BARGAINING UNDER INCOMPLETE INFORMATION WITH THE HELP OF DELAY

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Abstract. We study bargaining under incomplete information, with applications to trade and to provision of public good. In our setting, agents not only agree on how they share their surplus, but also when the trade takes place. We are interested in bargaining rules that do not depend on priors. We find a unique rule that satisfies a set of axioms. Under this rule, the higher the surplus, the sooner the agents agree to trade. Moreover, the surplus is shared as in the Nash bargaining solution. We present a dynamic protocol that implements this rule for any priors. Heterogeneous discount factors and degrees of risk aversion can be included.

Keywords: Bargaining, trade, uncertainty, asymmetric information, axiomatic approach, Nash program, Nash bargaining solution, alternating offers

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1. Introduction

Market economies feature goods and services that are exchanged for money. Ideally, trade between agents should occur whenever there is surplus, the price should be fair. The possibility of such trade has been formally established under complete information. The fair price can be identified by the solutions to axiomatic Nash bargaining. Various dynamic protocols have been provided that implement this solution in a subgame perfect equilibrium (for an overview, see Serrano, 2005). We add uncertainty and follow this agenda without relying on priors.

Uncertainty about values of others is omnipresent in markets. However, incorporating nontrivial uncertainty introduces two major obstacles. First, efficient trade can no longer be guaranteed when participation is voluntary and there are no subsidies. We recall the market for lemons (Akerlof, 1970) and the impossibility results of Myerson and Satterthwaite (1983) and Güth and Hellwig (1986) that prove nonexistence of market rules that lead to trade whenever there is surplus. Second, priors can influence how the market is designed, so a single ideal rule would no longer exist. Priors can also complicate how to behave under these rules and force agents to first agree on the priors in order for there to be trade as desired.

We overcome these obstacles by enriching the possible outcomes and focusing on rules where behavior does not depend on priors. Participants not only have to agree on how to share the surplus, they also have to agree on the time the trade takes place.

We start with a model of bargaining over how to share an output that results from the joint effort of a set of players. Individual effort costs are private information. Players are risk neutral. A bargaining rule determines not only how the output is shared but also when. Later we show that our results translate immediately to models of trade with private values and to public good provision (or collective action) with private values.

Without loss of generality we define bargaining rules using a direct mechanism. So, participants simultaneously report their private values. A bargaining rule then determines how and when the output is shared. To generate behavior independent of priors, we ensure that an agent's choice to participate and to report her private value does not depend on her prior. This leads to the four basic axioms: Dominant Strategy Incentive Compatibility, Renegotiation Proofness, Ex-post Individual Rationality, and Budget Balance. The remaining axioms are motivated by envisaging an impartial designer who is interested in the social surplus. We require that the output produced only depends on the total surplus (Impartiality), that the largest possible output is traded without delay (Weak Non-Wastefulness), and the agreement is based on the agents' actual reports (Independence of Irrelevant Information).

We find that there is a unique rule that satisfies our axioms. According to this rule, the time of trade is decreasing in the surplus deduced from the reported values. The output is divided among the agents according to the Nash bargaining solution, so each agent obtains equal share. We thus obtain the fair allocation by eliminating the importance of priors and only postulating conditions that relate to efficiency.

An important condition for the implementation of our solution is that participants can commit to it. This commitment assumption, albeit common in the literature on mechanism design, is often hard to justify. We demonstrate that our desired outcome can be supported by an alternating offers bargaining game, in which there is no commitment. A proposal consists of when the output should be traded and what shares each player should get. Proposals are made in a given order, where a proposal is implemented if and only if all participants agree. We show that our solution is implemented in a perfect Bayesian equilibrium of this game. On the equilibrium path, the proposal of each player reveals her true value. When the last participant has revealed her value, her proposal is accepted by all. Given this outcome, one is tempted to reject the last proposal and to propose instead the maximal quantity with the same shares as in the last proposal. This would be a Pareto improvement under complete information. However, if such a proposal were be accepted, then the original incentives to tell the truth would be destroyed. We deter this deviation as follows. The player who has deviated is punished by entering a path on which she gets none of the surplus, the maximal surplus is equally split among the others, and all are believed to have the lowest possible value.

This completes the so-called Nash program. We postulate axioms on the properties of the outcome, uncover a unique outcome that satisfies these axioms and then show how to implement this outcome with common market rules. We know of no other paper that pursues the Nash program under incomplete information.

Related Literature. We could not find any other paper that both postulates a desideratum and then shows how to implement it without commitment. In the existing literature we identify two papers that are closest to ours. Cramton (1992) is the only other paper that implements trade without commitment whenever there is surplus. Specifically, Cramton (1992) considers bilateral trade under two sided incomplete information without commitment. The agents are engaged in a continuous-time dynamic game and have the same discount factor. The paper constructs a sequential equilibrium of this game that features trade if and only if there is a positive surplus. However, there is efficiency loss, because this trade occurs with delay. The closed-form description of this sequential equilibrium is only available for uniform priors. We would like to point out that, curiously, the dynamic implementation of our bargaining rule presented in Section 3 below can be seen, with slight adjustments, as a different sequential equilibrium of Cramton's (1992) dynamic game, with the property that it does not depend on priors.

Čopič and Ponsati (2016) analyze probabilistic rules for bilateral trade and show that a rule satisfies Strategy Proofness, Voluntary Participation, and is ex-post Pareto undominated if and only if it can be implemented using randomized posted pricing. Our bargaining rule for $n = 2$ satisfies these axioms and, thus, can be represented using a randomized posted pricing mechanism. In fact, all of our results would go through if our axiom of Weak Efficiency is replaced by Čopič and Ponsati's (2016) axiom of ex-post Pareto undominance. Adding our axiom of Surplus Dependence selects the unique randomized posted pricing mechanism, namely the one with the uniform distribution of prices.

