mathbb{1}_{t\geq\Delta},t)$. In this situation, the attractiveness of investment μ_t is exactly p_t, thus player t invests whenever $p_t\geq\underline{p}$. Therefore, $\underline{p}$ is interpreted as the *single-player cutoff*. When $k_s>0$ for some of player t's predecessors, she invests if and only if her belief is larger than $\underline{p}e^{\lambda\int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds}$ which is strictly larger than $\underline{p}$: competition makes investment less attractive, thus the cutoff above which player t invests is larger than the single-player cutoff $\underline{p}$.

The behavior of μ_t is key to the construction of the equilibrium. Differentiating μ_t with respect to t, using (1) and simplifying, we obtain:

$$\dot{\mu}_t = -\mu_t\lambda(k_t-p_t k_{t-\Delta}\mathbb{1}_{t\geq\Delta}). \tag{5}$$

It follows directly from (5) that μ_t weakly decreases when $t\leq\Delta$ or $k_{t-\Delta}=0$. This implies that investment is continuously less and less attractive on the interval $[0,\Delta]$ and during periods with no competition.

[2] This occurs with probability 1 for players $t\leq\Delta$. For players $t>\Delta$, this occurs with probability 1 if the technology is bad, and with probability $e^{-\int_{\Delta}^{t}\lambda k_{s-\Delta}ds}$ if the technology is good, thus with probability $1-p_0+p_0e^{-\int_0^{t-\Delta}\lambda k_s ds}$, which reduces to $e^{-\lambda\int_{\Delta}^{t}p_s k_{s-\Delta}ds}$ by integrating (1).

Player 0 has no competitor and is the most optimistic of all players, thus she is the one with the most incentives to invest.[3] If $p_0 \leq \underline{p}$, even player 0 finds investment unattractive, thus it is dominant for all players to play $k_t = 0$. Therefore, if $p_0 \leq \underline{p}$, there in the unique equilibrium $k_t^* = 0$ and $\mu_t = p_0$ for all t.

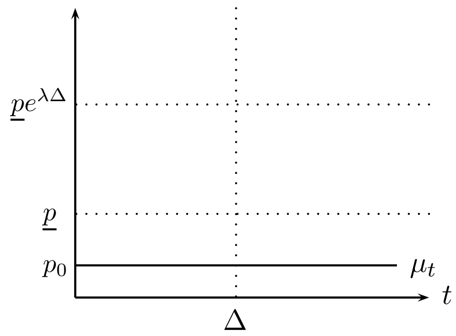

Figure 1: Dynamics of μ_t when $p_0 < \underline{p}$.

Let us now characterize the equilibrium profile when $p_0 > \underline{p}$. Since $\mu_0 = p_0 > \underline{p}$, player 0 invests all of her resource into the technology. As μ_t is weakly decreasing on $[0, \Delta]$, the immediate successors of player 0 also invest, up to some player $\tau > 0$ for which $\mu_\tau = \underline{p}$. Every player after player τ is indifferent about whether to invest, because $\mu_t = \underline{p}$ for every $t \geq \tau$.[4] Plugging $k_t = \mathbb{1}_{t \leq \tau}$ into (2), the attractiveness of investment becomes

$$\mu_t = \begin{cases} \dfrac{p_0 e^{-\lambda t}}{p_0 e^{-\lambda(t-\Delta)\mathbb{1}_{t\geq\Delta}} + 1 - p_0} & \text{if } t \leq \tau \\ \underline{p} & \text{if } t \geq \tau \end{cases}$$

By definition of τ, $\tau \leq \Delta$ if and only if $\mu_\Delta \geq \underline{p}$, that is if and only if $p_0 \geq \underline{p}e^{\lambda\Delta}$. The value of the cutoff τ thus depends on p_0 and is obtained by solving $\mu_\tau = \underline{p}$, which yields:

$$\tau(p_0) := \begin{cases} \Delta - \frac{1}{\lambda}\ln\left(\frac{\underline{p}e^{\lambda\Delta}}{p_0}\right) & \text{if } p_0 \in [\underline{p}, \underline{p}e^{\lambda\Delta}], \\ \Delta + \frac{1}{\lambda}\ln\left(\frac{\Omega(\underline{p}e^{\lambda\Delta})}{\Omega(p_0)}\right) & \text{if } p_0 \geq \underline{p}e^{\lambda\Delta}. \end{cases} \tag{6}$$

It is straightforward to establish that $\tau(p_0)$ decreases with Δ for every p_0.

Figure 3 represents the dynamics of μ_t when μ_t reaches $\underline{p}$ before Δ (i.e., when $p_0 \in [\underline{p}, \underline{p}e^{\lambda\Delta})$) and after Δ (i.e., when $p_0 \geq \underline{p}e^{\lambda\Delta}$).

[3] For every t, it holds that $\mu_t \leq p_t$ by definition. Moreover, $p_t \leq p_0$ because p_t is weakly decreasing.

[4] Let $\tau' = \inf\{t \geq \tau : \dot{\mu}_t \neq 0\}$. Suppose that $\tau' > \tau$. On $[\tau, \tau']$, $\mu_t = \underline{p}$, thus $\mu_{\tau'} = \underline{p}$. If $\dot{\mu}_{\tau'} < 0$ then $\mu_{\tau'_+} < \underline{p}$ which implies $k_{\tau'_+} = 0$. Plugging this into (5), it follows that $\dot{\mu}_{\tau'} = \mu_{\tau'}\lambda p_{\tau'} k_{\tau'-\Delta}$, which contradicts $\dot{\mu}_{\tau'} < 0$. If $\dot{\mu}_{\tau'} > 0$ then $\mu_{\tau'_+} > \underline{p}$ which implies $k_{\tau'_+} = 1$. Plugging this into (5), it follows that $\dot{\mu}_{\tau'} = -\mu_{\tau'}\lambda(1 - p_{\tau'}(k_{\tau'-\Delta})) \leq 0$, which contradicts $\dot{\mu}_{\tau'} > 0$.

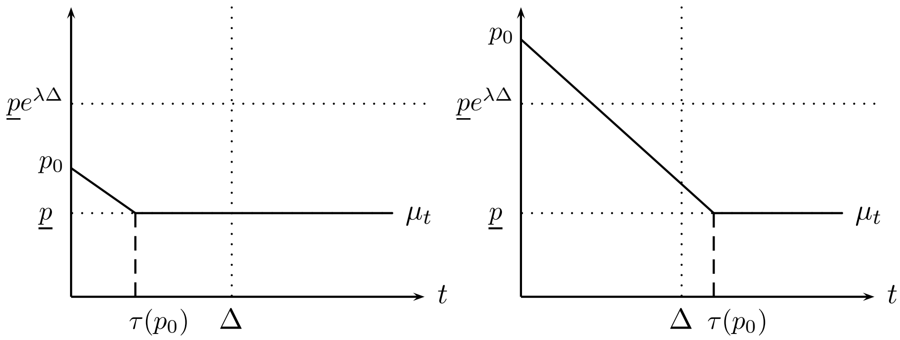

Figure 2: Dynamics of μ_t when $p_0 > \underline{p}$.

Players $t \leq \tau(p_0)$ play $k_t = 1$. Since $\mu_t = \underline{p}$ for every $t \geq \tau(p_0)$, players $t > \tau(p_0)$ are indifferent about whether to invest. Plugging the indifference condition $\dot{\mu}_t = 0$ into (5), we find that player t's best response at $t \geq \tau(p_0)$ depends on her predecessor' investment as follows:

$$k_t = p_t k_{t-\Delta} \mathbb{1}_{t \geq \Delta}. \tag{7}$$

Therefore, the best response of a player $t > \tau(p_0)$ whose predecessor at $t - \Delta$ either does not exist (when $t < \Delta$) or did not invest (when $k_{t-\Delta}) = 0$) is not to invest: $k_t = 0$. In contrast, a player t whose predecessor invested a positive fraction in the technology invests a strictly smaller, but strictly positive fraction of her own resource.

The equilibrium dynamics of investment thus qualitatively depends on p_0. When player 0 is very optimistic ($p_0 > \underline{p}e^{\lambda\Delta}$), the cutoff $\tau(p_0)$ is larger than Δ. Therefore, there is no period before Δ during which players do not invest, which implies that players always invest a positive and decreasing amount in the technology. When player 0 has more pessimistic beliefs ($p_0 \in [\underline{p}, \underline{p}e^{\lambda\Delta}]$), the cutoff $\tau(p_0)$ is smaller than Δ, which implies that players in $[\tau(p_0), \Delta]$ do not invest at all. This period without investment replicates identically Δ "periods" later, while investment periods replicate with a fraction of resource decreasing in time. This leads to a non-monotonic equilibrium strategy, in the sense that a less optimistic player may invest more in the technology than a more optimistic predecessor.

The following proposition describes the equilibrium dynamics of investment. For every $(n, t) \in \mathbb{N} \times \mathbb{R}_+$, let $\phi(n, t) := \dfrac{\Omega(\underline{p})}{\Omega(\underline{p}) + \Omega(\underline{p}^n)\Omega(p_0)e^{\lambda(t-n\Delta)}}$, with $\Omega(p) := \frac{1-p}{p}$.

