Approximability of Infinite Games

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Abstract

This paper examines the approximability of infinite games using finite discretizations. We identify simple conditions that guarantee that limits of approximate mutual best responses, aka *approximable equilibria*, do not depend on the particular way the infinite game is discretized. We show that these conditions are satisfied in many common infinite games, such as Bertrand competition, war of attrition, or auctions. We relate our results to those obtained for limit equilibria, approximate equilibria, or solutions for some sharing rule.

Key words: Infinite games, approximable equilibrium, finite discretizations.

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1 Introduction

The representation of strategic choices as continuous variables, such as payments, consumption, or effort, has been a common practice in economic theory. These modeling choices have proven to be a valuable tool for economists in exploring a vast array of economic settings, including pricing in monopolistic or oligopolistic markets, equilibria in auctions, signaling in labor markets, and moral hazard in firms.

While modeling variables as real numbers has numerous advantages, it also raises several challenges. Firstly, unlike finite games, infinite games may not possess a Nash equilibrium, requiring specific assumptions regarding the action space and payoff function to guarantee self-enforcing strategic behavior. Secondly, it is not always evident whether an equilibrium, if one exists, is the limit of equilibria of a sequence of finite games. This can be a critical issue when the continuum is used as a modeling convenience to study a setting with a fine but finite set of choices (e.g., payments). Thirdly, the dependence of the set of limit equilibria on the specific discretization of the game is often unclear, especially when actions are multi-dimensional or multiple players have infinite action spaces. This is of concern as one would not want the numerically obtained equilibria to be dependent on the specific discretization or the parameters used in the model.

In this paper, we delve into the approximability of infinite games using finite games. Specifically, we outline clear conditions that ensure the existence of limiting behavior that is the result of (almost) mutual best responses, regardless of how the infinite game is approximated by finite versions. Under these conditions, our predictions are not dependent on the specifics of the choice discretization or the numerical analysis method used. The main departure from the literature (discussed below) is the focus on robustness to *all* discretizations of the game.

We consider an infinite game in normal form, denoted by $G \equiv (S_i, u_i)_{i \in \mathcal{I}}$, where each player's action space S_i is a compact metric space. We define a *discretization* of this infinite game as a sequence of finite games whose action sets converge, under the Hausdorff distance, to the action set of the infinite game. Standard topology results imply the existence of a discretization in any game (even when the action space is infinite-dimensional). We show that for any strategy profile, there exists a sequence of the finite games' strategy profiles that converges weakly to it.

We define an *approximable equilibrium* for a given discretization as a strategy profile that is the limit of ε-equilibria of the finite games, where $\varepsilon \to 0$ along the sequence. Our first result establishes the existence of an approximable equilibrium for some discretization. We also provide examples of games with discretizations without a corresponding approximable equilibrium or strategy-payoff pairs which are approximable equilibria for some discretizations but not others.

We then investigate games where the set of approximable equilibria is independent of how they are discretized, which we call *approximable games.* The existence of an approximable equilibrium for some discretization implies that approximable games do feature behavior that can be approached by almost-optimal mutual behavior regardless of how the game is discretized. Conversely, in an approximable game, behavior that is the limit of almost-optimal mutual behavior under one discretization is also the limit of almost-optimal mutual behavior under *any* discretization. The main goal of our analysis is to provide simple and easy-to-verify conditions for approximability.

Our first result establishes two jointly-sufficient conditions for approximability. The first condition requires that any strategy profile and its corresponding payoff vector can be approximated under any discretization. The second condition requires that if a player can profitably deviate from a converging sequence of strategy profiles, then she can profitably deviate under any discretization and sequence of strategy profiles with the same limit. We call games satisfying these two conditions best-reply-secure (BRS) approximable games; hence a BRS-approximable game is also approximable.

We provide different characterizations of approximable equilibria in BRS-approximable games. We first show that a Nash equilibrium is an approximable equilibrium. We then show that the set of approximable equilibria of G coincides with the set of limits of ε-equilibria along any sequence of (finite or infinite games) approaching G as $\varepsilon \to 0$. We finally show that the approximable equilibria are solutions for some sharing rule and argue that, in most games, the set of approximable equilibria is independent of used the sharing rule.

We provide numerous examples illustrating the ease of use of our analysis. In particular, we show the approximability of a large class of discontinuous games, such as auctions, war of attrition, or Bertrand competition. We also analyze one-player games and show that, while some discontinuities may make the game not approximable, there is often a way to enhance the action space to make the game approximable.

The rest of the paper is organized as follows. After the literature review, Section 2 introduces the notation and the concepts of discretization and approximable equilibrium. Section 3 introduces approximable and BRS-approximable games and shows that all BRS-approximable games are approximable. It also provides sufficient conditions for BRS-approximability. In Section 4, we illustrate the analysis through numerous examples and show the approximability of a large class of discontinuous games. In Section 5, we provide different characterizations of approximable equilibria and relate these findings to previous literature. Finally, Section 6 discusses the results and concludes. Appendix A contains the omitted proofs, while Appendix B provides a short review of basic topological concepts and additional results on the convergence of strategy profiles.

1.1 Literature review

Existence of Nash equilibria: An important body of the literature studies conditions that permit extending Nash (1950)'s existence result to infinite games (see Reny, 2020, for a recent review). Fan (1952) and Glecksberg (1952) proved that compactness and metrizability of the action space and continuity of the payoff function guarantee the existence of Nash equilibria. Subsequent work has been devoted to finding more general conditions for existence, examples including Dasgupta and Maskin (1986a,b), Simon (1987), Reny (1999), or Prokopovych (2011). A related line of work has been further devoted to finding conditions for the existence of Nash equilibria which are the limit of (epsilon) equilibria along some discretization of the game. Examples include Dasgupta and Maskin (1986a), Simon and Stinchcombe (1995), De Castro (2011), Balder (2011), Carmona (2011, 2013), Reny (2011), and Bich and Laraki (2017). We instead focus our analysis on obtaining conditions for the existence of limit behavior (which may not be a Nash equilibrium of the infinite game) that is approximately optimal for all sequences of close-by finite versions of the game. As we will see, such conditions are reminiscent to Reny's concept of better-reply security; we discuss the connection in Section 5.

Approximability of equilibria: Fudenberg and Levine (1986) made an important step toward assessing the approximability of equilibria in infinite games. They did so by constructing a metric where payoff functions are uniformly continuous. They showed that, when finite approximations under this metric exist, ε-equilibria of the infinite game coincide with the limits of ε-equilibria of the finite games. Harris, Stinchcombe, and

Zame (2005) showed that only when the game is nearly compact and continuous, there are finite approximations under the previous metric. Stinchcombe (2005) studied games that are not nearly compact and continuous and defined and proved the existence of finitistic equilibria. The relation of these papers to ours is discussed in detail in Section 6.1. In there, we argue that our approach is more intuitive and simpler to use, as the conditions for the existence and approximability are easier to verify. Also, our approach permits discretizing the action space instead of the space of mixed strategies.

Alternative equilibrium concepts: An alternative to address the problem of the existence of Nash equilibria in infinite games has been proposing other equilibrium concepts. Simon and Zame (1990) incorporated sharing rules as part of the equilibrium concept. We show that, in BRS-approximable games, approximable equilibria are solutions for some sharing rule; hence, their equilibrium concept can be used to obtain approximable equilibria(see Section 5.3). Simon and Stinchcombe (1995) defined and compared different equilibrium refinements for normal-form games. In particular, they introduced *limit-of-finite perfect equilibria* as limits of ε_n-perfect equilibria for some discretization, for some $\varepsilon_n \to 0$, and they proved their existence in games with continuous payoffs. While their equilibrium concept is close to our concept of approximable equilibrium, our use of ε_n-perfect equilibria implies the existence of approximable equilibria in any game (see Section 2.2 for further discussion). Overall, the focus of their analysis and the nature of their results are different from ours: While their objective is to explore and compare various equilibrium refinements for infinite normal-form games, our focus is on determining general conditions for the existence of equilibria which are limits of ε_n-equilibria *for all* discretizations of the game.

2 Discretizations and approximable equilibria

In this section, we introduce infinite games, discretizations, and approximable equilibria. Throughout, we will use some standard results on weak convergence and compactness. The reader may check Section B.1 for a quick refresh.

We begin with the definition of a normal-form game with finitely many players but with potentially infinite action space.

Definition 2.1. A *game* is $G \equiv (S_i, u_i)_{i \in \mathcal{I}}$ where $\mathcal{I}$ is a finite set of players, $S \equiv \times_{i \in \mathcal{I}} S_i$, where each S_i is a compact metric space, and each $u_i : S \to \mathbb{R}$ is bounded.

From now on, we fix an infinite game G. We use the sup-norm distance in S, that is, $d(s,s')=\max_{i\in\mathcal{I}} d_i(s_i,s'_i)$, where d_i is the metric on S_i. Under this distance, S is a compact metric space. We let $\Sigma_i \equiv \Delta(S_i)$ be the *set of (mixed) strategies* of player i, and $\Sigma \equiv \times_{i\in\mathcal{I}}\Sigma_i$ be the *set of (mixed) strategy profiles.* Recall that Σ_i and Σ are compact metric spaces under the weak distance. As it is custom in the analysis of games, we will sometimes use an action $s_i \in S_i$ to denote the strategy assigning probability one to this action. For a given strategy profile $\sigma \in \Sigma$, we let $u_i(\sigma) \equiv \mathbb{E}_{\tilde{s}}[u_i(\tilde{s})|\sigma]$. An *$\varepsilon$-equilibrium* is a strategy profile σ such that $u_i(\sigma) \geq u_i(s_i,\sigma_{-i}) - \varepsilon$ for all $i \in \mathcal{I}$ and $s_i \in S_i$. A *Nash equilibrium* is a 0-equilibrium.

2.1 Discretizations

We aim to study sequences of finite games approaching G. We do so by discretizing the action space for each player, and then considering a sequence of finite versions of G.

Definition 2.2. A *discretization* of S_i is a sequence $(S_{i,n})_n$ of finite subsets of S_i converging to S_i.[1] A *discretization* (of G) is a sequence of finite games $(G_n)_n$, with $G_n \equiv (S_{i,n}, u_i)_{i\in\mathcal{I}}$ for all n, where each $(S_{i,n})_n$ is a discretization of S_i.

A discretization is a sequence of finite games with action spaces converging to their infinite counterparts. Given that we are considering potentially large (and infinite-dimensional) action spaces, one may wonder whether there is a discretization of G. It turns out that, as the following result establishes, our assumption that the action space S is metric and compact is enough to guarantee that G can be discretized.

Lemma 2.1. *Any game G has a discretization.*

Proof. By Proposition B.1, S_i is complete and totally bounded for each $i \in \mathcal{I}$. Hence, for each $n \in \mathbb{N}$, there is a cover of S_i containing finitely many sets, all of diameter lower than $1/n$. We let $S_{i,n}$ be a finite set of elements of S_i, each belonging to a different set of the partition. It then follows that, for each $s \in S$, there is a sequence $(s_{i,n})_n \in (S_{i,n})_n$ with $s_{i,n} \to s_i$. Hence, for all $s \in S$ there is a sequence $(s_n)_n \in (S_n)_n$ with $s_n \to s$, which concludes the proof. □

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[1] We use the usual Hausdorff distance between sets. We then have that $S_{i,n} \to S_i$ if and only if, for all $s_i \in S_i$, there is a sequence $(s_{i,n})_n \in (S_{i,n})_n$ such that $s_{i,n} \to s_i$.

In the arguments below, when a discretization $(G_n)_n$ is fixed, $S_{i,n}$ and S_n will signify the set of player i's actions and the set of action profiles, respectively, of the corresponding n-th finite game. Also, for each n, we will use Σ_n to denote the set of strategy profiles in Σ with support in S_n.[2]

Notation for sequences and sets of sequences

Since our definitions and arguments will often involve sequences, it will be useful to use the following notation. For a given set X (e.g., S, S_i, Σ,...), we will use $X^{\mathbb{N}}$ to denote the set of sequences in X. We will use both $(x_n)_n$ and $x_{\mathbb{N}}$ to denote a generic element of $X^{\mathbb{N}}$. A discretization of S_i will be denoted both as $(S_{i,n})_n$ and $S_{i,\mathbb{N}}$ (note that $S_{i,\mathbb{N}} \subset S_i^{\mathbb{N}}$), and the corresponding sets of sequences of strategy profiles both $(\Sigma_{i,n})_n$ and $\Sigma_{i,\mathbb{N}}$. Consistently, we will use $G_{\mathbb{N}}$ to denote a discretization of G.

2.2 Approximable equilibria

In this section, we provide additional notation for the set of strategy profiles, define approximable equilibria, and provide a preliminary existence result.

For given $\sigma \in \Sigma$ and $\sigma_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$, we will use $\sigma_n \to \sigma$ to denote that $\sigma_{\mathbb{N}}$ weakly converges to σ.[3] It is not difficult to see that, for each discretization $G_{\mathbb{N}}$, we have $\Sigma_n \to \Sigma$ as $n \to \infty$, that is, for each strategy profile $\sigma \in \Sigma$, there exists a sequence $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ such that $\sigma_n \to \sigma$ (see Proposition B.3). As usual, we will use $u : S \to \mathbb{R}^{\mathcal{I}}$ to denote the function $s \mapsto (u_i(s))_{i \in \mathcal{I}}$.

Definition 2.3. We say that $\sigma \in \Sigma$ is an *approximable equilibrium for a discretization* $G_{\mathbb{N}}$ if there are two sequences $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ and $\varepsilon_{\mathbb{N}} \to 0$ such that $\sigma_n \to \sigma$ and for each n, σ_n is an ε_n-equilibrium of G_n. We say that $\sigma \in \Sigma$ is an *approximable equilibrium* if it is an approximable equilibrium for *some* discretization.

Note that our definition of approximable equilibrium for a given discretization $G_{\mathbb{N}}$ requires that there is a sequence of strategy profiles $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ where players play asymp-

[2]Without risk of confusion, we abuse notation by letting Σ_n denote both the set $\times_{i \in \mathcal{I}} \Delta(S_{n,i})$ with the set of elements of $\times_{i \in \mathcal{I}} \Delta(S_i)$ with support on $\times_{i \in \mathcal{I}} S_{n,i}$.

[3]Recall that a sequence $\sigma_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ *weakly converges* to $\sigma \in \Sigma$ if for any bounded continuous function $f : S \to \mathbb{R}$ we have $\mathbb{E}_{\tilde{s}}[f(\tilde{s})|\sigma_n] \to \mathbb{E}_{\tilde{s}}[f(\tilde{s})|\sigma]$ or, equivalently, if $d_\Sigma(\sigma, \sigma_n) \to 0$, where d_Σ is the weak distance defined in Appendix B.

totic mutual best responses, and also that the corresponding payoffs converge. See Section 3.3 for a dicussion on an alternative definition where convergence is only required for strategy profiles. See Remark 2.1 and Section 4.1 for comparison with Simon and Stinchcombe (1995)'s concept of limit-of-finite perfect equilibrium. See also Section 6.1 for a discussion of the relationship between our approach and those in Fudenberg and Levine (1986), Harris, Stinchcombe, and Zame (2005), and Stinchcombe (2005).

Existence of approximable equilibria for some discretization

Nash (1950) showed that all finite games have a Nash equilibrium. We use this result to prove the following lemma, which establishes that all games have an approximable equilibrium for some discretization.