There are several alternative axiomatic approaches to bargaining under incomplete information when players have common priors. Unlike our approach, each of these solutions applies to a specific prior and deals with two players only. Harsanyi and Selten (1972) use axioms to select among the strict equilibrium points of a sequential two person Nash demand game. Their solution is to maximize a generalized Nash product whose exponential weights depend on the prior. Weidner (1992) extends this approach to select among incentive compatible individually rational mechanisms. Myerson (1984) selects among incentive compatible individually rational mechanisms that are consistent with the outcome of a random dictator whenever this is efficient. Osborne and Rubinstein (1990, Ch. 5) consider mechanisms for bilateral trade when there are two types of each player.

A typical assumption in the literature is to require allocation, provision or trade to be in a given quantity, immediate and with certainty or not at all. In this restrictive environment, only randomized posted pricing mechanisms satisfy Strategy

Proofness and Voluntary Participation (Kuzmics and Steg, 2017). Among these rules, the unique one that satisfies our axiom of Surplus Dependence is the rule that does not produce any output at all. Richer environments have been considered in related papers. Hartline and Roughgarden (2008) show how goods can be partially destroyed to create incentives. Probabilistic trade is used strategically in Čopič and Ponsati (2016) for trade and Mailath and Postlewaite (1990) for public good provision. The idea to use delay strategically under incomplete information appears in Mailath and Postlewaite (1990), Kennan and Wilson (1993), and Tóbiás (2018).

We proceed as follows. In Section 2, we introduce a bargaining problem and axioms, and we find the unique bargaining solution that satisfies these axioms. Section 3 describes an implementation of this solution via an alternating offers protocol. We apply our results to trade and to provision of public good in Section 4. Section 5 concludes. The proofs are in the Appendix.

2. Bargaining Problem

2.1. **Example.** A seller and a buyer negotiate a trade of an indivisible good. Provision of the good costs $c \in [0,1]$ to the seller. Consumption of the good has the value of $v \in [0,1]$ to the buyer. The seller's cost and the buyer's value are their private information. The traders do not have to trade instantly. If they agree to trade for a price p at time t, then the seller and buyer's payoffs are $\delta^t(p-c)$ and $\delta^t(v-p)$, respectively, where δ is a common discount factor. If they agree not to trade or if they fail reaching an agreement, then both get zero.

Market rules are defined as follows. The agents announce their information, after which it is determined whether the good is traded, and if so, the time of trade and the price are specified. Let $\hat{c}$ and $\hat{v}$ be the seller's and buyer's announcements of their cost and value, respectively. A market rule is a profile (κ, p, τ), where $\kappa(\hat{c},\hat{v}) \in \{0,1\}$ indicates whether the good is traded or not, $p(\hat{c},\hat{v}) \geq 0$ is a price, and $\tau(\hat{c},\hat{v}) \geq 0$ is a time of trade. The seller and buyer's payoffs are given by

$$u_s(\hat{c},\hat{v}|c) = \kappa(\hat{c},\hat{v})\delta^{\tau(\hat{c},\hat{v})}(p(\hat{c},\hat{v})-c) \quad \text{and} \quad u_b(\hat{c},\hat{v}|v) = \kappa(\hat{c},\hat{v})\delta^{\tau(\hat{c},\hat{v})}(v-p(\hat{c},\hat{v})).$$

We are interested in market rules that satisfy the following conditions (axioms). Our first requirement is that no player should be able to manipulate the outcome to her benefit by reporting any type profile other than the truth, irrespective of the reports of the others.

(DSIC) *Dominant strategy incentive compatibility.* Each player's dominant strategy is to report the value truthfully.

The next two axioms impose discipline on the agreement between the traders. These can be considered *ex-post* requirements that apply at the time when the agreement has been reached.

(RP) *Renegotiation proofness.* The players should not be able to find a Pareto superior deal ex post, after the conclusion of the agreement. In other words, the players must agree on an alternative that maximizes the trade surplus. In this case there are two alternatives: trade and no trade. This is analogous to the Pareto efficiency in bargaining under complete information.

(IR) *Ex-post individual rationality.* Whenever the good is agreed to be traded, no player prefers to walk away with zero payoff.

The last two axioms impose a discipline on the timing of the trade.

(WNW) *Weak Nonwastefulness.* There exist reports of the buyer and the seller such that an agreement (whether trade or no trade) is reached immediately, without delay. This is a very minimal requirement of non-wastefulness.

(IM) *Impartiality.* The bargaining mechanism is impartial, in the sense that the time of the agreement depend only on the surplus given by $\max\{v-c,0\}$, rather than on the individual reports. This requirement reflects the idea that the decision of whether and when trade takes place should depend only on the social benefit.

Proposition 1. *A market rule (κ^*, p^*, τ^*) satisfies DSIC, RP, IR, WNW, and IM if and only if it is as follows. When $v > c$, the good is traded at the price $p^*(c,v) = (c+v)/2$ and at the time $\tau^*(c,v)$ given by $\delta^{\tau^*(c,v)} = v - c$. When $v < c$, then the good is not traded. When $v = c$, then the good may be traded at price $p^*(c,v) = c$ and at any time, or not traded at all.*

Proposition 1 is a special case of Theorem 1 that we will present after introducing our general model of bargaining.

The market rule (κ^*, p^*, τ^*) is fair, in the sense that it shares the surplus equally and provides the same discounted expected utility of $(v-c)^2/2$ to each trader whenever there is a positive surplus.

The market rule (κ^*, p^*, τ^*) can be implemented as a perfect Bayesian equilibrium in a dynamic bargaining game where the players make alternating offers. Moreover, this equilibrium is prior-independent, so the players' equilibrium choices are independent of their beliefs.