Proposition 1. *There is a unique Nash equilibrium $(k_t^*)_t$ described as follows:*

- *if $p_0 \leq \underline{p}$, $k_t^* = 0$ for every t;*
- *if $p_0 \in (\underline{p}, \underline{p}e^{\lambda\Delta}]$, $k_t^* = 1$ if $t \in [0, \tau(p_0))$, and, $\forall\, n \in \mathbb{N}$,*

$$k_t^* = \begin{cases} 0 & \text{if } t \in [\tau(p_0) + n\Delta, (n+1)\Delta), \\ \phi(n, t) & \text{if } t \in [(n+1)\Delta, \tau(p_0) + (n+1)\Delta). \end{cases}$$

- *if* $p_0 > \underline{p}e^{\lambda\Delta}$, $k_t^* = 1$ *if* $t \in [0, \tau(p_0))$ *and,* $\forall\, n \in \mathbb{N}_+$,

$$k_t^* = \phi(n,t) \text{ if } t \in [\tau(p_0) + (n-1)\Delta, \tau(p_0) + n\Delta).$$

Figures (3) and (3) describe the equilibrium investment dynamics when $p_0 \in [\underline{p}, \underline{p}e^{\lambda\Delta})$ and $p_0 \geq \underline{p}e^{\lambda\Delta}$, respectively.

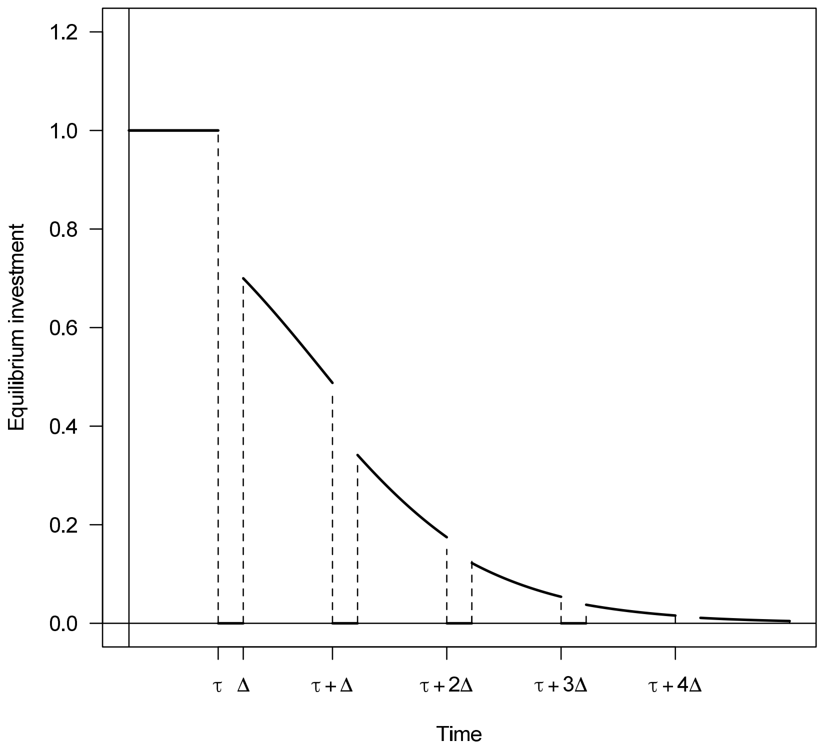

Figure 3: Equilibrium investment dynamics when $\alpha = 0.2$, $\lambda = 0.7$, $\Delta = 1.64$ and $p_0 = 0.7 < \underline{p}e^{\lambda\Delta} = 0.9$.

The attractiveness of investment increases with the confidence in the technology and decreases with the level of competition. Players cannot become more optimistic with time, whatever their strategies or history may be. However, the intensity of competition may not be monotone in time, which explains why investment is non-monotone in the public belief. Consider a period during which predecessors invested in the technology. As time passes within this period, the total amount of past investments increases, which increases the intensity of competition and the probability of preemption. Since players also become

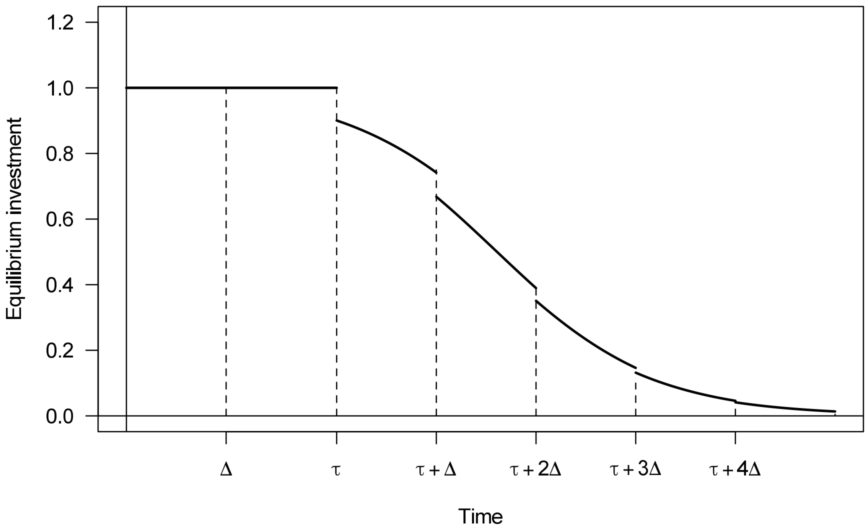

Figure 4: Equilibrium investment dynamics when $\alpha = 0.2$, $\lambda = 0.7$, $\Delta = 1.64$ and $p_0 = 0.97 > \underline{p}e^{\lambda\Delta} = 0.9$.

more pessimistic about the technology, they eventually stop investment. But then the probability of being the first to get a success increases again and the discouraging effects of competition fades away. When the intensity of competition is sufficiently low, investing in the risky technology becomes profitable again, even though the public belief is smaller.

3.1 Comparative statics

It follows directly from the expression of $\phi(n,t)$ that k_t^* converges to 0. This is the reason why, while players do not stop investment in finite time, the public belief converges to the single player cutoff but never falls below it. As a result, the total amount of investment only depends on $\underline{p}$ and p_0, not on Δ.

Lemma 1. *For every $p_0 \in [\underline{p}, 1)$, in equilibrium it holds that:*

(*i*) $\lim\limits_{t\to+\infty} p_t = \underline{p}$;

(*ii*) $\displaystyle\int_0^\infty k_s^* ds = \frac{1}{\lambda} \ln\left(\frac{\Omega(\underline{p})}{\Omega(p_0)}\right)$.

Proof. Since $\lim\limits_{t\to+\infty} \Phi(n,t) = 0$, it holds that $\lim\limits_{t\to+\infty} k_t^* = 0$ and $\lim\limits_{t\to+\infty} e^{-\lambda\int_{t-\Delta}^t k_s^* ds} = 0$. This implies that $\lim\limits_{t\to+\infty} \mu_t = \lim\limits_{t\to+\infty} p_t$. Yet $\lim\limits_{t\to+\infty} \mu_t = \underline{p}$. This proves (*i*). Expression (2) can be rewritten as $e^{\lambda\int_0^{t-\Delta} k_s ds} = \frac{\Omega(p_t)}{\Omega(p_0)}$ for $t \geq \Delta$. Taking the limit and using (*i*), we obtain (*ii*). □

The equilibrium payoff to player t is $u(k_t^*, k_{-t}^*) = \lambda k_t^*(\mu_t - \underline{p})$. Therefore,

- if $p_0 \leq \underline{p}$, then $u(k_t^*, k_{-t}^*) = 0$ for every t;

- if $p_0 \in \left[\underline{p}, \underline{p}e^{\lambda\Delta}\right]$, then $u(k_t^*, k_{-t}^*) = \begin{cases} \alpha\left(\frac{p_0}{\underline{p}}e^{-\lambda t} - 1\right) & \text{if } t \leq \tau(p_0), \\ 0 & \text{if } t \geq \tau(p_0). \end{cases}$

- if $p_0 \geq \underline{p}e^{\lambda\Delta}$, then $u(k_t^*, k_{-t}^*) = \begin{cases} \alpha\left(\frac{p_0}{\underline{p}}e^{-\lambda t} - 1\right) & \text{if } t \leq \Delta, \\ \alpha\left(\frac{1}{\underline{p}}\frac{e^{-\lambda\Delta}}{1+\Omega(p_0)e^{\lambda(t-\Delta)}} - 1\right) & \text{if } t \in [\Delta, \tau(p_0)], \\ 0 & \text{if } t \geq \tau(p_0). \end{cases}$

It is easy to see that early investors have an advantage, in the sense that $u(k_t^*, k_{-t}^*)$ decreases with t. The total payoff is $W(k^*) = \int_0^{+\infty} u(k_t^*, k_{-t}^*)dt$, which works out as follows;

- If $p_0 \leq \underline{p}$, then $W(k^*) = 0$.

- If $p_0 \in \left[\underline{p}, \underline{p}e^{\lambda\Delta}\right]$, then $W(k^*) = p_0 - \underline{p} + \underline{p}\ln\left(\frac{\underline{p}}{p_0}\right)$.

- if $p_0 \geq \underline{p}e^{\lambda\Delta}$, then $W(k^*) = p_0(1 - e^{-\lambda\Delta}) - e^{-\lambda\Delta}\ln\left(\frac{1-p_0}{1-\underline{p}e^{\lambda\Delta}}\right) - \alpha\tau(p_0)$.

Proposition 2. *The total payoff in equilibrium weakly decreases with* Δ.

Proof. See Section 5.1.2 in the Appendix. □

3.2 Disentangling the effects of outcome lag and uncertainty

In order to disentangle the respective effects of the innovation lag and uncertainty on the shape of the equilibrium dynamics, we apply Proposition 1 in the case where $p_0 = 1$ (no uncertainty) and $\Delta = 0$ (no outcome lag).

