Optimal Unemployment Insurance with Directed Search

Carlos da Costa[1], Lucas Maestri[1], and Cézar Santos[2]

[1] *EPGE FGV Brazilian School of Economics and Finance*
[2] *Inter-American Development Bank*

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Abstract

When searching for employment, workers take different job characteristics into account. We study an environment where unemployed workers search for jobs with different effort requirements in a labor market characterized by directed search. In equilibrium, firms are more likely to post vacancies for high-effort jobs, as these are more profitable. Hence, low-effort jobs are hard to come across. The optimal unemployment contract prescribes that the government should distort effort downwards through positive marginal tax rates on labor earnings, even when non-distortionary taxation is available. This result transpires both for the case of observable and, for GHH-CARA preferences, hidden savings.

Keywords: *Unemployment Insurance; Directed Search; Intensive Margin; Hidden Savings.*

JEL Codes: *H21; J64.*

1 Introduction

Unemployment insurance programs must strike a balance between the provision of insurance and the disincentives for working. Previous literature has addressed its optimal design by focusing on the extensive margin of labor supply — Hopenhayn and Nicolini (1997); Shimer and Werning (2007, 2008). This line of work has emphasized how insurance provision reduces the incentives for searching for jobs and accepting offers. But different jobs may require different levels of effort from workers. When searching for employment, workers can take these different job characteristics into account.

In this paper, we study the problem of a government that offers optimal unemployment insurance financed with income taxes in a dynamic environment featuring directed search. We innovate by taking into account the intensive margin of labor supply. By allowing for intensive margin adjustments, firms may expand the supply of vacancies by requiring agents to increase their output conditional on landing a job. From the worker's perspective, they can reduce the expected unemployment spell if they look for jobs that require more effort for the same level of earnings. An alternative interpretation is via the notion of equalizing differences.[1] What makes these non-pecuniary dimensions important for our analysis is that these adjustments in work conditions are neither observed nor controlled by the planner.

We characterize the optimum for general separable preferences when the planner controls the agent's savings. At the optimum, there is a positive wedge at the intensive margin of effort. This result materializes even though the planner can use non-distortionary instruments and there is no distributive motive. The logic is as follows. The planner observes earnings but neither effort nor the non-wage characteristics of a job. Hence, if an agent decides to deviate, they will do so by searching for a job that requires less effort but which they have a lower probability of finding. Then, conditional on landing the job, they will have a higher marginal willingness to work than someone who follows the strategy prescribed by the planner. Distorting effort downwards has second-order utility costs for equilibrium choices but first-order costs for one who deviates, as a deviator enjoys more

[1] In the real world, workers may adjust their search not only by becoming more selective with regard to wages but also with respect to how much effort they must exert once employed and the quality of their prospective work environment, neither of which is within the reach of policy. These equalizing differences surveyed by Rosen (1987) have been shown to be quantitatively important in recent work by Mas and Pallais (2017); Sorkin (2018); Hall and Mueller (2018).

leisure and attributes a lower marginal value to it. This relaxes incentive constraints and lowers the cost of providing insurance. Unemployment benefits and net earnings decline with the length of the unemployment spell. The repeated moral hazard nature of the problem implies that, at the optimum, the stochastic process governing consumption satisfies the inverse Euler Equation. In the long run, unemployment benefits converge to zero.

To implement the optimal allocation described above, the planner must control the agent's savings, which may not be possible in practice. We take the possibility of hidden savings and borrowing in perfect capital markets into account. For this case, we restrict our analysis to preferences of the Greenwood et al. (1988) type specialized to the case of Constant Absolute Risk Aversion, henceforth CARA-GHH preferences. The optimal allocation can be implemented by a simple stationary contract: an upfront unemployment installment, constant gross earnings, and taxes when the agent finds a job. The pattern of declining consumption in both employment and unemployment is achieved by the worker's (dis)savings along the unemployment spell. In this hidden-savings case too, a positive wedge on the intensive margin characterizes the optimum.

The rest of the paper is organized as follows. After a brief literature review, in Section 2, we describe the environment and offer a one-period account of the forces explaining our findings. We derive the properties of an optimal system under the assumption that the planner controls agents' savings in Section 3 and use Section 4 to do the same for the case of hidden savings. In Section 5 we assess the quantitative relevance of our theoretical results and offer a conclusion in Section 6.

Literature Review

The modern treatment of unemployment insurance program design has its roots in Shavel and Weiss (1979) and found its first canonical treatment in Hopenhayn and Nicolini (1997). We contribute by focusing on directed search and by introducing the possibility of selecting jobs according to their effort requirements. Acemoglu and Shimer (1999) consider a general equilibrium model of directed search with risk aversion. The static version of our model generalizes theirs by considering the possibility of adjusting the effort requirements of different jobs. Moreover, while their focus is on the general equilibrium aspects of unemployment insurance, we concentrate on the planner's solution to the optimal policy.

Shimer and Werning (2007, 2008) evaluate the consequences of allowing agents to borrow and save in perfect capital markets using McCall's (1970) model of sequential job search. Under CARA preferences, a policy comprised of a constant benefit during unemployment, a constant tax during employment, and free access to a riskless asset is optimal. In our directed search environment with the possibility of intensive margin adjustments in the amount of effort once employed, simple stationary policies are also optimal under CARA. We add to the prescription by proving the optimality of distorting effort downwards.

A strand of the literature investigates redistributive policies in the presence of labor market frictions. Golosov et al. (2013) consider the redistribution of residual income. Under directed search, the optimal redistribution of residual income can be attained with positive unemployment benefits and an increasing and regressive income tax schedule. As in our framework, a positive wedge obtains despite workers being homogeneous, albeit for a different reason. Kroft et al. (2020) focus on finding sufficient statistics for the optimal combination of income taxes and unemployment benefits but do not consider intensive margin adjustments as we do. da Costa et al. (2021) study optimal distributive policies in the presence of labor market frictions. While they focus on intensive margin choices, their model is static and focused on the interaction between distributive motives and unemployment insurance design. Here, we abstract from redistribution and focus on the dynamics of insurance when contracts are not observed and there is scope for adjustments in the intensive margin.

2 Environment

Time runs for $t = 0, 1, ...,$ and is discounted by $\beta \in (0, 1)$. Preferences are separable across time, states, and between consumption, c, and effort, n. The flow utility generated by (c, n) is given by $U(c, n) = \varphi(c) - \eta(n)$, with $\varphi', -\varphi'', \eta', \eta'' > 0$, satisfying the Inada conditions $\lim_{c\downarrow 0} \varphi'(c) = \infty$ and $\lim_{n\downarrow 0} \eta'(n) = 0$. One unit of effort, n, produces one unit of the consumption good, c, the price of which is normalized to one.

The economy starts with the worker in an unemployment state. A job offer is a contract specifying how much effort, n, the worker must make if hired and their earnings, y. A labor contract, consisting of the pair (n, y), defines a (sub)market. The probability, p, of receiving a job offer in any market depends on the market tightness, with the implied relationship captured by the function $\varrho : [0,1] \to [0,\infty)$. This function associates an employment probability, p, to the vacancy-to-workers ratio that generates it. That is, with some abuse in notation, let $p(\lambda)$ denote the probability that an agent gets an offer when the workers-to-vacancy ratio is λ. If $\lambda : [0,1] \to [0,\infty)$ is its inverse, then, for all p, define ϱ by $\varrho(p) := 1/\lambda(p)$. As in usual directed search specifications, we assume that ϱ is strictly increasing, twice differentiable, strictly convex, and satisfies $\varrho(0) = 0$, $\lim_{p\uparrow 1} \varrho(p) = \infty$. This implies that $\varrho(p)/p$ is strictly increasing and assume that $\phi = \lim_{p\downarrow 0} \varrho(p)/p > 0$.

The following assumption states that a worker who receives a sufficiently low constant lump-sum payment $\underline{c} > 0$ in every period would look for a job with a positive probability if labor was not taxed:

$$\varphi(\underline{c}) < \max_{y^e} \varphi(\underline{c} + y^e) - \eta(\phi + y^e).$$

This assumption always holds, for instance, when $\lim_{c\downarrow 0} \varphi(c) = -\infty$.

To model a firm's hiring decision, normalize the cost of posting a vacancy to $\kappa/(1-\beta)$. An unemployed worker who applies for a job at time t receives the answer at the beginning of the same period, before collecting unemployment insurance.

We study the cost minimization of a government that must guarantee a lifetime utility, W_0, to the worker. When offering insurance to the agent, the planner faces informational restrictions. First, the planner does not know whether the agent received a job offer and rejected it or whether no offer materialized. Second, conditional on the worker finding a job, the planner does not know the type of contract offered to the agent. More precisely, the planner observes earnings, y, but not effort n. Whereas the first source of informational asymmetry has been extensively studied, the second one is novel. To highlight its role, we first present a one-period version of our economy in which the heuristics for our main findings are simpler to convey.

2.1 A one-period economy

Consider a simplified version of our model in which an agent lives for a single period split into two sub-periods. In the first sub-period, the agent decides in which sub-market to search, i.e., they choose p. If they find a job, they earn y^e in exchange for producing $y^e + \kappa\varrho(p)/p$. If not, they are entitled to an unemployment benefit c^u.

Note two things about the worker's problem. First, contracts are not observable. That is, the planner observes how much an employed worker is paid, y^e, but it cannot monitor how much effort, n^e, a job demands. Second, from the zero profit condition, $p[n^e - y^e] = \kappa\varrho(p)$ must hold for any contract, on- and off-the-equilibrium path.

For an employed worker to consume c^e, they must earn $y^e = c^e + T$ and pay taxes, T, to the government. Since the output they produce must also cover the vacancy-related expenditures $n^e > y^e$. So, in what follows we relegate n^e and T to the background and write the planner's program with the controls c^e, c^u, and y^e.

The planner observes both c^e and y^e. Hence, the only margin in which deviation is possible is the choice of p. Moreover, the effort, $n^e = y^e + \kappa\varrho(p)/p$, is chosen when the worker decides which job to apply to. Because the government observes neither p nor n^e, it can only condition policy on the employment status and on earnings, y^e. A worker who chooses a higher matching probability and finds a job must exert more effort for the same level of earnings. The intuition is simple: worse jobs, shorter queues.

Note that (c^e, c^u, y^e) are controlled by the planner and define for any $\hat{p}$,

$$U(\hat{p}, c^u, c^e, y^e) := (1-\hat{p})\varphi(c^u) + \hat{p}\left[\varphi(c^e) - \eta\left(y^e + \kappa\frac{\varrho(\hat{p})}{\hat{p}}\right)\right],$$

and let the agent's optimal choice of $\hat{p}$ be

$$p \in \arg\max_{\hat{p}} U\left(\hat{p}, c^u, c^e, y^e\right). \tag{1}$$

Under the assumption that the solution to the agents' problem is interior, i.e., that the worker actively searches for a job, the solution must satisfy the following first-order condition:

$$\varphi(c^e) - \varphi(c^u) - \eta\left(y^e + \frac{\varrho(p)}{p}\right) - p\eta'\left(y^e + \frac{\varrho(p)}{p}\right)\left(\frac{\varrho(p)}{p}\right)' = 0, \tag{2}$$

where the notation

$$\left(\frac{\vartheta(p)}{p}\right)' = \frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right),$$

is used to simplify the expressions.

The Pareto frontier can be obtained by maximizing the planner's expected revenue

$$-(1-p)c^u + p\left(y^e - c^e\right), \tag{3}$$

subject to delivering utility U^* to the agent,

$$U\left(p, c^u, c^e, y^e\right) \geq U^*, \tag{4}$$

and to respecting the incentive-compatibility constraint (1).

Due to the concavity of the problem, (4) can be replaced by (2) whenever it is desirable to induce a positive search. If it is not desirable to induce positive search, the solution displays $c^u = c^e = \varphi^{-1}\left(U^*\right)$. Focus then on the case in which the optimal amount of search is positive.

To incentivize effort, the planner must ensure that $c^e > c^u$, which implies that both constraints bind. This fact, coupled with the concavity of the worker's problem with respect to p, confirms that $(c^e - y^e) > c^u$. Hence, the planner raises more revenues when the worker finds employment. This is the source of moral hazard in our model. The worker does not internalize the fiscal cost of the insurance provided by the government. Accordingly, the government finds it desirable to induce a higher matching probability.

Turn now to how this impacts the marginal rate of substitution between consumption and leisure. From the first-order conditions with respect to c^e and y^e, $\varphi'(c^e) > \eta'\left(y^e + \varrho(p)/p\right)$.

At the optimal allocation, labor effort is distorted downward. To better understand this property, consider an alternative allocation in which this margin is not distorted: $\varphi'(c^e) = \eta'\left(y^e + \varrho(p)/p\right)$. A small perturbation in which consumption when employed, c^e, and earnings, y^e, are both decreased by some small $\varepsilon > 0$ has no direct fiscal effect and only a second-order effect on the worker's utility. However, it changes the marginal incentive to search for a job. The convexity of the cost of labor and the fact that workers who intend to find a job with higher probability must provide higher effort once employed imply that this perturbation makes it relatively more attractive to search for a job. This relaxes the moral-hazard constraint and allows the planner to improve policy.

The planner wants the unemployed worker to search for jobs that are easier to find. As these jobs entail more effort once employed, the planner must provide incentives for the worker to work harder. By imposing an income tax, the planner creates a wedge where the marginal utility of consumption is higher than the marginal disutility of working. Hence, at the margin, exerting more effort is not so costly for the employed worker. Imposing such a wedge is thus optimal even when non-distortionary instruments are available. The next few sections show that this insight carries on to richer environments.

3 Optimal Unemployment Insurance

The one-period version of our model was proper to highlight the extra margin for deviation when some aspects of jobs cannot be controlled by the planner. Yet, it abstracts from an important dimension of real-world unemployment insurance: the time dimension of optimal policy. This section develops a dynamic version of our model and shows that it is optimal to distort effort downwards when the worker finds a job. We start by describing what would be possible if contract offers were observed. That is, we characterize the benchmark case in which a firm's posted contract is observable, but the planner cannot observe whether or not the worker received an offer.

3.1 Observable Contracts

For now, assume that the planner observes the details of contracts that are offered. A job contract is a pair, (c^e, n^e), where c^e denotes the consumption for an employed person and n^e the level of effort required from them.