The alternating offer bargaining protocol is described as follows. The negotiations proceed in continuous time. Players take turns. The player whose turn it is to move makes a price offer at the time of her choice. The other player then immediately decides to accept or reject the offer. If the offer is accepted, then the trade takes place at the agreed price, and the game ends. Otherwise the negotiations are resumed, and the next player makes a price offer at the time of her choice, etc. To ensure that there are finite number of offers in any interval of time, we assume that each player has a brief "cooldown" period between her own offers. In addition, any time any player can walk away, resulting in no trade. The negotiations proceed until an offer is accepted or a player walks away.

Suppose that the seller is the first mover. In equilibrium, the seller with cost c makes an offer $p_s = (c+1)/2$ at the time t_s such that $\delta^{t_s} = 1-c$. Note that the seller's offer corresponds to the outcome described in Proposition 1 when assuming that the buyer's value is $v=1$. If the buyer's value is indeed $v=1$, the the buyer accepts this offer, and the trade is concluded. If $v \leq c$, so there is no surplus, then the buyer walks away. Otherwise, if $c < v < 1$, then the buyer makes a counteroffer $p_b = (c+v)/2$ at the time t_b such that $\delta^{t_b} = v-c$, and the seller accepts this counteroffer.

The described strategies implement the outcome presented in Theorem 1. To see how these strategies can be supported as an equilibrium, note that each player's equilibrium offer reveals her private type. This is as if each player reports the type to a mechanism that chooses the time and the price offer. So, a player can deviate by following the above equilibrium strategy but pretending to have a different type. This is not a profitable deviation, because the equilibrium outcome satisfies IC. Alternatively, a player can deviate by making a choice that is inconsistent with the equilibrium play for any private type he or she can have. This deviation can be thwarted by assuming that the deviant has the highest possible value (if she is the buyer) or the lowest possible cost (if she is the seller), and is willing to give away the entire surplus. As long as the "cooldown" period between offers of the same player is small enough, such deviations cannot be profitable to the players.

2.2. **Model.** We now present a general bargaining problem.

There are $N \geq 2$ players and $K \geq 1$ alternatives. The players bargain about which alternative to choose. The players start with a status quo, interpreted as a null alternative and denoted by $k = 0$. A unanimous agreement of all the players is required for any alternative to be chosen instead of the status quo.

If the players do not agree on any alternative, then each player i obtains her status quo payoff $v_{i,0}$, which is normalized to zero, so $v_{i,0} = 0$ for each $i = 1, ..., N$. If the players agree on an alternative $k = 1, ..., K$, then each player $i = 1, ..., N$ obtains a value $v_{i,k}$ representing player i's net gain or loss relative to the status quo. The type of each player i is summarized by $v_i = (v_{i,1}, ..., v_{i,K})$. Let V_i be the set of types of player i. We assume that V_i is a subset of $\mathbb{R}^K$ that is convex, bounded from above, and has a nonempty interior. Let $V = V_1 \times ... \times V_N$, and let V_{-i} be the set of value profiles of all players except i. Let $\mathcal{V}$ be the set of all sets of value profiles V that satisfy the above assumptions.

The players do not have to agree instantly. They may reach an agreement later in time, where time is continuous. Delaying the agreement can be costly, because a certain fraction of the surplus can be lost due to discounting of the payoffs. If the players agree on an alternative k at time t, then the players enjoys the status quo up to time t. At time t they forgo the status quo for the sake of the agreed alternative k, and obtain the discounted payoffs from the agreement. Thus the discounted payoff of player i is given by

$$\delta^t (v_{i,k} + m_i),$$

where $\delta \in (0, 1)$ is a common discount factor and $m_i \in \mathbb{R}$ is a monetary transfer to player i made at the time of the agreement.

Each player's type is her private information, so a bargaining outcome must rely on the players' reports about their types. By invoking the revelation principle, we can restrict attention to direct mechanisms. A *bargaining mechanism* is a tuple $(\kappa, \mu, \tau) : V \to \{0, 1, ..., K\} \times \mathbb{R}^N \times \mathbb{R}_+$. For each profile of announced types $\hat{v} = (\hat{v}_1, ..., \hat{v}_N) \in V$, a bargaining mechanism determines an agreement $(\kappa(\hat{v}), \mu(\hat{v}))$ and a time $\tau(\hat{v})$ when this agreement is implemented. The agreement specifies an alternative $\kappa(\hat{v})$ to be implemented, and a vector of monetary transfers $\mu(\hat{v}) = (\mu_1(\hat{v}), ..., \mu_N(\hat{v}))$, where $\mu_i(\hat{v})$ is the transfer to player i.

Let (κ, μ, τ) be a bargaining mechanism. For each profile of reports $\hat{v}$, the payoff of each player i with type v_i is denoted by $u_i(\hat{v}|v_i)$ and is given by

$$u_i(\hat{v}|v_i) = \delta^{\tau(\hat{v})} \left(v_{i,\kappa(\hat{v})} + \mu_i(\hat{v}) \right).$$

A *bargaining problem* with N players and K alternatives is described by a set of value profiles $V \in \mathcal{V}$. A *solution* ψ to a bargaining problem is a mapping that associates with each $V \in \mathcal{V}$ a bargaining mechanism $\psi(V)$.

2.3. **Axioms.** Let ψ be a bargaining solution. As the set of value profiles $V \in \mathcal{V}$ is fixed throughout most of the paper, we suppress the dependence of the bargaining mechanism $\psi(V)$ on V and use the notation $\psi(V) = (\kappa, \mu, \tau)$. When we need to compare two value profiles V and $\tilde{V}$ in $\mathcal{V}$, we use the notation $\psi(\tilde{V}) = (\tilde{\kappa}, \tilde{\mu}, \tilde{\tau})$.

Let us consider seven desirable properties, or axioms, that a bargaining solution ψ should satisfy. The first five of these axioms are the IC, RP, IR, NW, and IM requirements specified in Section 2.1 above. We now formalize these axioms for our general setting.