Plugging $p_0 = 1$ into the expressions of ϕ and τ, we find that $\phi(n,t) = 1$ for every (n,t) and $\tau(1) = \begin{cases} -\frac{1}{\lambda}\ln(\underline{p}) & \text{if } \underline{p}e^{\lambda\Delta} > 1, \\ +\infty & \text{if } \underline{p}e^{\lambda\Delta} \leq 1 \end{cases}$. Therefore,

- if $\underline{p}e^{\lambda\Delta} \leq 1$, then $k_t^* = 1$ for every t;

- if $\underline{p}e^{\lambda\Delta} > 1$, then $k_t^* = \begin{cases} 1 & \text{if } t \in [n\Delta, (n+1)\tau(1)] \\ 0 & \text{if } t \in [\tau(1)(n+1), (n+1)\Delta] \end{cases}$

In other words, if the outcome lag is small enough, every player fully invests in the technology. Otherwise, the equilibrium dynamics is cyclical: players regularly alternate between full investment and no investment at all. In contrast, when $\Delta = 0$, the equilibrium dynamics is in threshold:

- if $p_0 \leq \underline{p}$, $k_t^* = 0$ for every t;

- if $p_0 > \underline{p}$, $k_t^* = \begin{cases} 1 & \text{if } t \leq \tau(p_0) \\ 0, & \text{if } t > \tau(p_0), \end{cases}$ where $\tau(p_0) = \frac{1}{\lambda}\ln\left(\frac{\Omega(\underline{p})}{\Omega(p_0)}\right)$ is the first time the public belief reaches $\underline{p}$.

Players fully invest when the public belief is larger than the single player cutoff $\underline{p}$, and 0 otherwise.

Even if players know that the project is feasible, the outcome lag creates competition between player t and every player in $[t-\Delta, t)$: the larger Δ, the more intense the competition. If Δ is small, the competition is so soft that every player fully invests. When competition is harder, players are too likely to be preempted by a past competitor and decide not to invest as a result. After a period with no investment, competition has softened enough to make investment valuable again. The non-monotonic feature of the equilibrium dynamics is thus driven by the outcome lag. Without an outcome lag, players use threshold strategies, but the equilibrium investment decreases with time, because players become more pessimistic about the feasibility of the project as time passes and no breakthrough occurs. Thus uncertainty drives the fact that, in periods where players invest in the technology, equilibrium investment is interior and decreasing with time.

4 Welfare criteria

4.1 Probability of breakthrough

The probability of breakthrough in equilibrium is $p_0(1 - e^{-\lambda \int_0^{+\infty} k_t^* dt})$. As the total investment is $\int_0^{+\infty} k_t^* dt = \frac{1}{\lambda} \ln \left(\frac{\Omega(\underline{p})}{\Omega(p_0)} \right)$ when $p_0 \in [\underline{p}, 1)$, the probability of breakthrough is

$$P(\text{Breakthrough}) = \frac{p_0 - \underline{p}}{1 - \underline{p}}$$

As with the total amount of investment, the probability of a breakthrough does not depend on the delay in equilibrium.

4.2 Expected time of completion

A social planner may want the breakthrough to occur as fast as possible, hence to maximize $E[e^{-rT}]$, where T is the random time of arrival of the breakthrough.

As $P(T \leq t) = 1 - e^{-\lambda \int_0^{t-\Delta} k_s ds}$ if the technology is good and $t \geq \Delta$, and $P(\tau \leq T) = 0$ otherwise, $P(T \leq t) = p_0 \left(1 - e^{-\lambda \int_0^{t-\Delta} k_s ds}\right) \mathbb{1}_{t \geq \Delta}$. Therefore, T is distributed according to the density function $f(t) = p_0 \lambda k_{t-\Delta} e^{-\lambda \int_0^{t-\Delta} k_s ds} \mathbb{1}_{t \geq \Delta}$.

$$E[e^{-rT}] = p_0 \int_{\Delta}^{\infty} e^{-rt} \lambda k_{t-\Delta} e^{-\lambda \int_0^{t-\Delta} k_s ds} dt$$

It is easy to prove that the optimal policy for this benchmark is to play $k_t = 1$ for every t, thus that the equilibrium is suboptimal. Moreover, in equilibrium the expected completion time increases with the delay.

4.3 Aggregate payoff

The social planner may want to maximize the sum of the players' payoffs, i.e. to determine the investment profile $\mathbf{k}$ that maximizes

$$W(\mathbf{k}) = \int_0^{+\infty} -\alpha k_t(1-p_0) + p_0 e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds} \left(\lambda k_t e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds} - \alpha k_t\right) dt$$

$$= \int_0^{+\infty} k_t \left(-\alpha(1-p_0) + p_0 e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds} \left(\lambda e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds} - \alpha\right)\right) dt \tag{8}$$

The lag in the state variable makes the problem of the social planner difficult to solve. However, we can prove that the equilibrium is inefficient, in the sense that it does not maximizes the aggregate payoff. This is because the social planner would prefer players to <
"take a break" to see if previous experimentation has yielded a success, something players are not willing to do in equilibrium.

Proposition 3. *The equilibrium k^* is inefficient.*

Proof. To prove the proposition we first note that there exists a cutoff strategy $\tilde{k}$ such that $W(\tilde{k}) = W(k^*)$, namely $\tilde{k}_t = \mathbb{1}_{t\leq\tau(p_0)}$. We now show that, for any given cutoff strategy, there exists another strategy that yields a strictly larger aggregate payoff.

As a preliminary, observe that, for every strategy profile $\mathbf{k}$ such that $\int_0^{+\infty} k_t dt < +\infty$, the aggregate payoff can be rewritten as

$$W(\mathbf{k}) = -\underline{p}(1-p_0)\lambda \int_0^{+\infty} k_t dt + p_0(1 - e^{-\lambda \int_0^{+\infty} k_t dt}) - \underline{p} p_0 \lambda \int_0^{+\infty} k_t e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds} dt.$$

Let us now prove that the optimal strategy is not in cutoff. Consider a cutoff strategy k defined by $k_t = \mathbb{1}_{t\leq\tau}$ and let us prove that there exists $\tilde{k}$ such that $W(\tilde{k}) > W(k)$.

If $\tau < \Delta$, let $\tilde{k}(\epsilon)$ be defined by $\tilde{k}(\epsilon)_t = \mathbb{1}_{t\leq\tau-\epsilon} + \mathbb{1}_{t\in[\Delta,\Delta+\epsilon]}$ with $0 < \epsilon < \tau - \epsilon$. As $\int_0^{+\infty} \tilde{k}_t dt = \tau$, one has

$$W(\tilde{k}(\epsilon)) = -\underline{p}(1-p_0)\lambda\tau + p_0(1-e^{-\lambda\tau}) - \underline{p}p_0\lambda\left(\tau - \epsilon + \int_\Delta^{\Delta+\epsilon} e^{-\lambda(t-\Delta)} dt\right)$$

Differentiating $W(\tilde{k}(\epsilon))$ with respect to ϵ, we obtain $\partial W(\tilde{k}(\epsilon))/\partial\epsilon = p_0\underline{p}\lambda(1-e^{-\lambda\epsilon}) > 0$. Moreover, $W(\tilde{k}(0)) = W(k)$. Therefore, there exists $\tilde{k}(\epsilon)$ such that $W(\tilde{k}(\epsilon)) > W(k)$.

If $\tau > \Delta$, let $\tilde{k}(\epsilon)$ be defined by $\tilde{k}(\epsilon)_t = \mathbb{1}_{t\leq\tau-\epsilon} + \mathbb{1}_{t\in[\tau,\tau+\epsilon]}$ with $\Delta < \tau - \epsilon$. As $\int_0^{+\infty} \tilde{k}_t dt = \tau$, one has

$$W(\tilde{k}(\epsilon)) = -\underline{p}(1-p_0)\lambda\tau + p_0(1-e^{-\lambda\tau}) - \underline{p}p_0\lambda\left(\int_0^{\tau-\epsilon} e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds} + \int_\tau^{\tau+\epsilon} e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds} dt\right).$$

Differentiating $W(\tilde{k}(\epsilon))$ with respect to ϵ, we obtain

$$\partial W(\tilde{k}(\epsilon))/\partial\epsilon = p_0\underline{p}\lambda\left(e^{-\lambda \int_0^{\tau-\epsilon-\Delta} k_s\, ds} - e^{-\lambda \int_0^{\tau+\epsilon-\Delta} k_s\, ds}\right).$$

This is strictly greater than 0 if and only if $\int_{\tau-\epsilon-\Delta}^{\tau+\epsilon-\Delta} k_s\, ds > 0$, which is satisfied for any $\Delta > 0$. Moreover, $W(\tilde{k}(0)) = W(k)$. Therefore, there exists $\tilde{k}(\epsilon)$ such that $W(\tilde{k}(\epsilon)) > W(k)$.