The planner cannot force the agent to find a job if

$$\varphi(c^u) > \varphi(c^e) - \eta\left(n^e\right).$$

Hence, the program that the planner solves is

$$C(W) = \max_{p,c^e,c^u,n,\tilde{W}} \frac{p}{1-\beta}\left[n - c^e - \kappa\frac{\vartheta(p)}{p}\right] + (1-p)\left[-c^u + \beta C(\tilde{W})\right],$$

subject to the promise-keeping,

$$W = \frac{p}{1-\beta}\left[\varphi(c^e) - \eta\left(n\right)\right] + (1-p)\left[\varphi(c^u) + \beta\tilde{W}\right], \tag{5}$$

and the incentive constraint,

$$\frac{\varphi(c^e) - \eta\left(n\right)}{1-\beta} \geq \varphi(c^u) + \beta\tilde{W}. \tag{6}$$

We can write the program above as the following Kuhn-Tucker problem,[2]

$$C(W) = \max_{p,c^e,c^u,n,\tilde{W}} \frac{p}{1-\beta}\left[n - c^e - \kappa\frac{\vartheta(p)}{p}\right] + (1-p)\left[-c^u + \beta C(\tilde{W})\right] +$$
$$\mu\left[\frac{p}{1-\beta}\left[\varphi(c^e) - \eta(n)\right] + (1-p)\left[\varphi(c^u) + \beta\tilde{W}\right] - W\right] +$$
$$\psi\left[\frac{\varphi(c^e) - \eta(n)}{1-\beta} - \varphi(c^u) - \beta\tilde{W}\right]$$

To proceed, we first assess whether the moral hazard and the promise-keeping constraints bind at the optimum. Lemma 3.1 below states that whenever agents search for a job, they are indifferent between doing so and remaining unemployed for another period.

Lemma 3.1 *The promise-keeping constraint (5) binds in every period, and* $\mu > 0$*. In any period in which there is positive search, the moral-hazard constraint binds,* $\varphi(c^e) - \eta(n) = [1-\beta]\left[\varphi(c^u) + \beta\tilde{W}\right]$*, and* $\psi > 0$*.*

For every period t in which the moral-hazard constraint binds we have

$$\mu_{t+1} = \mu_t - \frac{\psi_t}{1-p_t},$$

which implies that unemployment consumption decreases over time,

$$c^u_{t-1} = (\varphi')^{-1}\left(\mu_t^{-1}\right) > (\varphi')^{-1}\left(\mu_{t+1}^{-1}\right) = c^u_t.$$

Moreover, the consumption process is described by an inverse Euler equation,

$$\frac{1}{\varphi'\left(c^u_{t-1}\right)} = \mu_t = p_t\left[\mu_t + \psi_t p_t^{-1}\right] + (1-p_t)\left[\mu_t + \psi_t p_t^{-1}\right] = \frac{p_t}{\varphi'\left(c^e_t\right)} + \frac{1-p_t}{\varphi'\left(c^u_t\right)}.$$

Also, the first-order conditions with respect to c^e and n imply that, in contrast to our one-period model with non-observed contracts, the effort is not distorted

[2] We can rely on Lemma A.3 to write the problem as such. This lemma refers to the case in which contracts are not observed, but the argument is easily adapted to the case with observed contracts.

at the optimum in the dynamic model with observable contracts. We gather these findings in Proposition 3.1.

Proposition 3.1 *The solution for the planner's problem when contracts are observable has the following properties:*

i) It entails a zero marginal income tax rate.

ii) The unemployment insurance is decreasing over time. Moreover, if there is a search at period t then the unemployment insurance is strictly lower than the one from the previous period.

iii) The consumption process is described by an inverse Euler equation.

The incentive-compatibility constraint (6) only depends on the agent's utility when employed, not on how it is generated. Since the government observes contracts, it can choose them to minimize the cost of providing this utility. That is, given any utility level, there is no reason for the government to distort effort, which implies i). Second, unemployment insurance should decrease over time in order to make it more costly to turn down employment opportunities, which is the content of ii). Finally, similar to several dynamic moral-hazard models – e.g., Rogerson (1985) –, the consumption process is described by an inverse Euler equation.

3.2 Non-observable Contracts

Section 3.1 adopted the strong assumption that the government observes the contracts chosen by workers and hence the disutility of effort from a particular job. We now consider optimal policies under non-observable labor contracts. With non-observable contracts, the optimal policy must be based only on whether or not the agent is employed, on their earnings, and on the length of the unemployment spell.

If the principal promises a utility sufficiently high for the agent, the solution to the planning problem implies that there is no search in equilibrium. I.e., it is cheaper to deliver the promised utility if the agent remains unemployed forever. This is an uninteresting case and we focus, instead, on the case in which utility is not too high.

To characterize the optimal unemployment insurance program in this case, we rely on a first-order approach. Lemma A.2 shows that the solution for this relaxed problem is the solution to the original program. Hence, the planner's problem has a recursive structure and can be written as follows,

$$C\left(W_0\right)=\max_{p,c^e,c^u,y^e,W_1}\frac{p}{1-\beta}\left(y^e-c^e\right)+(1-p)\left[-c^u+\beta C\left(W_1\right)\right],$$

subject to a promise-keeping constraint

$$\frac{p}{1-\beta}\left[\varphi(c^e)-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]+(1-p)\left[\varphi(c^u)+\beta W_1\right]-W_0\geq 0, \tag{7}$$

and an incentive compatibility constraint

$$\frac{1}{1-\beta}\left[\varphi(c^e)-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]-\varphi(c^u)-\beta W_1=\frac{p\kappa}{1-\beta}\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\left(\frac{\vartheta(p)}{p}\right)'. \tag{8}$$

Lemma A.3 shows that the planner's problem is differentiable, and hence the optimum must satisfy a constraint optimization in which we write μ and λ for the multipliers relative to the constraints (7) and (8). Both multipliers are strictly positive. If μ were not strictly positive, the planner would be able to save resources by lowering the utility promised to the agent in both states with no consequences for incentives. λ is strictly positive because the worker does not internalize the fiscal externality when unemployed.

Combining the first order conditions with respect to y^e and c^e, one obtains

$$\varphi'(c^e)-\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)=\frac{\lambda\kappa p}{\mu p+\lambda}p\eta''\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\left(\frac{\vartheta(p)}{p}\right)'>0. \tag{9}$$

The optimal allocation now displays a positive wedge at the intensive margin. The dynamic model inherits the finding from our one-period model. If a firm offers a better job, i.e., one requiring less effort for the same earnings, then it will attract more job candidates. Workers, in turn, will find it harder to land such a job, thus remaining unemployed for more periods. Conditional on getting one of these jobs a worker would have a higher willingness to exert effort compared to someone who got one of the jobs offered by firms along the equilibrium path. To make these deviations less attractive, the planner distorts effort downwards by taxing earnings at the margin.

Since preferences are separable in consumption and effort, it is always feasible to vary the unemployment consumption utility in a period and compensate for it by varying the consumption utility in all states of nature in subsequent periods. Such a strategy changes neither incentives nor expected utility. Thus, these perturbations cannot save resources at the optimum. Because the marginal cost of delivering utility is $1/\varphi'$, the inverse Euler equation ensues.

These findings are summarized in Theorem 3.1, which is proved in the appendix.

Theorem 3.1 *At the optimum, in every period in which there is a positive search,*

i) the moral-hazard constraint (8) binds, and the government benefits from strictly increasing p;

ii) the marginal income tax rate is always positive, and;

iii) conditional on not finding a job at period t, the worker's marginal utility of consumption satisfies the inverse Euler equation,

$$\frac{1}{\varphi'\left(c_t^u\right)} = \mathbb{E}\left[\frac{1}{\varphi'\left(c_{t+1}\right)}\right].$$

The planner may avoid distorting the effort margin. Taxes may be based on employment, independently of earnings. Moreover, the utility conditional on finding a job depends on $\varphi(c^e) - \eta(n^e)$, regardless of whether c^e and n^e are efficiently chosen. What is then the rationale for distorting the intensive margin prescribed in Proposition 3.1? It is the same that we have seen in a static setting.

Consider a worker deciding whether to apply for a job in a slightly tighter submarket, $\hat{p} > p$. The planner controls y^e and c^e, but not the amount of effort the agent is making. Upon landing a job in a tighter market, the worker is required to supply effort, $\hat{n} = y^e + \kappa\varrho(\hat{p})/\hat{p} < y^e + \kappa\varrho(p)/p = n$ while receiving the same c^e. This worker, therefore, has a lower marginal disutility of effort than agents who followed the optimal policy. To make this downward deviation less valuable – this is the relevant deviation according to *i*) – the planner distorts effort downward by introducing a positive wedge. A little less surprising is the fact that, as in Rogerson (1985); Atkeson and Lucas (1995), the Inverse Euler Equation characterizes the dynamics of consumption for the unemployed.

Proposition 3.2 *The unemployment benefit is decreasing over time with $c_t^u > c_{t+1}^u$ whenever the worker searches in period $t+1$.*

Moreover, whenever the worker searches in period $t+1$, their consumption from employment is strictly greater than the unemployment benefit from any period $\tau \geq t$.

Due to income effects, the optimal contract provides constant benefits and asks the worker never to search for a job when the promised utility is very high. On the other hand, job search must be incentivized when the government promises a sufficiently low utility to the worker. These two possibilities render the government's cost of providing utility W to the worker not convex in W, in general. As a consequence, we cannot rule out the possibility that the worker does not search for a job in the first period of the optimal contract.

To better understand when it is optimal to search in every period, let W^* be the supremum over all utilities that induce efficient search in a static economy. Lemma A.1 shows that $W^* \in \mathbb{R}$. Moreover, Lemma 3.2 below shows that, if the initial unemployment insurance provides less utility than W^*, then the worker must search for a job in every period.

Lemma 3.2 *Assume that $\varphi\left(c_0^u\right) < W^*$. Then, there is a positive search in every period.*

When the initial utility is smaller than W^*, the initial contract must induce search in some period. Moreover, whenever the worker searches in some period the unemployment benefits eventually fall so that $\varphi\left(c_t^u\right) < W^*$ for some period t. Hence, the worker searches in every period, $\tau > t$.

Lemma 3.3 *The following conditions hold in any optimal contract:*

a) Assume that $W_0 < W^*$. Then, there is $t > 0$ such that $\varphi\left(c_t^u\right) < W^*$. Hence, the worker who is unemployed in any period $\tau > t$ searches for a job.

b) Assume that the worker searches for a job in some period t. Then there is $T > t$ such that the unemployed worker searches in any period $\tau > t$.

In this case, according to Proposition 3.2, $c_t^u > c_{t+1}^u$ for all t. Therefore, the unemployment benefit converges to a non-negative number. Proposition 3.3 shows that this number is 0.

Proposition 3.3 *Assume that $W_0 < W^*$.Then unemployment benefits converge to zero.*

We have focused thus far on the case of separable preferences between consumption and effort. This has been the most frequently studied case in the literature. In Appendix B, we study non-separability for the case of GHH-CARA utility $\mathcal{U}(c,n) = -\exp\left\{-\alpha\left[c-\eta(n)\right]\right\}$.[3] These preferences will be the focus of our analysis when we assume that savings cannot be controlled by the planner. The results of this section carry over to the GHH-CARA case. In particular, the optimal policy for this case also prescribes a positive wedge between effort and consumption.

4 Hidden Savings

In Section 3 we have followed most of the literature, and Hopenhayn and Nicolini (1997), in particular, in assuming that the planner controls the worker's savings. This allowed us to define a one-to-one mapping from after-tax earnings, $y^e - T$, to

[3]The constant absolute risk aversion (CARA) case is the only one for which Shimer and Werning (2007) have theoretical results for the non-observable savings scenario. They offer numerical explorations for the constant relative risk aversion (CRRA) case. Because we are also interested in understanding choices at the intensive margin, we suppress income effects at this margin through the assumption of quasi-linearity, as in Greenwood et al. (1988).

consumption, c^e. What happens if this is not the case? If the government does not control savings, do these results remain valid? We answer these questions next.[4]

To study the optimal design of an unemployment insurance program for the case in which the government does not observe agents' savings, we assume that the agents have access to perfect capital markets with an interest rate $r = \beta^{-1} - 1$. Because consumption and earnings need not coincide due to the possibility of borrowing and saving a crucial distinction between the optimal consumption path and the optimal transfer path arises, as highlighted by Shimer and Werning (2007) in the context of unemployment insurance. We follow their lead in restricting our analysis to the case of preferences that do not exhibit income effects, i.e., we assume that preferences are of the GHH-CARA.

The planner's program is to minimize the expected cost of delivering utility W_0 for the unemployed agent subject to providing incentives for him to follow the optimal search strategy.

Assume that the worker starts with assets, A_0. In a deterministic mechanism, the government adds liquidity, $a_0 - A_0$, at time $t = 0$ and transfers, b_t, to the unemployed in period $t \geq 0$. If a job is found at period t the government demands the amount of work y_t^e and makes a net transfer T_t^e (which may be negative) in every subsequent period.[5]

Recall that $p \to (\varrho(p)/p)$ is strictly increasing and strictly convex. We make the following additional assumption, which guarantees that it can be optimal for the agent to search for a job.

Assumption H1: *There exists y and $p > 0$ such that*

$$y > \eta \left(y + \kappa \frac{\vartheta(p)}{p} \right).$$

[4]As we know from Allen (1985); Cole and Kocherlakota (2001), the existence of hidden savings represents an important constraint for the design of optimal policies, in general.

[5]Once the worker lends a job, there is no further need to provide incentives and utility provision is optimally accomplished by a time-invariant allocation. Hence, the time index in y_t^e and T_t^e refers to the period t in which the job is found.

Intuitively, if H1 were violated, the benefit of making effort would never compensate for its disutility. Optimal programs would entail no vacancy creation.

A ***policy*** is a tuple, $(a_0, \{y_t^e, T_t^e, b_t\}_{t=0}^{\infty})$, where a_0 is the agent's initial asset holdings, y_t^e is the amount of effort that is demanded in every future period if a job is found at period t, T_t^e are the taxes to be paid in every future period if a job is found at t, and b_t is the value of unemployment insurance at period t.