Axiom 1 (Dominant Strategy Incentive Compatibility, DSIC)**.** For each $i \in \{1, ..., N\}$, each $v_i, \hat{v}_i \in V_i$, and each $\hat{v}_{-i} \in V_{-i}$,

$$u_i(v_i, \hat{v}_{-i}|v_i) \geq u_i(\hat{v}_i, \hat{v}_{-i}|v_i).$$

Axiom 2 (Renegotiation Proofness, RP)**.** For all $\hat{v} \in V$,

$$\kappa(\hat{v}) \in \underset{k \in \{0,1,...,K\}}{\arg\max} \sum\nolimits_{i=1}^{N} \hat{v}_{i,k}.$$

Axiom 3 (Ex-post Individual Rationality, IR)**.** For each $i = \{1, ..., N\}$ and each $\hat{v} \in V$,

$$v_{i,\kappa(\hat{v})} + \mu_i(\hat{v}) \geq 0.$$

Axiom 4 (Weak Non-Wastefulness, WNW)**.** There exists $\hat{v} \in V$ such that $\tau(\hat{v}) = 0$.

To formalize the next axiom, let us first introduce the following notation. Given a profile of types v, the surplus $S(v)$ is defined as the maximal added value:

$$S(v) = \max_{k \in \{0,1,...,K\}} \sum\nolimits_{i=1}^{N} v_{i,k}.$$

Axiom 5 (Impartiality, IM)**.** For all $\hat{v}', \hat{v}'' \in V$,

$$S(\hat{v}') = S(\hat{v}'') \text{ implies } \tau(\hat{v}') = \tau(\hat{v}'').$$

Two additional axioms can be attributed to the generality of our setting. We require that the bargaining mechanism is budget balanced, in the sense that the payments to the players must add up to zero. So, neither players are subsidized, nor money is destroyed. This axiom rules out the feasibility of Vickrey-Clarke-Groves mechanisms that rely on subsidies.

Axiom 6 (Budget Balance, BB)**.** For each $\hat{v} \in V$,

$$\sum_{i=1}^{N} \mu_i(\hat{v}) = 0.$$

Our last requirement says that the agreement, which specifies the chosen alternative and the transfers, should depend only on the reports made by the players. It should not depend on the set of possible values V that the players can have. This reflects the idea that the agreement is based on the actual reports, rather on some hypothetical values.

Axiom 7 (Independence of Irrelevant Information, III)**.** For each $V, \tilde{V} \in \mathcal{V}$, the mechanisms $(\kappa, \mu, \tau) = \psi(V)$ and $(\tilde{\kappa}, \tilde{\mu}, \tilde{\tau}) = \psi(\tilde{V})$ satisfy

$$\hat{v} \in V \cap \tilde{V} \text{ implies } \kappa(\hat{v}) = \tilde{\kappa}(\hat{v}) \text{ and } \mu(\hat{v}) = \tilde{\mu}(\hat{v}).$$

2.4. **Solution.** Before presenting our axiomatic solution, we introduce the following notation.

We say that a bargaining solution ψ is a *Nash bargaining solution with delay* if for each $V \in \mathcal{V}$ the mechanism $(\kappa, \mu, \tau) = \psi(V)$ maximizes the Nash product (Nash, 1950) at the time of agreement. Specifically, for each $v \in V$, the agreement $(\kappa(v), \mu(v))$ is a solution of

$$\max_{(k,m)\in\{0,1,...,K\}\times\mathbb{R}^N} \prod_{i=1}^{N} (v_{i,k} + m_i) \text{ subject to } \sum_{i=1}^{N} m_i = 0.$$

Any such mechanism chooses a surplus-maximizing alternative, and divides the surplus equally among the players. So, $\kappa(v)$ and $\mu(v)$ are given by

$$\kappa(v) \in \arg\max_{k\in\{0,1,...,K\}} \sum_{i=1}^{N} v_{i,k}, \tag{1}$$

$$\mu_i(v) = \frac{S(v)}{N} - v_{i,\kappa(v)} \text{ for each } i = 1, ..., N. \tag{2}$$

Note that $\kappa(v)$ and $\mu(v)$ are uniquely defined, except when there is more than one surplus-maximizing alternative. The induced payoffs are unique. The payoff of each player i is her status quo value plus the equal share of the discounted surplus, so

$$u_i(v|v_i) = \frac{\delta^{\tau(v)} S(v)}{N} \quad \text{for each } i = 1, ..., N. \tag{3}$$

The Nash bargaining solution with delay is *fair* in the sense that the discounted surplus is shared equally among the players.

We state our main result.

Theorem 1. *A bargaining solution satisfies Axioms 1–7 if and only if it is a Nash bargaining solution with delay where for each $V \in \mathcal{V}$ the delay function τ satisfies*

$$\delta^{\tau(\hat{v})} = \left(\frac{S(\hat{v})}{\max_{v \in V} S(v)} \right)^{N-1} \quad \textit{for each } \hat{v} \in V \textit{ such that } S(\hat{v}) > 0. \tag{4}$$

We prove Theorem 1 in Appendix A.1, and show that Axioms 1–7 are independent in Appendix A.2.

2.5. **Discussion.** We now discuss some properties of our solution and its underlying assumptions.

2.5.1. *Incentives.* In the delayed Nash bargaining solution presented in Theorem 1, the responsiveness of the time of agreement to the players' reports is instrumental to ensure that the players have incentives to report their costs truthfully. If instead the agreement was always instant, then the players could announce higher costs to get higher monetary transfers from the others. Unlike in the Vickrey-Clarke-Groves (VCG) setting where the traders are subsidized, here the budget is balanced. The time of the agreement provides the needed strategic variable for the incentive compatibility that replaces the lack of freedom in the choice of monetary transfers.