□

5 An alternative reward scheme.

In this section, we compare the equilibrium we characterized in the previous section with the equilibrium obtained under another reward scheme called *hidden-equal-sharing* (**HES**). Under **HES**, players do not observe past outcomes and do not communicate with each other; instead, there is a mediator who observes all the breakthroughs and keeps them secret until some commonly known deadline $T > \Delta$. At time T, the mediator splits the payoff 1 among all the players who obtained a breakthrough, if any. The difference with the winner-takes-all reward scheme is that a player does not know what payoff she will receive from obtaining a success. Let r_t stand for the random reward of player t in case of success. As successes are hidden, the public belief is p_0 at every time, thus the expected payoff to player t given a strategy profile $\mathbf{k}$ is

$$u(k_t, \mathbf{k}) = p_0 \lambda k_t E[r_t \mid \mathbf{k}] - \alpha k_t$$

The best response of player t is as follows:

$$k_t \begin{cases} = 1 & \text{if } p_0 E[r_t \mid \mathbf{k}] > \underline{p}, \\ \in [0,1] & \text{if } p_0 E[r_t \mid \mathbf{k}] = \underline{p}, \\ = 0 & \text{if } p_0 E[r_t \mid \mathbf{k}] < \underline{p} \end{cases}$$

Let us compute $E[r_t \mid \mathbf{k}]$. Since players $t \geq T - \Delta$ will observe the output of their investment after the deadline T, they will not receive any payoff, hence $r_t = 0$ for $t \geq T - \Delta$. Now consider $t < T - \Delta$. If $n \geq 0$ other players among $[0, T - \Delta]$ obtain a success, player t will receive a payoff of $1/(n+1)$. Her expected reward is thus

$$E[r_t \mid \mathbf{k}] = \sum_{n=0}^{\infty} \frac{1}{n+1} P(n \text{ other players obtain a success})$$

Yet the number of success occurrences during the interval $[0, T - \Delta]$ follows a Poisson law of intensity $\lambda \int_0^{T-\Delta} k_s d$. Therefore,

$$\begin{aligned} E[r_t \mid \mathbf{k}] &= \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{\left(\lambda \int_0^{T-\Delta} k_s ds\right)^n}{n!} e^{-\lambda \int_0^{T-\Delta} k_s ds} \\ &= \frac{1 - e^{-\lambda \int_0^{T-\Delta} k_s ds}}{\lambda \int_0^{T-\Delta} k_s ds} := \Gamma(\mathbf{k}) \end{aligned}$$

Proposition 4. *Let T^H be defined by* $\dfrac{1-e^{-\lambda(T_H-\Delta)}}{\lambda(T_H-\Delta)} = \dfrac{\underline{p}}{p_0}$.

(*i*) *If $T \leq T^H$, there is an (essentially) unique equilibrium $\mathbf{k}^*$ such that $k_t^* = 1$ for $t \leq T-\Delta$ and $k_t^* = 0$ if $t > T-\Delta$. It is unique if $T < T^H$.*

(*ii*) *If $T > T^H$ then in every equilibrium $\mathbf{k}^*$, it holds that $\int_{t=0}^{T-\Delta} k_s^* ds = T^H - \Delta$ and $k_t^* = 0$ if $t > T-\Delta$.*

Proof. As a preliminary, observe that the function $x \mapsto \frac{1-e^{-\lambda x}}{\lambda x}$ is strictly decreasing.

- Proof of (*i*). Suppose that $T \leq T^H$ and consider the strategy profile $\tilde{k}$ defined by $\tilde{k}_t = \mathbb{1}_{t \leq T-\Delta}$. Let us prove that no player has a profitable deviation from $\tilde{k}$. We already know that players $t > T-\Delta$ have no profitable deviation from $k_t = 0$. Fix some player $t \leq T-\Delta$. Straightforwardly, $\Gamma(\tilde{k}) = \frac{1-e^{-\lambda(T-\Delta)}}{\lambda(T-\Delta)}$. As $\Gamma(\tilde{k}))$ decreases with T, the fact that $T \leq T^H$ implies that $\Gamma(\tilde{k}) \geq \frac{1-e^{-\lambda(T^H-\Delta)}}{\lambda(T^H-\Delta)} = \underline{p}/p_0$. Playing $k_t = 1$ is thus a best response to $\tilde{k}$. This proves that $\tilde{k}$ is an equilibrium. Let us now prove that it is the unique equilibrium. This is immediate if $T < T^H$. Now we shall prove ad absurdum that it remains essentially unique (i.e., but for deviations on a null set) if $T = T^H$. Suppose that $\mathbf{k}$ is an equilibrium such that $k_t < 1$ for some subset of $[0, T-\Delta]$ that has positive measure. It follows that $\int_0^{T-\Delta} k_s ds < T-\Delta$, hence that $\Gamma(\mathbf{k}) > \Gamma(\tilde{k})$. As $\Gamma(\tilde{k}) \geq \underline{p}/p_0$, this implies that $\Gamma(\mathbf{k}) > \underline{p}/p_0$, and that player t's best response is to play $k_t = 1$, which is a contradiction.

- Proof of (*ii*). Suppose now that $T > T^H$ and consider an equilibrium strategy $\tilde{k}$. If $\int_{t=0}^{T-\Delta} \tilde{k}_s ds < T^H - \Delta$, then $\Gamma(\tilde{k}) > \frac{1-e^{-\lambda(T_H-\Delta)}}{\lambda(T_H-\Delta)} = \frac{\underline{p}}{p_0}$, which implies that there is at least a player $t < T-\Delta$ such that $\tilde{k}_t < 0$ who has a profitable deviation to $k_t = 1$. Therefore, $\int_{t=0}^{T-\Delta} \tilde{k}_s ds \geq T^H - \Delta$. If $\int_{t=0}^{T-\Delta} \tilde{k}_s ds > T^H - \Delta$, then $\Gamma(\tilde{k}) < \frac{1-e^{-\lambda(T_H-\Delta)}}{\lambda(T_H-\Delta)} = \frac{\underline{p}}{p_0}$, hence there is at least a player $t < T-\Delta$ such that $\tilde{k}_t > 0$ who has a profitable deviation to $k_t = 0$. This proves that $\int_{t=0}^{T-\Delta} \tilde{k}_s ds = T^H - \Delta$.

□

What is the best **HES** mechanism? If the mediator seeks to maximize the probability of breakthrough, then she should design T so as to maximize the total amount of investment. It follows from Proposition 4 that, in every equilibrium under **HES**,

$$\int_0^\infty k_t^* dt = \begin{cases} T-\Delta & \text{if } T \leq T^H, \\ T^H - \Delta & \text{if } T > T^H, \end{cases} = \min\{T, T^H\} - \Delta.$$

Clearly, the total amount of investment is maximum when the mediator sets the deadline T after T^H. However, even if all deadlines $T \geq T^H$ entail the same probability of breakthrough, deadlines $T > T^H$ allow for equilibria where investment is dispersed over time, which increases the expected time of completion.

Proposition 5. *The probability of breakthrough is maximal when $T \geq T^H$. The expected time of completion is minimal when $T = T^H$.*

We now compare the welfare properties of the optimal **HES** with the winner-takes-all reward scheme. The total investment in equilibrium is $T^H - \Delta$ with **HES** and $\frac{1}{\lambda}\ln\left(\frac{\Omega(\underline{p})}{\Omega(p_0)}\right)$ with the winner-takes-all mechanism by Lemma 1. As $x \mapsto \frac{1-e^{-\lambda x}}{\lambda x}$ is strictly decreasing,

$$T^H - \Delta > \frac{1}{\lambda}\ln\left(\frac{\Omega(\underline{p})}{\Omega(p_0)}\right) \Leftrightarrow \frac{1-e^{-\lambda(T^H-\Delta)}}{\lambda(T^H-\Delta)} < \frac{1-\Omega(p_0)/\Omega(\underline{p})}{\ln(\Omega(\underline{p}))-\ln(\Omega(p_0))} \Leftrightarrow \frac{\underline{p}}{p_0} < \frac{1-\Omega(p_0)/\Omega(\underline{p})}{\ln(\Omega(\underline{p}))-\ln(\Omega(p_0))}$$

This allows to prove the following result.

Proposition 6. *It holds that (i) $\Leftrightarrow$ (ii).*

(i) $\underline{p} \leq 1/2$ *and* $p_0 \leq \tilde{p_0}$*, where* $\tilde{p_0}$ *is the unique solution of* $\Omega(\underline{p}) - \Omega(p_0) = \ln\left(\frac{\Omega(\underline{p})}{\Omega(p_0)}\right)\frac{1-\underline{p}}{p_0}$.

(ii) The probability of breakthrough is larger under the ***HES*** *mechanism than the winner-takes-all mechanism.*

Moreover, (i) implies that the expected time of completion is smaller under ***HES****.*

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Appendix

5.1 Proof of Proposition 1

Proof. Differentiating $\mu_t = p_t e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds}$ with respect to t, we obtain

$$\dot{\mu}_t = e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds} \left(\dot{p}_t - p_t \lambda (k_t - k_{t-\Delta}\mathbb{1}_{t\geq\Delta})\right),$$

which reduces to

$$\dot{\mu}_t = -\lambda \mu_t (k_t - p_t k_{t-\Delta}\mathbb{1}_{t\geq\Delta}) \tag{9}$$

after using (1).