We say that the consumption sequence $\{c_t^u, c_t^e\}_{t=0}^{\infty}$ is feasible under the policy $(a_0, \{y_t^e, T_t^e, b_t\}_{t=0}^{\infty})$ if $\lim \beta^t c_t^u = \lim \beta^t c_t^e = 0$ and there exists $\{a_t\}_{t=0}^{\infty}$ with $\lim \beta^t a_t = 0$ such that

$$a_{t+1} = \beta^{-1} \left(a_t - c_t^u + b_t \right),$$

and

$$c_t^e = (1 - \beta)\, a_t + y_t^e - T_t^e.$$

Without loss of generality, we restrict our attention to policies that generate feasible consumption sequences.

Next, we define a ***simple policy***, which will play an important role in all that follows.

Definition 4.1 *A simple policy is a triple (a_0, y^e, T^e) in which the earnings, y_t, of an employed agent, are constant, $y_t^e = y^e$ for all t, and the transfers, T_t^e, that the agent makes to the government once employed are also constant, $T_t^e = T^e$.*

With Lemma 4.1 we explain how an agent optimally responds to a simple policy. We then show – Theorem 4.2 – that the constrained efficient allocation can be implemented by a simple contract which we fully characterize in Section 4.1.

When facing a simple policy the worker's problem can be written in a recursive form as

$$W_t(a_t) = \max_{a_{t+1} \in R,\ p \in [0,1]} \Big\{ -\exp\left\{ -\alpha \left(a_t - \beta^{-1} a_{t+1} \right) \right\} \\ + \beta \left\{ p W_{t+1}^e \left(a_{t+1}, p \right) + (1-p) W_{t+1}\left(a_{t+1} \right) \right\} \Big\}, \quad (10)$$

subject to

$$W_{t+1}^e(a_{t+1},p) = -\frac{1}{1-\beta}\exp\left\{-\alpha\left[y^e - T^e + \left(\beta^{-1}-1\right)a_{t+1} - \eta\left(y^e + \kappa\frac{\varrho(p)}{p}\right)\right]\right\}.$$

For short, we write $W_t = W_t(a_t)$, $W_t^e = W_{t+1}^e(a_{t+1},p_t)$, and $W_t^u = W_{t+1}(a_{t+1})$, at the optimal (a_{t+1},p_t).

Let $\hat{c}_{t,\tau}^e$ be the consumption at τ for an agent who found a job in period $t<\tau$. It is immediate to see that $\hat{c}_{t,\tau}^e = c_t^e$ for all $\tau > t$. Hence, in what follows we omit the current period subscript, τ, and write c_t^e to denote the time-invariant consumption of an agent who found a job in period t.

Lemma 4.1 *Assume that the planner offers a simple contract, (a_0, y^e, T^e), to the agent. In this case, the agent chooses,*

i) a stationary p;

ii) $c_{t+1}^e - c_t^e = -\Delta_c$ and $c_{t+1}^u - c_t^u = -\Delta_c$, for a constant, $\Delta_c > 0$;

iii)

$$\frac{W_t^e}{W_t} = \left\{1 + \alpha p(1-p)\kappa\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\left(\frac{\vartheta(p)}{p}\right)'\right\}^{-1} = k_e < 1,$$

and

$$\frac{W_t^u}{W_t} = \left(1 + \frac{\alpha p^2\kappa\eta'\left(y^e + \kappa\vartheta(p)/p\right)\left(\vartheta(p)/p\right)'}{1 + \alpha^2 p^2(1-p)\kappa\eta'\left(y^e + \kappa\vartheta(p)/p\right)\left(\vartheta(p)/p\right)'}\right) = k_u > 1;$$

iv)

$$\frac{p}{(1-\beta)W_t}\exp\left\{-\alpha\left[c_e - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right]\right\} = 1 - (1-p)\,k_u.$$

According to Lemma 4.1, when facing a simple contract, the agent chooses a constant p, and, for every additional period in which he is unemployed, he reduces both the consumption while unemployed, c_t^u, and the planned consumption after finding a job, c_t^e, by the same amount Δ_c.[6] As a result, the ratios W_t^e/W_t and

[6] Note that consumption is kept constant after the agent finds a job.

W_t^u/W_t remain constant at k_e and k_u, respectively.[7]

Theorem 4.2 *There exists a simple policy that implements the optimal allocation.*

The optimal unemployment contract implements an allocation characterized by a constant search, p, and a constant work effort, $n^e = y^e + \kappa\varrho(p)/p$ after a job is found. To provide incentives for agents to keep searching one must guarantee that $c_t^e > c_t^u$ in every period t. However, spreading consumption across the two states, unemployment, and employment, within a single period, is a costly way of delivering promised utility. To reduce this cost, incentives are back-loaded; the promised utility is reduced every time an agent fails to find a job, as stated in (*iii*). For the specification of preferences that we adopt, a lower utility promise with the same p and y^e can be made incentive compatible by an equal reduction in c^e and c^u, which we show to be optimal in the proof of Theorem 4.2.

Next, we explain the rationale for how the optimal allocation can be implemented with a simple contract where the agent is given assets a_0 and is promised a labor contract (y^e, c^e) if he manages to land a job. Agents' (dis)savings choices guarantee that c_t^e and c_t^u will follow the path prescribed in Theorem 4.2.

Under Assumption H1, and with GHH-CARA preferences, changes in asset positions have no impact on agents' search choices. However, they imply an adjustment in consumption during the unemployment and after a job is found which leads to a simple scaling of expected utility. Given this simple response to asset position, a Ricardian-equivalence result obtains. Alternative paths are fully characterized by the time in which the worker finds a job and the worker's decision only depends on the present value of transfers associated with each path. By performing simple changes in the timing of payments one can show that simple insurance schemes are optimal.

4.1 The Optimal Policy

Now that we have established that the optimal contract is stationary and of the form (a_0, c^e, y^e), we rely on this simple structure to provide its complete character-

[7]Because $W \in (-\infty, 0)$, expected utility decreases with the unemployment spell — $k_u > 1$ — and increases once a job is found — $k_e < 1$.

ization.

We can restrict the search for the optimal contract to that of finding a triple (a_0, c^e, y^e) that solves the problem

$$\max_{(a_0,c^e,y^e)} \left\{ \frac{p\left(y^e, c^e\right)}{1-\beta\left(1-p\left(y^e, c^e\right)\right)} \frac{y^e-c^e}{1-\beta} - a_0 \right\},$$

subject to

$$U\left(y^e, c^e, a_0\right) \geq U_0,$$

where $U\left(y^e, c^e, a_0\right)$ is the value of the worker's program (10) under the simple policy, $\left(y^e, c^e, a_0\right)$. Note that the incentive constraint is summarized by the dependence of p on y^e and c^e.

The promise-keeping constraint can equivalently be written as

$$U\left(y^e, c^e\right) \exp\left\{-\alpha\left(1-\beta\right) a_0\right\} \geq U_0,$$

where we use the simplified form $U\left(y^e, c^e\right)$ for $U\left(y^e, c^e, 0\right)$.

The first-order condition with respect to a_0 allows us to eliminate the Lagrange multiplier and the promise-keeping constraint. We may, then, write the planner's objective as

$$\mathcal{L} = \frac{p\left(y^e, c^e\right)}{1-\beta\left(1-p\left(y^e, c^e\right)\right)} \frac{y^e-c^e}{1-\beta} - a_0 + \frac{U\left(y^e, c^e\right)}{\exp\left\{-\alpha\left(1-\beta\right) a_0\right\} \alpha\left(1-\beta\right)\left|U_0\right|}.$$

The first-order condition with respect to c^e is

$$-\frac{p\left(y^e, c^e\right)}{1-\beta\left(1-p\left(y^e, c^e\right)\right)} + \frac{\partial U\left(y^e, c^e\right)/\partial c^e}{\exp\left\{-\alpha\left(1-\beta\right) a_0\right\} \alpha\left(1-\beta\right)\left|U_0\right|} + \frac{\partial}{\partial p}\left[\frac{p}{1-\left(1-p\right)\beta}\right] \frac{y^e-c^e}{1-\beta} \frac{\partial p}{\partial c^e} = 0, \quad (11)$$

and, with respect to y^e is

$$\frac{p\left(y^e, c^e\right)}{1-\beta\left(1-p\left(y^e, c^e\right)\right)}+\frac{\partial U\left(y^e, c^e\right)/\partial y^e}{\exp\left\{-\alpha\left(1-\beta\right)a_0\right\}\alpha\left(1-\beta\right)\left|U_0\right|} + \frac{\partial}{\partial p}\left[\frac{p}{1-\left(1-p\right)\beta}\right]\frac{y^e-c^e}{1-\beta}\frac{\partial p}{\partial y^e}=0. \quad (12)$$

Consider the optimality conditions above. The first term regards the direct fiscal cost of an increase in y^e. The second term is the impact on the worker's utility. Both are purely mechanical impacts. The third term summarizes the indirect, behavioral fiscal effects which are present because p is not observable.

As we show, $\partial p/\partial c^e > 0$ and $\partial p/\partial y^e < 0$: the worker's best response to a higher disposable income is to increase the job-finding probability and, to a higher gross income is to decrease the job-finding probability. Of course, the fiscal effect also depends on the sign of $y^e - c^e$; whether the behavioral response translates into a positive or a negative fiscal effect.

Theorem 4.3 *The efficient allocation is characterized by:*

i) $y^e - c^e$ *is strictly positive;*

ii) The labor wedge,

$$1+\frac{\partial U(y^e, c^e)/\partial y^e}{\partial U(y^e, c^e)/\partial c^e},$$

is strictly positive;

iii) The utility of the agent who does not get a job by period t diverges to minus infinity as well as the utility of the agent who gets a job at period t.

According to (*i*), $y^e - c^e > 0$; when the worker finds a job he pays net taxes. An increase in y^e increases the job-finding probability, while an increase in c^e decreases it, i.e., a worker that provides lower effort responds better to incentives, being more prone to increase his job-finding rate due to an increase in employment consumption. As a result, we obtain (*ii*) which shows that, as in the model with observable savings, the moral-hazard problem implies that effort should be discouraged at the margin.

It is important to emphasize that the moral hazard problem does not arise because of positive taxes, $y^e - c^e > 0$. The planner can make taxes dependent only on whether the agent is employed regardless of how much he or she earns thus avoiding the distortions at the work effort margin. The fiscal externality is important because it makes it desirable for the planner to induce agents to search harder. What ultimately makes it optimal for the government to distort the effort margin is the fact that a positive effort wedge increases the cost of downward deviation of the search margin.

Finally, the last point of the theorem. The worker always expects to find a job with a constant probability in every period. Because of that, he dis-saves, and hence his unemployment consumption decreases along the duration of the unemployment spell. The absence of income effects in our specification implies that his consumption diverges to minus infinity as the unemployment spell becomes arbitrarily long.

5 Quantitative Exploration

[TO BE DONE]

6 Conclusion

We study the consequences of intensive margin adjustments for optimal unemployment insurance design. We add to an otherwise standard search problem the real-world feature that an important dimension of labor contracts is not observed by the policy maker: how hard an agent is required to work in each job.

We find that it is always optimal to distort downward the intensive margin by imposing a positive marginal income tax rate. This is true regardless of whether savings are controlled by the planner or not. While framed as an effort margin, a similar conclusion applies to amenities: in a second-best world their supply should be discouraged when compared to the first-best.

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A Appendix

A.1 Proofs of Section 3.1

Proof of Lemma 3.1. First, we show that the constraint (5) binds. The first-order condition with respect to c^u reads

$$\varphi'(c^u) = \frac{1-p}{\mu(1-p)-\psi} > 0.$$

If $\mu \leq 0$ then $\psi < 0$ and thus $p > 0$ and then using the first-order condition w.r.t. c^e we obtain

$$\varphi'(c^e) = \frac{p}{p\mu+\psi} < 0.$$

A contradiction.

Next, towards a contradiction, assume that, without loss of generality, the constraint (6) does not bind at $t = 0$,

$$\varphi'(c_0^u) = \varphi'(c_0^e) = \mu_0^{-1} = \eta'(n_0).$$

In this case,

$$\varphi(c_0^e) - \eta(n_0) < \varphi(c_0^u). \tag{13}$$

Notice that the moral-hazard constraint must bind for some $t > 0$, otherwise,

$$\varphi(c_t^u) = \mu_0^{-1},$$

for every t. This means that getting a job in period zero is worse than being unemployed forever.

Assume that the first period in which the constraint binds is $t = 1$ (the other case is analogous). We have $\mu_1 = \mu_0$, $\psi_1 > 0$ and, hence,

$$\varphi'(c_1^e) = \frac{p_1}{p_1\mu_0+\psi_1} = \eta'(n_1).$$

Therefore,

$$\varphi(c_0^e) - \eta(n_0) < \varphi(c_1^e) - \eta(n_1) \tag{14}$$

Hence, using (13) and (14) we obtain

$$\frac{\varphi(c_0^e) - \eta(n_0)}{1-\beta} < \varphi(c_0^u) + \beta\frac{\varphi(c_1^e) - \eta(n_1)}{1-\beta},$$

which, using the fact that the moral-hazard constraint was binding in the second period implies that worker strictly prefers being unemployed than getting a job at zero. A contradiction. ■

A.2 Proofs of Section 3.2

Let $\phi := \lim_{p\downarrow 0} \varrho(p)/p$ and define $z(W)$ by

$$z(W) \equiv \operatorname*{argmin}_z z \quad \text{s.t.} \quad v(z) \geq W,$$

for

$$v(z) \equiv \max_{y^e} \left[\varphi(y^e + z) - \eta(y^e + \phi)\right].$$

Intuitively, $z(W)$ is the minimum amount of resources that would cost for the government to motivate the worker to search for employment if his unemployment continuation utility were W, assuming that the labor market was competitive. To see this, we use the fact that $\varrho(p)/p$ is increasing in p. Hence, to find a job with probability p the worker would have to pay $\varrho(p)/p > \phi$ to the firm upon the job arrival.

We also define $c^u(W)$ by

$$\varphi(c^u(W)) = W,$$

the cost of providing utility W for a worker who never searches for a job.