Lemma 2.2. *An approximable equilibrium for some discretization exists.*

Proof. By Prokhorov's theorem (see Proposition B.2 and Corollary B.1), Σ is a compact metric space (with the weak distance) and so is sequentially compact. Fix a discretization $G_{\mathbb{N}}$. Let $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ be such that σ_n is a Nash equilibrium of G_n for each n. Then, $\sigma_{\mathbb{N}}$ has a subsequence $\sigma_{k_{\mathbb{N}}}$ satisfying that $\sigma_{k_n} \to \sigma$ for some $\sigma \in \Sigma$. As a result, σ is an approximable equilibrium for the discretization $G_{k_{\mathbb{N}}}$. □

While Lemma 2.2 ensures the existence of an approximable equilibrium for some discretization, it does not guarantee that all discretizations have some approximable equilibrium. There are games and discretizations without an approximable equilibrium. Consider, for example, a one-player game with $S=[0,1]$, with $u(s)=s$ when s is rational and $u(s)=-s$ otherwise. In this case, there is no approximable equilibrium for a discretization where each S_n only contains rational numbers when n is odd and irrational numbers when n is even.

Remark 2.1. Simon and Stinchcombe (1995) define *limit-of-finite (lof) perfect equilibria* as limits of ε_n-perfect equilibria along some discretization and sequence $\varepsilon_n \to 0$. (Recall that ε_n-perfect equilibria are full-support strategy profiles assigning a probability at most ε_n to actions that are not best responses, see Myerson, 1978.) While we believe that limit-of-finite perfect equilibria coincide with approximable equilibria (for *some* discretization), the result that the set of approximable equilibria is independent of the discretizations for approximable games does *not* extend to lof perfect equilibria, even when u is continuous (see Example 4.1).

3 Approximable and BRS-approximable games

In this section, we introduce the concepts of approximable and BRS-approximable games and we show that they are immune to some of the approximability concerns posed in the Introduction. We will then provide weaker but easier-to-verify conditions, which will be used in Section 4 to show that many games of interest are both approximable and BRS-approximable.

We then begin with an important concept.

Definition 3.1. We say that G is *approximable* if the set of approximable equilibria is independent of the discretization.

Approximable games have two important properties. First, any approximable equilibrium for a discretization is an approximable equilibrium for all discretizations. Hence, it suffices to compute the set approximable equilibria for a given discretization to obtain the set of approximable equilibria. Second, as an immediate corollary of Lemma 2.2, an approximable equilibrium exists for all discretizations. That is, an approximable game contains behavior where players play asymptotic best responses to each other's behavior independently of the discretization of the game. Hence, approximable games are immune to many concerns that approximations through a sequence of finite versions of it may raise (see the Introduction).

3.1 BRS-approximable games

We now introduce two conditions each $\sigma \in \Sigma$ may satisfy or not, and we define BRS-approximable games as those games that satisfy them. We then show that all BRS-approximable games are approximable.

The first condition, denoted (APP$_\sigma$), holds if, for any discretization, there is a sequence approaching σ with corresponding payoffs approaching $u(\sigma)$ (recall a strategy profile σ can always be approached in through in any discretization).

(APP$_\sigma$) For any discretization $G_{\mathbb{N}}$ there is some $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ with $(\sigma_n, u(\sigma_n)) \to (\sigma, u(\sigma))$.

The second condition holds if a player i can asymptotically benefit from deviating from a sequence of strategy profiles converging to σ with payoffs converging to some $v \in$

$\mathbb{R}^{\mathcal{I}}$, then a profitable deviation exists under any sequence of strategy profiles converging to σ with payoffs converging to $v \in \mathbb{R}^{\mathcal{I}}$ and discretization of i's action space. We will see in Section 5 that this property is related to Reny (1999)'s condition of better-reply security, so we will denote this property (BRS$_\sigma$).

(BRS$_\sigma$) If $\sigma_{\mathbb{N}}, \hat{\sigma}_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ are such that $(\sigma_n, u(\sigma_n)), (\hat{\sigma}_n, u(\hat{\sigma}_n)) \to (\sigma, v)$ for some $v \in \mathbb{R}^{\mathcal{I}}$, and there are $i \in \mathcal{I}$ and $s_{i,\mathbb{N}} \in S_i^{\mathbb{N}}$ with $\limsup_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) > v_i$, then, for any discretization $G_{\mathbb{N}}$, there are $\hat{\imath} \in \mathcal{I}$ and $\hat{s}_{\hat{\imath},\mathbb{N}} \in S_{\hat{\imath},\mathbb{N}}$ with $\limsup_{n\to\infty} u_{\hat{\imath}}(\hat{s}_{\hat{\imath},n}, \hat{\sigma}_{-\hat{\imath},n}) > v_{\hat{\imath}}$.

We see (APP$_\sigma$) and (BRS$_\sigma$) as natural requirements which hold in many games of economic interest. Section 3.2 provides results that simplify proving that (APP$_\sigma$) and (BRS$_\sigma$) hold for all $\sigma \in \Sigma$. Section 4 shows that, indeed, (APP$_\sigma$) and (BRS$_\sigma$) hold for all strategy profiles in many relevant games. We also illustrate when these conditions may fail, and we provide procedures to enhance the action space when they do not.

Definition of BRS-approximable equilibria and main result

We now define BRS-approximable games and prove that they are approximable.

Definition 3.2. G is *BRS-approximable* if (APP$_\sigma$) and (BRS$_\sigma$) hold for all $\sigma \in \Sigma$.

Theorem 3.1. *If G is BRS-approximable, then it is approximable.*

Proof. We first show the following useful lemma:

Lemma 3.1. *If* (APP$_\sigma$) *holds for all $\sigma \in \Sigma$ and there is some $\sigma_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$ then, for all discretizations $G_{\mathbb{N}}$, there is $\hat{\sigma}_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ with $(\hat{\sigma}_n, u(\hat{\sigma}_n)) \to (\sigma, v)$.*

Proof. Let $\sigma_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ be such that $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$. Fix some discretization $G_{\mathbb{N}}$. Fix some $n \in \mathbb{N}$. By (APP$_{\sigma_n}$), there is some sequence $\hat{\sigma}_{n,\mathbb{N}} \in \Sigma_{\mathbb{N}}$ such that $(\hat{\sigma}_{n,\hat{n}}, u(\hat{\sigma}_{n,\hat{n}})) \to (\sigma, v)$ as $\hat{n} \to \infty$. We now use a (standard) diagonal argument to show that there exists some sequence $\hat{n}_{\mathbb{N}}$ such that $(\sigma_{n,\hat{n}_n}, u(\sigma_{n,\hat{n}_n})) \to (\sigma, v)$; hence $\hat{\sigma}_n \equiv \sigma_{n,\hat{n}_n}$ will be our desired sequence. We then define $\hat{n}_{\mathbb{N}}$ by setting $\hat{n}_0 = 0$ and, for each $n \in \mathbb{N}$, we let $\hat{n}_n$ be the smallest natural larger than $\hat{n}_{n-1}$ satisfying that[4]

$$d((\hat{\sigma}_{n,\hat{n}}, u(\hat{\sigma}_{n,\hat{n}})), (\sigma_n, u(\sigma_n))) < 1/n \quad \text{for all } \hat{n} \geq \hat{n}_n .$$

It is then clear that $(\hat{\sigma}_{n,\hat{n}_n}, u(\hat{\sigma}_{n,\hat{n}_n})) \to (\sigma, v)$ as $n \to \infty$. □

[4]The distance between two pairs $(\sigma, v), (\hat{\sigma}, \hat{v}) \in \Sigma \times \mathbb{R}^{\mathcal{I}}$ is $\max\{d_\Sigma(\sigma, \hat{\sigma}), \max_{i\in\mathcal{I}} |v_i - \hat{v}_i|\}$ (recall that d_Σ is the weak distance defined in Appendix B).

Take a discretization $G_{\mathbb{N}}$ and let σ be an approximable equilibrium of $G_{\mathbb{N}}$. Let $\varepsilon_{\mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$ and $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ be such that $\varepsilon_n \to 0$, σ_n is an ε_n-equilibrium of G_n for all n, and $\sigma_n \to \sigma$. Take another discretization $\hat{G}_{\mathbb{N}}$. We want to show that σ is an approximable equilibrium of $\hat{G}_{\mathbb{N}}$.

Let $v \in \mathbb{R}^{\mathcal{I}}$ be such that $u(\sigma_{k_n}) \to v$ for some stricly increasing $k_{\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. By Lemma 3.1, there is some sequence $\hat{\sigma}_{\mathbb{N}} \in \hat{\Sigma}_{\mathbb{N}}$ such that $(\hat{\sigma}_n, u(\hat{\sigma}_n)) \to (\sigma, v)$. We now verify that there exists $\hat{\varepsilon}_{\mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$ converging to 0 such that $\hat{\sigma}_n$ is an $\hat{\varepsilon}_n$-equilibrium of $\hat{G}_n$ for all $n \in \mathbb{N}$. Assume, for the sake of contradiction, that there is no such sequence. There must then bes a player $i \in \mathcal{I}$ and a sequence $\hat{s}_{i,\mathbb{N}} \in \hat{S}_{i,\mathbb{N}}$ such that $\limsup_{n\to\infty} u_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}) > v_i$. By (BRS$_\sigma$), there is a player $\hat{\imath} \in \mathcal{I}$ and sequence $s_{\hat{\imath},\mathbb{N}} \in S_{\hat{\imath},\mathbb{N}}$ such that $\limsup_{n\to\infty} u_{\hat{\imath}}(s_{\hat{\imath},n}, \sigma_{\hat{\imath},n}) > v_{\hat{\imath}}$. This contradicts that σ_n is an ε_n-equilibrium for all n and $\varepsilon_n \to 0$. □

Remark 3.1. In Condition (BRS$_\sigma$), $\sigma_{\mathbb{N}}$ and $\hat{\sigma}_{\mathbb{N}}$ are sequences of strategy profiles with potentially infinite support. It is easy to see that Theorem 3.1 holds under the weaker condition that requires the condition to hold for $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ and $\hat{\sigma}_{\mathbb{N}} \in \hat{\Sigma}_{\mathbb{N}}$, for some discretizations $G_{\mathbb{N}}$ and $\hat{G}_{\mathbb{N}}$. The reasons we choose a stronger sufficient condition are that (1) in most games, they are equally difficult to show that they hold, and (2) the stronger condition permits establishing clear results relating approximable equilibria Nash equilibria, approximate equilibria, and better reply security (see Section 5).

Continuous games

A corollary from the previous result is that games with continuous payoffs are BRS-approximable, and hence their sets of Nash and limit equilibria coincide. It is also easy to see that (APP$_\sigma$) when u is continuous at σ; so, to prove BRS-approximability, (APP$_\sigma$) only needs to be checked at points where u is discontinuous.

Corollary 3.1. *If u is continuous at σ, then* (APP$_\sigma$) *holds. If u is continuous, then G is BRS-approximable, and its sets of Nash and approximable equilibria coincide.*

3.2 Sufficient conditions for BRS-approximability

In this section, we provide results that make proving the BRS-approximability (and hence the approximability) of a game simpler in practice. In Section 4, we illustrate their use through some examples, and we provide an additional result showing the approximability of a large class of discontinuous infinite games.

A simpler sufficient condition for (APP_σ)

The first result establishes that to prove that (APP_σ), it is sufficient to prove that it holds for a much smaller subset of strategy profiles.

Proposition 3.1. *Assume there is some* $i \in \mathcal{I}$ *such that* $(\mathrm{APP}_{(s_i,\sigma_{-i})})$ *holds for all* $s_i \in S_i$ *and* $\sigma_{-i} \in \Sigma_{-i}$ *with finite support. Then,* (APP_σ) *holds for all* $\sigma \in \Sigma$.

Proposition 3.1 simplifies proving that (APP_σ) holds for all σ, as showing the approximability of $(\sigma, u(\sigma))$ tends to be easier when $\sigma_i \in S_i$ and σ_{-i} has finite support. Example 4.5 illustrates why Proposition 3.1 cannot be further generalized to only verifying that (APP_s) holds for all $s \in S$.

A simpler sufficient condition for (BRS_σ)

We continue by defining another property that each $\sigma \in \Sigma$ may or may not satisfy.

$(\mathrm{BRS}'_{\sigma,i})$ For all $\sigma_{\mathbb{N}}, \hat{\sigma}_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $\sigma_n, \hat{\sigma}_n \to \sigma$, and all $G_{\mathbb{N}}$, there is $\hat{s}_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ such that $\limsup_{n\to\infty} u_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}) \geq \limsup_{n\to\infty} u_i(\sigma_n)$.

The following result establishes that checking $(\mathrm{BRS}'_{\sigma,i})$ in a convenient subset of strategy profiles is enough to prove that (BRS_σ) holds for all σ.

Proposition 3.2. *If* $(\mathrm{BRS}'_{\sigma,i})$ *holds for all* $i \in \mathcal{I}$ *and* $\sigma \in S_i \times \Sigma_{-i}$*, then* (BRS_σ) *holds for all* $\sigma \in \Sigma$.

Proposition 3.2 is useful for verifying that (BRS_σ) holds for all $\sigma \in \Sigma$. Indeed, as we shall see, many games of interest (e.g, Bertrand competition, auctions, or war of attrition) are such that, for each $\sigma \in S_i \times \Sigma_{-i}$, it is easy to compute the *maximum securable payoff* player i can achieve,

$$\overline{u}_i(\sigma) = \sup\left\{ \limsup_{n\to\infty} u_i(\sigma_n) \,\middle|\, \sigma_{\mathbb{N}} \in \Sigma^{\mathbb{N}} \text{ with } \sigma_n \to \sigma \right\}. \tag{3.1}$$

Furthermore, when s_i is interior of S_i, it is often the case that for any given $\sigma_{-i,n} \to \sigma$ and $S_{i,\mathbb{N}}$, there is some $s_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ such that $\lim_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) = \overline{u}_i(\sigma)$. Together, these observations can be used to show that $(\mathrm{BRS}'_{(s_i,\sigma_{-i}),i})$ holds for all $i \in \mathcal{I}$ and $(s_i, \sigma_{-i}) \in S_i \times \Sigma_{-i}$ whenever $s_i \in S_i$ is interior, and proving $(\mathrm{BRS}'_{(s_i,\sigma_{-i}),i})$ holds when s_i is not interior is typically straightforward.

3.3 Discussion

In this section, we discuss the concepts of approximable equilibrium and approximable game, and present some corollaries to Theorem 3.1.

Approximable equilibrium pairs

Some authors use equilibrium concepts containing both a strategy profile and a payoff vector (e.g., Bich and Laraki, 2017). In this section, we briefly discuss how this approach can be used in our analysis.

Definition 3.3. We say that $(\sigma,v)\in\Sigma\times\mathbb{R}^{\mathcal{I}}$ is an *approximable equilibrium pair for a discretization* $G_{\mathbb{N}}$ if there are two sequences $\sigma_{\mathbb{N}}\in\Sigma_{\mathbb{N}}$ and $\varepsilon_{\mathbb{N}}\to 0$ such that $(\sigma_n,u(\sigma_n))\to(\sigma,v)$ and for each n, σ_n is an ε_n-equilibrium of G_n.