- **Case when** $p_0 \leq \underline{p}$

The objective is to prove by induction on n that the proposition $\mathcal{P}(n)$: "$k_t^* = 0 \ \forall\ t \in [n\Delta, (n+1)\Delta]$" is true for every $n \in \mathbb{N}$. By definition of μ_t, for every $t \in [0, \Delta]$ and every k_{-t}, $\mu_t \leq p_t = p_0 \leq \underline{p}$. Therefore, it is dominant for every player in $[0, \Delta]$ to play $k_t = 0$, which implies that $\mathcal{P}(0)$ is true. Suppose now that $\mathcal{P}(n)$ holds for some $n \in \mathbb{N}$, i.e., that every player $t \in [n\Delta, (n+1)\Delta]$ plays $k_t^* = 0$. Plugging this into (9), it comes that $\dot{\mu}_t \leq 0$ for every $t \in [(n+1)\Delta, (n+2)\Delta]$. Moreover, $k_{(n+1)\Delta} = 0$ implies that $\mu_{(n+1)\Delta} < \underline{p}$ by (4). Therefore, $\mu_t < \underline{p}$ for every $t \in [(n+1)\Delta, (n+2)\Delta]$, which implies that $\mathcal{P}(n+1)$ is true.

- **Case when** $p_0 > \underline{p}$

As $\mu_0 = p_0$, $\mu_0 > \underline{p}$. Therefore, there exists $\tau > 0$ such that $\mu_s > \underline{p}$ for every $s \leq \tau$, hence such that $k_s = 1$ for every $s \leq \tau$ by (4). Plugging this into the expression of μ_t and into (1), it comes that

$$\mu_\tau = p_\tau e^{-\lambda(\tau - (\tau-\Delta)\mathbb{1}_{\tau\geq\Delta})}$$

and $\dot{p}_\tau = -\lambda p_\tau (1-p_\tau)\mathbb{1}_{\tau\geq\Delta}$.

Straightforwardly, the function $h(\tau) := p_\tau e^{-\lambda(\tau-(\tau-\Delta)\mathbb{1}_{\tau\geq\Delta})}$ is continuous and strictly decreasing in τ. Moreover, h takes the value $p_0 > \underline{p}$ in $\tau = 0$ and the value $\underline{p}e^{-\lambda\Delta} < \underline{p}$ in $\tau = \Delta + \frac{1}{\lambda}\ln\left(\frac{\Omega(\underline{p})}{\Omega(p_0)}\right)$. This implies that there exists a unique $\bar{\tau}$ such that $h(\bar{\tau}) = \underline{p}$, whose expression depends on whether it is larger than Δ or not.

If $h(\Delta) > \underline{p}$, then $\bar{\tau} > \Delta$, hence

$$\begin{aligned} h(\bar{\tau}) = \underline{p} \quad &\Leftrightarrow p_{\bar{\tau}} e^{-\lambda\Delta} = \underline{p} \\ &\Leftrightarrow p_{\bar{\tau}} = \underline{p}e^{\lambda\Delta} := \tilde{p} \end{aligned}$$

Integrating the law of motion of p_t $\dot{p}_\tau = -\lambda p_\tau(1-p_\tau)$ between Δ and $\bar{\tau}$, we obtain $e^{\lambda(\bar{\tau}-\Delta)} = \frac{\Omega(\tilde{p})}{\Omega(p_0)}$, which is rewritten

$$\bar{\tau} = \Delta + \frac{1}{\lambda}\ln\left(\frac{\Omega(\tilde{p})}{\Omega(p_0)}\right).$$

If $h(\Delta) < \underline{p}$, then $\bar{\tau} < \Delta$, hence

$$\begin{aligned} h(\bar{\tau}) = \underline{p} \quad &\Leftrightarrow p_0 e^{-\lambda\bar{\tau}} = \underline{p} \\ &\Leftrightarrow e^{\lambda(\Delta-\bar{\tau})} = \frac{\tilde{p}}{p_0} \end{aligned}$$

which is rewritten

$$\bar{\tau} = \Delta - \frac{1}{\lambda}\ln\left(\frac{\tilde{p}}{p_0}\right).$$

Therefore, there exists $\tau(p_0)$ such that $k_t^* = 1$ for every $t < \tau(p_0)$ and $\mu_{\tau(p_0)} = \underline{p}$. We now use the next lemma.

Lemma 2. *In equilibrium, if $\mu_t = \underline{p}$, then $\mu_s = \underline{p}$ for every $s \geq t$.*

Proof. Fix some player t and suppose that $\mu_t = \underline{p}$. As μ_t is continuous, if there is $t' > t$ such that $\mu_{t'} > \underline{p}$, then there is an open interval $S \subset [t, t']$ such that $\mu_s' > 0$ and $\mu_s > \underline{p}$ for every $s \in S$. This implies that that $k_s = 1$ for every $s \in S$ by (4), thus contradicts $\mu_s' > 0$ by (9). Also, if there is $t' > t$ such that $\mu_{t'} < \underline{p}$, then there is an open interval $S \subset [t, t']$ such that $\mu_s' < 0$ and $\mu_s < \frac{\alpha}{\lambda}$ for every $s \in S$. This implies that that $k_s = 0$ for every $s \in S$ by (4), thus contradicts $\mu_s' < 0$ by (9). $\square$

As a consequence, in equilibrium $\mu_t = \underline{p}$ for every $t \geq \tau(p_0)$, hence $\dot{\mu}_t = 0$ for every $t \geq \tau(p_0)$. By (9), this implies that in equilibrium, satisfies $k_t^* = 1$ if $t < \tau(p_0)$, and $k_t^* = p_t k_{t-\Delta}^* \mathbb{1}_\Delta$ if $t \geq \tau(p_0)$.

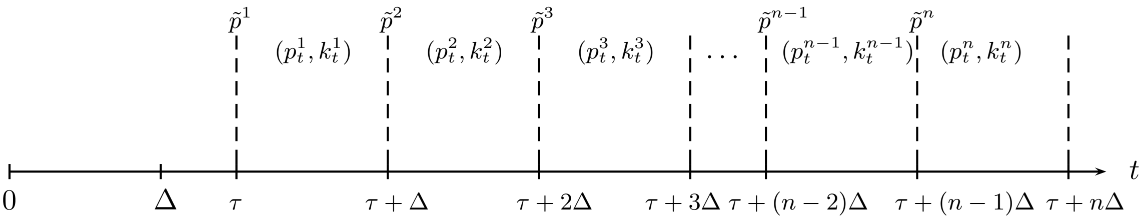
Uniqueness Note that the best response of player t is a function of $\{k_t\}_{s=\mathbb{1}_{t>\Delta}t-\Delta}^t$. Because k_0 does not depend on the other players' strategy, the equilibrium is unique. $\square$

5.1.1 Analytic expressions of k^*

Analytic expression of k^* and p when $p_0 \geq \tilde{p}$.

Let p_t^n and k_t^n respectively denote the public belief and player t's action in time $t \in [\tau + (n-1)\Delta, \tau + n\Delta]$.

Also, $\tilde{p}^n := p^n_{\tau+(n-1)\Delta} = p^{n-1}_{\tau+(n-1)\Delta}$. By definition of τ, $\tilde{p}^1 = \underline{p}e^{\lambda\Delta}$.



Step 1 The first step is to establish by induction on n that, for every $n \geq 2$,

$$\frac{\Omega(p_t^n)}{\Omega(\tilde{p}^n)} = \frac{1}{1-(1-e^{-\lambda(t-(n-1)\Delta-\tau)})\prod_{k=1}^{n-1}\tilde{p}^k} \tag{10}$$

1) **Relation between p_t^n and $p_{t-\Delta}^{n-1}$**

Let $n \geq 2$. By definition, $\dot{p}_t^n = -\lambda p_t^n(1-p_t^n)k_{t-\Delta}^{n-1}$ and $\dot{p}_t^{n-1} = -\lambda p_t^{n-1}(1-p_t^{n-1})k_{t-\Delta}^{n-2}$. Moreover, in equilibrium, $k_{t-\Delta}^{n-1} = p_{t-\Delta}^{n-1}k_{t-2\Delta}^{n-2}$. Therefore, $k_{t-\Delta}^{n-1} = -\frac{\dot{p}_{t-\Delta}^{n-1}}{\lambda(1-p_{t-\Delta}^{n-1})}$. It follows that, for every $n \geq 2$ and every t,

$$\frac{\dot{p}_t^n}{p_t^n(1-p_t^n)} = \frac{\dot{p}_{t-\Delta}^{n-1}}{1-p_{t-\Delta}^{n-1}} \tag{11}$$

Integrating (11) between $\tau + (n-1)\Delta$ and $t \in [\tau + (n-1)\Delta, \tau + n\Delta]$, we obtain:

$$\ln\left(\frac{\Omega(p_t^n)}{\Omega(p^n_{\tau+(n-1)\Delta})}\right) = \ln\left(\frac{1-p_{t-\Delta}^{n-1}}{1-p^{n-1}_{\tau+(n-2)\Delta}}\right),$$

which, by definition of $\tilde{p}^n$, becomes

$$\frac{\Omega(p_t^n)}{\Omega(\tilde{p}^n)} = \frac{1-p_{t-\Delta}^{n-1}}{1-\tilde{p}^{n-1}} \tag{12}$$

2) Let us prove that (10) is true for $n = 2$. As $k_t = 1$ for every $t \leq \tau$, $\Omega(p_t^1) = \Omega(\tilde{p}^1)e^{\lambda(t-\tau)}$, i.e.,

$$1 - p_1^t = \frac{\Omega(\tilde{p}^1)}{\Omega(\tilde{p}^1) + e^{-\lambda(t-\tau)}} = (1-\tilde{p}^1)\frac{1}{1-\tilde{p}^1+\tilde{p}^1e^{-\lambda(t-\tau)}}$$

By (12), $\frac{\Omega(p_t^2)}{\Omega(\tilde{p}^2)} = \frac{1-p_{t-\Delta}^1}{1-\tilde{p}^1}$, hence

$$\frac{\Omega(p_t^2)}{\Omega(\tilde{p}^2)} = \frac{1}{1-(1-e^{-\lambda(t-\Delta-\tau)})\tilde{p}^1}.$$

3) Suppose that (10) is true for some given n, i.e.,

$$1-p_t^n = \frac{\Omega(\tilde{p}^n)}{\Omega(\tilde{p}^n)+1-(1-e^{-\lambda(t-(n-1)\Delta-\tau)})\prod_{k=1}^{n-1}\tilde{p}^k},$$

and let us prove it is true for $n+1$. By (12), $\dfrac{\Omega(p_t^{n+1})}{\Omega(\tilde{p}^{n+1})} = \dfrac{1-p_{t-\Delta}^n}{1-\tilde{p}^n}$. As

$$1-p_{t-\Delta}^n = \frac{\Omega(\tilde{p}^n)}{\Omega(\tilde{p}^n)+1-(1-e^{-\lambda(t-n\Delta-\tau)})\prod_{k=1}^{n-1}\tilde{p}^k},$$

$$\frac{1-p_{t-\Delta}^n}{1-\tilde{p}^n} = \frac{1}{1-(1-e^{-\lambda(t-n\Delta-\tau)})\prod_{k=1}^{n}\tilde{p}^k)},$$

hence (10) is true for $n+1$.