Lemma A.1 *There exists W^* such that $z(W^*) = c^u(W^*)$. Moreover, $z(W) > c^u(W)$, for all $W > W^*$, and $z(W) < c^u(W)$, for all $W < W^*$. Moreover, both mappings, $z(\cdot)$ and $c^u(\cdot)$, are strictly increasing, twice differentiable, and strictly convex.*

Proof of Lemma A.1. Let $y^e(W)$ be given by

$$\underset{y^e}{\operatorname{argmax}} \left[\varphi\left(y^e + z(W)\right) - \eta\left(y^e + \phi\right)\right],$$

and notice that if $\varphi'(z(W)) - \eta'(\phi) \leq 0$, then $y^e(W) = 0$. Otherwise, $y^e(W)$ is given by

$$\varphi'\left(y^e + z(W)\right) - \eta'\left(y^e + \phi\right) = 0.$$

Hence, we have

$$z'(W) = \frac{1}{\varphi'\left(z(W) + y^e(W)\right)} > \frac{1}{\varphi'\left(c^u(W)\right)} = c^{u\prime}(W),$$

because $z(W) + y^e(W) > c^u(W)$. This implies that if $z(\cdot)$ and $c^u(\cdot)$ cross at most once, and $z(W) > c^u(W)$ (resp. $z(W) < c^u(W)$) for every utility greater (resp. lower) than this utility level.

b) Since $z(W) \to \infty$ as $W \to \infty$, we have $y^e(W) = 0$ for W large enough, which implies $z(W) > c^u(W)$. The existence of a small W such that $z(W) < c^u(W)$ holds by assumption. Therefore, W^* exists by continuity.

It remains to show that both mappings are strictly convex. Since $c^e(W) := z(W) + y^e(W)$ is strictly increasing with positive derivative, we have

$$z''(W) = \frac{-\varphi''\left(c^e(W)\right)}{\varphi'\left(c^e(W)\right)^2} c^{e\prime}(W) > 0,$$

and

$$c^{u\prime\prime}(W) = \frac{-\varphi''\left(c^u(W)\right)}{\varphi'\left(c^u(W)\right)^2} c^{u\prime}(W) > 0.$$

■

Proof of Lemma 3.2. First, we claim that unemployment insurance benefits are weakly decreasing over time. Consider two subsequent periods s and $s+1$. First notice that if there is no search at period $s+1$ then the concavity of φ implies that $\varphi\left(c^u_{s+1}\right) = \varphi\left(c^u_s\right)$.On the other hand, if there is search at period $s+1$ the result follows from Lemma A.4.

Therefore, since unemployment benefits are weakly decreasing, it suffices to show that if there is no search in period t then the planner could profitably deviate by offering a contract in which the worker also searches at t. Notice that the (normalized) utility $(1-\beta) W_t$ can be written as a convex combination of the terms:

i)
$$\varphi\left(c^e_{t+k}\right) - \eta\left(y^e_{t+k} + \frac{\varrho(p_{t+k})}{p_{t+k}}\right),$$
which are obtained if the worker finds a job at period $t+k$, and;

ii) $\varphi\left(c^u_{t+k}\right)$, which are obtained if the worker does not get a job by period $t+k$.

Since $c^u_{t+k} \leq c^u_0$, this implies that $\varphi\left(c^u_{t+k}\right) < W^*$. Hence, the cost of delivering $\varphi\left(c^u_{t+k}\right)$ is less than $Z\left(\varphi\left(c^u_{t+k}\right)\right)$ by Lemma A.1. Notice also that the cost of providing utility,
$$\varphi\left(c^e_{t+k}\right) - \eta\left(y^e_{t+k} + \frac{\varrho(p_{t+k})}{p_{t+k}}\right),$$
is less than
$$Z\left(\varphi\left(c^e_{t+k}\right) - \eta\left(y^e_{t+k} + \frac{\varrho(p_{t+k})}{p_{t+k}}\right)\right).$$
Since the function, Z, is strictly convex, by Jensen's inequality and a continuity argument, there exists $\varepsilon > 0$ such that, if the planner offers the contract in which payments $y^e(W_t + \varepsilon)$ are required from and consumption $c^e(W_t + \varepsilon)$, is extended to the worker, he will search for a job with positive probability, will obtain a utility $\tilde{W} > W_t$ from this search, and the government will incur a strictly lower cost.

This strategy makes both the worker as well as the planner better-off at period t, but may decrease the worker's incentives at period $t-1$. To avoid that, the planner decreases the worker's unemployment consumption at period $t-1$ up to the point at which the worker is indifferent at period $t-1$. This further improves the planner's utility at $t-1$ by showing a strictly more profitable contract. ■

Proof of Lemma 3.3. a) Assume towards a contradiction that $\varphi\left(c^u_t\right) \geq W^*$ for every t. Therefore, the worker can guarantee a utility at least as large as W^* at every period. Then, if we let

$$w_t^e := \varphi\left(c_t^e\right) - \eta\left(y_t^e + \frac{\varrho(p_t)}{p_t}\right),$$

we observe that incentive-compatibility implies that $w_e^t > W^*$ for every t for which there is a positive search, in which case $C^u\left(w_e^t\right) < Z^e\left(w_e^t\right)$. Thus since $W_0(1-\beta)$ is a convex combination of $\{\varphi\left(c_0^u\right), w_0^e, \varphi\left(c_1^u\right), w_1^e, .., \}$ and $C^u(\cdot)$ is strictly convex, applying Jensen's inequality we conclude that the government's cost is strictly less than

$$\frac{C^u(W_0(1-\beta))}{(1-\beta)},$$

which can be achieved by offering constant unemployment insurance equal to $C^u(W_0(1-\beta))$ and never having the worker search for a job. A contradiction.

Next, notice that since c_t^u is decreasing, a) implies that there exists t such that $\varphi\left(c_t^u\right) < W^*$. Then apply Lemma 3.2. ■

Lemma A.2 *Suppose that if a worker gets a job then he must earn $c^e + T$, paying T to the government, to consume c^e whereas if the worker fails to get a job then he obtains the continuation utility W. Then this problem admits a unique solution. If the solution is interior, it is given by the associated first-order conditions.*

Proof of Lemma A.2. Consider the problem

$$\max p\left[\varphi(c^e) - \eta\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right) - W\right]$$

This problem admits an interior solution if and only if

$$\varphi(c^e) - \eta\left(c^e + T\right) > W.$$

Assume that this is the case and consider p that makes its derivative equal to zero:

$$\varphi(c^e) - \eta\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right) - W - p\eta'\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right) = 0$$

Differentiate the left-hand side again to obtain

$$\begin{aligned} &- 2\eta'\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right) \\ &\qquad - p\eta'\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d^2}{dp^2}\left(\frac{\vartheta(p)}{p}\right) \\ &\qquad\qquad - p\eta''\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right)\kappa\left[\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right)\right]^2. \end{aligned}$$

To show that the expression above is negative, it suffices to show that

$$\begin{aligned} &- 2\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right) - p\frac{d^2}{dp^2}\left(\frac{\vartheta(p)}{p}\right) < 0 \Leftrightarrow \\ &\qquad - 2\left(\frac{\vartheta'(p)p - \vartheta(p)}{p^2}\right) - p\frac{d}{dp}\left(\frac{\vartheta'(p)p - \vartheta(p)}{p^2}\right) < 0 \Leftrightarrow \\ &- 2\left(\frac{\vartheta'(p)p - \vartheta(p)}{p^2}\right) - \left(\frac{\frac{d}{dp}\left[\vartheta'(p)p - \vartheta(p)\right]p^2 - 2p\left[\vartheta'(p)p - \vartheta(p)\right]}{p^3}\right) < 0 \Leftrightarrow \\ &\qquad\qquad - \left(\frac{\vartheta''(p)}{p}\right) < 0. \end{aligned}$$

■

Lemma A.3 *For every W, let $C(W)$ be the planner's cost of providing utility W. The mapping $C(\cdot)$ is differentiable at W_t for every $t > 0$.*

Proof of Lemma A.3. We prove that C is differentiable at W_t. For that we assume that $p_t > 0$ as the other case is analogous. Consider any small $\epsilon \in \mathbb{R}$ and notice that the following perturbation is feasible:

$$\left(\underline{\tilde{u}}_{t-1}, \underline{\tilde{u}}_t, \tilde{c}_t^e\right) = \left(\underline{u}_{t-1} + \epsilon, \underline{u}_t - \epsilon\beta^{-1}, \varphi^{-1}\left(\varphi(c_t^e) + \epsilon\right)\right).$$

One can thus apply the argument in Clausen and Strub (2020) to conclude that

$$C'(W_t) = -c'(\underline{u}_t) = \frac{1}{\varphi'(\underline{u}_t)}.$$

■

Proof of Lemma 3.1. Follows from Lemma A.4 below. ■

Lemma A.4 *The multipliers, μ and λ, are strictly positive if there is a search.*

Proof of Lemma A.4. First, notice that

$$[\mu(1-p) - \lambda]\,\varphi'(c^u) = 1 - p$$

and

$$\frac{p\mu + \lambda}{1-\beta}\varphi'(c^e) = \frac{p}{1-\beta}$$

Hence, $\mu = 0$ implies $\varphi'(c^u)\varphi'(c^e) \leq 0$, which is absurd.

Hence assume towards a contradiction that $\lambda_0 \leq 0$. Clearly, there is a last period at which $\lambda_t \leq 0$ and $\lambda_{t+1} > 0$, otherwise, as we will verify below, $c_t^u \geq c_t^e$ for every t, and hence there is no search. Assume that $\lambda_1 > 0$ (case in which $\lambda_s \leq 0$ for all $s < t$ and $\lambda_t > 0$ for some $t > 1$ can be analogously handled).

From the first-order condition with respect to p we get

$$\varphi'(c^u) = \frac{1}{\mu - \lambda(1-p)^{-1}} \leq \frac{1}{\mu + \lambda p^{-1}} = \varphi'(c^e).$$

Hence, $c^u \geq c^e$.

Moreover, notice that from the first order condition we have

$$C'(W_1) = -\mu_0 + \frac{\lambda_0}{(1-p)} = -\mu_1,$$

which implies

$$\mu_1 = \mu_0 - \frac{\lambda_0}{(1-p)} \geq \mu_0.$$

This, and $\lambda_0 \leq 0 < \lambda_1$ imply

$$\varphi'(c_1^e) = \frac{1}{\mu_1 + p_1^{-1}\lambda_1} < \frac{1}{\mu_0 + p_0^{-1}\lambda_0} = \varphi'(c_0^e).$$

Hence,

$$c_1^e > c_0^e. \tag{15}$$

We can rearrange the first order condition with respect to y^e to get

$$\mu\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right) = 1 - \lambda\eta''\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right) - \frac{\lambda}{p}\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right).$$

Therefore, $\lambda_0 \leq 0 < \lambda_1$ imply

$$\eta'\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right) < \mu_1^{-1}.$$

Similarly,

$$\eta'\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right) \geq \mu_0^{-1}.$$

Since $\mu_1 \geq \mu_0$, this implies

$$y_1^e + \kappa\frac{\vartheta(p_1)}{p_1} < y_0^e + \kappa\frac{\vartheta(p_0)}{p_0},$$

and

$$\eta\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right) < \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right),$$

because η is strictly convex.

Since $p_0 > 0$, by the assumption of the lemma, we have

$$
\begin{aligned}
0 < \frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] - \left[\varphi(c_0^u) + \beta W_1\right] \\
= \frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] - \varphi(c_0^u) \\
- \beta\left[p_1\frac{1}{1-\beta}\left[\varphi(c_1^e) - \eta\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\right] + (1-p_1)\left[\varphi(c_1^u) + \beta W_2\right]\right] \\
= \varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right) - \varphi(c_0^u) + \beta\Bigg[\frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] \\
- p_1\frac{1}{1-\beta}\left[\varphi(c_1^e) - \eta\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\right] - (1-p_1)\left[\varphi(c_1^u) + \beta W_2\right]\Bigg] \\
= \frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] - \varphi(c_0^u) \\
- \beta\left[p_1\frac{1}{1-\beta}\left[\varphi(c_1^e) - \eta\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\right] + (1-p_1)\left[\varphi(c_1^u) + \beta W_2\right]\right] \quad (16)
\end{aligned}
$$

Since $p_1 > 0$, due to $\lambda_1 > 0$, we have

$$
\frac{1}{1-\beta}\left[\varphi(c_1^e) - \eta\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\right] > \varphi(c_1^u) + \beta W_2
$$

Hence,

$$
\begin{aligned}
\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right) - \varphi(c_0^u) + \beta\Bigg\{\frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] \\
- p_1\frac{1}{1-\beta}\left[\varphi(c_1^e) - \eta\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\right] - (1-p_1)\left[\varphi(c_1^u) + \beta W_2\right]\Bigg\} \\
< \varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right) - \varphi(c_0^u) \\
+ \beta\left[\frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] - \left[\varphi(c_1^u) + \beta W_2\right]\right]
\end{aligned}
$$

Since the first line from the last term is negative, the entire term is less than

$$\beta \left[\frac{1}{1-\beta} \left[\varphi(c_0^e) - \eta \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right) \right] - \left[\varphi(c_1^u) + \beta W_2 \right] \right],$$

which is less than,

$$\frac{1}{1-\beta} \left[\varphi(c_0^e) - \eta \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right) \right] - \left[\varphi(c_1^u) + \beta W_2 \right],$$

since the term is positive.