For every approximable equilibrium $\sigma\in\Sigma$, there is some $v\in\mathbb{R}^{\mathcal{I}}$ such that (σ,v) is an approximable equilibrium pair for some discretization, but it maybe that no guarantee that there is some $v'\in\mathbb{R}^{\mathcal{I}}$ such that (σ,v') is an approximable equilibrium pair for all discretizations (see Example 4.5). In this later case, approximable equilibrium may seem unnatural, as the payoff profiles along sequences of ε-equilibria converging to σ may be rather different across different discretizations, hence providing a sense of fragility to the predictions. The following result states that, in BRS-approximable games, the two equilibrium concepts are equivalent.

Corollary 3.2. *Assume G is BRS-approximable. Then, (σ,v) is an equilibrium pair for some discretization, then it is an equilibrium pair for all discretizations.*

Sequences of Nash equilibria

Some previous work (see the literature review) has studied conditions for Nash equilibria to be limits of Nash equilibria along some discretization, which are often quite restrictive (see Sections 5 and 6 for some discussions). It can be anticipated that the conditions for Nash equilibria to be the limit of Nash equilibria along all discretizations.[5] Instead, focussing on limits of sequences of ε_n-equilibria for some $\varepsilon_n\to 0$ guarantees the existence of approximable equilibria for some discretization (recall Lemma 2.2) and also provides

[5] Continuity of u, for example, is not sufficient, as it can be seen in Example 4.1.

simple conditions to ensure that the set of approximable equilibria is independent of the discretizations (Theorem 3.1), which are written in terms of weak convergence of strategy profiles. The following result establishes that, under some conditions, both approaches coincide.[6]

Corollary 3.3. *If G is approximable and has a unique approximable equilibrium, then any sequence of Nash equilibria along some discretization converges to σ.*

Approximable sets

We now argue that sets that can be approximated through a discretization are not relevant to determine the set of approximable equilibria of the game.

We say that a set of actions $S' \equiv \times_{i \in \mathcal{I}} S'_i \subset S$ is *approximable* if there is a discretization $G_{\mathbb{N}}$ such that $S_n \cap S' = \emptyset$ for all $n \in \mathbb{N}$. The following is a corollary of Theorem 3.1.

Corollary 3.4. *Fix S and $\mathcal{I}$, and assume $u, \hat{u} : S \to \mathbb{R}^{\mathcal{I}}$ differ only on an approximable set. Then, if both $(S_i, u_i)_{i \in \mathcal{I}}$ and $(S_i, \hat{u}_i)_{i \in \mathcal{I}}$ are approximable, their sets of approximable equilibria coincide.*

An important implication of Corollary 3.4 is that, when the set of discontinuity points of u is approximable, then the value of u at such points is irrelevant to determine the set of approximable equilibrium of the game, as long as the game is approximable. This is the case in many games of important economic relevance. For example, in a Bertrand game (see Examples 4.6 and 4.7), u is discontinuous at a price profile s only if it satisfies that $s_i = s_{\hat{\imath}}$ for some $i, \hat{\imath} \in \mathcal{I}$. It is then easy to construct discretizations where $S_{i,n} \cap S_{\hat{\imath},n} = \emptyset$ for all $n \in \mathbb{N}$ and $i, \hat{\imath} \in \mathcal{I}$ with $i \neq \hat{\imath}$. Hence, the tie-breaking rule used to determine the outcome when two of the actions coincide is irrelevant to determine the set of approximable equilibria.

Restricted discretizations

Theorem 3.1 provides sufficient conditions ensuring that the set of approximable equilibria is the same for all possible discretizations. In some applications, nevertheless, it

[6]Coincide in the sense that if there is an outcome of approximable equilibria, this is the limit of all sequences of Nash equilibria. Nevertheless, as Example 4.2 shows, the limit strategy profile may not be a Nash equilibrium.

may be reasonable to restrict attention to a smaller set of discretizations. For example, it may be natural to require that the bids available to each bidder in an auction are the same; hence one may want to consider only discretizations $G_{\mathbb{N}}$ where $S_{i,n}=S_{\hat{\imath},n}$ for all $n\in\mathbb{N}$ and $i,\hat{\imath}\in\mathcal{I}$. One can further restrict attention to evenly spaced discretizations (where the increments are interpreted as the smallest monetary unit). We now briefly discuss how such a possibility may be incorporated into our analysis.

Let $\mathcal{G}$ be the set of all discretizations of G and $\hat{\mathcal{G}}\subset\mathcal{G}$ be a set of discretizations. We say that G is $\hat{\mathcal{G}}$-approximable if all discretizations in $\hat{\mathcal{G}}$ have the same set of approximable equilibria. Note that ($\mathcal{G}$-)approximable games are $\hat{\mathcal{G}}$-approximable for all sets $\hat{\mathcal{G}}\subset\mathcal{G}$. We can then define ($\text{APP}_\sigma^{\hat{\mathcal{G}}}$) like ($\text{APP}_\sigma$) but instead requiring approximability for only discretizations in $\hat{\mathcal{G}}$. Similarly, we can define ($\text{BRS}_\sigma^{\hat{\mathcal{G}}}$) like ($\text{BRS}_\sigma$) but instead requiring that the discretization is in $\hat{\mathcal{G}}$. Note that ($\text{APP}_\sigma^{\hat{\mathcal{G}}}$) and ($\text{BRS}_\sigma^{\hat{\mathcal{G}}}$) are weaker than ($\text{APP}_\sigma$) and ($\text{BRS}_\sigma$), respectively, as they only impose conditions to a subset of discretizations. Using the same arguments as before, we now have the following result.

Corollary 3.5. *Fix some set of discretizations $\hat{\mathcal{G}}\subset\mathcal{G}$. Then, if* ($\text{APP}_\sigma^{\hat{\mathcal{G}}}$) *and* ($\text{BRS}_\sigma^{\hat{\mathcal{G}}}$) *hold for all $\sigma\in\Sigma$, G is $\hat{\mathcal{G}}$-approximable.*

Corollary 3.5 indicates that most of our analysis can be generalized to only allow a small set of discretizations. Still, most relevant games are BRS-approximable, they are also $\hat{\mathcal{G}}$-BRS-approximable.

4 Examples

In this section, we use different examples to illustrate the previous concepts and results and show the approximability of some commonly-used infinite games. Examples in Sections 4.1 and 4.2 are of little economic relevance, but they are illustrative of some of the difficulties of finding conditions that are not restrictive that guarantee approximability. We illustrate in Section 4.3 how our conditions can be used to show the approximability of important infinite games with discontinuous payoffs.

4.1 Illustrative one-player games

We first provide some examples and discussions on the BRS-approximability (and approximability) of one-player games, where a single player takes an action $s\in S$ and

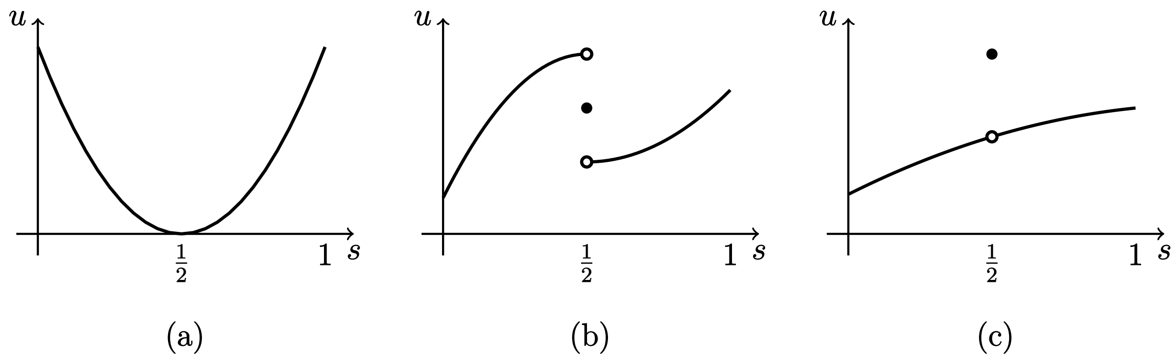

Figure 1

obtains a corresponding payoff $u(s) \in \mathbb{R}$. As we shall afterward see, such examples and discussions are also useful for multi-player games.

Before presenting the examples, we provide two useful observations. First, by Proposition 3.1 implies that, in one-player games, (APP_σ) holds for all $\sigma \in \Sigma$ if and only if it holds for all pure strategies. Second, Lemma 3.1 implies that for one-player games, when (BRS_σ) holds for all $\sigma \in \Sigma$ whenever (APP_σ) holds for all $\sigma \in \Sigma$. Then, by Corollary 3.1, proving the BRS-approximability of a one-player game reduces to proving that (APP_s) holds for all pure strategies $s \in S$ where u is discontinuous.

Example 4.1. Continuous payoff

Consider a one-player game with $S = [0,1]$ and $u(s) = (s - 1/2)^2$, depicted in Figure 1(a). Since u is continuous, the game is approximable (by Corollary 3.1). It is easy to see that the set of approximable equilibria of this game is $\Delta(\{0,1\})$.

We now use this example to illustrate the advantage of using limits of ε_n-equilibria in our definition of approximable equilibrium (Definition 3.3) instead of using ε_n-perfect equilibria as in the definition of lof perfect equilibria by Simon and Stinchcombe (1995) (recall Remark 2.1). We do so by constructing a discretization with no lof perfect equilibria. Such discretization is given by

$$S_n = \begin{cases} \{1/n, 2/n, ..., 1-1/n, 1\} & \text{if } n \text{ is even,} \\ \{0, 1/n, ..., 1-2/n, 1-1/n\} & \text{if } n \text{ is odd.} \end{cases}$$

Since, for each n, $s_n^* \equiv (1 + (-1)^n)/2$ is the only maximizer of u on S_n, any ε_n-perfect equilibrium of G_n assigns probability at least $1 - \varepsilon_n$ to s_n^*; hence, there is no lof perfect equilibrium for this discretization. In particular, this implies that there are discretizations with no convergent sequences of Nash equilibria.

Example 4.2. Jump discontinuities

Consider a one-player game with $S=[0,1]$ and u as depicted in Figure 1(b). We now argue that even though u features a "jump discontinuity" at $s=1/2$, the game is BRS-approximable. Indeed, since u is continuous at all $s \neq 1/2$, Proposition 3.1 implies that (APP$_s$) holds for all $s \neq 1/2$. See that, since $u(1/2) \in [u((1/2)^-), u((1/2)^+)]$, there is some $\alpha \in [0,1]$ such that $u(1/2) = \alpha u((1/2)^-) + (1-\alpha) u((1/2)^+)$. It is also clear that, for any discretization, one can take a strictly increasing sequence $s_n^{\uparrow} \nearrow 1/2$ and a strictly decreasing sequence $s_n^{\downarrow} \searrow 1/2$ so that

$$u(\alpha \circ s_n^{\uparrow} + (1-\alpha) \circ s_n^{\downarrow}) \to \alpha u((1/2)^-) + (1-\alpha) u((1/2)^+) = u(1/2)\ .$$

Since, by Proposition 3.1, (APP$_s$) holds for all pure strategies $s \in S$, the above observation implies that the game is approximable. The only approximable equilibrium pair is $(1/2, u((1/2)^-))$ (note that, nonetheless, $1/2$ is not a Nash equilibrium).

Games with jump discontinuities where payoffs can be approximated by a convex combination of close-by payoffs tend to be BRS-approximable. This occurs in many games of interest, where some "splitting rule" is used when players choose the same action. Examples include auctions, Bertrand competition, or war of attrition (see Section 4.3). When instead payoffs at discontinuous action profiles can *not* be approximated by convex combinations of close-by payoffs (e.g., under removable discontinuities, see the next example and Example 4.5), approximability typically fails.

Example 4.3. Removable discontinuities

Consider a one-player game with $S=[0,1]$ and u as depicted in Figure 1(c). Note that u features a "removable discontinuity" at $s=1/2$, which makes the game not approximable. Indeed, (APP$_{1/2}$) (where $1/2$ is the distribution assigning probability one to $s=1/2$) does not hold, because the payoff $u(1/2)$ cannot be approximated via the discretization $(\{1/n, ..., 1-1/n, 1\})_n$.

Stinchcombe (2005) argues using a similar example (see his Example 2.2, where the removable discontinuity is at $s=0$) that "the unique continuous selection [á la Simon and Zame (1990)] fails to capture the crucial strategic aspect of $s=0$, player 1's ability to guarantee her/himself a payoff of 2" and uses that to defend that "finitistic equilibria capture $s=0$ being available to player 1." While, in our example, $s=1/2$ is an approximable equilibrium for some discretizations (e.g., $(\{0,1/(2n),...,1-1/(2n),1\})_n$), we argued before it is not an approximable equilibrium for other discretizations. There is, nevertheless, a simple way to enhance the action space to obtain an equivalent game that is approximable. Consider the game $(\hat{S},\hat{u})$ defined by $\hat{S}\equiv[0,1]\cup\{2\}$ and

$$\hat{u}(\hat{s})\equiv\begin{cases} u(\hat{s}) & \text{if } \hat{s}\in[0,1]\backslash\{1/2\}, \\ u((1/2)^-) & \text{if } \hat{s}=1/2, \\ u(1/2) & \text{if } \hat{s}=2, \end{cases}$$

for all $\hat{s}\in\hat{S}$. Note that $\hat{u}$ is continuous on $\hat{S}$, hence now the game is approximable. In the new game, the unique approximable equilibrium assigns probability one to action 2. The enhancement of the action space (by adding the new action 2) incorporates that the payoff from choosing 2 (or 0 in the original game) is significantly different from that of choosing any other action. While this procedure can also be applied to games with more than one player (e.g., in Stinchcombe's example), it may not be applicable in some games. For example, in games with multiple players, the place of the removable discontinuity may depend on the actions of the other players.

It is finally worth stating an intuitive corollary to Theorem 3.1, which holds for games with any number of players. It states that games with removable discontinuities that "jump down" are approximable as long as $(\mathrm{BRS}'_{\sigma,i})$ holds for all i and σ.

Corollary 4.1. *Assume that, for any discretization $G_{\mathbb{N}}$ and $\sigma\in\Sigma$ (1) there is some $\sigma_{\mathbb{N}}\in\Sigma_{\mathbb{N}}$ with $\sigma_n\to\sigma$ and $\liminf_{n\to\infty}u_i(\sigma_n)\geq u_i(\sigma)$ for all $i\in\mathcal{I}$, and (2) $(\mathrm{BRS}'_{\sigma,i})$ holds. Then, G is approximable.*

4.2 Illustrative two-player games

In this section, we provide two examples of multiplayer games which are not approximable. The first example illustrates that (BRS_σ) may fail for strategy profiles σ satisfying that that set of best responses of one player vanishes as the other players' strategy approach σ_{-i}. The second example illustrates that (APP_σ) may fail when there are actions $s_i\in S_i$ such that u is discontinous at multiple elements of $\{s_i\}\times S_{-i}$. As the previous examples, they will be illustrative of the difficulties that proving approximability may pose.