Step 2 The second step is to establish by induction on n that, for every $n \geq 1$,

$$\Omega(\tilde{p}^n) = \frac{(1-\tilde{p}^1)(1-\underline{p})}{\underline{p}^{n-1}(\tilde{p}^1-\underline{p})+\underline{p}(1-\tilde{p}^1)}. \tag{13}$$

This is straightforward for $n=1$; Moreover, $\Omega(\tilde{p}^2) = \Omega(p_{\tau+\Delta}^1)$ and, as $k_t = 1$ for $t \leq \tau$, $\Omega(p_{\tau+\Delta}^1) = \Omega(\tilde{p}^1)e^{\lambda(\tau+\Delta-\tau)}$. Therefore, $\Omega(\tilde{p}^2) = \Omega(\tilde{p}^1)e^{\lambda\Delta}$. As we obtain $\Omega(\tilde{p}^2) = \frac{1-\underline{p}}{\tilde{p}^1}$ by plugging $n=2$ into (13), and s $\tilde{p}^1 = \underline{p}e^{\lambda\Delta}$, (13) is true for $n=2$.

Fix $n \geq 3$ and suppose that (13) is true for every $k \leq n$, i.e., suppose that

$$\tilde{p}^k = \frac{\underline{p}^{k-1}(\tilde{p}^1-\underline{p})+\underline{p}(1-\tilde{p}^1)}{\underline{p}^{k-1}(\tilde{p}^1-\underline{p})+1-\tilde{p}^1} \ \forall\, k \leq n.$$

The aim is to establish that (13) is true for $n+1$. As $\tilde{p}^{n+1} = p_{\tau+n\Delta}^{n+1} = p_{\tau+n\Delta}^n$ by definition, taking (10) for $t = \tau+n\Delta$, we obtain:

$$\Omega(\tilde{p}^{n+1}) = \frac{\Omega(\tilde{p}^n)}{1-(1-e^{-\lambda\Delta})\displaystyle\prod_{k=1}^{n-1}\tilde{p}^k}$$

As $\underline{p} = e^{-\lambda\Delta}\tilde{p}^1$, the latter expression is rewritten:

$$\Omega(\tilde{p}^{n+1}) = \frac{\tilde{p}^1\Omega(\tilde{p}^n)}{\tilde{p}^1-(\tilde{p}^1-\underline{p})\displaystyle\prod_{k=1}^{n-1}\tilde{p}^k} \tag{14}$$

Let us compute $\prod_{k=1}^{n-1}\tilde{p}^k$ under the induction hypothesis. Noticing that, for every $k \leq n$, $\tilde{p}^k = \underline{p} \times \dfrac{A(k-1)}{A(k)}$, with $A(k) = \underline{p}^{k-1}(\tilde{p}^1-\underline{p})+1-\tilde{p}^1$, we can simplify the product as follows:

$$\prod_{k=1}^{n-1}\tilde{p}^k = \prod_{k=1}^{n-1}\underline{p}\times\frac{A(k-1)}{A(k)} = \underline{p}^{n-1}\frac{A(0)}{A(n-1)} = \underline{p}^{n-1}\frac{\underline{p}^{-1}(\tilde{p}^1-\underline{p})+1-\tilde{p}^1}{\underline{p}^{n-2}(\tilde{p}^1-\underline{p})+1-\tilde{p}^1}$$ Therefore,

$$\prod_{k=1}^{n-1}\tilde{p}^k = \underline{p}^{n-2}\frac{\tilde{p}^1(1-\underline{p})}{\underline{p}^{n-2}(\tilde{p}^1-\underline{p})+1-\tilde{p}^1} \tag{15}$$

Plugging this into (14) and simplifying by $\tilde{p}^1$, we obtain:

$$\Omega(\tilde{p}^{n+1}) = \Omega(\tilde{p}^n)\frac{\underline{p}^{n-2}(\tilde{p}^1 - \underline{p}) + 1 - \tilde{p}^1}{\underline{p}^{n-2}(\tilde{p}^1 - \underline{p}) + 1 - \tilde{p}^1 - (\tilde{p}^1 - \underline{p})\underline{p}^{n-2}(1 - \underline{p})}$$

As $\Omega(\tilde{p}^n) = \dfrac{(1-\tilde{p}^1)(1-\underline{p})}{\underline{p}}\dfrac{1}{\underline{p}^{n-2}(\tilde{p}^1 - \underline{p}) + 1 - \tilde{p}^1}$ under the induction hypothesis, we obtain:

$$\Omega(\tilde{p}^{n+1}) = \frac{(1-\tilde{p}^1)(1-\underline{p})}{\underline{p}}\frac{1}{\underline{p}^{n-1}(\tilde{p}^1 - \underline{p}) + 1 - \tilde{p}^1},$$

hence (13) is true for $n+1$.

Step 3 Plugging (15) and (13) into (10), we obtain

$$\Omega(p_t^n) = \frac{\Omega(\underline{p})\Omega(\tilde{p}^1)}{\Omega(\tilde{p}^1)(1-\underline{p}^{n-1}) + \underline{p}^{n-2}(1-\underline{p})e^{-\lambda(t-(n-1)\Delta-\tau)}} \tag{16}$$

Step 4: Equilibrium action

It is straightforward to show by induction that, in equilibrium, $k_t^n = \prod_{i=0}^{n-1} p_{t-i\Delta}^{n-i}$. Yet, by (16),

$$p_t^n = \underline{p}\frac{\Omega(\tilde{p}^1)(1-\underline{p}^{n-1}) + \underline{p}^{n-2}(1-\underline{p})e^{-\lambda(t-(n-1)\Delta-\tau)}}{\Omega(\tilde{p}^1)\Omega(\underline{p}) + \Omega(\tilde{p}^1)(1-\underline{p}^{n-1}) + \underline{p}^{n-2}(1-\underline{p})e^{-\lambda(t-(n-1)\Delta-\tau)}}$$

Because $t - i\Delta - (n-i-1)\Delta - \tau = t - (n-1)\Delta - \tau$, for every $i \leq n-2$,

$$p_{t-i\Delta}^{n-i} = \underline{p}\frac{A(i+1)}{A(i)},$$

with $A(i) = \Omega(\tilde{p}^1)(1-\underline{p}^{n-i}) + \underline{p}^{n-1-i}(1-\underline{p})e^{-\lambda(t-(n-1)\Delta-\tau)}$. Therefore,

$$k_t^n = \prod_{i=0}^{n-1}\left(\underline{p}\frac{A(i+1)}{A(i)}\right) = \underline{p}^n\frac{A(n)}{A(0)},$$

which simplifies to

$$k_t^n = \frac{\underline{p}^n\Omega(\underline{p})}{\underline{p}^{n-1}(1-\underline{p}) + \Omega(p_1)(1-\underline{p}^n)e^{\lambda(t-(n-1)\Delta-\tau)}}$$

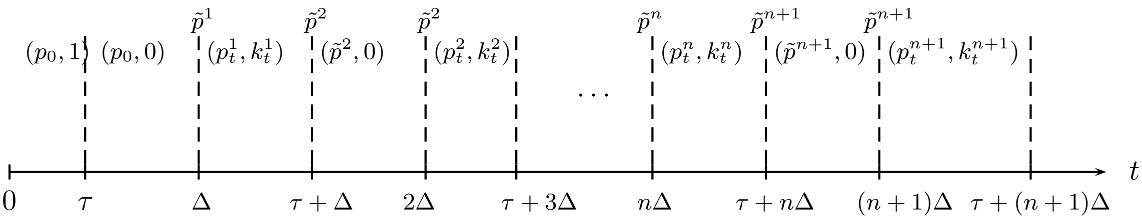
As $\Omega(\tilde{p}^1) = \Omega(p_0)e^{\lambda(t-\Delta)}$,

$$k_t^n = \frac{\underline{p}^n\Omega(\underline{p})}{\underline{p}^n\Omega(\underline{p}) + (1-\underline{p}^n)\Omega(p_0)e^{\lambda(t-n\Delta)}} \tag{17}$$

Analytic expression of k^* and p when $p_0 \in (\underline{p}, \tilde{p})$.