Since $\varphi(c_0^e) < \varphi(c_1^e)$, and

$$\eta \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right) > \eta \left(y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} \right),$$

this is less than

$$\frac{1}{1-\beta} \left[\varphi(c_1^e) - \eta \left(y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} \right) \right] - \left[\varphi(c_1^u) + \beta W_2 \right].$$

Hence, using the first-order conditions with respect to p, the algebra just performed means that

$$\frac{p_1}{1-\beta} \eta' \left(y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} \right) \kappa \left(\frac{\vartheta(p_1)}{p_1} \right) > \frac{p_0}{1-\beta} \eta' \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right) \kappa \left(\frac{\vartheta(p_0)}{p_0} \right). \tag{17}$$

Since

$$y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} < y_0^e + \kappa \frac{\vartheta(p_0)}{p_0},$$

if $y_1^e \geq y_0^e$, we will have $p_1 < p_0$ which together contradict (17). We conclude that

$$y_1^e < y_0^e. \tag{18}$$

Finally, notice that $\lambda_1 > 0$ and the first order condition with respect to p and the fact that p is a local maximum imply

$$\frac{y_0^e - c_0^e}{1-\beta} \leq -c_0^u + \beta C\left(W_1\right). \tag{19}$$

Analogously, in period 1, using $\lambda_0 \leq 0$, the first order condition with respect to p implies

$$\frac{y_1^e - c_1^e}{1-\beta} \geq -c_1^u + \beta C\left(W_2\right).$$

But notice that

$$C\left(W_1\right) = p_1 \frac{y_1^e - c_1^e}{1-\beta} + \left(1-p_1\right)\left[-c^u + \beta C\left(W_2\right)\right] \leq \frac{y_1^e - c_1^e}{1-\beta} \tag{20}$$

Hence, using (19), we have

$$c_0^u \leq \beta C\left(W_1\right) - \frac{y_0^e - c_0^e}{1-\beta} \Leftrightarrow c_0^u + \left(y_0^e - c_0^e\right) \leq \beta C\left(W_1\right) - \frac{\beta\left(y_0^e - c_0^e\right)}{1-\beta}$$

Since $c_0^u \geq c_0^e$ and $y_0^e \geq 0$ we have

$$0 \leq \beta\left[C\left(W_1\right) - \frac{y_0^e - c_0^e}{1-\beta}\right].$$

Using (20), the last term is less than

$$\beta\left[\frac{\left(y_1^e - c_1^e\right)}{1-\beta} - \frac{\left(y_0^e - c_0^e\right)}{1-\beta}\right] = \beta\left[\frac{\left(y_1^e - y_0^e\right)}{1-\beta} + \frac{\left(c_0^e - c_1^e\right)}{1-\beta}\right].$$

Hence, using (17) and (18), we see that $\left(y_1^e - y_0^e\right) + \left(c_0^e - c_1^e\right) < 0$.

A contradiction. ■

Proof of Theorem 3.1. i) The facts that the moral-hazard constraint (8) binds, and the government benefits from strictly increasing p follow immediately from Lemma A.4.

ii) We have

$$
\begin{aligned}
(\varphi'^{e}) &= \frac{p}{\mu p+\lambda} \\
\eta'\left(y^{e}+\kappa\frac{\vartheta(p)}{p}\right) &= \frac{p}{\mu p+\lambda}-\frac{p\lambda}{\mu p+\lambda}\eta''\left(y^{e}+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'
\end{aligned}
$$

Hence,

$$
1-\frac{\eta'\Big(y^{e}+\kappa\vartheta(p)/p\Big)}{(\varphi'^{e})}=\lambda\eta''\left(y^{e}+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'>0
$$

iii) Using the first-order conditions we have

$$
\frac{1}{\varphi'(c^{u}_{t-1})}=\mu_{t-1}+p_{t-1}^{-1}\lambda_{t-1}=\mu_{t-1}-\lambda(1-p_t)^{-1}=\mu_t,
$$

and

$$
\left(\frac{p_t}{\varphi'(c^{e}_{t})}\right)+\left(\frac{1-p_t}{\varphi'(c^{u}_{t})}\right)=(p_t\mu_t+\lambda_t)+(\mu_t(1-p_t)-\lambda_t)=\mu_t,
$$

Hence,

$$
\frac{1}{\varphi'(c^{u}_{t-1})}=\frac{p_t}{\varphi'(c^{e}_{t})}+\frac{1-p_t}{\varphi'(c^{u}_{t})}.
$$

■

Proof of Proposition 3.2. Notice that

$$
\varphi'(c^{u}_{t})=\frac{1}{\mu_t-\frac{\lambda_t}{1-p_t}}
$$

and

$$
\mu_{t+1}=\mu_t-\frac{\lambda_t}{(1-p_t)},
$$

hence

$$
\varphi'(c^{u}_{t+1})-\varphi'(c^{u}_{t})=\frac{1}{\mu_{t+1}-\frac{\lambda_{t+1}}{1-p_{t+1}}}-\frac{1}{\mu_t-\frac{\lambda_t}{1-p_t}}=\frac{1}{\mu_{t+1}-\frac{\lambda_{t+1}}{1-p_{t+1}}}-\frac{1}{\mu_{t+1}}\geq 0,
$$

with strict inequality whenever the worker searches in $t+1$ and hence $\frac{\lambda_{t+1}}{1-p_{t+1}} > 0$.

Finally, for the last claim assume that the worker actively searches in period $t+1$, use

$$\frac{p_{t+1}}{\varphi'(c^e_{t+1})} + \frac{1-p_{t+1}}{\varphi'(c^u_{t+1})} = \frac{1}{\varphi'(c^u_t)}$$

and $c^u_{t+1} > c^u_t$ to conclude that $c^e_{t+1} > c^u_\tau$ for every $\tau \geq t$. ∎

Proof of Proposition 3.3. Notice that unemployment insurance is decreasing. Moreover, since $W_0 < W^*$, there is a period T such that it is strictly decreasing from T on. Suppose towards a contradiction that it converges to $c^u_\infty > 0$. If it does not converge to zero, since $\varphi'(c^u_t) = \frac{1}{\mu_{t+1}}$, we conclude that $\mu_t \to (\varphi'(c^u_\infty))^{-1}$. Therefore,

$$\frac{\lambda_t}{1-p_t} \to 0.$$

We claim that $p_t \to 0$. Suppose towards a contradiction that there is a subsequence $p_{t_r} \to \hat{p} > 0$ and notice that, since

$$\varphi'(c^e_t) = \frac{1}{\mu_t + p_t^{-1}\lambda_t},$$

we have along the subsequence $\varphi'(c^e_{t_r}) \to \varphi'(c^u_\infty)$, implying $c^e_{t_r} \to c^u_\infty$. By incentive compatibility,

$$\eta\left(y^e_{t_r} + \kappa\frac{\vartheta(p_{t_r})}{p_{t_r}}\right) \to 0,$$

which is not possible, a contradiction.

But then by a continuity argument, for every $\varepsilon > 0$, there exists a period t^* such that $t \geq t^*$ implies that the government's utility is ε away from $-c^u_\infty/(1-\beta)$ while the worker's utility is ε away from $\varphi(c^u_\infty)/(1-\beta)$.

Since $\varphi(c^u_\infty) < W^*$, there exists $\varepsilon > 0$ such that $Z(\varphi(c^u_\infty) + \varepsilon) < c^u_\infty$. Hence if the government deviates toward a stationary employment contract in which he demands $y^e(\varphi(c^u_\infty) + \varepsilon)$ (see notation of Lemma A.1) and provides consumption $y^e(\varphi(c^u_\infty) + \varepsilon) + Z(\varphi(c^u_\infty) + \varepsilon)$ then worker searches for a job with probability bounded away from zero (for t sufficiently large). Moreover, for t sufficiently large this deviation makes both the worker and the government better off. A contradiction. ■

A.3 Non-observable Savings

Lemma A.5 *Consider any deterministic mechanism. Assume that the agent starts with income a_0. Let (c_t^u, p_t) be the optimal choices of the agent at period t. The agent who starts with income $\tilde{a}_0$ chooses $(c_t^u + (\tilde{a}_0 - a_0)(1-\beta), p_t)$ in every period t and obtains $e^{-\alpha(1-\beta)(\tilde{a}_0 - a_0)} W_t$ where W_t is the utility obtained at t by the agent who starts the game with assets a_t.*

Proof. The proof will be based on the principle of optimality. We will guess and verify that if W_t is the agent's continuation utility when period t is started with income a_t^1, then $e^{-\alpha(1-\beta)(a_t^2 - a_t^1)} W_t$ is the continuation utility when starts period t with a_t^2. Take any optimal strategy $\{(c_\tau^u(a_t), p_\tau(a_t))\}_{\tau \geq t}$ when period t starts with income $a_t \in \{a_t^1, a_t^2\}$ and let W_t^i be its value. Notice that the worker that starts with assets a_t^2 can follow strategy $\{(c_\tau^u(a_t^1) + (a_t^1 - a_t^2)(1-\beta), p_\tau(a_t^1))\}_{\tau \geq t}$. Hence, by revealed preference,

$$W_t^1 \geq e^{-\alpha(1-\beta)(a_t^2 - a_t^1)} W_t^2.$$

Similarly,

$$W_t^2 \geq e^{-\alpha(1-\beta)(a_t^1 - a_t^2)} W_t^1,$$

and thus

$$W_t^1 = e^{-\alpha(1-\beta)(a_t^2 - a_t^1)} W_t^2.$$

Finally, let W_0 be the value from following the optimal strategy when the initial asset is a_0 and observe that strategy $(c_t^u + (\tilde{a}_0 - a_0)(1-\beta), p_t)$ is feasible and it leads to $e^{-\alpha(1-\beta)(\tilde{a}_0 - a_0)} W_0$. Hence this strategy is optimal. ■

We start with a lemma that establishes a Ricardian equivalence result for this setting. Take a feasible sequence $\{c_t^u, c_t^e, p_t\}_{t=0}^{\infty}$ under the policy $(a_0, \{y_t^e, T_t^e, b_t\}_{t=0}^{\infty})$. Let $\{W_t, W_t^e\}_{t=0}^{\infty}$ the sequence of indirect utilities when employed and unemployed respectively which is generated by $\{c_t^u, c_t^e, p_t\}_{t=0}^{\infty}$. Moreover, let W_t^u the utility from failing to find a job at period t.

Lemma A.6 *The sequence* $\{c_t^u, c_t^e, p_t\}_{t=0}^{\infty}$ *is optimal if and only if:*

i)

$$p_t = \arg\max p \left[-\frac{e^{-\alpha\left[c_t^e - \eta\left[y_t^e + \kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta} \right] - (1-p)\, W_{t+1} \tag{21}$$

ii)

$$e^{-\alpha c_t^u} = -(1-\beta)\, W_{t+1}. \tag{22}$$

Proof. In light of Lemma A.5, $\{c_t^u, c_t^e, p_t\}_{t \geq T}^{\infty}$ is optimal at period T if and only if

$$\{c_t^u + (1-\beta)(\tilde{a}_T - a_T), c_t^e + (1-\beta)(\tilde{a}_T - a_T), p_t\}_{t \geq T}^{\infty}$$

when period T starts with assets $\tilde{a}_T$. This allows us to apply the one-shot deviation principle. In our context, this asserts that $\{c_t^u, c_t^e, p_t\}_{t=0}^{\infty}$ is optimal if and only if the optimality conditions w.r.t. p_t and c_t^u hold, which are given by (21) and (22). ∎

The following Lemma states a Ricardian equivalence result for our environment:

Lemma A.7 *Assume that the sequence* $\{c_t^u, c_t^e, p_t\}_{t=0}^{\infty}$ *is feasible under the policies*

$$(a_0, \{y_t^e, T_t^e, b_t\}_{t=0}^{\infty}) \quad \text{and} \quad \left(\tilde{a}_0, \left\{\tilde{y}_t^e, \tilde{T}_t^e, \tilde{b}_t\right\}_{t=0}^{\infty}\right).$$

The sequence, $\{c_t^u, c_t^e, p_t\}_{t=0}^{\infty}$*, is optimal under* $(a_0, \{y_t^e, T_t^e, b_t\}_{t=0}^{\infty})$ *if and only if it is optimal under* $\left(\tilde{a}_0, \left\{\tilde{y}_t^e, \tilde{T}_t^e, \tilde{b}_t\right\}_{t=0}^{\infty}\right)$.

Proof. Follows immediately from Lemma A.6. ∎

Lemma A.8 *Let* $(c_t^u, c_t^e, y_t^e, p_t)_{t=0}^{\infty}$ *solve the government's problem when the agent starts with utility* W_0*. Then* $(c_t^u + \Delta, c_t^e + \Delta, y_t^e, p_t)_{t=0}^{\infty}$ *solves the government's problem when the agent who starts with utility* $\tilde{W} = e^{-\alpha\Delta} W_0$.

Proof. We claim that $(c_t^u + \Delta, c_t^e + \Delta, y_t^e, p_t)_{t=0}^{\infty}$ is at least as good as any allocation $(\tilde{c}_t^u, \tilde{c}_t^e, \tilde{y}_t^e, \tilde{p}_t)_{t=0}^{\infty}$ that yields utility $\tilde{W}$. Indeed, take $(\tilde{c}_t^u, \tilde{c}_t^e, \tilde{y}_t^e, \tilde{p}_t)_{t=0}^{\infty}$ and notice that $(\tilde{c}_t^u - \Delta, \tilde{c}_t^e - \Delta, \tilde{y}_t^e, \tilde{p}_t)_{t=0}^{\infty}$ generates utility W_0. Hence, the optimality of

$(c_t^u, c_t^e, y_t^e, p_t)_{t=0}^{\infty}$ implies

$$\sum_{t=0}^{\infty} \beta^t \left(\sum_{\tau \leq t} \left(\Pi_{s<\tau} \left(1 - p_s\right)\right) p_\tau \left[y_\tau^e - c_\tau^e\right] + \left(1 - \sum_{\tau \leq t} \left(\Pi_{s<\tau} \left(1 - p_s\right)\right) p_\tau \right) \left[-c_t^u\right] \right)$$
$$\geq \sum_{t=0}^{\infty} \beta^t \left(\sum_{\tau \leq t} \left(\Pi_{s<\tau} \left(1 - \tilde{p}_s\right)\right) \tilde{p}_\tau \left[\tilde{y}_\tau^e - \left(\tilde{c}_\tau^e - \Delta\right)\right] + \left(1 - \sum_{\tau \leq t} \left(\Pi_{s<\tau} \left(1 - \tilde{p}_s\right)\right) \tilde{p}_\tau \right) \left[-\left(\tilde{c}_t^u - \Delta\right)\right] \right),$$

which holds if and only if

$$\sum_{t=0}^{\infty} \beta^t \left(\sum_{\tau \leq t} \left(\Pi_{s<\tau} \left(1 - p_s\right)\right) p_\tau \left[y_\tau^e - \left(c_\tau^e + \Delta\right)\right] + \left(1 - \sum_{\tau \leq t} \left(\Pi_{s<\tau} \left(1 - p_s\right)\right) p_\tau \right) \left[-\left(c_t^u + \Delta\right)\right] \right)$$
$$\geq \sum_{t=0}^{\infty} \beta^t \left(\sum_{\tau \leq t} \left(\Pi_{s<\tau} \left(1 - \tilde{p}_s\right)\right) \tilde{p}_\tau \left[\tilde{y}_\tau^e - \tilde{c}_\tau^e\right] + \left(1 - \sum_{\tau \leq t} \left(\Pi_{s<\tau} \left(1 - \tilde{p}_s\right)\right) \tilde{p}_\tau \right) \left[-\tilde{c}_t^u\right] \right),$$

which proves the optimality of $(c_t^u + \Delta, c_t^e + \Delta, y_t^e, p_t)_{t=0}^{\infty}$ when the promised utility is $\tilde{W}$. ■

Proof of Lemma 4.1. Consider a simple policy (a_0, y^e, T^e). Let (p_0, c_0^u) be the first-period choices and a_1 be the corresponding level of assets if the agent is unemployed in period 1. Lemma A.5 implies that

$$W_1 = e^{-\alpha(1-\beta)(a_1 - a_0)} W_0.$$

Moreover, $p_1 = p_0$, $c_1^e = c_0^e - (1-\beta)(a_1 - a_0)$, and $c_1^u = c_0^u - (1-\beta)(a_1 - a_0)$. Hence, if we let

$$\Delta := \alpha^{-1} \log\left(\frac{W_1}{W_0}\right),$$

and apply Lemma A.5 inductively we see that $p_t = p_0$ for every t, $c_t^e = c_0^e - \Delta t$ and $c_t^u = c_0^u - \Delta t$.