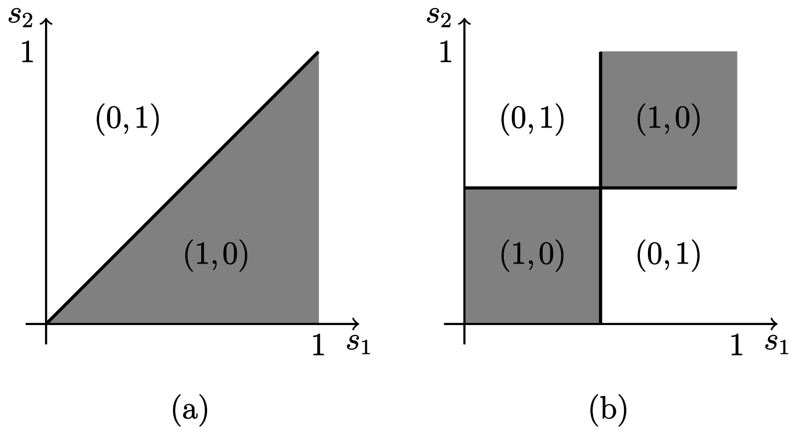

Figure 2

Example 4.4. The largest-number-wins game

We now study a classical example of a discontinuous game. In this game, two players (i.e., $\mathcal{I} \equiv \{1,2\}$) simultaneously choose a number in $[0,1]$. If the choice of a player is the largest number below 1 between the two choices, she obtains a payoff of 1, if the choices coincide, she obtains 1/2, and she obtains 0 otherwise. Hence, the payoff for each player i, depicted in Figure 2(a), is

$$u_i(s_i, s_{-i}) = \begin{cases} 1 & \text{if } s_i > s_{-i}, \\ 1/2 & \text{if } s_{-i} = s_i, \\ 0 & \text{otherwise.} \end{cases}$$

We argue that this game is *not* BRS-approximable. To see that, consider the sequences of pure strategy profiles $\sigma_{\mathbb{N}} = (1-4/n, 1-3/n)_n \in S^{\mathbb{N}}$ and $\hat{\sigma}_{\mathbb{N}} = (1-4/n, 1-1/n)_n \in S^{\mathbb{N}}$, both converging to $(1,1)$ and with corresponding payoffs converging to $(1,0)$. Note that the sequence $s_{1,\mathbb{N}} \equiv (1-2/n)_n \in S_1^{\mathbb{N}}$ satisfies

$$\limsup_{n\to\infty} u_1(s_{1,n}, \sigma_{2,n}) = 1 > 0 \ .$$

Consider the discretization

$$S_{1,n} = \{0, 2/n, 4/n, ..., 2-2/n, 2\} \ .$$

Note that, for all sequences $\hat{s}_{1,\mathbb{N}} \in S_{1,\mathbb{N}}$, we have $\lim_{n\to\infty} u_1(\hat{s}_{1,n}, \hat{\sigma}_{2,n}) = 0$; hence, $(\mathrm{BRS}_{(1,1)})$ does not hold.

Note that (APP_σ) holds for all σ. Hence, the lack of BRS-approximability of the largest-number-wins game arises from the fact that the set of best responses of player 2 shrinks as player 1 chooses a larger number (and the other way around). Hence, while there are strategy profiles converging to $s=(1,1)$ where player 2 obtains a payoff 1 along the sequence, player 2 cannot obtain such payoff when player 1 chooses, for each n, a number higher than the highest value available to player 2 in $\Sigma_{2,\mathbb{N}}$.

As noted in Section 3.3, the strategy profile $\sigma=(1,1)$ is the limit of ε_n-equilibria for all discretizations. Nevertheless, as we have shown, these equilibria may look quite different depending on the discretization considered, so the largest-number-wins game does not have any approximable equilibrium pair. Under some discretizations, player 1 obtains 1 along all ε_n-equilibria, while she obtains 0 along all ε_n-equilibria of some other discretizations.

Example 4.5. Counterexample to Proposition 3.1 with only pure strategies

In this example, we illustrate why Proposition 3.1 cannot be generalized to the statement "If (APP_s) holds for all $s\in S$, then (APP_σ) holds for all $\sigma\in\Sigma$." Such a result would be intuitive because, ultimately, elements of Σ are distributions over S, so it seems natural that if $u(s)$ can be approached through any discretization for all $s\in S$, then $u(\sigma)$ should be approachable through any discretization for all $\sigma\in\Sigma$. To see why this is not the case, consider a two-player game where $\mathcal{I}=\{1,2\}$ and $S_1=S_2=[0,1]$. The payoff function, depicted in Figure 2(b), is given by

$$u(s_1,s_2)=\begin{cases}(1,0) & \text{if } (s_1-1/2)(s_2-1/2)\geq 0,\\ (0,1) & \text{if } (s_1-1/2)(s_2-1/2)<0,\end{cases}$$

It is clear that (APP_s) holds for all $s\in S$. Consider now the strategy profile $\sigma\equiv(1/2,1/2\circ 0+1/2\circ 1)$, with corresponding payoff $u(\sigma)=(1,0)$. Define

$$S_{1,n}=S_{2,n}=\{-1,-1+1/(2n+1),...,1-1/(2n+1),1\}$$

for all $n\in\mathbb{N}$. It is clear that each $S_{i,\mathbb{N}}$ is a discretization of S_i and that $1/2\notin S_{i,n}$ for all n and i. Assume, for the sake of contradiction, there is a sequence $\sigma_{\mathbb{N}}\in\Sigma_{\mathbb{N}}$ such that $(\sigma_n,u(\sigma_n))\to(\sigma,u(\sigma))$. Since $1/2\notin S_{i,n}$ for all n and i, we have

$$u_1(\sigma_n)=\sigma_{1,n}([0,1/2))\,\sigma_{2,n}([0,1/2))+\sigma_{1,n}((1/2,0])\,\sigma_{2,n}((1/2,0])\ .$$

Weak convergence of $\sigma_{2,n}$ to σ_2 requires that $\sigma_{2,n}([0,1/2))$ and $\sigma_{2,n}((1/2,1])$ tend to 1/2 and 1/2, respectively. We then have $u_1(\sigma_n)\to 1/2$. This contradicts that $u(\sigma_n)\to u(\sigma)=(1,0)$; hence, (APP$_\sigma$) does not hold. Intuitively, when $s_1=1/2$, player 1's payoff is 1 when player 2 plays according to $1/2\circ 0+1/2\circ 1$. For all other values s_1, nevertheless, player 1 obtains 1/2. This type of discontinuity, discussed in more detail above (recall Example 4.3), makes the game not BRS-approximable.

If one insists on the prominence of actions $s_1=s_2=1/2$, one can expand the game to remove the removable discontinuity using the technique discussed above. For example, we can let $\hat{S}_1\equiv\hat{S}_2\equiv[0,1]\cup\{2\}$ and define

$$\hat{u}(s_1,s_2)\equiv\begin{cases}(1,0) & \text{if } (s_1-1/2)\,(s_2-1/2)>0 \text{ or } 2\in\{s_1,s_1\},\\ (0,1) & \text{if } (s_1-1/2)\,(s_2-1/2)<0 \text{ and } 2\notin\{s_1,s_1\},\\ (1/2,1/2) & \text{otherwise.}\end{cases}$$

Now, by playing action 2, each player can "induce" the payoff (1,0) (as could do by playing 1/2 in the original game). It follows that $(\hat{S}_i,\hat{u}_i)_{i\in\mathcal{I}}$ is now BRS-approximable.

4.3 Approximability in multiplayer games

We now show that even though many games of interest feature payoffs with a similar set of discontinuity points than the largest-number-wins game discussed before, they are approximable. Most of these results arise from the following proposition.

Proposition 4.1. *Assume that, for all $i\in\mathcal{I}$, we can write S_i as $S_i^1\times S_i^2$, where $S_i^1=[0,1]$. Assume also that, for each $i\in S$ and $s\in S$, we have*[7]

$$u_i(s)=u_i^1(s)+\frac{\mathbb{I}_{s_i^1=\max_{i\in\mathcal{I}}s_i^1}}{|\{\hat{\imath}|s_{\hat{\imath}}^1=\max_{i'\in\mathcal{I}}s_{i'}^1\}|}\,u_i^2(s)\ , \tag{4.1}$$

where $u_i^1,u_i^2:S\to\mathbb{R}$ are continuous. Then, (APP$_\sigma$) holds for all $\sigma\in\Sigma$. Assume that, additionally, for each $s_i\in S_i$, either $u_i^2(s_i,s_{-i})\geq 0$ or $u_i^2(s_i,s_{-i})\leq 0$ for all $s_{-i}\in S_{-i}$ satisfying $\max_{\hat{\imath}\in\mathcal{I}}s_{\hat{\imath}}^1=s_i^1$, and also

$$u_i^2(s)\leq 0 \text{ when } s_i^1=\max_{\hat{\imath}\neq i}s_{\hat{\imath}}^1=1 \quad\text{and}\quad u_i^2(s)\geq 0 \text{ when } s_i^1=\max_{\hat{\imath}\neq i}s_{\hat{\imath}}^1=0\ . \tag{4.2}$$

Then, (BRS$_\sigma$) holds for all $\sigma\in\Sigma$.

[7] For a predicate P, we have $\mathbb{I}_P=1$ if P is true and $\mathbb{I}_P=1$ if P is false.

We provide an intuition for the proof of Proposition 4.1 in Example 4.6. Proposition 4.1 is general enough to imply the approximability of the most common discontinuous infinite games, and it can be generalized in several directions (e.g., more general tie-breaking rule or allowing different S_i^1 across players).

Example 4.6. Bertrand competition between two sellers

Consider a Bertrand game between two buyers, $\mathcal{I} \equiv \{1,2\}$, with corresponding valuations $v_1, v_2 \in (0,1]$.[8] We assume that supply is inelastic (see next example for the general case). We assume buyers set prices in $S_1 = S_2 = [0,1]$. Player i's payoff is given for a price profile s by equation (4.1) with

$$u_i^1(s) = 0 \quad \text{and} \quad u_i^2(s) = v_i - s_i \ .$$

Given that the conditions of Proposition 4.1 are satisfied, the game is approximable. We now provide an intuition why this is the case. We first fix some $(s_1, \sigma_2) \in S_1 \times \Sigma_2$, and we will prove that both $(\text{APP}_{(s_1,\sigma_2)})$ and $(\text{BRS}_{(s_1,\sigma_2)})$ hold. By Propositions 3.1 and 3.2, and by the fact that an analogous argument can be made for strategy profiles in $\Sigma_1 \times S_2$, we will conclude that the game is BRS-approximable, and hence it is approximable. We note that $u(s_1, \sigma_2)$ is equal to

$$\int_{[0,1]\setminus\{s_1\}} u(s_1, \tilde{s}_1)\, \sigma_2(\mathrm{d}\tilde{s}_2) + \sigma_2(\{s_1\})\left(\tfrac{1}{2}(v_1 - s_1, 0) + \tfrac{1}{2}(0, v_2 - s_1)\right) ,$$

and also that

$$\lim_{s_1' \nearrow s_1} u(s_1', \sigma_2) = \int_{[0,1]\setminus\{s_1\}} u(s_1, \tilde{s}_2)\, \sigma_2(\mathrm{d}\tilde{s}_2) + \sigma_2(\{s_1\})\,(v_1 - s_1, 0) \ ,$$
$$\lim_{s_1' \searrow s_1} u(s_1', \sigma_2) = \int_{[0,1]\setminus\{s_1\}} u(s_1, \tilde{s}_2)\, \sigma_2(\mathrm{d}\tilde{s}_2) + \sigma_2(\{s_1\})\,(0, v_2 - s_1) \ .$$

Assume first $s_1 \notin \{0,1\}$. It is then not difficult to see that, for each $\sigma_{2,\mathbb{N}} \in \Sigma_2^{\mathbb{N}}$ with $\sigma_{2,n} \to \sigma_2$, discretization $S_{1,\mathbb{N}}$, and $\sigma_{1,\mathbb{N}} \in \Sigma_{1,\mathbb{N}}$ with $\sigma_{1,n} \to s_1$, we have that if the limit of $u(\sigma_n)$ as $n \to \infty$ exists, then there is some $\alpha \in [0,1]$ with

$$u(\sigma_n) \to \int_{[0,1]\setminus\{s_1\}} u(s_1, \tilde{s}_2)\, \sigma_2(\mathrm{d}\tilde{s}_2) + \alpha\,(v_1 - s_1, 0) + (1-\alpha)\, u(0, v_2 - s_1) \ . \tag{4.3}$$

[8] We study Bertrand competition between buyers instead of between sellers so that the highest price obtains the gooda and we can apply Proposition 4.1 directly. A straightforward renormaliation allows obtaining the same result for the case with two competing sellers.

Conversely, for each $\alpha \in [0,1]$, $\sigma_{2,\mathbb{N}} \in \Sigma_2^{\mathbb{N}}$ with $\sigma_{2,n} \to \sigma_2$, and discretization $S_{1,\mathbb{N}}$, there is a sequence $\sigma_{1,\mathbb{N}} \in \Sigma_{1,\mathbb{N}}$ with $\sigma_{1,n} \to s_1$ such that equation (4.3) holds. Intuitively, for a given sequence $\sigma_{2,\mathbb{N}}$ with $\sigma_{2,n} \to \sigma_2$ and $S_{1,\mathbb{N}}$, one can choose two sequences $s_{1,\mathbb{N}}^-, s_{1,\mathbb{N}}^- \in S_{1,\mathbb{N}}$ converging to s_1 from below and above, respectively, at a slow rate so that

$$\lim_{n\to\infty} u(s_{1,n}^-, \sigma_{2,n}) = \lim_{s_1' \nearrow s_1} u(s_1', \sigma_2) \quad \text{and} \quad \lim_{n\to\infty} u(s_{1,n}^+, \sigma_{2,n}) = \lim_{s_1' \searrow s_1} u(s_1', \sigma_2) \ .$$

For each $\alpha \in [0,1]$, we can then define $\sigma_{1,n} \equiv \alpha \circ s_{1,n}^- + (1-\alpha) \circ s_{1,n}^+$, so equation (4.3) holds. From these observations, it follows that both $(\text{APP}_{(s_1,\sigma_2)})$ and $(\text{BRS}_{(s_1,\sigma_2)})$ hold.

Consider now the case $s_1 = 1$. This case has the potential to be problematic for the same reasons that made the largest-number-wins game not BRS-approximable. Indeed, we can use the sequences considered in the analysis of the largest-number-wins game above to show that, while there are sequences $\sigma_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $\sigma_n \to (1,1)$ with $u_1(\sigma_n) \to v_1 - 1$, there are partitions $S_{1,\mathbb{N}}$ such that there is no $s_{1,\mathbb{N}} \in S_{1,\mathbb{N}}$ with $u_1(s_{1,n}, \sigma_{2,n}) \to v_1 - 1$. Nevertheless, differently from the largest-number-wins game, player 1's inability to choose a higher action than player 2 along the sequence does not hurt him: by the previous argument, it is easy to see that, independently of $S_{1,\mathbb{N}}$, there is a sequence $\sigma_{1,\mathbb{N}} \in S_{1,\mathbb{N}}$ slowly converging toward 1 that gives player 1 a limit payoff of $0 > v_1 - 1$. We can then use a similar argument to the one for the case $s_1 \in (0,1)$ to show that $(\text{APP}_{(s_1,\sigma_2)})$ and $(\text{BRS}_{(s_1,\sigma_2)})$ hold when $s_1 = 1$. A similar argument also works for $s_1 = 0$.[9]

Example 4.7. Bertrand competition between more than two sellers, with entry decisions and general demand

Consider now a Bertrand game between $|\mathcal{I}| \geq 2$ buyers. Now, we assume that the action space of each seller i is $[0,1] \cup \{-1\}$, where $s_i = -1$ signifies that buyer i does not enter the market, while $s_i \in [0,1]$ is interpreted as the price buyer i sets if she enters the market at cost c_i. Again, buyer i's valuation is v_i. Buyer i's payoff is then given by equation (4.1) with

$$u_i^1(s) = \mathbb{I}_{s_i \in [0,1]} - c_1 \quad \text{and} \quad u_i^2(s) = \mathbb{I}_{s_i \in [0,1]} \, Q(s_i) \, (v_i - s_i) \ ,$$

[9]Condition (4.2) ensures that the same logic can be applied in the more general case. When, for example, $s_i^1 = \max_{\hat{\imath} \neq i} s_{\hat{\imath}}^1 = 1$, player i benefits by setting an action slightly below 1, as such an action eliminates the second term on the right side of (4.1), making him weakly better off because $u_i^2(s) \leq 0$.