2.5.2. *Efficiency.* In the delayed Nash bargaining solution presented in Theorem 1, the agreement is always Pareto efficient. The inefficiency emerges because of the delay. To understand how inefficient our solution is, consider the efficiency loss measure $L_{(\kappa,\mu,\tau)}$ given by the maximum difference between the aggregate payoff under the efficient allocation and that under a given mechanism (κ, μ, τ) and normalized to be in $[0,1]$, so

$$L_{(\kappa,\mu,\tau)} = \frac{1}{\max_{v'\in V} S(v')} \max_{v\in V} \left(S(v) - \sum\nolimits_{i=1}^{N} u_i(v|v_i) \right).$$

In other words, $L_{(\tau,\kappa,\mu)}$ is the upper bound on how large fraction of the maximum surplus can be lost.

Let us derive the efficiency loss measure for the delayed Nash bargaining solution.

Proposition 2. *Let (κ,μ,τ) be the delayed Nash bargaining solution with delay τ given by Theorem 1. Then*

$$L_{(\kappa,\mu,\tau)} = \frac{N-1}{N^{\frac{N}{N-1}}}.$$

Proof. Let $S_{max} = \max_{v'\in V} S(v')$. By Theorem 1, using (3) and (4), we have

$$\frac{1}{S_{max}} \left(S(v) - \sum\nolimits_{i=1}^{N} u_i(v|v_i) \right) = \frac{S(v)}{S_{max}} - \delta^{\tau(v)} \frac{S(v)}{S_{max}} = \frac{S(v)}{S_{max}} - \left(\frac{S(v)}{V_{max}} \right)^N.$$

Let $x = S(v)/S_{max}$. Note that $x \in [0,1]$ by the definition of $S(v)$. Maximizing $x - x^N$ w.r.t. x over the interval $[0,1]$ yields the maximum value $(N-1)N^{-\frac{N}{N-1}}$. □

When there are $N = 2$ players, the maximum efficiency loss is $L_{(\kappa,\mu,\tau)} = 0.25$. This means the players never lose more than $1/4$ of the maximal surplus due to the delayed agreement. The maximum efficiency loss is approximately 0.38 for $N = 3$ and 0.47 for $N = 4$, and it is increasing in N. We would like to stress, however, that this is the upper bound on the efficiency loss. So, for any priors on the players' costs, the expected efficiency loss is smaller.

2.5.3. *Other Interpretations of Delay.* Observe that the role of the delay $\tau(v)$ is to cause the decay of the value of the pie that the players share. Let

$$q(v) = \delta^{\tau(v)}.$$

So, $q(v)$ is the remaining part of the value after the delay $\tau(v)$. However, there are ways of destroying the value other than decaying it over time. For example, $q(v)$ can be the probability of the agreement. For another example, $q(v)$ can be the quantity of production (or equivalently, $1 - q(v)$ the fraction of the good to be destroyed). Thus, instead of specifying the time $\tau(v)$ of the agreement, the bargaining mechanism can directly specify the quantity $q(v)$ as a function of the players' reported costs.

The delay can be substantial when the surplus is very small. It makes little sense that market participants will wait so long for trade or production. This is easily accommodated by introducing an exogenous deterioration process of the output value, or a limited window of opportunity for the trade surplus to be realized. Consider some process under which the expected value of the output is strictly decreasing over time and becomes zero within a finite time horizon T. This can be anticipated, for example, by choosing $q(v) = (T - \tau(v))/T$, as q determines the deterioration of the good, regardless of whether it is driven by impatience or exogenous decay. In this model, the delay will never exceed T.

3. Implementation

Above we designed a bargaining solution that satisfies a set of axioms. In this section, we follow the so-called Nash program (see, e.g., the survey of Serrano, 2020) to show how one can obtain this outcome in a perfect Bayesian equilibrium of a non-cooperative Bayesian game. Moreover, we will consider a prior-free implementation, where each player's equilibrium strategy is independent of her prior about the costs of the other players. For this we adapt the alternating offers bargaining game of Rubinstein (1982) to our setting.

The negotiations proceed in continuous time. The players make offers in a predetermined order. Without loss of generality, let player i be i-th in the order. We will refer to the player making an offer as the *proposer* and the other players as the *responders*. When player i becomes the proposer, she does not have to make an offer immediately. We will refer to the time interval between when a player becomes the proposer and when she makes an offer as a *round*. Round 1 starts at time $t = 0$. Round $r > 1$ starts immediately after an offer has been made (and rejected) in round $r-1$. Let t_{r-1} denote the time when round $r - 1$ ends and round r begins. Note that the proposer in round $r = 1, 2, ...$ is player $i = r \pmod N$.

We now describe what happens between the start and end of each round $r = 1, 2, ...$. At time t_{r-1} when round r starts, all players simultaneously choose whether to stay or to exit. If at least one player exits, then the game ends, and everyone receives zero payoff. Otherwise the proposer chooses what offer to make and when. Specifically, the proposer chooses a tuple (k_r, m_r, t_r) that specifies an alternative $k_r \in \{0, 1, ..., K\}$ and a vector of transfers $m_r = (m_{1,r}, ..., m_{N,r})$ to be offered, as well as a time t_r of the offer. Immediately after the offer has been made, all the responders simultaneously decide whether to accept or reject it. If the offer is unanimously accepted, then the game ends, and each player $j = 1, ..., N$ receives the payoff $\delta^{t_r}(v_{j,k_r} + m_{j,r})$. If the offer is rejected by someone, then the negotiations proceed to round $r + 1$.

There are three constraints that the proposer's choice must satisfy. First, the proposer must be able to honor the transfers specified in her offer, so if player i is the proposer in round r, then

$$\sum\nolimits_{j \neq i} m_{j,r} = -m_{i,r}.$$

Second, the time of the offer t_r cannot be earlier than the round starts, so

$$t_r \geq t_{t-1}.$$

Third, each player can make at most K offers within a given time interval $\lambda > 0$, so

$$t_r \geq t_{r-NK} + \lambda \text{ for } r > NK.$$

We assume that interval λ is positive, but small enough.

Observe that there can be a succession of several instantaneous rounds, where one offer succeeds another immediately, at the same time. However, the last assumption ensures that at any time t the number of rounds preceding t is finite.