The last part of the lemma follows immediately from the agent's first-order condition w.r.t. p_t and straightforward algebra. ■

Proof of Theorem 4.2. Let $\{(p_t^*, y_t^{e*}, c_t^{u*}, c_t^{e*})\}_{t=0}^{\infty}$ be the optimal allocation.

Notice that

$$W_0^* = p_0^* W_0^{e*} + (1-p_0^*) W_0^{u*}.$$

If $W_0^{e*} \leq W_0^{u*}$ then $p_0^* = 0$. In this case, the optimal allocation can be implemented by asset

$$a_0 = \frac{-\alpha^{-1}\log\left(-(1-\beta)W_0\right)}{1-\beta}$$

as well as some pair (y^e, T^e) with $y^e = T^e$. The worker best responds by never searching for a job and consuming $-(1-\beta)\alpha^{-1}\log\left(-(1-\beta)W_0\right)$ in every period. By Lemma A.8, this is optimal.

Next assume that $W_0^{e*} > W_0^{u*}$. Consider the first-order condition:

$$-\frac{e^{-\alpha\left[c_0^{e*} - \eta\left[y_0^{e*} + \kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)\right]\right]}}{1-\beta} - W_0^{u*} - \alpha p_0^{e*} \frac{e^{-\alpha\left[c_0^{e*} - \eta\left[y_0^{e*} + \kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)\right]\right]}}{1-\beta} \eta' \left[y^e + \kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)\right] \kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)' = 0,$$

and the following promise-keeping condition:

$$W_0^* = p_0^* W_0^{e*} + (1-p_0^*) W_0^{u*}.$$

Solving these two equations we obtain:

$$
\begin{aligned}
W_0^{e*} &= \frac{W_0^*}{1+\alpha p_0^{e*}(1-p_0^*)\eta'\left[y_0^{e*}+\kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)\right]\kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)'} \\
W_0^{u*} &= W_0^*\left(1+\frac{\alpha p_0^{*2}\eta'\kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)}{1+\alpha p_0^*(1-p_0^*)\alpha\eta'\kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)}\right)
\end{aligned}
$$

Next notice that W_0^{e*} delivers c_0^{e*} by

$$
-\frac{e^{-\alpha\left[c_0^{e*}-\eta\left[y_0^{e*}+\kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)\right]\right]}}{1-\beta}=W_0^{e*},
$$

which implies

$$
c_0^{e*}=-\alpha^{-1}\log\left(-\left(1-\beta\right)W_0^{e*}\right)+\eta\left[y_0^{e*}+\kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)\right].
$$

We claim that there exists (a_0^*, T^{e*}) that solves the system:

$$
c_0^{e*}=\left(1-\beta\right)a_0+y_0^{e*}+T^e \tag{23}
$$

$$
W_0^{u*}=\max_c -e^{-\alpha c}+\beta U\left(\beta^{-1}\left(a_0-c\right),y_0^{e*},T^e\right), \tag{24}
$$

where $U\left(a, y_0^{e*}, T^e\right)$ is the utility of an agent who starts a period unemployed and faces a simple policy, (a, y_0^{e*}, T^e). Notice that if $T^e = y^e$ and $a_0 = \frac{c_0^{e*}}{1-\beta}$ then

$$
W_1^* < \max_c -e^{-\alpha c}+\beta U\left(\beta^{-1}\left(a_0-c\right),y_0^{e*},T^e\right) \tag{25}
$$

as the agent can keep consumption constant at c^{e*} even without taking a job. In fact, he best responds to that contract by choosing $p = 0$ in every period. From this point, notice that if we decrease a_0 by $-\frac{\varepsilon}{1-\beta}$ and decrease T^e by ε the planner's payoff is increased by

$$\frac{\varepsilon}{1-\beta}\left(1-\frac{p\left(a, y_0^{e*}, T^e\right)}{1-\left(1-p\left(a, y_0^{e*}, T^e\right)\right)\beta}\right)>0. \tag{26}$$

Next notice that by construction

$$-\frac{e^{-\alpha\left[\left((1-\beta)a_0+y_0^{e*}+T^e\right)-\eta\left[y_0^{e*}+\kappa\left(\frac{\vartheta(p_0^*)}{p_0^*}\right)\right]\right]}}{1-\beta}=W_0^{e*}.$$

Hence, meanwhile the inequality (25) the worker's optimality condition w.r.t. p implies $p\left(a, y_0^{e*}, T^e\right)<p^*$. We claim that if we keep decreasing a_0 by $-\frac{\varepsilon}{1-\beta}$ and T^e by ε we generate (a_0^*, T^{e*}) satisfying (23) and (24). Otherwise, as we take a_0 to $-\infty$ the planner's revenue goes to infinity while the worker's utility at the beginning remains above W_0^*, a contradiction. From the first order condition, we know that p remains bounded below p^* (and by lemma A.5 this holds in every future period) and the principal obtains infinite profits because of (26). At the same time the worker' s utility remains greater than $pW_0^{e*}+(1-p)W_0^{u*}$. A contradiction.

The reasoning above shows that offering (a_0^*, y^{e*}, T^{e*}) in the first period is optimal to generate utility W_0^*. In this case, Lemma A.8 implies that (a_1^*, y^{e*}, T^{e*}) is optimal to generate utility W_1^*, where a_1^* is the asset holdings chosen by the agent. Inductively, we conclude that (a_t^*, y^{e*}, T^{e*}) is optimal to generate utility W_t^* for every t and hence the simple policy (a_0^*, y^{e*}, T^{e*}) is optimal. ■

Lemma A.9 *We have* $\frac{\partial p}{\partial c^e}>0$ *and* $\frac{\partial p}{\partial y^e}<0.$

Proof. We must calculate $\frac{\partial p}{\partial c^e}$ and $\frac{\partial p}{\partial y^e}$. Let $c^e:=y^e-T^e$, assume without a loss that the agent starts with zero assets (Lemma A.5) and write W_1 for the payoff of an agent who starts a period of unemployed with zero assets. Start with the f.o.c.

w.r.t. p,

$$-\frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}-\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]$$
$$-\alpha p\frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)'=0. \quad (27)$$

Next, we remark that the problem is strictly concave in p and hence the derivative of (27) w.r.t. p is strictly negative. Differentiating this condition w.r.t. c^e we obtain

$$\alpha\frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}-\frac{d}{dc^e}\left[\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]\right]$$
$$+\alpha^2 p\frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)'$$

Now notice that

$$\frac{d}{dc^e}\left[\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]\right]<-\alpha\max_{a'}\left[e^{\alpha a'\beta}-e^{-\alpha a'(1-\beta)}\beta W_1\right] \quad (28)$$

as the last number is obtained by the derivative of an increase in c in every state of nature. Therefore, we have

$$
\begin{aligned}
&\alpha\frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}-\frac{d}{dc^e}\left[\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]\right]\\
&\qquad+\alpha^2 p\frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)'=\\
&\qquad\alpha\frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}+\alpha\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]\\
&\qquad+\alpha^2 p\frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)'\\
&-\alpha\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]-\frac{d}{dc^e}\left[\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]\right]=\\
&-\alpha\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]-\frac{d}{dc_e}\left[\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]\right]>0,
\end{aligned}
$$

where we have used (27) and (28). Therefore,

$$
\frac{\partial p}{\partial c^e}>0.
$$

Next, differentiating the f.o.c. w.r.t. y^e we get

$$
\begin{aligned}
&-\alpha\eta'\frac{e^{-\alpha\left[c_e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}-\frac{d}{dy^e}\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]\\
&\qquad-\alpha^2 p\eta'\frac{e^{-\alpha\left[c_e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)'\\
&\qquad\qquad-\alpha p\eta''\frac{e^{-\alpha\left[c_e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)'.
\end{aligned}
$$

Notice that

$$
\frac{d}{dy_e}\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]>\alpha\eta'\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right]\tag{29}
$$

Hence,

$$
\begin{aligned}
&-\alpha\eta' \frac{e^{-\alpha\left[c_e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta} - \frac{d}{dy_e}\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right] \\
&\qquad -\alpha^2 p\eta' \frac{e^{-\alpha\left[c_e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)' \\
&\qquad -\alpha p\eta'' \frac{e^{-\alpha\left[c_e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)' = \\
&\qquad -\alpha\eta' \frac{e^{-\alpha\left[c_e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta} - \alpha\eta'\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right] \\
&\qquad -\alpha^2 p\eta' \frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)' \\
&\quad \alpha\eta'\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right] - \frac{d}{dy_e}\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right] \\
&\qquad -\alpha p\eta'' \frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)' = \\
&\quad \alpha\eta'\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right] - \frac{d}{dy_e}\max_{a'}\left[-e^{\alpha a'\beta}+e^{-\alpha a'(1-\beta)}\beta W_1\right] \\
&\qquad\qquad -\alpha p\eta'' \frac{e^{-\alpha\left[c^e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\eta'\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\kappa\left(\frac{\vartheta(p)}{p}\right)' < 0,
\end{aligned}
$$

where we have used (27) and (29). ■

Lemma A.10 *We have*

$$
\sum_{t=0}^{\infty} p\beta^t (1-p)^{t-1}\left[1+\frac{1}{W_0}e^{-\alpha(1-\beta)a_t}\frac{e^{-\alpha\left[c_e-\eta\left[y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}\right] > 0.
$$

Proof. We have

$$\sum_{t=0}^{\infty} p\beta^t (1-p)^{t-1} \left[\frac{-1}{1-\beta} - \frac{e^{-\alpha(1-\beta)a_t} \frac{e^{-\alpha\left[c_e - \eta\left[y^e + \kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}}{(1-\beta) W_0} \right] < 0$$

$$\Leftrightarrow \sum_{t=0}^{\infty} p\beta^t (1-p)^{t-1} \left[- \frac{e^{-\alpha(1-\beta)a_t} \frac{e^{-\alpha\left[c_e - \eta\left[y^e + \kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]}}{1-\beta}}{\sum_{t=0}^{\infty} p\beta^t (1-p)^{t-1}} \right] > W_0.$$

Since $z \to -e^{-\alpha z}$ is strictly increasing. Notice that U_0 is the mixture of the distribution F^e over employed payoffs defined above and the distribution over $-e^{-\alpha c_t^u}$, which we call F^u. It follows that if F^e first-order stochastic dominates F^u :

$$\int x dF^e(x) > \int x dF^u(x)$$

and hence for any $\lambda \in (0,1)$,

$$\int x d\left[\lambda F^e(x) + (1-\lambda) F^e(x)\right] < \int x dF^e(x).$$

Hence it suffices to show that

$$W_0 < \int x dF^e(x).$$

We have

$$W_0 (1-\beta) = p(1-\beta) W_e^0 + (1-p)(1-\beta) \left[-e^{-\alpha c_0^u} + \beta \left[p W_1^e + (1-p)\left[-e^{-\alpha c_1^u} + \beta W_2^u\right]\right]\right]$$

Using $W_e^0 > -\frac{e^{-\alpha c_0^u}}{1-\beta}$ and

$$-\frac{e^{-\alpha c_0^u}}{1-\beta} = p W_1^e + (1-p)\left[-e^{-\alpha c_1^u} + \beta W_2^u\right],$$

we have

$$W_0 < \frac{pW_e^0 + \beta(1-p)\left[pW_1^e + (1-p)\left[-e^{-\alpha c_1^u} + \beta W_2^u\right]\right]}{1-(1-p)(1-\beta)}.$$

Proceeding analogously, it follows that the last expression is less than

$$\frac{pW_e^0 + \beta(1-p)\left[pW_1^e + (1-p)\,\beta W_2^u\right]}{1-(1-p)(1-\beta)-(1-p)^2\beta^2},$$

Proceeding analogously and taking the limit we obtain the desired inequality.
■

Proof of Theorem 4.3. Part (i). Recall from (11)

$$\frac{\partial}{\partial p}\left[\frac{p}{1-(1-p)\beta}\right]\left(\frac{y^e-c^e}{1-\beta}\right)\frac{\partial p}{\partial c^e} = \frac{p\,(y^e,c^e)}{1-(1-p\,(y^e,c^e))\,\beta} + \frac{U_{c^e}\,(y^e,c^e)}{e^{\alpha(1-\beta)a_0}\alpha\,(1-\beta)\,W_0}$$

Since

$$\frac{\partial}{\partial p}\left[\frac{p}{1-(1-p)\beta}\right] > 0 \quad \text{and} \quad \frac{\partial p}{\partial c^e} > 0,$$

$y^e - c^e$ has the same sign as

$$-\sum_{t=0}^{\infty} p\beta^t\,(1-p)^{t-1}\left[\frac{-1}{1-\beta} - \frac{\exp\left\{-\alpha\left\{c_e - \eta\big(y^e + \kappa\vartheta(p)/p\big)\right\}\right\}}{(1-\beta)^2 W_0}\right],$$

by Lemma A.9, which is strictly positive by Lemma A.10.