Auction	$u_i^1(s)$	$u_i^2(s)$
First price	0	$v_i - s_i$
Second price	0	$v_i - \max_{\hat{\imath}\neq i} s_{\hat{\imath}}$
All pay	$-s_i$	v_i

Table 1

where $Q:[0,1]\to[0,1]$ is a decreasing right-continuous function interpreted as the supply.

Given that the conditions of Proposition 4.1 are satisfied, the game is approximable. The intuition is analogous to the proof for the simple two-buyer model with inelastic supply. Intuitively, the role of σ_2 is replaced by the distribution of the largest price offered by buyer 1's opponents. The addition of the choice of entering or not the market also does not affect the arguments.

Example 4.8. Auctions

Consider now an auction of a single indivisible good with N bidders. Each bidder i values the good at $v_i \in (0,1]$ and chooses a bid $s_i \in [0,1]$. The object is allocated uniformly among the bidders with the highest bid. It is easy to see that for all standard auction formats (first price, second price, and all pay), bidder i's payoff is given by Equation (4.1), where the different u_i^1 and u_i^2 are provided in Table 1. It is also easy to see that the conditions of Proposition 4.1 are satisfied; hence all standard auctions are approximable.

Example 4.9. War of attrition

We now consider a game of war of attrition between two players, $i \in \{1,2\}$, each with discount rate r_i. For simpliity, we normalize the action space so that each player chooses $s_i \in [0,1]$, which is interpreted as choosing to stop at time $t_i \in [0,+\infty]$ satisfying $s_i = 1 - e^{-t_i}$. The payoff from an action profile s is given by equation (4.1) with

$$u_i^1(s) = e^{r_i \log(1-s_i)} R_i^1 \quad \text{and} \quad u_i^2(s) = e^{r_i \log(1-s_{-i})} R_i^2 - e^{r_i \log(1-s_i)} R_i^1 \ .$$

with $R_i^2 > R_i^1 > 0$. That is, player i obtains $e^{-r_i t_i} R_i^1$ when $t_i < t_{-i}$, $e^{-r_i t_{-i}} R_i^2$ when $t_i > t_{-i}$, and half of each when $t_i = t_{-i}$. Given that the conditions of Proposition 4.1 are satisfied, the game is approximable. This result can be generalize to allow for more players, random exogenous exit times, time-dependent rewards, etc.

5 Characterizations of approximable equilibria

In this section, we provide different characterizations of approximable equilibria in approximable games. We first generalize the concept of discretization and approximable equilibrium to allowing sequences of games with infinite action spaces. We then show that Nash equilibria are approximable equilibria, that approximable equilibria coincide with limits of ε-equilibria as $\varepsilon \to 0$, and that we characterize approximable equilibrium in terms of a local incentive compatibility condition.

5.1 Fully-approximable equilibria

We say that $S_{i,\mathbb{N}}$ is an *approximation* of S_i if $S_{i,n} \to S_i$ (under the Hausdorff metric) and, for each n, $S_{i,n} \subset S_i$. We say that $G_{\mathbb{N}} \equiv ((S_{i,n}, u_i)_{i\in\mathcal{I}})_n$ is an approximation of G if, for each $i \in \mathcal{I}$, the $S_{i,\mathbb{N}}$ is an *approximation* of S_i. We say that $\sigma \in \Sigma$ is a *fully-approximable equilibrium for an approximation* $G_{\mathbb{N}}$ if there are $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ and $\varepsilon_{\mathbb{N}} \to 0$ such that $\sigma_n \to \sigma$ and for each n, σ_n is an ε_n-equilibrium of G_n. We say that G is *fully-approximable* if the set of fully-approximable equilibria is the same for all of the approximations.

We use ($\text{APP}_\sigma^{\text{full}}$) and ($\text{BRS}_\sigma^{\text{full}}$) to denote the properties analogous to (APP_σ) and (BRS_σ) allowing $G_{\mathbb{N}}$ in them to be approximations instead of discretizations. If ($\text{APP}_\sigma^{\text{full}}$) and ($\text{BRS}_\sigma^{\text{full}}$) hold for all σ, we say that G is *BRS-fully-approximable.* Analogous arguments to those in the proof of Theorem 3.1 show that if G is BRS-fully-approximable then it is fully-approximable. We now further show that discretizations are sufficient to characterize BRS-full-approximability.

Theorem 5.1. *G is BRS-approximable if and only if it is BRS-fully-approximable.*

As indicated in Section 3.3, that BRS-full-approximability implies BRS-approximability follows from the fact that (APP_σ) and (BRS_σ) are easier to satisfy than ($\text{APP}_\sigma^{\text{full}}$) and ($\text{BRS}_\sigma^{\text{full}}$), respectively. The reverse direction follows from the fact that all approximations contain a discretization. As we will now see, Theorem 5.1 implies that, if G is BRS-approximable, Nash equilibria are approximable equilibria, and approximable equilibria are approximate equilibria and are solutions for some sharing rule.

5.2 Relationship to Nash equilibria

A first implication from Theorem 5.1 is that Nash equilibria are approximable equilibria.

Corollary 5.1. *Assume G is BRS-approximable. Then, σ is a Nash equilibrium if and only if $(\sigma, u(\sigma))$ is an approximable equilibrium pair.*

BRS-approximability is then sufficient for Nash equilibria te be approximable through almost-optimal behavior in finite games. Intuitively, if there is a player who can profitably deviate from a given strategy profile σ, then (BRS_σ) guarantees that she can benefit from deviating from a close-by strategy profile in any close-by finite game. Conversely, if a player can profitably deviate in a convergent sequence of strategy profiles, she can also profitably deviate from the limit strategy profile.

Remark 5.1. When, for a discretization $G_\mathbb{N}$, there is a sequence $\sigma_\mathbb{N} \in \Sigma_\mathbb{N}$ with $\sigma_n \to \sigma$ where each σ_n is a Nash equilibrium of G_n, we have that σ is an approximable equilibrium. It is nonetheless *not* true that all approximable equilibria can be approximated by Nash equilibria of *some* discretization, not even in BRS-approximable games. To see that, consider a one-player game with $\hat{S} = [0,1] \cup \{2\}$ and payoff given by Figure 1(b) when $s \in [0,1]$ and $u(2) = u((1/2)^-)$. It is easy to prove that this game is approximable (see Section 4.1 for analogous arguments), and its set of approximable equilibria is $\Delta(\{1/2, 2\})$. Nevertheless, any discretization $G_\mathbb{N}$ has the property that any sequence $\sigma_\mathbb{N} \in \Sigma_\mathbb{N}$ where each σ_n is a Nash equilibrium of G_n converges to 2. Still, we believe that $s = 1/2$ (or mixings between 1/2 and 2) is also a plausible prediction since for any $\varepsilon > 0$, there is some $\hat{u}$ with $\|\hat{u} - u\| < \varepsilon$ (under the sup-norm) such that the unique Nash/approximable equilibrium of $(\hat{S}, \hat{u})$ is ε-close to $s = 1/2$ (under the weak distance).

5.3 Relationship to approximate equilibria

Because the constant sequence $(G)_n$ is an approximation of G, it follows from Theorem 5.1 that, in BRS-approximable games, the set of approximable equilibria coincides with the set of *approximate equilibria*, that is, the set of limits of ε-equilibria (of G) as $\varepsilon \to 0$.

Corollary 5.2. *If G is BRS-approximable, then σ is an approximable equilibrium if and only it is an approximate equilibrium.*

Corollary 5.2 is useful in that it permits characterizing approximable equilibria without using discretizations. Furthermore, as we will now see, it permits showing that approximable equilibria are solutions for some sharing rule.

Relationship to Simon and Zame (1990). Given a game G, one can define a corresponding *game with endogenous sharing rules* by first defining the correspondence

$$U(s) = \mathrm{co}(\{v \in \mathbb{R}^{\mathcal{I}} | (s,v) \in \overline{\Gamma}\}) \quad \text{where } \overline{\Gamma} \equiv \overline{\{(s,u(s)) | s \in S\}}.$$

A *sharing rule* is a function $\hat{u} : S \to \mathbb{R}^{\mathcal{I}}$ such that $\hat{u}(s) \in U(s)$ for all $s \in S$. A *solution* for a sharing rule $\hat{u}$ is a Nash equilibrium of the game $\langle \mathcal{I}, S, \hat{u} \rangle$. Simon and Zame (1990) show that all games with an endogenous sharing rule have a solution. The following corollary follows from Bich and Laraki (2017)'s result showing that approximate equilibria are solutions for some sharing rule.

Corollary 5.3. *If G is BRS-approximable, an approximable equilibrium is a solution for some sharing rule.*

Corollary 5.3 establishes the candidates for approximable equilibria are solutions for some sharing rules, which do not require considering sequences of strategy profiles or discretizations. Such a necessary condition is nevertheless not sufficient. The reason is, in part, that, as Stinchcombe (2005) points out, "It allows for profitable deviations and introduces spurious correlation between players' choices". See Example 4.5 for an example of a non-BRS-approximable game with a Nash equilibrium (which is also a solution for some sharing rule) that is not an approximable equilibrium.

5.4 Securable payoffs

For the final characterization of approximable equilibria, we define the vector of *securable payoffs* for each pair (σ, v), denoted $\overline{u}(\sigma|v)$, as

$$\overline{u}_i(\sigma|v) \equiv \inf_{\sigma_{\mathbb{N}} \in \Sigma^{\mathbb{N}} | (\sigma_n, u(\sigma_n)) \to (\sigma, v)} \; \sup_{s_{i,\mathbb{N}} \in S_i^{\mathbb{N}}} \; \limsup_{n \to \infty} \overline{u}_i(s_{i,n}, \sigma_{-i,n}) \quad \text{for all } i \in \mathcal{I},$$

and $\overline{u}_i(\sigma|v) = -\infty$ if there is no $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ with $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$. That is, $\overline{u}_i(\sigma|v)$ is the lowest limit payoff that player i can secure from deviating from a sequence of strategy profiles and payoff vectors converging to (σ, v). Note that while $\overline{u}_i(\sigma|v)$ is defined using sequences of strategy profiles, its definition does not involve discretizations of G. The following proposition shows that (σ, v) is an approximable equilibrium pair if and only if each player i obtains $\overline{u}_i(\sigma|v)$.

Proposition 5.1. *If G is BRS-approximable, (σ, v) is an approximable equilibrium pair if and only if $\overline{u}(\sigma|v) = v$.*

Relationship to better-reply secure. Proposition 5.1 is reminiscent of Reny (1999)'s condition named "better-reply secure", which is used (together with other conditions) to ensure the existence of Nash equilibria in infinite games with discontinuous payoffs.

Reny (1999) says that player i can *secure* a payoff $v_i \in \mathbb{R}$ at $s \in S$ if there is $\tilde{s}_i$ such that $u_i(\tilde{s}_i, s'_{-i}) \geq v_i$ for all s'_{-i} in some neighborhood of s_{-i}. He then says that G is *better-reply secure* if whenever (s^*, u^*) is in the closure of the graph of its vector payoff function and s^* is not a Nash equilibrium, some player can secure a payoff strictly above u_i^* at s^*. He shows, among other results, that if the game is better-reply secure and quasiconcave, it admits a Nash equilibrium in pure strategies.[10] For non-quasiconcave games, this result is usually applied to the mixed extension of the game (since each Σ_i is convex and $u_i : \Sigma_i \times \{\sigma_{-i}\} \to \mathbb{R}$ is quasiconcave for all $\sigma_{-i} \in \Sigma_{-i}$). Nevertheless, showing better-reply security may be difficult in those games, since verifying the required condition in a neighborhood of the opponent's strategy profile (using the distance d_Σ defined in Appendix B) may difficult because Σ infinite-dimensional.

An important step in the proof of Proposition 5.1 shows that (BRS_σ) implies that any player i can achieve her securable payoff $\overline{u}_i(\sigma|v)$ across any sequence converging to (σ, v); that is, for any $\sigma_\mathbb{N} \in \Sigma^\mathbb{N}$ with $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$, we have

$$\limsup_{n\to\infty} \sup_{s_{i,\mathbb{N}} \in S_i^\mathbb{N}} u_i(s_{i,n}, \sigma_{-i,n}) = \overline{u}_i(\sigma|v) \ .$$

It then follows from (BRS_σ) that, similarly to Reny's condition, if $\overline{u}(\sigma|v) > v$, then (σ, v) is *not* an approximable equilibrium pair since, for all strategy profiles close to σ with payoff close to v, some player i can secure a payoff strictly higher than v_i.

Even though the uses of better-reply security and BRS-approximability are used for different purposes (showing the existence of Nash equilibria versus showing approximability), it is perhaps insightful to compare them. We first note that Theorem 3.1 holds when (BRS_σ) is replaced by a property similar to better-reply secure, namely "If v is such that (σ, v) is *not* an approximable equilibrium pair for some discretization then, for any discretization $G_\mathbb{N}$ and $\sigma_\mathbb{N} \in \Sigma_\mathbb{N}$ with $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$, there is some $i \in \mathcal{I}$ and $s_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ such that $\limsup_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) > v_i$." Like better-reply secure, this formulation only requires showing the property for non-approximable equilibria. Still, we perceive (BRS_σ) as easier to use, especially using the simplification provided by Proposition 3.2.

[10] G is quasi-concave if each S_i is convex and $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R}$ is quasiconcave.

6 Discussion and conclusion

6.1 Discretizations with the "most difference it can make to anyone" metric

In this section, we compare our approach and results with those in Fudenberg and Levine (1986), Harris, Stinchcombe, and Zame (2005), and Stinchcombe (2005).

Fudenberg and Levine (1986) studied discretizations of infinite normal-form games. Instead of using the inherent metric in the action spaces (they also do not assume they are compact), they defined the distance between any two $\sigma_i, \sigma_i' \in \Sigma_i$ as[11]

$$d_{FL}(\sigma_i, \sigma_i') = \sup_{\sigma_{-i} \in \Sigma_{-i}} \max_{\hat{\imath} \in \mathcal{I}} \left| u_{\hat{\imath}}(\sigma_i, \sigma_{-i}) - u_{\hat{\imath}}(\sigma_i', \sigma_{-i}) \right| .$$

As they point out, the main advantage of using this distance is that the "product topology from d_{FL} is the coarsest product uniformity such that the u_i are uniformly continuous" (their Proposition 3.5). Their definition of discretization (or approximation) coincides with ours, but using d_{FL}. Their main result (their Proposition 3.2) states that if $G_{\mathbb{N}}$ approximates G (under d_{FL}), then the limit of any converging sequence of ε_n-equilibria of each G_n for some $\varepsilon_{\mathbb{N}} \to \varepsilon$ is an ε-equilibrium of G, and also that if σ is an ε-equilibrium of G and $\sigma_n \to \sigma$, then there is a sequence $\varepsilon_{\mathbb{N}} \to \varepsilon$ of such that each σ_n is an ε_n-equilibrium of G_n.