At the start of the game, each player is equipped with a prior about the values of the others. There can be a common prior, or players can have subjective priors. The solution concept is perfect Bayesian equilibrium.

Let us formally describe the players' strategies. For $r \geq 1$ let h_r be the history of everything that the players have observed up to the conclusion of round r, including times when offers were made, contents of the offers, and acceptance decisions. The history h_0 that precedes round 1 contains no information. A strategy of each player specifies how she makes offers in rounds when she is the proposer, how she responds to offers in rounds when she is a responder, and how she makes exit decisions in each round. Consider player $i = 1, ..., N$. An offer strategy of player i is a tuple $(\kappa_i, \mu_i, \tau_i)$ that specifies an alternative $\kappa_i(v_i, h_{r-1}) \in \{0, 1, ..., K\}$, and a vector of monetary payments $\mu_i(v_i, h_{r-1}) \in \mathbb{R}^N$, and a time the offer $\tau_i(v_i, h_{r-1}) \in \mathbb{R}_+$ in each round r where i is the proposer. An acceptance strategy α_i specifies a choice $\alpha_i(v_i, h_{r-1}, k_r, m_r, t_r) \in \{0, 1\}$ each round r where i is a responder, where 1 signifies acceptance and 0 signifies rejection. Note that this choice depends on an offer (k_r, m_r) made at time t_r by the proposer in round r. Finally, an exit strategy e_i specifies a choice $e_i(v_i, h_{r-1}) \in \{0, 1\}$, where 1 signifies exit and 0 signifies stay.

We now describe a perfect Bayesian equilibrium of this game. The equilibrium strategy is symmetric across the players, so it is independent of their order of moves, and it is denoted by $E^* = (\kappa^*, \mu^*, \tau^*, \alpha^*, e^*)$. The equilibrium belief profile is denoted by B^*.

The equilibrium strategy describes the play in two phases, an *equilibrium phase* and a *punishment phase*. The play starts in the equilibrium phase until a deviation triggers the punishment phase.

We begin the description with the strategy profile in the equilibrium phase. For this purpose, we introduce the following notation. Let $(\hat{S}_r)_{r=0,1,...}$ be a sequence of *admissible surpluses* and let $(\hat{V}_r)_{r=0,1,...}$ be a sequence of *admissible sets of value profiles*. Each admissible set $\hat{V}_r$ is a product of the admissible sets of individual players, so $\hat{V}_r = \times_{i=1}^{N} \hat{V}_{i,r}$. At the start of the game, the admissible surplus is equal to the maximum surplus, and the admissible set is equal to the set of value profiles, so

$$\hat{S}_0 = S_{\max} \text{ and } \hat{V}_{j,0} = V_j \text{ for each } j = 1, ..., N.$$

For each subsequent round $r = 1, 2, ...$, we define $\hat{S}_r$ as follows. Let $\sigma(t)$ be the solution of (4) for $s = S(\hat{v})$ as a function of $t = \tau(\hat{v})$, and let $\sigma^{-1}(s)$ be its inverse, so

$$\sigma(t) = \delta^{\frac{t}{N-1}} S_{\max} \text{ and } \sigma^{-1}(s) = \frac{(N-1)(\ln s - \ln S_{\max})}{\ln \delta}.$$

Given the time t_r of the offer in round r, let

$$\hat{S}_r = \sigma(t_r).$$

The interpretation is that, in equilibrium, at a time t_r the surplus cannot exceed $\sigma(t_r)$, thus ruling out the possibility of a greater surplus. Had the surplus been greater, the offer would have been made earlier in equilibrium.

Next, we define $\hat{V}_r$ as follows. Let i be the proposer in round r. For each responder $j \neq i$ in the admissible set remains the same as in the previous round,

$$\hat{V}_{j,r} = \hat{V}_{j,r-1} \text{ for each } j \neq i.$$

For the proposer i, the admissible set $\hat{V}_{i,r}$ includes all profiles of i's values in $\hat{V}_{i,r-1}$ such that the maximum surplus does not exceed $\hat{S}_r$, so

$$\hat{V}_{i,r} = \left\{ \hat{v}_i \in \hat{V}_{i,r-1} : \max_{k=0,1,...,K,\ \hat{v}_{-i} \in \hat{V}_{-i,r-1}} \sum\nolimits_{j=1}^{N} \hat{v}_{j,k} \leq \hat{S}_r \right\}.$$

The values that lead to any higher surplus than $\hat{S}_r$ are ruled out in equilibrium, and thus excluded from the admissible set.

Fix a player $i = 1, ..., N$. The exit strategy e^* of player i in the equilibrium phase is as follows. For each round $r = 1, 2, ...$ let $s^*_{i,r}$ be the highest surplus from perspective of player i, provided the value profiles of the other players are in the admissible set $\hat{V}_{r-1}$, so

$$s^*_{i,r} = \max_{k=0,1,\dots,K} \left(v_{i,k} + \max_{\hat{v}_{-i} \in \hat{V}_{-i,r-1}} \sum\nolimits_{j \neq i} \hat{v}_{j,k} \right).$$

Player i exits (so $e^*(v_i, h_{r-1}) = 1$) if $s^*_{i,r} = 0$ and stays if $s^*_{i,r} > 0$.