Part (ii). Consider the problem

$$C(W_0) = \max_{W_1,c_e,y_e} p\left[\frac{y^e-c_e}{1-\beta}\right] + (1-p)\,\beta C(e^{-\alpha a(1-\beta)}W_1),$$

subject to

$$-\frac{p}{1-\beta}\exp\left\{-\alpha\left[c_e - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right]\right\} + \\ (1-p)\max_{a'}\left\{-\exp\{\alpha a'\beta\} + \exp\{-\alpha a'\,(1-\beta)\}\beta W_1\right\} - W_0 = 0$$

and

$$
\begin{aligned}
&-\frac{1}{1-\beta}\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\} \\
&\qquad -\max_{a'}\left\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\right\} \\
&-\alpha p\frac{1}{1-\beta}\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\}\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'=0.
\end{aligned}
$$

Plugging the last constraint into the problem, one obtains the following Lagrangian

$$
\begin{aligned}
C(W_0)=\max_{W_1,c_e,y_e} & p\left[\frac{y^e-c_e}{1-\beta}\right]+(1-p)\,\beta C(e^{-\alpha a(1-\beta)}W_1)+ \\
&\mu\left[-\frac{1}{1-\beta}\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\}-\right. \\
&\left.\alpha(1-p)\,p\frac{1}{1-\beta}\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\}\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'-W_0\right]
\end{aligned}
$$

Therefore, we have the first-order conditions with respect to c^e,

$$
p=\mu\alpha\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\}\left[1+\alpha(1-p)\,p\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'\right]
$$

with respect to y^e,

$$
\begin{aligned}
p=\mu\alpha\exp&\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\}\left[\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)+\right. \\
&\left.\alpha(1-p)\,p\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)^2\kappa\left(\frac{\vartheta(p)}{p}\right)'\right] \\
&+\mu(1-p)\,p\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\}\eta''\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'
\end{aligned}
$$

Therefore, we have

$$\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)=1-\frac{(1-p)\,p\eta''\Big(y^e+\kappa\vartheta(p)/p\Big)\kappa\left(\frac{\vartheta(p)}{p}\right)'}{\alpha\left[1+\alpha\,(1-p)\,p\eta'\Big(y^e+\kappa\vartheta(p)/p\Big)\kappa\left(\frac{\vartheta(p)}{p}\right)'\right]}.$$

Part (iii). Notice that $W_t^* < W_t^{e*}$ and hence it suffices to show that $\lim W_t^{e*} = -\infty$. We have

$$\lim(1-\beta)W_t^{e*} =$$
$$-\lim\exp\left\{-\alpha\left[c_0^*+(1-\beta)\,\bar{a}_0-\eta\left[y^{e*}+\kappa\left(\frac{\vartheta(p^*)}{p^*}\right)\right]\right]\alpha\,(t-1)\,\Delta_c\right\}=-\infty.$$

■

B Extension: GHH-CARA type and Observable Savings

In this section, we consider the case of observable savings with period utility of the form

$$\mathcal{U}(c,n) = -\exp\left\{-\alpha\left[c-\eta(n)\right]\right\}.$$

We can write the Lagrangean as

$$C\left(W_0\right) = \max \frac{p}{1-\beta}\left(y^e - c^e\right) + (1-p)\left[-c^u + \beta C\left(W_1\right)\right],$$

subject to

$$\frac{p}{1-\beta}\left[-\exp\left\{-\alpha\left[c^e-\eta(n^e)\right]\right\}\right] + (1-p)\left[-\exp\left\{-\alpha\left[c_u\right]\right\} + \beta W_1\right] - W_0 \geq 0,$$

and

$$\begin{aligned}\frac{1}{1-\beta}&\left[-\exp\left\{-\alpha\left[c-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\}\right] + \exp\left\{-\alpha\left[c_u\right]\right\} - \beta W_1 \\ &= \frac{1}{1-\beta}\alpha\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\left(\frac{\kappa\vartheta(p)}{p}\right)'\exp\left\{-\alpha\left[c-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\}.\end{aligned}$$

Let

$$\begin{aligned}U^e &:= \exp\left\{-\alpha\left[c-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\} \\ U^u &:= \exp\left\{-\alpha c^u\right\}\end{aligned}$$

The first order condition for c_0^e is

$$-p + \mu p\alpha U^e + \lambda\alpha U^e = 0.$$

The first order condition for c_0^u is

$$-(1-p)+(1-p)\,\mu U^u-\lambda\alpha U^u=0.$$

The first order condition for y^e is

$$p-p\mu\alpha U^e\eta'(n^e)-\lambda\alpha U^e\eta'(n^e)$$
$$-\lambda U^e\left[\alpha\eta''(n^e)\kappa\left(\frac{\vartheta(p)}{p}\right)'+\alpha^2U^e\left[\eta'(n^e)\right]^2\kappa\left(\frac{\vartheta(p)}{p}\right)'\right]=0$$

From these, we have

$$U^e=\frac{p}{\mu p\alpha+\lambda\alpha}$$
$$U^u=\frac{1-p}{\mu\,(1-p)\,\alpha-\lambda\alpha}$$
$$p-\eta'(n^e)U^e\left[p\mu\alpha+\lambda\alpha\right]=\lambda U^e\left[\alpha\eta''(n^e)\kappa\left(\frac{\vartheta(p)}{p}\right)'+\alpha^2U^e\left[\eta'(n^e)\right]^2\kappa\left(\frac{\vartheta(p)}{p}\right)'\right]$$
$$1-\eta'(n^e)=\frac{\lambda U^e}{p}\left[\alpha\eta''(n^e)\kappa\left(\frac{\vartheta(p)}{p}\right)'+\alpha^2\left[\eta'(n^e)\right]^2\kappa\left(\frac{\vartheta(p)}{p}\right)'\right] \tag{30}$$
$$C'\left(W_1\right)=-\mu_0+\frac{\lambda_0}{(1-p_0)}=-\mu_1,$$

which implies

$$\mu_1=\mu_0-\frac{\lambda_0}{(1-p)}.$$

Moreover, the derivative with respect to p implies

$$\frac{p}{1-\beta}\left(y^e-c^e\right)=\frac{\lambda\alpha U^e}{1-\beta}\Bigg[\eta'(n^e)\left(\kappa\frac{\vartheta(p)}{p}\right)'+\eta''(n^e)\left(\frac{\kappa\vartheta(p)}{p}\right)'$$
$$+\eta'(n^e)\left(\kappa\frac{\vartheta(p)}{p}\right)''+\left[\eta'(n^e)\left(\kappa\frac{\vartheta(p)}{p}\right)'\right]^2\Bigg] \tag{31}$$

Lemma B.1 *The multipliers μ and λ are strictly positive if there is search.*

Proof of Lemma B.1. First notice that

$$\begin{aligned} U_0^u &= \frac{1 - p_0}{\mu_0 (1 - p_0) \alpha - \lambda_0 \alpha} \\ U_0^e &= \frac{p_0}{\mu_0 p_0 \alpha + \lambda_0 \alpha} \end{aligned}$$

hence $\mu_0 = 0$ implies $U_0^u U_0^e \leq 0$ which is an absurd.

Hence assume towards a contradiction that $\lambda_0 \leq 0$. Clearly, there is a last period at which $\lambda_t \leq 0$ and $\lambda_{t+1} > 0,$ otherwise, as we will verify below, $c_t^u \geq c_t^e$ for every t, and hence there is no search. Assume that $\lambda_1 > 0$ (case in which $\lambda_s \leq 0$ for all $s < t$ and $\lambda_t > 0$ for some $t > 1$ can be analogously handled).

From the first-order conditions, we have:

$$- U_1^e > -U_0^u \geq -U_0^e. \tag{32}$$

From the first-order condition with respect to p we get

Since the agent searches with positive probability

$$\frac{-U_0^e}{1-\beta} > -U_0^u + \beta \left[p_1 \frac{-U_1^e}{1-\beta} + (1-p_1) \left[-U_1^e + \beta W_2 \right] \right] \tag{33}$$

$$\frac{-U_1^e}{1-\beta} > -U_1^u + \beta W_2 \tag{34}$$

Notice that

$$\begin{aligned} \left(U_1^e + \beta W_2\right) &< \frac{U_1^e}{1-\beta} \\ \left(U_1^e + \beta W_2\right) &< \frac{U_0^u}{1-\beta} \end{aligned}$$

Therefore

$$U_1^e + \beta W_2 < U_0^u + \beta \left[p_1 \left(\frac{U_1^e}{1-\beta} \right) + (1-p_1) \left(U_1^e + \beta W_2 \right) \right] \tag{35}$$

$$\frac{y_0^e - c_0^e}{1-\beta} \leq -c_0^u + \beta C \left(W_1 \right). \tag{36}$$

$$\frac{y_1^e - c_1^e}{1-\beta} > -c_1^u + \beta C\left(W_2\right). \tag{37}$$

Using (30) and $\lambda_0 \leq 0 < \lambda_1$ we obtain

$$y_1^e + \kappa\frac{\vartheta(p_1)}{p_1} < y_0^e + \kappa\frac{\vartheta(p_0)}{p_0} \tag{38}$$

Using (32), (35) and the agent's f.o.c. w.r.t. p we obtain

$$\begin{aligned}
\frac{1}{1-\beta}\alpha\eta'\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\left(\frac{\kappa\vartheta(p_1)}{p_1}\right)' U_1^e &= \frac{-U_1^e}{1-\beta} - \left[-U_1^e + \beta W_2\right] \\
&> \frac{-U_0^e}{1-\beta} - \left[U_0^u + \beta\left[p_1\left(\frac{U_1^e}{1-\beta}\right) + (1-p_1)\left(U_1^e + \beta W_2\right)\right]\right] \\
&= \frac{1}{1-\beta}\alpha\eta'\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\left(\frac{\kappa\vartheta(p_0)}{p_0}\right)' U_0^e,
\end{aligned}$$

and therefore

$$\eta'\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\left(\frac{\kappa\vartheta(p_1)}{p_1}\right)' U_1^e > \eta'\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\left(\frac{\kappa\vartheta(p_0)}{p_0}\right)' U_0^e.$$

Since $-U_1^u > -U_0^e$ we have $U_1^u < U_0^e$ and hence

$$\eta'\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\left(\frac{\kappa\vartheta(p_1)}{p_1}\right)' > \eta'\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\left(\frac{\kappa\vartheta(p_0)}{p_0}\right)'. \tag{39}$$

We claim that (38) and (39) imply $y_1 < y_0$ and $p_1 > p_0$. If $y_1 \geq y_0$ then (38) imply $p_1 > p_0$. These imply

$$\eta'\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\left(\frac{\kappa\vartheta(p_1)}{p_1}\right)' < \eta'\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\left(\frac{\kappa\vartheta(p_0)}{p_0}\right)',$$

a contradiction. Thus $y_1 < y_0$ and using (38) we also have $p_1 > p_0$. Hence

$$y_1 < y_0 \tag{40}$$

and

$$p_1 > p_0 \tag{41}$$

$$U_0^u \geq U_0^e$$

Finally, we have

$$\begin{aligned}\frac{y_0^e - c_0^e}{1-\beta} &\leq -c_0^u + \beta C\left(W_1\right) \\ &= -c_0^u + \beta\left[p_1 \frac{y_1^e - c_1^e}{1-\beta} + \left(1-p_1\right)\left[-c_1^u + \beta C\left(W_2\right)\right]\right] \\ &< -c_0^u + \beta\left[p_1 \frac{y_1^e - c_1^e}{1-\beta} + \left(1-p_1\right)\left[\frac{y_1^e - c_1^e}{1-\beta}\right]\right] = -c_0^u + \beta \frac{y_1^e - c_1^e}{1-\beta}\end{aligned} \tag{42}$$

To finish the proof, we consider two cases.

Case 1: $y_0^e - c_0^e \geq y_1^e - c_1^e$ **or** $y_0^e - c_0^e < y_1^e - c_1^e$ **and** $y_1^e - c_1^e < -c_0^u$.

We claim that the planner as well a the agent are better off if the planner pays a constant unemployment insurance equal to c_0^u in each period. In response, the agent never searches. First, (32) and (33) imply that the agent is better off.

To see that the planner is better off, notice that (42) imply that

$$-c_0^u > \max\left\{y_0^e - c_0^e, y_1^e - c_1^e\right\}.$$

. This and (37) imply

$$-c_0^u > p_0\left(\frac{y_0^e - c_0^e}{1-\beta}\right) + \left(1-p_0\right)\left[-c_0^u + \beta\left[\begin{array}{c} p_1 \frac{y_1^e - c_1^e}{1-\beta} \\ + \left(1-p_1\right)\left[-c_1^u + \beta C\left(W_2\right)\right]\end{array}\right]\right].$$

Case 2: $y_0^e - c_0^e < y_1^e - c_1^e$ **and** $y_1^e - c_1^e \geq -c_0^u$.

Consider a deviation in which the planner gives (y_1^e, c_1^e) to the agent in the first period.