Harris, Stinchcombe, and Zame (2005) show that a discretization of S exists (under d_{FL}) if and only if the game is nearly compact and continuous, that is, if it can be understood as a game played on strategy spaces that are dense subsets of the strategy spaces of larger compact games (under d_{FL}) with jointly continuous payoffs.

Stinchcombe (2005) studies games that are not nearly compact and continuous. He says (his Section 2.3) that $S_{i,\mathbb{N}}$ *converges* to S_i if for all finite $S_i' \subset S_i$, for all sufficiently large n, $S_i' \subset S_{i,n}$. Note that if $S_{i,\mathbb{N}}$ converges to S_i (under Stinchcombe's definition) then either S_i is discrete or $S_{i,n}$ is infinite (and has the same cardinality of S_i) for n is large enough. That is, if, for example, S_i has the cardinality of the continuum (or higher), a sequence of finite sets cannot converge to S_i. He then says that σ is a *finitistic*

[11] As Fudenberg and Levine note, d_{FL} is not a metric because $d_{FL}(\sigma_i, \sigma_i')$ may be 0 for strategically equivalent pairs of strategies σ_i and σ_i' (in fact, d_{FL} is a *pseudometric*). Harris, Stinchcombe, and Zame (2005) call this pseudometric the "most difference it can make to anyone."

equilibrium (see his Definition 2.2) if there is some $G_{\mathbb{N}}$ converging to G, $\sigma_{\mathbb{N}}$ converging to σ (under the d_{FL} distance), and $\varepsilon_{\mathbb{N}}$ converging to 0, such that σ_n is an ε_n-equilibrium of G_n for all n. Similar to our Lemma 2.2 and Corollary 5.3, he shows that all games have a finitistic equilibrium and, if each S_i is a compact Hausdorff space, then every finitistic equilibria is a solution for some sharing rule.

Comparison: The continuity of payoff functions under d_{FL} eases the theoretical analysis (in a similar way that, by our Corollary 3.1, G is approximablewhen u is continuous). Still, we believe our approach offers numerous advantages with respect to theirs. First, our distance is defined independently for each S_i. This permits easily assessing whether S is compact, because many games have well-known action sets with corresponding inherent distances, such as real intervals (e.g., in pricing games) or sets of bounded functions (e.g., in auctions). In contrast, assessing the compactness of S under d_{FL} is difficult in general. In fact, proving that S is d_{FL}-compact is, in general, difficult, and hence the existence of discretization of S is not guaranteed, as well as the existence of approximable equilibria (proving that a game is nearly compact and continuous is difficult in general). Second, d_{FL} may be neither easy to compute nor intuitive. Instead, inherent distances provide an intuitive idea of proximity for real variables (e.g., prices, effort, signals, or types) or even function-valued variables (e.g., bidding in auctions), which is independent of the other aspects of the game; if $S_i \subset \mathbb{R}^k$ for some $k \in \mathbb{N}$, close (far away) actions under any inherent distance may be far away (close) in the d_{LP} distance. For example, in the largest-number-wins game described above, $d_{FL}(s_1, s_1') = 1$ for all $s_1, s_1' \in (0,1)$ with $s_1 \neq s_1'$. In the Bertrand game described above, the distance between $s_1, s_1' \in [c_1, 1]$ with $s_1 \neq s_1'$ is

$$d_{FL}(s_1, s_1') = \max\{s_1, s_1'\} - c_1 \ .$$

As a result, neither of these two games is compact under d_{FL}.[12] Third, the papers above focus on pure strategies, while our analysis allows for studying mixed strategies. Of course, mixed strategies are actions of the mixed strategy extension of the game. Nevertheless, while working with the mixed strategy extension is elegant, discretizing of the set of mixed strategies seems rather unnatural.

[12]Fudenberg and Levine (1986) provide some results using the inherent topology. They show, for example, that if S_n approximates S (under d_{FL}), then the limit of a sequence of sets of ε-best responses to a sequence $s_{\mathbb{N}} \in S_{\mathbb{N}}$ converging to some s coincides with the set of ε-best responses to $s \in S$.

6.2 Conclusions

The use of infinite games to model strategic interactions is prevalent in economics. While working with infinite action spaces is technically convenient, doing so raises a number of concerns. This paper's main contribution is to obtain simple conditions that permit addressing two important concerns: the approximability of optimal behavior and the independence of the approximation used for numerical analysis.

Overall, our work contributes to the study of the approximability of infinite normal-form games. Under our conditions, predictions are independent of how the game is discretized, and hence robust to possible approximability concerns. In the follow-up paper Dilmé (2023), we use similar techniques to study the approximability of infinite extensive-form games.[13]

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[13]See Reeves and Wellman (2012), Kroer and Sandholm (2015), and Carbonell-Nicolau (2021), for some related work.

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A Omitted proofs

Proof of Corollary 3.1

Proof. Assume first that u is continuous at $\sigma \in \Sigma$. To prove that (APP$_\sigma$) holds, fix a discretization $G_{\mathbb{N}}$. By Proposition B.3, there is a sequence $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ with $\sigma_n \to \sigma$. By continuity of u, we have that $(\sigma_n, u(\sigma_n)) \to (\sigma, u(\sigma))$, hence (APP$_\sigma$) holds.

We now assume now u is continuous and we prove that G is approximable. Fix some $\sigma \in \Sigma$. By the previous argument, (APP$_\sigma$) holds. To prove that (BRS$_\sigma$) holds, consider two sequences $\sigma_{\mathbb{N}}, \hat{\sigma}_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$ and $(\hat{\sigma}_n, u(\hat{\sigma}_n)) \to (\sigma, v)$, for some $v \in \mathbb{R}^{\mathcal{I}}$, and assume there is some $s_{i,\mathbb{N}} \in S_i^{\mathbb{N}}$ with $\limsup_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) > v_i$. Note that, by continuity, $v = u(\sigma)$. There is then a strictly increasing sequence of indexes $\hat{n}_{\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $s_{i,\hat{n}_{\mathbb{N}}}$ converges to some $s_i \in S_i$ and

$$(s_{i,\hat{n}_n}, \sigma_{-i,\hat{n}_n}, u(s_{i,\hat{n}_n}, \sigma_{-i,\hat{n}_n})) \to (s_i, \sigma_{-i}, u(s_i, \sigma_{-i})) \ ,$$

where $u_i(s_i, \sigma_{-i}) > u_i(\sigma)$. Fix then some discretization $G_{\mathbb{N}}$, and a sequence in $\hat{s}_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ with $\hat{s}_{i,n} \to s_i$. By continuity, we have that

$$(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}, u(\hat{s}_{i,n}, \hat{\sigma}_{-i,n})) \to (s_i, \sigma_{-i}, u(s_i, \sigma_{-i})) \ .$$

The result then holds because $u_i(s_i, \sigma_{-i}) > u_i(\sigma)$.

We now assume that u is continuous and prove that sets of Nash and approximable equilibria coincide. Let first $\sigma \in \Sigma$ be a Nash equilibrium, $G_{\mathbb{N}}$ be a discretization, and $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ be such that $\sigma_n \to \sigma$. Assume, for the sake of contradiction, that there are some $i \in \mathcal{I}$ and a sequence $s_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ such that $\limsup_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) > u_i(\sigma)$. By (BRS$_\sigma$), there is a player $\hat{\imath} \in \mathcal{I}$ and sequence $s_{\hat{\imath},\mathbb{N}} \in S_{\hat{\imath},\mathbb{N}}$ such that $\limsup_{n\to\infty} u_{\hat{\imath}}(s_{\hat{\imath},n}, \sigma_{-\hat{\imath}}) > u_i(\sigma)$, which contradicts that σ is a Nash equilibrium. Assume now that $\sigma \in \Sigma$ is an approximable equilibrium for a discretization $G_{\mathbb{N}}$, and let $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ be such that $\sigma_n \to \sigma$ and each σ_n is an ε_n-equilibrium of G_n for some $\varepsilon_{\mathbb{N}} \to 0$. If there were some $i \in \mathcal{I}$ and $s_i \in S_i$ such that $u_i(s_i, \sigma_{-i}) > u_i(\sigma)$, then (BRS$_\sigma$) would imply that there is some $s_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ such that $\limsup_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) > u_i(\sigma)$, which would contradict that each σ_n is an ε_n-equilibrium of G_n and $\varepsilon_n \to 0$. Hence, σ is a Nash equilibrium. □

Proof of Proposition 3.1

Proof. Assume there is some $i \in \mathcal{I}$ such that (APP$_\sigma$) holds for all σ satisfying that $\sigma_i \in S_i$ and σ_{-i} has finite support. Fix some $\sigma \in \Sigma$ satisfying that $\sigma_i \in S_i$ and σ_{-i} has finite support, and also fix some discretization $G_\mathbb{N}$. By Proposition B.4, there is a discretization $\hat{G}_\mathbb{N}$ and some $\hat{\sigma}_\mathbb{N} \in \hat{\Sigma}_\mathbb{N}$ with $(\hat{\sigma}_n, u(\hat{\sigma}_n)) \to (\sigma, u(\sigma))$. Fix some $\hat{n} \in \mathbb{N}$. By our initial assumption, and since $\hat{\sigma}_{\hat{n},-i}$ has finite support, we have that for each $\hat{s}_i \in \hat{S}_{i,\hat{n}}$, there is a sequence $\sigma^{\hat{s}_i}_{\hat{n},\mathbb{N}} \in \Sigma_\mathbb{N}$ such that

$$(\sigma^{\hat{s}_i}_{\hat{n},n}, u(\sigma^{\hat{s}_i}_{\hat{n},n})) \to ((\hat{s}_i, \hat{\sigma}_{\hat{n},-i}), u(\hat{s}_i, \hat{\sigma}_{\hat{n},-i})) \quad \text{as } n \to \infty.$$

We then define

$$\sigma_{\hat{n},n} \equiv \sum_{\hat{s}_i \in \hat{S}_{i,\hat{n}}} \hat{\sigma}_{i,\hat{n}}(\{\hat{s}_i\})\, \sigma^{\hat{s}_i}_{\hat{n},n} \;\in \Sigma_n \;.$$

It is then clear that

$$(\sigma_{\hat{n},n}, u(\sigma_{\hat{n},n})) \to (\hat{\sigma}_{\hat{n}}, u(\hat{\sigma}_{\hat{n}})) \quad \text{as } n \to \infty \;.$$

Because $(\hat{\sigma}_{\hat{n}}, u(\hat{\sigma}_{\hat{n}})) \to (\sigma, u(\sigma))$ as $\hat{n} \to \infty$, a standard diagonal argument (similar to that in the proof of Lemma 3.1) implies that there exists an sequence of indexes $\hat{n}_\mathbb{N}$ such that $\hat{n}_n \to \infty$, $\sigma_{\hat{n}_n,n} \in \Sigma_n$ for all n, and also $(\sigma_{\hat{n}_n,n}, u(\sigma_{\hat{n}_n,n})) \to (\sigma, u(\sigma))$, so the proof is complete. □

Proof of Proposition 3.2

Proof. Fix $\sigma \in \Sigma$ and assume that (BRS$'_{\sigma,i}$) holds for all $i \in \mathcal{I}$ and $\sigma \in S_i \times \Sigma_{-i}$. Take some $\sigma_\mathbb{N}, \hat{\sigma}_\mathbb{N} \in \Sigma^\mathbb{N}$ such that $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$ and $(\hat{\sigma}_n, u(\hat{\sigma}_n)) \to (\sigma, v)$, for some $v \in \mathbb{R}^\mathcal{I}$. Assume also there is $i \in \mathcal{I}$ and $s_{i,\mathbb{N}} \in S_i^\mathbb{N}$ with $\limsup_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) > v_i$. Note that there is a sequence of indexes $k_\mathbb{N}$ such that $s_{i,k_\mathbb{N}}$ converges to some $s_i \in S_i$ and $\limsup_{n\to\infty} u_i(s_{i,k_n}, \sigma_{-i,k_n}) > v_i$. Fix some discretization $G_\mathbb{N}$. Using that (BRS$'_{(s_i,\sigma_{-i}),i}$) holds by assumption, there is some sequence $\hat{s}_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ with

$$\limsup_{n\to\infty} u_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}) \geq \limsup_{n\to\infty} u_i(s_{i,k_n}, \sigma_{-i,k_n}) \;.$$

Then, $\limsup_{n\to\infty} u_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}) > v_i$, so (BRS$_\sigma$) holds. □

Proof of Corollary 3.2

Proof. The "if" part of the statement is trivial. To prove the "only if" part, assume (σ, v) is an approximable equilibrium pair for some discretization $G_{\mathbb{N}}$, and let $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ be such that each σ_n is a ε_n-equilibrium G_n, for some $\varepsilon_{\mathbb{N}} \to 0$. Fix another discretization $\hat{G}_{\mathbb{N}}$. Lemma 3.1 implies that there is some $\hat{\sigma}_{\mathbb{N}} \in \hat{\Sigma}_{\mathbb{N}}$ such that $(\hat{\sigma}_n, v(\hat{\sigma}_n)) \to (\sigma, v)$. Assume, for the sake of contradiction, that there is no sequence $\hat{\varepsilon}_{\mathbb{N}} \to 0$ such that each $\hat{\sigma}_n$ is a $\hat{\varepsilon}_n$-equilibrium of G_n. This implies that there is a player $i \in \mathcal{I}$ and $\hat{s}_{i,\mathbb{N}} \in \hat{S}_{i,\mathbb{N}}$ such that $\limsup_{n\to\infty} u_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}) > v_i$. By (BRS$_\sigma$) we have that there is a player $\hat{\imath}$ and a sequence $s_{\hat{\imath},\mathbb{N}} \in S_{\hat{\imath},\mathbb{N}}$ such that $\limsup_{n\to\infty} u_{\hat{\imath}}(s_{\hat{\imath},n}, \sigma_{-\hat{\imath},n}) > v_{\hat{\imath}}$, but this contradicts that σ_n is a ε_n-equilibrium G_n and $\varepsilon_{\mathbb{N}} \to 0$. □

Proof of Corollary 3.3

Proof. Assume that there is a unique an approximable equilibrium $\sigma \in \Sigma$. Fix a discretization $G_{\mathbb{N}}$. Let $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ be a sequence of Nash equilibria. We show that we must have $\sigma_n \to \sigma$. To see this note that, otherwise, there would be some $\sigma' \neq \sigma$ and a strictly increasing sequence $k_{\mathbb{N}} \subset \mathbb{N}^{\mathbb{N}}$ with $\sigma_{k_n} \to \sigma'$. Nonetheless, this would imply that σ' is an approximable equilibrium of $G_{k_{\mathbb{N}}}$, contradicting our original assumption about σ. □

Proof of Corollary 3.4

Proof. Fix S and $\mathcal{I}$, and assume $u, \hat{u} : S \to \mathbb{R}^{\mathcal{I}}$ differ only on an approximable set. Assume also that both $(S_i, u_i)_{i\in\mathcal{I}}$ and $(S_i, \hat{u}_i)_{i\in\mathcal{I}}$ are approximable. Let $G_{\mathbb{N}}$ be such that $S_n \cap S' = \emptyset$ for all $n \in \mathbb{N}$. Since the set of approximable equilibria for $G_{\mathbb{N}}$ is the same for both $(S_i, u_i)_{i\in\mathcal{I}}$ and $(S_i, \hat{u}_i)_{i\in\mathcal{I}}$, and since both are approximable, we have that the set of approximable equilibria of these games coincide. □

Proof of Corollary 3.5

Proof. The proof of Theorem 3.1 can be easily modified to prove this result. □

Proof of Corollary 4.1

Proof. The proof of Theorem 3.1 can be easily modified to prove this result. □

Proof of Proposition 4.1

Proof. The proof is divided into 2 parts.