The offer strategy $(\kappa^*, \mu^*, \tau^*)$ of player i in the equilibrium phase is as follows. Consider a round r where player i is the proposer. Let

$$\kappa^*(v_i, h_{r-1}) \in \arg\max_{k=0,1,\dots,K} \left(v_{i,k} + \max_{\hat{v}_{-i} \in \hat{V}_{-i,r-1}} \sum\nolimits_{j \neq i} \hat{v}_{j,k} \right) \tag{5}$$

$$\mu^*_j(v_i, h_{r-1}) = \frac{s^*_{i,r}}{N} - \max_{\hat{v}_j \in \hat{V}_{j,r-1}} \hat{v}_{j,\kappa^*_i(v_i,h_{r-1})} \quad \text{for each } j \neq i, \tag{6}$$

$$\mu^*_i(v_i, h_{r-1}) = -\sum\nolimits_{j \neq i} \mu^*_j(v_i, h_{r-1}), \tag{7}$$

$$\tau^*(v_i, h_{r-1}) = \begin{cases} \sigma^{-1}(s^*_{i,r}) & \text{if } r \leq NK, \\ \max\left\{\sigma^{-1}(s^*_{i,r}), t_{r-NK} + \lambda\right\} & \text{if } r > NK. \end{cases} \tag{8}$$

Note that this offer strategy corresponds to that in (1), (2), and (4) under the assumption that player i behaves as if for each player other than i, the value of each alternative k is the maximal in $\hat{V}_{r-1}$.

The acceptance strategy α^* of player i in the equilibrium phase is as follows. Consider a round r where player i is a responder. Let (k_r, m_r, t_r) be an offer made in round r by the proposer in that round. Player i accepts the offer, so $\alpha^*(v_i, h_{r-1}, k_r, m_r, t_r) = 1$ if and only if the payoff of player i in the continuation game of round $r+1$ would have been smaller than or equal to the payoff offered in round r.

A detectable deviation from the equilibrium offer strategy triggers the punishment phase. We say that an offer (k_r, m_r, t_r) is *admissible* if there exists the proposer's value profile $\hat{v}_i$ admissible in round $r-1$, so $\hat{v}_i \in \hat{V}_{i,r-1}$, such that (k_r, m_r, t_r) is the equilibrium offer for this proposer,

$$k_r = \kappa^*(\hat{v}_i, h_{r-1}), \quad m_r = \mu^*(\hat{v}_i, h_{r-1}), \quad t_r = \tau^*(\hat{v}_i, h_{r-1}).$$

Intuitively, a non-admissible offer cannot be rationalized by the responders as being the equilibrium choice of the proposer i for any possible value profile in $\hat{V}_{i,r-1}$.

We now describe the punishment phase. Suppose a non-admissible offer has been made at the end of a round $r^* - 1$. The play then enters the punishment phase in round r^*. In this phase, all the players stay in the game in round r^*, but exit the game in every subsequent round, so for each $j = 1, ..., N$

$$e^*(v_j, h_{r^*-1}) = 0 \text{ and } e^*(v_j, h_{r-1}) = 1 \text{ and each } r = r^* + 1, r + 2, ...$$

So an offer in each round $r = r^*, r^* + 1, ...$ is a take-it-or-leave-it offer by the proposer in that round.

Let player i be a responder in round $r \geq r^*$. She accepts the offer made in this round if and only if her payoff from this offer is nonnegative.

Let player i be the proposer in round $r \geq r^*$. Define the take-it-or-leave-it offer as follows:

$$\kappa^*(v_i, h_{r-1}) \in \underset{k=0,1,...,K}{\arg\max} \left(v_{i,k} + \max_{\hat{v}_{i^*} \in V_{-i}} \sum\nolimits_{j \neq i} \hat{v}_{j,k} \right) \tag{9}$$

$$\mu_j^*(v_i, h_{r-1}) = - \max_{\hat{v}_j \in V_j} \hat{v}_{j, \kappa_i^*(v_i, h_{r-1})} \text{ for each } j \neq i, \tag{10}$$

$$\mu_i^*(v_i, h_{r-1}) = - \sum\nolimits_{j \neq i} \mu_j^*(v_i, h_{r-1}), \tag{11}$$

$$\tau^*(v_i, h_{r-1}) = \begin{cases} t_{r-1} & \text{if } r \leq NK, \\ \max \left\{ t_{r-1}, t_{r-NK} + \lambda \right\} & \text{if } r > NK. \end{cases} \tag{12}$$

So player i forms an (out-of-equilibrium) posterior belief about each player that her value of the chosen alternative the maximal possible, and then asks to obtain the entire surplus.

This punishment deters non-admissible offers, because the deviant expects the next player in the order, to reject the deviant's offer and then to make a take-it-or-leave-it counteroffer by asking for the entire surplus. After that the game will end, because all players will exit. So the deviant cannot get more than zero after making a non-admissible offer.

To complete the description of the perfect Bayesian equilibrium, we specify the equilibrium belief profile B^*. In the equilibrium phase, the beliefs are determined by Bayes' rule. In the punishment phase, the beliefs are as described above.

The next theorem shows that the strategy and belief profiles implement the Nash bargaining outcome as specified in Theorem 1, irrespective of the players' priors.

Theorem 2. *The strategy profile E^*, together with the belief profile B^*, is a perfect Bayesian equilibrium for all priors. For each $v \in V$ this equilibrium implements the delayed Nash bargaining solution.*

The proof is in Appendix. Intuitively, the idea is very simple. The equilibrium outcome coincides with that of the delayed Nash bargaining solution presented in Theorem 1 by construction. A deviation by any player i, for example, an attempt to reach an agreement earlier than prescribed by the equilibrium, leads to the punishment, where the next player in the order obtains the absolute bargaining power.

4. Applications

In this section, we apply the model in Section 2 in the settings of public good provision and joint trade.

4.1. **Public Good Provision.** Consider a public good problem with $N \geq 2$ participants. The participants decide about buying one of K possible alternatives of a public good that potentially benefits all of them. The cost of each alternative $k = 1, ..., K$ is C_k, which is common knowledge. The value of the alternative k to participant i is $v_{i,k} \leq 1$, which is her private information. We assume that $0 < C_k < N$, so the total value of all the participants can be greater than the cost of the alternative.