Let $\tilde{p}_0$ be the best response of the agent. The planner's payoff is

$$\tilde{p}_0\left(\frac{y_1^e-c_1^e}{1-\beta}\right)+(1-\tilde{p}_0)\left[-c_0^u+\beta\left[p_1\frac{y_1^e-c_1^e}{1-\beta}+(1-p_1)\left[-c_1^u+\beta C\left(W_2\right)\right]\right]\right]$$
$$>-c_0^u+\beta\left[p_1\frac{y_1^e-c_1^e}{1-\beta}+(1-p_1)\left[-c_1^u+\beta C\left(W_2\right)\right]\right]$$
$$>p_0\left(\frac{y_0^e-c_0^e}{1-\beta}\right)+(1-p_0)\left[-c_0^u+\beta\left[p_1\frac{y_1^e-c_1^e}{1-\beta}+(1-p_1)\left[-c_1^u+\beta C\left(W_2\right)\right]\right]\right],$$

where the last line uses (36). To show that the agent is better off, notice that p_1 is available and yields

$$p_1\frac{U_1^e}{1-\beta}+(1-p_1)\left(U_0^u+\beta\left[p_1\frac{U_1^e}{1-\beta}+(1-p_1)\left[U_1^e+\beta W_2\right]\right]\right)$$
$$>p_1\frac{U_0^e}{1-\beta}+(1-p_1)\left(U_0^u+\beta\left[p_1\frac{U_1^e}{1-\beta}+(1-p_1)\left[U_1^e+\beta W_2\right]\right]\right)$$
$$>p_0\frac{U_0^e}{1-\beta}+(1-p_0)\left(U_0^u+\beta\left[p_1\frac{U_1^e}{1-\beta}+(1-p_1)\left[U_1^e+\beta W_2\right]\right]\right),$$

which completes the proof. ■

Recall that

$$\phi=\lim_{p\downarrow 0}\vartheta(p)/p>0.$$

We make the following assumption (otherwise working is always inefficient):

Assumption DS (desirable search) Search is desirable,

$$\max y-\eta\left(y+\kappa\phi\right)>0.$$

Let

$$y^*=\arg\max y-\eta\left(y+\kappa\phi\right)$$

or

$$1 = \eta' \left(y^* + \kappa\phi \right)$$

or

$$y^* = \left(\eta' \right)^{-1} (1) - \kappa\phi.$$

Lemma B.2 *There is positive search in every period.*

Proof of Lemma B.2. Assume towards a contradiction that there is no search at period t. Let W_t be the agent's utility at t. We have

$$W_t = \mathbb{E} \left[- \sum_{t=0}^{\infty} \beta^t \exp \left(-\alpha c_t + \alpha\eta \left(y_t + \kappa\vartheta(p)/p \right) \right) \right].$$

Let $\chi (W)$ be given by

$$-\frac{\exp \left(-\alpha\chi (W) \right)}{1-\beta} = W,$$

or

$$\chi (W) = -\frac{\log \left(-W (1-\beta) \right)}{\alpha}.$$

Let $c (W)$ be given by

$$c (W) = \eta \left(y^* + \kappa\phi \right) + \chi (W).$$

Since $x \to -\exp (-\alpha x)$ is strictly concave, it follows that

$$C(W) > \frac{\eta \left(y^* + \kappa\phi \right) + \chi (W) - y^*}{1-\beta}.$$

From Assumption DS, there is $\varepsilon > 0$ such that if the planner demands production y^* in exchange for consumption $\eta \left(y^* + \kappa\phi \right) + \chi (W) + \varepsilon$ at period t then the agent searches for a job at t and conditionally on finding a job and both players are better off. If $t > 0$ then the planner can decrease the unemployment insurance at $t-1$ to keep the agent indifferent. This decreases the planner´s cost and establishes a contradiction. ■

Theorem B.1 *At the optimum, in every period in which there is positive search,*

1. *the moral-hazard constraint (8) binds, and the planner benefits from strictly increasing p;*

2. *the marginal income tax rate is always positive, and;*

3. *conditional on not finding a job at period t, the worker's marginal utility of consumption satisfies the inverse Euler equation,*

$$\frac{1}{\varphi'\left(c_t^u\right)} = \mathbb{E}\left[\frac{1}{\varphi'\left(c_{t+1}\right)}\right].$$

Proof of Theorem B.1. i) The fact that the moral-hazard constraint (8) binds, and the planner benefits from strictly increasing p follows immediately from Lemma B.1.

ii) From (30) and $\lambda > 0$ we have

$$1 - \eta'(n^e) = \frac{\lambda U^e}{p}\left[\alpha\eta''(n^e)\kappa\left(\frac{\vartheta(p)}{p}\right)' + \alpha^2\left[\eta'(n^e)\right]^2\kappa\left(\frac{\vartheta(p)}{p}\right)'\right] > 0.$$

iii) Using the first-order conditions we have

$$\frac{1}{\varphi'(c_{t-1}^u)} = \mu_{t-1} - \frac{\lambda_{t-1}}{(1-p_{t-1})} = \mu_t.$$

Hence,

$$\frac{p_t}{\varphi'(c_t^e)} + \frac{1-p_t}{\varphi'(c_t^u)} = (p_t\mu_t + \lambda_t) + (\mu_t(1-p_t) - \lambda_t) = \mu_t = \frac{1}{\varphi'(c_{t-1}^u)}$$

■

Proposition B.1 *The unemployment benefit is decreasing over time with $c_t^u > c_{t+1}^u$.*

Moreover, the worker´s consumption from employment at period t is strictly greater than the unemployment benefit from any future period $\tau \geq t$.

Proof of Proposition B.1. Notice that $\lambda_{t+1} > 0$ implies

$$\varphi'(c^u_{t+1}) - \varphi'(c^u_t) = \frac{1}{\mu_{t+1} - \frac{\lambda_{t+1}}{1-p_{t+1}}} - \frac{1}{\mu_t - \frac{\lambda_t}{1-p_t}} = \frac{1}{\mu_{t+1} - \frac{\lambda_{t+1}}{1-p_{t+1}}} - \frac{1}{\mu_{t+1}} > 0,$$

and hence $c^u_{t+1} < c^u_t$.

Next notice that

$$U^u - U^e = \frac{1}{\mu\alpha - \frac{\lambda\alpha}{1-p}} - \frac{1}{\mu\alpha + \lambda\alpha p^{-1}} > 0,$$

which implies $c^e_t > c^u_t + \eta(n^e_t) > c^u_t$. ∎

Proposition B.2 *Unemployment benefits converge to zero.*

Proof of Proposition B.2. Suppose towards a contradiction that $c^u_t \to c^u_\infty > 0$. If it does not converge to zero, since $\varphi'(c^u_t) = \frac{1}{\mu_{t+1}}$, we conclude that $\mu_t \to (\varphi'(c^u_\infty))^{-1}$. Therefore,

$$\frac{\lambda_t}{1-p_t} \to 0.$$

We claim that $p_t \to 0$. Suppose towards a contradiction that there is a subsequence $p_{t_r} \to \hat{p} > 0$ and notice that, since

$$\varphi'(c^e_t) = \frac{1}{\mu_t + p_t^{-1}\lambda_t},$$

we have along the sub-sequence $\varphi'(c^e_{t_r}) \to \varphi'(c^u_\infty)$, implying $c^e_{t_r} \to c^u_\infty$. By incentive compatibility,

$$\eta\left(y^e_{t_r} + \kappa\frac{\vartheta(p_{t_r})}{p_{t_r}}\right) \to 0,$$

which is not possible, a contradiction.

But then by a continuity argument, for every $\varepsilon > 0$, there exists a period t^* such that $t \geq t^*$ implies that the planner's utility is ε away from $-c^u_\infty/(1-\beta)$ while the worker's utility is ε away from $\varphi(c^u_\infty)/(1-\beta)$.

It follows by Assumption DS that there is $\varepsilon > 0$ such that if the planner demands production y^* in exchange for consumption $\eta(y^* + \kappa\phi) + \chi(\varphi(c^u_\infty)/(1-\beta)) +$

ε then the worker searches with probability bounded away from some $\underline{p} > 0$ for every t large enough. Moreover, this ε can be chosen to make both players are better off. A contradiction. ■

C Internal Appendix

Start by noting that the planner's program has a recursive structure,

$$C\left(W_0\right) = \max \frac{p}{1-\beta}\left(y^e - c^e\right) + (1-p)\left[-c^u + \beta C\left(W_1\right)\right],$$

subject to

$$\frac{p}{1-\beta}\left[\varphi(c^e) - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right] + (1-p)\left[\varphi(c^u) + \beta W_1\right] - W_0 \geq 0, \tag{43}$$

and

$$\frac{1}{1-\beta}\left[\varphi(c^e) - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right] - \varphi(c^u) - \beta W_1 = \frac{p\kappa}{1-\beta}\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\left(\frac{\vartheta(p)}{p}\right)'. \tag{44}$$

We show – Lemma A.3 – that the planner's problem is differentiable, and hence the optimum must satisfy a constraint optimization maximization in which we write μ and λ for the multipliers relative to the constraints (43) and (44) respectively. The Lagrangian for the problem is, then,

$$\begin{aligned}\mathcal{L} = {} & \frac{p}{1-\beta}\left(y^e - c^e\right) + (1-p)\left[c^u + \beta C\left(W_1\right)\right] + \\ & \mu\left\{\frac{p}{1-\beta}\left[\varphi(c^e) - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right] + (1-p)\left[\varphi(c^u) + \beta W_1\right] - W_0\right\} + \\ & \lambda\left\{\frac{1}{1-\beta}\left[\varphi(c^e) - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right] - \varphi(c^u) - \beta W_1 \right. \\ & \left. - \frac{p\kappa}{1-\beta}\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\left(\frac{\vartheta(p)}{p}\right)'\right\}\end{aligned}$$

The first-order conditions are

$$\frac{p}{1-\beta}-\mu\frac{p}{1-\beta}\eta'\left(n^{e}\right)-\lambda\left[\frac{1}{1-\beta}\eta'\left(n^{e}\right)+\frac{p\kappa}{1-\beta}\eta''\left(n^{e}\right)\left(\frac{\vartheta(p)}{p}\right)'\right]=0 \tag{45}$$

$$-\frac{p}{1-\beta}+\mu\frac{p}{1-\beta}\varphi'\left(c^{e}\right)+\lambda\frac{1}{1-\beta}\varphi'\left(c^{e}\right)=0 \tag{46}$$

$$\begin{aligned}\frac{1}{1-\beta}\left(y^{e}-c^{e}\right)-\left[-c^{u}+\beta C\left(W_{1}\right)\right]+\frac{\mu}{1-\beta}\Bigg[&\varphi(c^{e})-\eta\left(n^{e}\right)\\ &-p\kappa\eta'\left(n^{e}\right)\left(\frac{\vartheta(p)}{p}\right)'-(1-\beta)\left[\varphi(c^{u})+\beta W_{1}\right]\Bigg]\\ -\frac{\lambda\kappa}{1-\beta}\left\{2\eta'\left(n^{e}\right)\left(\frac{\vartheta(p)}{p}\right)'+p\left[\kappa\eta''\left(n^{e}\right)\left[\left(\frac{\vartheta(p)}{p}\right)'\right]^{2}+\eta'\left(n^{e}\right)\left(\frac{\vartheta(p)}{p}\right)''\right]\right\}&=0,\end{aligned} \tag{47}$$

$$-(1-p)+\mu(1-p)\varphi'(c^{u})=\lambda\varphi'(c^{u}), \tag{48}$$

and

$$(1-p)C'(W_{1})+\mu(1-p)=\lambda. \tag{49}$$

Combining (48) and (49),

$$C'(W_{1})=-\frac{1}{\varphi'(c^{u})}.$$

Adding (45) and (46) (repeated below for ease),

$$p-\mu p\eta'\left(n^{e}\right)-\lambda\left[\eta'\left(n^{e}\right)+p\kappa\eta''\left(n^{e}\right)\left(\frac{\vartheta(p)}{p}\right)'\right]=0,$$

and

$$-p+\mu p\varphi'\left(c^{e}\right)+\lambda\varphi'\left(c^{e}\right)=0,$$

gives us

$$\mu p\varphi'(c^e) - \mu p\eta'(n^e) - \lambda\left[\eta'(n^e) - \varphi'(c^e) + p\kappa\eta''(n^e)\left(\frac{\vartheta(p)}{p}\right)'\right] = 0,$$

which we rearrange to obtain

$$\varphi'(c^e) - \eta'(n^e) = \frac{\lambda p}{\mu p + \lambda}\kappa\eta''(n^e)\left(\frac{\vartheta(p)}{p}\right)',$$

or

$$1 - \frac{\eta'(n^e)}{\varphi'(c^e)} = \frac{\lambda p}{\varphi'(c^e)\{\mu p + \lambda\}}\kappa\eta''(n^e)\left(\frac{\vartheta(p)}{p}\right)'.$$

Using (46),

$$\mu p + \lambda = \frac{p}{\varphi'(c^e)},$$

we get

$$\tau = 1 - \frac{\eta'(n^e)}{\varphi'(c^e)} = \lambda\kappa\eta''(n^e)\left(\frac{\vartheta(p)}{p}\right)'.$$

Now, combining (46) and (48), gives us an expression for μ,

$$\mu = \frac{p}{\varphi'(c^e)} + \frac{1-p}{\varphi'(c^u)}, \tag{50}$$

and another for λ,

$$\left[\frac{1}{\varphi'(c^e)} - \frac{1}{\varphi'(c^u)}\right]p\,[1-p] = \lambda > 0, \tag{51}$$

which, then, implies $\tau > 0$.

Next, recall that

$$\begin{aligned} & y^e - c^e - (1-\beta)\left[-c^u + \beta C(W_1)\right] + \\ & \quad \mu\left[\varphi(c^e) - \eta(n^e) - p\kappa\eta'(n^e)\left(\frac{\vartheta(p)}{p}\right)' - (1-\beta)\left[\varphi(c^u) + \beta W_1\right]\right] \\ & \quad - \lambda\eta'(n^e)\kappa\left\{2\left(\frac{\vartheta(p)}{p}\right)' + p\left[\kappa\frac{\eta''(n^e)}{\eta'(n^e)}\left[\left(\frac{\vartheta(p)}{p}\right)'\right]^2 + \left(\frac{\vartheta(p)}{p}\right)''\right]\right\} = 0 \end{aligned}$$

The term in brackets multiplied by μ is the worker's first order condition with respect to p, which implies

$$n^e - c^e - \kappa \frac{\vartheta(p)}{p} + (1-\beta)\left[\varphi(c^u) - \beta C\left(W_1\right)\right] = \\ \lambda \eta'\left(n^e\right)\kappa\left\{2\left(\frac{\vartheta(p)}{p}\right)' + p\left[\kappa\frac{\eta''\left(n^e\right)}{\eta'\left(n^e\right)}\left[\left(\frac{\vartheta(p)}{p}\right)'\right]^2 + \left(\frac{\vartheta(p)}{p}\right)''\right]\right\},$$

or

$$\left(n^e - c^e - \kappa\frac{\vartheta(p)}{p}\right) + (1-\beta)\left[c^u - \beta C\left(W_1\right)\right] = \\ \eta'\left(n^e\right)\left[\frac{1}{\varphi'\left(c^e\right)} - \frac{1}{\varphi'\left(c^u\right)}\right] p\left[1-p\right]\kappa\left\{2\left(\frac{\vartheta(p)}{p}\right)' \right. \\ \left. + p\left[\kappa\frac{\eta''\left(n^e\right)}{\eta'\left(n^e\right)}\left[\left(\frac{\vartheta(p)}{p}\right)'\right]^2 + \left(\frac{\vartheta(p)}{p}\right)''\right]\right\}$$