Part 1. Assume first that the first conditions (in the first two sentences) of the statement hold. To ease notation, we assume that $S_i = [0,1]$ for all $i \in \mathcal{I}$, that is, $S_i^2 = \emptyset$ (the arguments when $S_i^2 \neq \emptyset$ are analogous because of the continuity assumptions on u_i^1 and u_i^2). Take some $\sigma \in \Sigma$ with finite support. Let $S' = \times_{i \in \mathcal{I}} S_i'$ be the support of σ. We fix a discretization $G_{\mathbb{N}}$. Our aim is to find a sequence $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ such that $(\sigma_n, u(\sigma_n)) \to (\sigma, u(\sigma))$ and then apply Proposition 3.1 to show that (APP$_\sigma$) holds for all $\sigma \in \Sigma$.

The complication for the proof is that may be such that players do not have common actions, that is, it may be that $S_{i,n} \cap S_{\hat{\imath},n} = \emptyset$ for some $i, \hat{\imath} \in \mathcal{I}$ and $n \in \mathbb{N}$. In this case, i and $\hat{\imath}$ cannot "draw", so the "allocation probability' (the term in front of u_i^2 in equation (4.1)) has to be reproduced by independent randomizations of each player over close-by actions. We overcome this difficulty using the following sequence of strategy profiles:

1. Let Δ be the minimum distance between two actions of the same player, that is,

$$\Delta \equiv \min_{i \in \mathcal{I}} \min_{s_i' \in S_i'} \min_{s_i'' \in S_i' \backslash \{s_i'\}} |s_i' - s_i''| \ .$$

   For each $x \in [0,1]$, let $B^\varepsilon(x)$ be an interval of length ε in $[0,1]$ containing x. Note that, for each $\varepsilon \in (0, \Delta/2)$, we have that $B^\varepsilon(s_i') \cap B^\varepsilon(s_i'') = \emptyset$ for all $s_i', s_i'' \in S_i'$ with $s_i' \neq s_i''$. We define $\sigma^\varepsilon \in \Sigma$ by defining, for each player $i \in \mathcal{I}$ and $x \in [0,1]$,

$$\sigma_i^\varepsilon([0,x)) \equiv \sum_{s_i' \in S_i'} \sigma_i(\{s_i'\}) \, \frac{\mu(B^\varepsilon(s_i') \cap [0,x))}{\varepsilon} \ ,$$

   where μ is the usual Lebesgue measure. Essentially, σ_i^ε is a distribution which is 0 everywhere except for the neighborhoods $\{B^\varepsilon(s_i') | s_i' \in S_i'\}$. In these neighborhoods, it is uniform, and the probability of a neighborhood $B^\varepsilon(s_i')$ is $\sigma_i(\{s_i'\})$. Hence, for a sequence $\varepsilon_{\mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$ with $\varepsilon_n \to 0$, we have $\sigma^{\varepsilon_n} \to \sigma$. It is easy to see that σ^ε reproduces the allocation probabilities under σ. For example, if there is an action $s_i' \in S_i'$ such that $s_i' \in S_{\hat{\imath}}'$ for all players $\hat{\imath}$ in some set $\hat{\imath} \subset \mathcal{I} \backslash \{i\}$, then the probability under σ^ε that $s_i > s_{\hat{\imath}}$ for all $\hat{\imath} \in \hat{\imath}$ conditional on all players in $\{i\} \cup \hat{\imath}$ choosing an action in $B^\varepsilon(s_i')$ is $1/(1+|\hat{\imath}|)$. Furthermore, σ^ε is a continuous distribution, so the probability that $s_i = s_{\hat{\imath}}$ is 0.

2. We now fix some sequence $\varepsilon_{\mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$ with $\varepsilon_n \to 0$. For all $\hat{n} \in \mathbb{N}$, we let $\sigma_{\hat{n},\mathbb{N}} \in \Sigma_{\mathbb{N}}$ be such that $\sigma_{\hat{n},n} \to \sigma^{\varepsilon_{\hat{n}}}$, which exists by Proposition B.3. Following a usual diagonal

argument, and using that u^1 and u^2 are conintuous functions, it follows that there is a strictly increasing sequence $\hat{n}_{\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $(\sigma_{\hat{n}_n,n}, u(\sigma_{\hat{n}_n,n})) \to (\sigma, u(\sigma))$. Since, by construction, each $\sigma_{\hat{n}_n,n}$ belongs to Σ_n, the proof of Part 1 is concluded.

Part 2. Assume now that all conditions in the statement of Proposition 4.1 hold. As before, to ease notation, we assume that $S_i = [0,1]$ for all $i \in \mathcal{I}$, that is, $S_i^2 = \emptyset$. We will use Proposition 3.2 to show that (BRS$_\sigma$) holds for all σ. Take then some $i \in \mathcal{I}$ and $(s_i, \sigma_{-i}) \in S_i \times \Sigma_{-i}$. Take also some $\sigma_{\mathbb{N}}, \hat{\sigma}_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $\sigma_n, \hat{\sigma}_n \to \sigma$ and some $S_{i,\mathbb{N}}$. We want to show that there is some $\hat{s}_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ such that $\limsup_{n\to\infty} u_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}) \geq \limsup_{n\to\infty} u_i(\sigma_n)$. The argument then is analogous to that in Example 4.6. Indeed, it is easy to see that $\limsup_{n\to\infty} u_i(\sigma_n)$ is not larger than

$$\int_{S_{-i}} \left(u_i^1(s_i, \tilde{s}_{-i}) + \mathbb{I}_{s_i > \max_{\hat{i} \neq i} \tilde{s}_{\hat{i}}} u_i^2(s_i, \tilde{s}_{-i}) + \mathbb{I}_{s_i = \max_{\hat{i} \neq i} \tilde{s}_{\hat{i}}} \max\{0, u_i^2(s_i, \tilde{s}_{-i})\} \right) \sigma_{-i}(\mathrm{d}\tilde{s}_{-i}) \ .$$

Then, as argued in Exercise 4.6, if $u_i^2(s_i, s_{-i}) \geq 0$ for all $s_{-i} \in S_{-i}$ satisfying $\max_{\hat{i} \in \mathcal{I}} s_{\hat{i}}^1 = s_i^1$, there is a slowly decreasing sequence $\hat{s}_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ so that $\limsup_{n\to\infty} u_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n})$ tends to the expression above, while if $u_i^2(s_i, s_{-i}) \leq 0$ for all $s_{-i} \in S_{-i}$ satisfying $\max_{\hat{i} \in \mathcal{I}} s_{\hat{i}}^1 = s_i^1$, there is a slowly increasing sequence $\hat{s}_{i,\mathbb{N}} \in S_{i,\mathbb{N}}$ so that $\limsup_{n\to\infty} u_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n})$ tends to the expression above. The proof is then concluded. □

Proof of Theorem 5.1

Proof. It is clear that if G is BRS-fully-approximable then it is BRS-approximable. Assume then that G is BRS-approximable. Note that, for any approximation $G_{\mathbb{N}}$, there is a discretization $\hat{G}_{\mathbb{N}}$ such that $\hat{G}_n \subset G_n$ for all $n \in \mathbb{N}$. Then, it is clear that if (APP$_\sigma$) holds then (APP$_\sigma$) holds as well. Now, take $\sigma_{\mathbb{N}}, \hat{\sigma}_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ such that $(\sigma_n, u(\sigma_n)), (\hat{\sigma}_n, u(\hat{\sigma}_n)) \to (\sigma, v)$ for some $v \in \mathbb{R}^{\mathcal{I}}$, and assume there are $i \in \mathcal{I}$ and $s_{i,\mathbb{N}} \in S_i^{\mathbb{N}}$ with $\limsup_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) > v_i$. Fix some approximation $G_{\mathbb{N}}$, and let $\hat{G}_{\mathbb{N}}$ be such that $\hat{G}_n \subset G_n$ for all $n \in \mathbb{N}$. By (BRS$_\sigma$), we have that there are some $\hat{\imath} \in \mathcal{I}$ and $\hat{s}_{\hat{\imath},\mathbb{N}} \in \hat{S}_{\hat{\imath},\mathbb{N}}$ with $\limsup_{n\to\infty} u_{\hat{\imath}}(\hat{s}_{\hat{\imath},n}, \hat{\sigma}_{-\hat{\imath},n}) > v_{\hat{\imath}}$. Since $\hat{s}_{\hat{\imath},\mathbb{N}} \in S_{\hat{\imath},\mathbb{N}}$, we have that (BRS$_\sigma$) holds. □

Proof of Corollary 5.1

Proof. **Part 1:** Assume σ is Nash equilibrium. Since $(G)_n$ is an approximation of G, we have that $(\sigma, u(\sigma))$ is an fully-approximable equilibrium pair, and hence an fully-approximable equilibrium pair.

Part 2: Assume $(\sigma, u(\sigma))$ is an approximable equilibrium and let $G_{\mathbb{N}}$ be a discretization. Let $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ be such that, for each n, σ_n is an ε_n-equilibrium of G_n for some $\varepsilon_n \to 0$. Assume, for the sake of contradiction, that σ is *not* a Nash equilibrium. Let $i \in \mathcal{I}$ and $s_i \in S_i$ be such that $u_i(s_i, \sigma_{-i}) > u_i(\sigma)$. By (BRS$_\sigma$), that there are some $\hat{\imath} \in \mathcal{I}$ and $\hat{s}_{\hat{\imath},\mathbb{N}} \in \hat{S}_{\hat{\imath},\mathbb{N}}$ such that $\limsup_{n\to\infty} u_{\hat{\imath}}(\hat{s}_{\hat{\imath},n}, \sigma_{-\hat{\imath},n}) > u_{\hat{\imath}}(\sigma)$, but this contradicts that, for each n, σ_n is an ε_n-equilibrium of G_n and $\varepsilon_n \to 0$. □

Proof of Corollary 5.2

Proof. **"If" part:** Assume σ is an approximate equilibrium. Since $(G)_n$ is an approximation of G, we have that σ is an fully-approximable equilibrium, and hence an approximable equilibrium.

"Only if" part: Assume (σ, v) is an approximable equilibrium. Fix some discretization $G_{\mathbb{N}}$. Let $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ satisfying that $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$ and there is some sequence $\varepsilon_{\mathbb{N}} \to 0$ such that, for each n, σ_n is an ε_n-equilibrium of G_n. Assume, for the sake of contradiction, that there is no $\hat{\varepsilon}_{\mathbb{N}}$ satisfying that $\hat{\varepsilon}_n \to 0$ and, for all n, σ_n is an $\hat{\varepsilon}_n$-equilibrium of G. This implies that there is some player $i \in \mathcal{I}$ and a sequence $\hat{s}_{i,\mathbb{N}} \in S_i^{\mathbb{N}}$ such that $\limsup_{n\to\infty} u_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}) > v_i$. By (BRS$_\sigma$), that there are some $\hat{\imath} \in \mathcal{I}$ and $\hat{s}_{\hat{\imath},\mathbb{N}} \in \hat{S}_{\hat{\imath},\mathbb{N}}$ such that $\limsup_{n\to\infty} u_{\hat{\imath}}(\hat{s}_{\hat{\imath},n}, \hat{\sigma}_{-\hat{\imath},n}) > v_{\hat{\imath}}$, contradicting that there is some sequence $\varepsilon_{\mathbb{N}} \to 0$ such that, for each n, σ_n is an ε_n-equilibrium of G_n. □

Proof of Proposition 5.1

Proof. **Part 1:** We begin with a useful lemma.

Lemma A.1. *If* (BRS$_\sigma^{\text{full}}$) *holds for all* $\sigma \in \Sigma$ *then, for all* $(\sigma, v) \in \Sigma \times \mathbb{R}^{\mathcal{I}}$ *and all* $\sigma_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ *with* $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$*, we have*

$$\limsup_{n\to\infty} \sup_{s_{i,n}\in S_i} \overline{u}_i(s_{i,n}, \sigma_{-i,n}) = \overline{u}_i(\sigma|v) \quad \textit{for all } i \in \mathcal{I}. \tag{A.1}$$

Proof. Assume, for the sake of contradiction, that the statement of the lemma is false. Let $i \in \mathcal{I}$ and $\sigma_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$ be such that equation (A.1) holds with "$<$" instead of "$=$". There is then some $\hat{\sigma}_{\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $(\hat{\sigma}_n, u(\hat{\sigma}_n)) \to (\sigma, v)$ such that

$$\lim_{n\to\infty} \sup_{\hat{s}_{i,n}\in S_i} \overline{u}_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}) > \lim_{n\to\infty} \sup_{s_{i,n}\in S_i} \overline{u}_i(s_{i,n}, \sigma_{-i,n}) \ .$$

By ($\mathrm{BRS}^{\mathrm{full}}_{\sigma}$) we have that there is a sequence $s_{i,\mathbb{N}} \in S_i^{\mathbb{N}}$ such that

$$\limsup_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) \geq \lim_{n\to\infty} \sup_{\hat{s}_{i,n}\in S_i} \overline{u}_i(\hat{s}_{i,n}, \hat{\sigma}_{-i,n}) \ .$$

But this is a clear a contradiction. □

Let (σ, v) be such that $\overline{u}(\sigma, v) = v$ and let $G_{\mathbb{N}}$ be a discretization. Since $\overline{u}(\sigma, v) > -\infty$, there is some sequence $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ such that $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$. By Lemma 3.1, there is a sequence $\hat{\sigma}_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ with $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$. Given that $\overline{u}(\sigma, v) = v$, it follows that there is a sequence $\varepsilon_{\mathbb{N}}$ with $\varepsilon_n \to 0$ such that each $\hat{\sigma}_n$ is an ε_n-equilibrium of G_n. Hence (σ, v) is an approximable equilibrium pair, and σ is an approximable equilibrium.

Part 2: Let (σ, v) be an approximable equilibrium pair, $G_{\mathbb{N}}$ a discretization, and $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ be such that each σ_n is an ε_n-equilibrium of G_n and $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$, for some $\varepsilon_n \to 0$. We assume, for the sake of contradiction, that $\overline{u}_i(\sigma|v) > v_i$ for some $i \in \mathcal{I}$ (note that it can not be that $\overline{u}_i(\sigma|v) < v_i$). By Lemma A.1, there is some sequence $s_{i,\mathbb{N}} \in S_i^{\mathbb{N}}$ such that $\limsup_{n\to\infty} u_i(s_{i,n}, \sigma_{-i,n}) > v_i$. By ($\mathrm{BRS}_{\sigma}$), that there are some $\hat{\imath} \in \mathcal{I}$ and $\hat{s}_{\hat{\imath},\mathbb{N}} \in \hat{S}_{\hat{\imath},\mathbb{N}}$ such that $\limsup_{n\to\infty} u_{\hat{\imath}}(\hat{s}_{\hat{\imath},n}, \sigma_{-\hat{\imath},n}) > u_{\hat{\imath}}(\sigma)$, contradicting that each σ_n is an ε_n-equilibrium of G_n and $(\sigma_n, u(\sigma_n)) \to (\sigma, v)$ and $\varepsilon_n \to 0$. □

B Results on convergence

In this section, we summarize the topological results that are useful for studying the approximability of infinite games. They can be found in many standard texbooks, such as Billingsley (2013). We also present our findings on the approximability of strategy profiles.