Let p_t^n and k_t^n respectively denote the public belief and player t's action in time $t \in [n\Delta, \tau + n\Delta]$.

Also, $\tilde{p}^n := p^n_{n\Delta}$. $\tilde{p}^1 = p_0$. Finally, by definition of $\tau(p_0)$, $e^{\lambda(\Delta-\tau(p_0))} = \frac{\tilde{p}}{p_0}$, thus $e^{-\lambda\tau(p_0)} = \frac{\underline{p}}{p_0}$.



Step 1 The first step is to establish by induction on n that, for every $n \geq 2$,

$$\frac{\Omega(p_t^n)}{\Omega(\tilde{p}^n)} = \frac{1}{1-(1-e^{-\lambda(t-n\Delta)})\prod_{k=1}^{n-1}\tilde{p}^k} \tag{18}$$

and $\frac{\Omega(p_t^1)}{\Omega(\tilde{p}^1)} = e^{\lambda(t-\Delta)}$.

1) **Relation between p_t^n and $p_{t-\Delta}^{n-1}$**

Let $n \geq 2$. By definition, $\dot{p}_t^n = -\lambda p_t^n(1-p_t^n)k_{t-\Delta}^{n-1}$ and $\dot{p}_t^{n-1} = -\lambda p_t^{n-1}(1-p_t^{n-1})k_{t-\Delta}^{n-2}$. Moreover, in equilibrium, $k_{t-\Delta}^{n-1} = p_{t-\Delta}^{n-1}k_{t-2\Delta}^{n-2}$. Therefore, $k_{t-\Delta}^{n-1} = -\frac{\dot{p}_{t-\Delta}^{n-1}}{\lambda(1-p_{t-\Delta}^{n-1})}$. It follows that, for every $n \geq 2$ and every t,

$$\frac{\dot{p}_t^n}{p_t^n(1-p_t^n)} = \frac{\dot{p}_{t-\Delta}^{n-1}}{1-p_{t-\Delta}^{n-1}} \tag{19}$$

Integrating (19) between $n\Delta$ and $t \in [n\Delta, \tau+n\Delta]$, we obtain:

$$\ln\left(\frac{\Omega(p_t^n)}{\Omega(p_{n\Delta}^n)}\right) = \ln\left(\frac{1-p_{t-\Delta}^{n-1}}{1-p_{n\Delta}^{n-1}}\right),$$

which, by definition of $\tilde{p}^n$, becomes

$$\frac{\Omega(p_t^n)}{\Omega(\tilde{p}^n)} = \frac{1-p_{t-\Delta}^{n-1}}{1-\tilde{p}^{n-1}} \tag{20}$$

2) Let us prove that (18) is true for $n=2$. As $k_t = 1$ for every $t \leq \tau$, $\Omega(p_t^1) = \Omega(p_0)e^{\lambda(t-\Delta)}$, i.e.,

$$1-p_1^t = \frac{\Omega(p_0)e^{\lambda(t-\Delta)}}{1+\Omega(p_0)e^{\lambda(t-\Delta)}} = \frac{1-p_0}{1-p_0+p_0e^{-\lambda(t-\Delta)}}$$

By (20), $\frac{\Omega(p_t^2)}{\Omega(\tilde{p}^2)} = \frac{1-p_{t-\Delta}^1}{1-p_0}$, hence

$$\frac{\Omega(p_t^2)}{\Omega(\tilde{p}^2)} = \frac{1}{1-(1-e^{-\lambda(t-2\Delta)})p_0}.$$

3) Suppose that (18) is true for some given n, i.e.,

$$1-p_t^n = \frac{\Omega(\tilde{p}^n)}{\Omega(\tilde{p}^n)+1-(1-e^{-\lambda(t-n\Delta)})\prod_{k=1}^{n-1}\tilde{p}^k},$$

and let us prove it is true for $n+1$. After rearrangement,

$$1-p_t^n = \frac{1-\tilde{p}^n}{1-(1-e^{-\lambda(t-n\Delta)})\prod_{k=1}^n \tilde{p}^k},$$

By (20), $\dfrac{\Omega(p_t^{n+1})}{\Omega(\tilde{p}^{n+1})} = \dfrac{1-p_{t-\Delta}^n}{1-\tilde{p}^n}$. As

$$1-p_{t-\Delta}^n = \frac{1-\tilde{p}^n}{1-(1-e^{-\lambda(t-(n+1)\Delta)})\prod_{k=1}^n \tilde{p}^k},$$

$$\frac{1-p_{t-\Delta}^n}{1-\tilde{p}^n} = \frac{1}{1-(1-e^{-\lambda(t-(n+1)\Delta)})\prod_{k=1}^n \tilde{p}^k)},$$

hence (18) is true for $n+1$.

Step 2 The second step is to establish by induction on n that, for every $n \geq 1$,

$$\tilde{p}^n = \underline{p}\frac{\underline{p}^{n-2}(p_0-\underline{p})+1-p_0}{\underline{p}^{n-1}(p_0-\underline{p})+1-p_0}. \tag{21}$$

This is straightforward for $n=1$; Moreover, $\tilde{p}^2 = p_{\tau+\Delta}^1)$ and, as $k_t = 1$ for $t \leq \tau$, $\Omega(p_{\tau+\Delta}^1) = \Omega(p_0)e^{\lambda(\tau)}$. Therefore, $\tilde{p}^2 = \frac{\underline{p}}{\underline{p}+1-p_0}$, which implies that (21) is true for $n=2$.

Fix $n \geq 3$ and suppose that (21) is true for every $k \leq n$. Observing that, under the induction hypothesis, $\tilde{p}^k = \underline{p}\frac{A(k-2)}{A(k-1)}$ with $A(u) = \underline{p}^u(p_0-\underline{p})+1-p_0$, we can write

$$\prod_{k=1}^{n-1}\tilde{p}^k = \prod_{k=1}^{n-1}\underline{p}\frac{A(k-2)}{A(k-1)} = \underline{p}^{n-1}\frac{A(-1)}{A(n-2)} = \underline{p}^{n-2}\frac{p_0(1-\underline{p})}{\underline{p}^{n-2}(p_0-\underline{p})+1-p_0} \tag{22}$$

As $\tilde{p}^{n+1} = p_{(n+1)\Delta}^{n+1} = p_{\tau+n\Delta}^n$ by definition, taking (18) for $t = \tau+n\Delta$, we obtain, for every $n \geq 2$:

$$\Omega(\tilde{p}^{n+1}) = \frac{\Omega(\tilde{p}^n)}{1-(1-e^{-\lambda\tau})\displaystyle\prod_{k=1}^{n-1}\tilde{p}^k}$$

As $e^{-\lambda\tau} = \frac{\underline{p}}{p_0}$, the latter expression is rewritten:

$$\Omega(\tilde{p}^{n+1}) = \frac{p_0\Omega(\tilde{p}^n)}{p_0-(p_0-\underline{p})\displaystyle\prod_{k=1}^{n-1}\tilde{p}^k} \tag{23}$$

Plugging (22) into the (23) and simplifying by p_0, we obtain:

$$\Omega(\tilde{p}^{n+1}) = \Omega(\tilde{p}^n)\frac{\underline{p}^{n-2}(p_0-\underline{p})+1-p_0}{\underline{p}^{n-1}(p_0-\underline{p})+1-p_0}$$

Using the induction hypothesis, the latter expression becomes:

$$\Omega(\tilde{p}^{n+1}) = \frac{(1-p_0)(1-\underline{p})}{\underline{p}}\frac{1}{\underline{p}^{n-1}(p_0-\underline{p})+1-p_0},$$

hence (21) is true for $n+1$.

Step 3 Plugging (22) and (21) into (18), we obtain

$$\Omega(p_t^n) = \frac{\Omega(\underline{p})\Omega(p_0)}{\Omega(p_0)(1-\underline{p}^{n-1}) + \underline{p}^{n-2}(1-\underline{p})e^{-\lambda(t-n\Delta)}} \tag{24}$$

Step 4: Equilibrium action

It is straightforward to show by induction that, in equilibrium, $k_t^n = \prod_{i=0}^{n-1} p_{t-i\Delta}^{n-i}$. Yet, by (24),

$$p_t^n = \underline{p}\frac{\Omega(p_0)(1-\underline{p}^{n-1}) + \underline{p}^{n-2}(1-\underline{p})e^{-\lambda(t-n\Delta)}}{\Omega(p_0)(1-\underline{p}^{n}) + \underline{p}^{n-1}(1-\underline{p})e^{-\lambda(t-n\Delta)}}$$

Because $t - i\Delta - (n-i)\Delta = t - n\Delta$, for every $i \leq n-2$,

$$p_{t-i\Delta}^{n-i} = \underline{p}\frac{A(n-i-1)}{A(n-i)},$$

with $A(u) = \Omega(p_0)(1-\underline{p}^u) + \underline{p}^{u-1}(1-\underline{p})e^{-\lambda(t-n\Delta)}$. Therefore,

$$k_t^n = \prod_{i=0}^{n-1}\left(\underline{p}\frac{A(n-i-1)}{A(n-i)}\right) = \underline{p}^n\frac{A(0)}{A(n)},$$

which simplifies to

$$k_t^n = \frac{\Omega(\underline{p})}{\Omega(\underline{p}) + \Omega(p_0)\Omega(\underline{p}^n)e^{\lambda(t-n\Delta)}}$$

5.1.2 Proof of Proposition 2

Proof. $W(k^*)$ does not depend on Δ when $p_0 \leq \underline{p}e^{\lambda\Delta}$. When $p_0 > \underline{p}e^{\lambda\Delta}$,

$$\frac{\partial W(k^*)}{\partial\Delta} = \lambda p_0 e^{-\lambda\Delta} + \lambda e^{-\lambda\Delta}\ln\left(\frac{1-p_0}{1-\underline{p}e^{\lambda\Delta}}\right) - \lambda\frac{\underline{p}}{1-\underline{p}e^{\lambda\Delta}} - \alpha\frac{\partial\tau(p_0)}{\partial\Delta},$$

and $\partial\tau(p_0)/\partial\Delta = -\underline{p}e^{\lambda\Delta}/(1-\underline{p}e^{\lambda\Delta})$, thus

$$\frac{\partial W(k^*)}{\partial\Delta} = \lambda\left(-\underline{p} + p_0e^{-\lambda\Delta} + e^{-\lambda\Delta}\ln\left(\frac{1-p_0}{1-\underline{p}e^{\lambda\Delta}}\right)\right).$$

Differentiating $\frac{\partial W(k^*)}{\partial\Delta}$ with respect to p_0, we obtain $-\lambda e^{-\lambda\Delta}p_0/(1-p_0)$. Moreover, it equals 0 when $p_0 = \underline{p}e^{\lambda\Delta}$. Therefore, $\frac{\partial W(k^*)}{\partial\Delta}$ is negative. $\square$

5.2 Proof of Proposition 5

Proof. Let $\tau(k)$ the random time of arrival of the first success given investment profile k. For every t,

$$P(\tau(k) \leq t) = p_0(1 - e^{-\lambda \int_0^{t-\Delta} k_s ds})\mathbb{1}_{t \geq \Delta}$$

If $T = T^H$, then the unique equilibrium is k^* such that $k_t^* = \mathbb{1}_{t \leq T^H - \Delta}$. At this equilibrium,

$$\begin{aligned} P(\tau(k^*) \leq t) &= 0 && \text{if } t \leq \Delta \\ &= p_0(1 - e^{-\lambda(t-\Delta)}) && \text{if } t \in [\Delta, T^H] \\ &= p_0(1 - e^{-\lambda(T^H-\Delta)}) && \text{if } t \geq T^H \end{aligned}$$

If $T > T^H$, all equilibria are such that $\tilde{k}_t = 0$ if $t \geq T - \Delta$ and $\int_0^{T-\Delta} \tilde{k}_s ds = T^H - \Delta$. Consider one of these equilibria such that $\tilde{k} \neq k^*$ a.s. At this equilibrium,

$$\begin{aligned} P(\tau(\tilde{k}) \leq t) &= 0 && \text{if } t \leq \Delta \\ &= p_0(1 - e^{-\lambda \int_0^{t-\Delta} \tilde{k}_s ds}) && \text{if } t \in [\Delta, T] \\ &= p_0(1 - e^{-\lambda(T^H-\Delta)}) && \text{if } t \geq T \end{aligned}$$

For every $t \leq \Delta$ or $t \geq T$, then $P(\tau(\tilde{k}) \leq t) = P(\tau(k^*) \leq t)$

For every $t \in [T^H, T]$, $P(\tau(\tilde{k}) \leq t) \leq P(\tau(k^*) \leq t)$ because $\int_0^{t-\Delta} \tilde{k}_s ds \leq \int_0^{T-\Delta} \tilde{k}_s ds = T_H - \Delta$.

For every $t \in [\Delta, T^H]$, $P(\tau(\tilde{k}) \leq t) \leq P(\tau(k^*) \leq t)$ because $\int_0^{t-\Delta} \tilde{k}_s ds \leq t - \Delta$. Moreover, $\tilde{k} \neq k^*$ a.s. implies that $\exists\, t \in [\Delta, T^H]$ such that $\int_0^{t-\Delta} \tilde{k}_s ds < t - \Delta$, hence there is $t \in [\Delta, T^H]$ such that $P(\tau(\tilde{k}) \leq t) < P(\tau(k^*) \leq t)$.

Therefore, $\tau(k^*)$ first-order stochastically dominates $\tau(\tilde{k})$. As a consequence, $E[e^{-r\tau(\tilde{k})}] \leq E[e^{-r\tau(k^*)}]$. □

5.3 Proof of Proposition ??

Proof. We seek conditions on p_0 such that

$$(I) := \frac{\underline{p}}{p_0} < \frac{1 - \Omega(p_0)/\Omega(\underline{p})}{\ln(\Omega(\underline{p})) - \ln(\Omega(p_0))}$$

is true. Multiplying both sides by $\Omega(\underline{p})$ and using that $\frac{1}{p_0} = 1 + \Omega(p_0)$, (I) is rewritten

$$\underline{p}(1 + \Omega(p_0)) < \frac{\Omega(\underline{p}) - \Omega(p_0)}{\ln(\Omega(\underline{p})) - \ln(\Omega(p_0))},$$

which is equivalent to $f(\Omega(p_0)) > 0$ with $f(x) := \frac{\Omega(\underline{p})-x}{\ln(\Omega(\underline{p}))-\ln(x)} - (1+x)(1-\underline{p})$. Let us study the function f for $x \in [0, \Omega(\underline{p})]$.

Let us first establish that $f(\Omega(\underline{p})) = 0$. The limit of $\frac{\Omega(\underline{p})-x}{\ln(\Omega(\underline{p}))-\ln(x)}$ in $\Omega(\underline{p})$ is undetermined as both the numerator and the denominator converge to 0. Applying L'Hôpital's rule, we find that $\lim_{x\to\Omega(\underline{p})} \frac{\Omega(\underline{p})-x}{\ln(\omega(\underline{p}))-\ln(x)} = \Omega(\underline{p})$. As a result, $\lim_{x\to\Omega(\underline{p})} f(x) = 0$.

Differentiating f with respect to x, we obtain

$$f'(x)) = \frac{1}{(\ln(\Omega(\underline{p})) - \ln x)^2} \underbrace{(-(\ln(\Omega(\underline{p})) - \ln x) + \frac{\Omega(\underline{p})}{x} - 1 - (1-\underline{p})(\ln(\Omega(\underline{p})) - \ln x)^2)}_{:=g(x)}$$

Differentiating g with respect to x, we obtain

$$g'(x) = \frac{1}{x^2} \underbrace{(x - \Omega(\underline{p}) + 2(1-\underline{p})x(\ln(\Omega(\underline{p})) - \ln x))}_{:=h(x)}$$

Differentiating h with respect to x, we obtain

$$h'(x) = 2\underline{p} - 1 + 2(1-\underline{p})(\ln \Omega(\underline{p}) - \ln x)$$

• If $\underline{p} \geq 1/2$, then $h'(x) > 0$ for every x. As $h(\Omega(\underline{p})) = 0$, this implies that $h(x) < 0$, thus $g'(x) < 0$, for every x. As $g(\Omega(\underline{p})) = 0$, this implies that $g(x) > 0$, thus $f'(x) > 0$, for every x. As $f(\Omega(\underline{p})) = 0$, this implies $f(x) < 0$ for every $x \in [0, \Omega(\underline{p}))$. Therefore, (I) cannot be satisfied when $\underline{p} \geq 1/2$.

• If $\underline{p} < 1/2$, $h'(x) > 0 \Leftrightarrow x < \Omega(\underline{p})e^{(2\underline{p}-1)/(2(1-\underline{p}))} := \tilde{x}_1 (< \Omega(\underline{p}))$. Hence, h is increasing on $[0, \tilde{x}_1]$ and decreasing on $[\tilde{x}_1, \Omega(\underline{p})]$. As $h(0) = -\Omega(\underline{p})$ and $h(\Omega(\underline{p})) = 0$, there exists $\tilde{x}_2 \in (0, \tilde{x}_1)$ such that $h(x) < 0$ (thus $g'(x) < 0$) on $[0, \tilde{x}_2)$ and $h(x) \geq 0$ (thus $g'(x) \geq 0$) on $[\tilde{x}_2, \Omega(\underline{p})]$. This implies that $g(x)$ is decreasing on $[0, \tilde{x}_2)$ and increasing on $[\tilde{x}_2, \Omega(\underline{p})]$. As $g(\Omega(\underline{p})) = 0$ and $\lim_{x \to 0} g(x) = +\infty$, there exists $\tilde{x}_3 \in (0, \tilde{x}_2)$ such that $g(x) > 0$ (thus $f'(x) > 0$) on $[0, \tilde{x}_3)$ and $g(x) \leq 0$ (thus $f'(x) \leq 0$) on $[\tilde{x}_3, \Omega(\underline{p})]$. As a consequence, $f(x)$ is increasing on $[0, \tilde{x}_3)$ and decreasing on $[\tilde{x}_3, \Omega(\underline{p})]$. As $f(0) = -(1-\underline{p})$ and $f(\Omega(\underline{p})) = 0$, there exists $\tilde{x}_4 \in (0, \tilde{x}_3)$ such that $f(x) < 0$ on $[0, \tilde{x}_4]$ and $f(x) \geq 0$ otherwise.

Therefore, (I) is satisfied if and only if $\underline{p} < 1/2$ and $x \geq \tilde{x}_4$, where $\tilde{x}_4$ is the unique solution to $f(x) = 0$, that is

$$\frac{\Omega(\underline{p}) - \tilde{x}_4}{\ln(\Omega(\underline{p})) - \ln(\tilde{x}_4)} = (1 + \tilde{x}_4)(1 - \underline{p})$$

□