B.1 Weak convergence and sequential compactness

We now briefly review standard results on compactness of metric spaces and weak convergence. We write them for the action space of a player in a game, S_i, endowed with an inherent distance, d_i.

Let (S_i, d_i) be a metric space. We say that (S_i, d_i) is *compact* if every open cover of S_i has a finite subcover. We say that (S_i, d_i) is *sequentially compact* if any sequence in $S_i^{\mathbb{N}}$ has a subsequence converging to a set. We say that (S_i, d_i) *totally bounded* (or *precompact*) if, for every $\varepsilon > 0$, S_i can be covered by finitely many subsets that each have diameter smaller than ε.

Proposition B.1. (S_i, d_i) *is compact if and only if it is sequentially compact. Also,* (S_i, d_i) *is sequentially compact if and only if it is complete and totally bounded.*

Let $C_b(S_i)$ denote the set of all functions from S_i to $\mathbb{R}$ that are both continuous and bounded. Let Σ_i the set of Borel probability measures on X. We say that $\sigma_{i,\mathbb{N}} \in \Sigma_i^{\mathbb{N}}$ *converges weakly* to $\sigma \in \Sigma_i$, denoted $\sigma_{i,n} \to \sigma_i$, if

$$\int f \mathrm{d}\sigma_{i,n} \to \int f \mathrm{d}\sigma \quad \text{as } \sigma \to \infty \text{ for all } f \in C_b(S_i) \ . \tag{B.1}$$

A standard result states that the $\sigma_{i,n} \to \sigma_i$ if and only if (B.1) holds for all bounded and uniformly continuous functions.

We now define the *weak (or Prokhorov's) distance* between $\sigma_i, \sigma_i' \in \Sigma_i$,

$$d_{\Sigma_i}(\sigma_i, \sigma_i') \equiv \inf \left\{ \alpha > 0 \,\middle|\, \sigma_i(A) \leq \sigma_i'(A_\alpha) + \alpha \text{ and } \sigma_i'(A) \leq \sigma_i(A_\alpha) + \alpha \ \forall A \in \mathcal{B}(S_i) \right\}, \ .$$

where, for each $\alpha > 0$ and $S \in \mathcal{B}(S_i)$, we have

$$A_\alpha \equiv \{ s_i \in S_i \,|\, d(s_i, A) < \alpha \} \text{ if } A \neq \emptyset, \quad \emptyset_\alpha := \emptyset \text{ for all } \alpha > 0 \ .$$

Proposition B.2. *Assume (S_i, d_i) is compact. Then, (Σ_i, d_{Σ_i}) is compact metric space. Furthermore, for all $\sigma_{i,\mathbb{N}} \in \Sigma_i^{\mathbb{N}}$ and $\sigma_i \in \Sigma_i$, we have that $d_{\Sigma_i}(\sigma_{i,n}, \sigma_i) \to 0$ if and only if $\sigma_{i,n} \to \sigma_i$.*[14]

For the analysis of games, it is sometimes useful to rely on the compactness of $\Sigma \equiv \times_{i\in\mathcal{I}} \Sigma_i$, where $\mathcal{I}$ is finite and each Σ_i is the set of Borel probability distributions on some compact metric space (S_i, d_i). We define d_Σ as

$$d_\Sigma(\sigma, \sigma') \equiv \max_{i\in\mathcal{I}} d_{\Sigma_i}(\sigma_i, \sigma_i') \quad \text{for all } \sigma, \sigma' \in \Sigma \ .$$

We now provide a corollary of Proposition B.2.

Corollary B.1. *(Σ, d_Σ) is a compact metric space.*

Proof. It is clear that d_Σ is a metric on Σ. To prove that Σ is compact, consider a sequence $\sigma_\mathbb{N} \in \Sigma_\mathbb{N}$. We argue that it has a convergent subsequence. By the compactness of each Σ_i, we have that there is a strictly increasing sequence of indexes $k_\mathbb{N}$ such that each $\sigma_{i,k_\mathbb{N}}$ converges weakly to some $\sigma_i \in \Sigma_i$. Trivially, $\sigma \equiv (\sigma_i)_{i\in\mathcal{I}}$ is an element of Σ. Also, since $d_{\Sigma_i}(\sigma_{i,n}, \sigma_i) \to 0$ for all $i \in \mathcal{I}$, we have $d_\Sigma(\sigma_n, \sigma) \to 0$, hence the result holds. □

B.2 Approximability of strategy profiles

In this section, we provide two technical results. The first establishes that for each discretization $G_\mathbb{N}$ and strategy profile $\sigma \in \Sigma$, there is a sequence $\sigma_\mathbb{N} \in \Sigma_\mathbb{N}$ with $\sigma_n \to \sigma$. The second result establishes that for all $\sigma \in \Sigma$, there is a discretization $G_\mathbb{N}$ and a sequence $\sigma_\mathbb{N} \in \Sigma_\mathbb{N}$ with $(\sigma_n, u(\sigma_n)) \to (\sigma, u(\sigma))$.

Proposition B.3. *Fix a discretization $G_\mathbb{N}$. For each $\sigma \in \Sigma$, there is a sequence $\sigma_\mathbb{N} \in \Sigma_\mathbb{N}$ such that $\sigma_n \to \sigma$.*

Proof. We divide this proof into three parts.

Part 1: Definition of $\sigma_\mathbb{N}^i$. Let $G_\mathbb{N}$ be a discretization. Fix some $i \in \mathcal{I}$. For each $n \in \mathbb{N}$, it is convenient to denote player i's actions as $S_{i,n} = \{s_{i,n,j_i} | j_i \in J_{i,n}\}$, where $J_{i,n} =$

[14] While the implication "$d_{\Sigma_i}(\sigma_{i,n}, \sigma_i) \to 0$ implies $\sigma_{i,n} \to \sigma_i$" holds regardless the compactness of S_i, the implication "$\sigma_{i,n} \to \sigma_i$ implies $d_{\Sigma_i}(\sigma_{i,n}, \sigma_i) \to 0$" requires compactness of S_i. (Note that this second implication is often stated for separable spaces, but all compact metric spaces are separable.)

$\{1,...,|J_{i,n}|\}$ is a finite set of indexes. By Proposition B.1 there is a sequence $\varepsilon_{\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$ strictly decreasing towards 0 such that, for each n, the set of balls $\{B_{i,\varepsilon_n}(s_{i,n,j_i}) | j \in J_{i,n}\}$ cover S_i (where $B_{i,\varepsilon_n}(s_{i,n,j_i}) \equiv \{s_i \in S_i | d_i(s_i, s_{i,n,j_i}) < \varepsilon_n\}$). We use this cover to define the following partition of S_i for each n:

1. $B'_{i,n,1} = B_{\varepsilon_n}(s_{i,n,1})$.
2. $B'_{i,n,2} = B_{\varepsilon_n}(s_{i,n,2}) \backslash B'_{i,n,1}$.
3. $B'_{i,n,3} = B_{\varepsilon_n}(s_{i,n,3}) \backslash \cup_{j=1}^{2} B'_{i,n,2}$.
4. ....

Note that $\{B'_{n,j_i} | j_i \in J_{n,i}\}$ is a partition of S_i. We let $\sigma_{i,n}$ be the strategy for player i defined by assigning, to each $S'_i \subset S_i$, the probability

$$\sigma^i_{i,n}(S'_i) \equiv \sum_{j_i \in J_{i,n}} \sigma_i(B'_{i,n,j_i}) \, \mathbb{I}_{s_{i,n,j_i} \in S'_i} \ .$$

We let $\sigma^i_n \equiv (\sigma^i_{i,n}, \sigma_{-i})$.

Part 2: Proof that $\sigma^i_n \to \sigma$. To prove that $\sigma^i_n \to \sigma$, we fix an absolutely continuous and bounded function $f: S \to \mathbb{R}$, hence satisfying that for each $\delta > 0$ there is some $\overline{n}$ such that, for all $s \in S$, if $f(B_{\varepsilon_n}(s)) \subset B_\delta(f(s))$. Note that

$$\begin{aligned} \int_{B'_{i,n,j_i} \times S_{-i}} f(s_{i,n,j_i}, s_{-i}) \, \sigma(\mathrm{d}s) &= \int_{B'_{i,n,j_i} \times S_{-i}} f(s_{i,n,j_i}, s_{-i}) \, \sigma^i_n(\mathrm{d}s) \\ &= \int_{B'_{i,n,j_i} \times S_{-i}} f(s) \, \sigma^i_n(\mathrm{d}s) \ . \end{aligned}$$

Since $|f(s_i, s_{-i}) - f(s_{i,n,j_i}, s_{-i})| < \delta_{\varepsilon_n}$ for all $s_i \in B'_{i,n,j_i}$, we have that

$$\left| \int_{B'_{i,n,j_i} \times S_{-i}} f(s) \, \sigma(\mathrm{d}s) - \int_{B'_{i,n,j_i} \times S_{-i}} f(s_{i,n,j_i}, s_{-i}) \, \sigma^i_n(\mathrm{d}s) \right| < \sigma(B'_{i,n,j_i} \times S_{-i}) \, \delta_{\varepsilon_n} \ ,$$

where we used that $\sigma(B'_{i,n,j_i} \times S_{-i}) = \sigma^i_n(B'_{i,n,j_i} \times S_{-i})$. Hence, we have that

$$\begin{aligned} &\left| \int_S f(s) \, \sigma(\mathrm{d}s) - \int_S f(s) \, \sigma^i_n(\mathrm{d}s) \right| \\ &\quad = \sum_{j_i \in J_{i,n}} \left| \int_{B'_{i,n,j_i} \times S_{-i}} f(s) \, \sigma(\mathrm{d}s) - \int_{B'_{i,n,j_i} \times S_{-i}} f(s_{i,n,j_i}, s_{-i}) \, \sigma^i_n(\mathrm{d}s) \right| \\ &\quad < \sum_{j_i \in J_{i,n}} \sigma(B'_{i,n,j_i} \times S_{-i}) \, \delta_{\varepsilon_n} \\ &\quad = \delta \ . \end{aligned}$$

This proves that $\lim_{n\to\infty}\int f\,\mathrm{d}\sigma_n^i = \int f\,\mathrm{d}\sigma$ for all uniformly continuous functions; hence, $\sigma_n^i \to \sigma$.

Part 3: Conclusion of the proof. Part 2 proves that when only S_i is discretized, we have that there is a sequence $\sigma_n^i \to \sigma$. We can now take some player $\hat{\imath} \neq i$ and define $\sigma_n^{i,\hat{\imath}}$ as before replacing σ by σ_n^i. We let $\sigma_n^{i,\hat{\imath}} \equiv (\sigma_{i,n}^i, \sigma_{\hat{\imath},n}^{i,\hat{\imath}}, \sigma_{-i,\hat{\imath}})$, where $\sigma_{-i,\hat{\imath}}$ is the components of σ different from i and $\hat{\imath}$. A similar argument of Part 1.2 proves that

$$\lim_{n\to\infty}\int f\,\mathrm{d}\sigma_n^{\hat{\imath}i} = \lim_{n\to\infty}\int f\,\mathrm{d}\sigma_n^i = \int f\,\mathrm{d}\sigma$$

for all f absolutely continuous. Hence, $\sigma_n^{i,\hat{\imath}} \to \sigma$. Proceeding iteratively over the players in $\mathcal{I}$ we obtain that $\sigma_n^{\mathcal{I}} \to \sigma$. □

Proposition B.4. *For each $\sigma \in \Sigma$, there is a discretization $G_{\mathbb{N}}$ and $\sigma_{\mathbb{N}} \in \Sigma_{\mathbb{N}}$ such that $(\sigma_n, u(\sigma_n)) \to (\sigma, u(\sigma))$.*

Proof. Fix $\sigma \in \Sigma$. We will construct the discretization by discretizing each S_i in order. To do so, for this proof, we index the players so that $\mathcal{I} \equiv \{1, ..., |\mathcal{I}|\}$. We divide the proof into four parts.

Part 1: Partition of $u(S)$. Recall that $u(S)$ is a bounded set of $\mathbb{R}^{\mathcal{I}}$. This implies that, for each $n \in \mathbb{N}$, there is a cover of $u(S)$ composed of sets of diameter lower than $1/n$. We denote one such cover by $\{B''_{n,k} | k \in K_n\}$, where K_n is some finite set, and we let $v_{n,k}$ be an element of $B''_{n,k}$ for each k. Note that if $v \in B''_{n,k}$ then $d(v, v_{n,k}) \leq \varepsilon_n$ (where $d(v, v') \equiv \max_{i\in\mathcal{I}} |v_i - v'_i|$ for all $v, v' \in \mathbb{R}^{\mathcal{I}}$).

Part 2: Partition of S_1. Fix some n. Since S_1 is metric and compact, there is a partition of S_1 satisfying that each element of the partition has a diameter at most equal to $1/n$. We let $\{B'_{i,n,j_1} | j_1 = 1, ..., J_{1,n}\}$ denote such partition.

Part 3: Definition and convergence of σ_n^1. We define, for each $j_1 \in J_{1,n}$ and $k \in K_n$,

$$\hat{B}_{1,n,j_1,k} \equiv B'_{1,n,j_1} \cap \{s_1 \in S_1 \,|\, u(s_1, \sigma_{-1}) \in B''_{n,k}\} \subset S_1 \ .$$

It is clear that $\{\hat{B}_{i,n,j_1,k} \,|\, j_1 \in J_1 \text{ and } k \in K\}$ is a cover of S_1. For each j_1 and k, we let $s_{1,j_1,k,n}$ be an element of $\hat{B}_{i,n,j_i,k}$. Now, now define

$$\sigma_{1,n}^1(S'_1) \equiv \sum_{j_1\in J_{1,n}} \sum_{k\in K} \sigma_1(\hat{B}_{1,n,j_1,k})\, \mathbb{I}_{s_{1,n,j_1,k}\in S'_1} \ .$$

We define $\sigma_n^1 = (\sigma_{1,n}^1, \sigma_{-i})$. It is then easy to see, proceeding as in the proof of Theorem B.4, that $(\sigma_n^{\{1\}}, u(\sigma_n^1)) \to (\sigma, u(\sigma))$.

Part 4: Recursive definition and convergence of σ_n^i. The proof proceeds by iterating over the players. Indeed, proceeding as in Parts 2 and 3, now for player 2, it follows that for each n there is a discretization $S_{2,\mathbb{N}}$ and corresponding sequence $\sigma_{2,\mathbb{N}} \in S_{2,\mathbb{N}}$ such that

$$((\sigma_{2,n'}, \sigma_{-2,n}^1), u(\sigma_{2,n'}, \sigma_{-2,n}^1)) \to (\sigma_n^1, u(\sigma_n^1))$$

as $n' \to \infty$. A standard diagonal argument shows that there is some discretization $S_{2,\mathbb{N}}$ and sequence $\sigma_{2,\mathbb{N}}^2 \in \Sigma_{2,\mathbb{N}}$ such that, defining $\sigma_n^2 \equiv (\sigma_{2,n}^2, \sigma_{-2,n}^1)$ for all n, we have that $(\sigma_n^2, u(\sigma_n^2)) \to (\sigma, u(\sigma))$. The same procedure can be applied to all players until $|\mathcal{I}|$, so the corresponding partition is $S_{\mathbb{N}}$ and the desired sequence is $\sigma_{\mathbb{N}}^{|\mathcal{I}|} \in \Sigma_{\mathbb{N}}$. □