Provision rules are defined as follows. The participants announce their values, after which the time of the provision, the alternative, and the participants' payments are determined. Let $\hat{v}_i$ be an announcement of agent i about her value v_i. A provision rule is a triple (τ, κ, m), where $\tau(\hat{v}) \geq 0$ is the time when the good is to be provided, $\kappa(\hat{v}) \in \{0, 1, ..., K\}$ is the alternative, and $m_i(\hat{v}) \in \mathbb{R}$ is the monetary payment by participant i. The payoff of each player i is given by

$$u_i(\hat{v}|v_i) = \delta^{\tau(\hat{v})}(v_{i,\kappa(\hat{v})} - m_i(\hat{v})).$$

The monetary transfers must add up to the cost of the provided public good, so

$$\sum\nolimits_{i=1}^{n} m_i(\hat{v}) = C_{\kappa(\hat{v})}. \tag{13}$$

It can be easily seen that the public good provision problem is equivalent to the bargaining problem, up to relabeling of the variables. Let

$$H = \max_{k=1,...,K} (N - C_k).$$

The value of each alternative $k = 1, ..., K$ is given by $V_k = (N - C_k)/H$, and cost of each participant $i = 1, ..., N$ is given by $c_{i,k} = (1 - v_{i,k})/H$. These values are rescaled by $1/H$ to ensure that the maximum alternative value is 1. The surplus is given by

$$S(v) = \max_{k=0,1,...,K} \left(V_k - \sum\nolimits_{i=1}^{N} c_{i,k} \right) = \frac{1}{H} \max_{k=0,1,...,K} \left(\sum\nolimits_{i=1}^{N} v_{i,k} - C_k \right).$$

We again apply Theorem 1 to find the public good provision rule that satisfies our Axioms 1–5.

Corollary 1. *Let $v = (v_1, ..., v_N) \in (-\infty, 1]^N$. If $\sum_{i=1}^{N} v_{i,k} > C_k$ for some k, then at time $\tau^*(v)$ the surplus-maximizing alternative $\kappa^*(v)$ is provided, and each participant i pays $m_i^*(v)$ given by*

$$\delta^{\tau^*(v)} = (S(v))^{N-1}, \tag{14}$$

$$\kappa^*(v) \in \arg\max_{k=1,...,K} \left(\sum\nolimits_{i=1}^{N} v_{i,k} - C_k \right), \tag{15}$$

$$m_i^*(v) = v_{i,\kappa(v)} - \frac{S(v)}{N} \quad \text{for each } i = 1, ..., N. \tag{16}$$

Alternatively, if there is no surplus, then no public good is provided.

This provision rule is fair, in the sense that it provides equal utility to each player i:

$$u_i(v|v_i) = \delta^{\tau^*(v)} \frac{S(v)}{N}$$

Note that the monetary payment $m_i^*(v)$ to player i can be negative when $v_{i,\kappa^*(v)}$ is sufficiently small relative the average value $\frac{1}{N} \sum_{j=1}^{n} v_{j,\kappa^*(v)}$. So, players with low values can be subsidized by the provision rule. Subsidizing some players is the price paid for the ability to elicit truthful information about the participants' values while guaranteeing that no player prefers to block the public good provision.

4.2. **Joint Trade.** This section extends the model of bilateral trade to the case of multiple buyers and sellers who trade a single indivisible good. In this model, there are $n_b \geq 1$ buyers who benefit from the good if it is provided to them, and there are $n_s \geq 1$ sellers whose joint effort is required to produce the good.

Provision of the good has the per-unit cost of $c_i \geq 0$ to each seller $i = 1, ..., n_s$. Consumption of the good has the per-unit value of $v_j \in [0,1]$ to each buyer $j = 1, ..., n_b$. The sellers' costs and the buyers' values are their private information.

Market rules are defined as follows. The agents announce their values, after which the time of the trade and the monetary payments are determined. Let $\hat{c}_i$ and $\hat{v}_j$ be seller i's and buyer j's announcements of their cost and value, respectively. Let $\hat{c} = (\hat{c}_1, ..., \hat{c}_{n_s})$ and $\hat{v} = (\hat{v}_1, ..., \hat{v}_{n_b})$. A market rule is a tuple (κ, p_s, p_b, τ), where $\kappa(\hat{c}, \hat{v}) \in \{0, 1\}$ indicates whether the good is traded, $p_{s,i}(\hat{c}, \hat{v})$ is the payment to seller i, $p_{b,j}(\hat{c}, \hat{v})$ is the payment by buyer j, and $\tau(\hat{c}, \hat{v}) \geq 0$ is the time of the trade. Seller i's payoff is given by

$$u_{s,i}(\hat{c}, \hat{v}|c_i) = \kappa(\hat{c}, \hat{v})\delta^{\tau(\hat{c},\hat{v})}(p_{s,i}(\hat{c}, \hat{v}) - c_i).$$

Buyer j's payoff is given by

$$u_{b,j}(\hat{c}, \hat{v}|v_j) = \kappa(\hat{c}, \hat{v})\delta^{\tau(\hat{c},\hat{v})}(v_j - p_{b,j}(\hat{c}, \hat{v})).$$

The total amount paid by the buyers has to be equal to the total amount received by the sellers, so the following market clearing condition must hold:

$$\sum\nolimits_{i=1}^{n_s} p_{s,i}(\hat{c}, \hat{v}) = \sum\nolimits_{j=1}^{n_b} p_{b,j}(\hat{c}, \hat{v}) \text{ for all } (\hat{c}, \hat{v}) \in [0,1]^{n_s} \times [0,1]^{n_b}. \tag{17}$$

It can be easily seen that the joint trade problem is equivalent to the bargaining problem with $N = n_s + n_b$ players and $K = 1$ alternative, up to relabeling and rescaling the variables. Let status quo correspond to no trade, and let alternative $k = 1$ correspond to trade. Let sellers be labeled by $i = 1, ..., n_s$ and buyers by $j = n_s + 1, ..., n_s + n_b$. Each seller's value is the negative of her cost. The surplus is

$$S(c, v) = \max\left\{\sum\nolimits_{j=1}^{n_b} v_j - \sum\nolimits_{i=1}^{n_s} c_i, 0\right\}.$$

The maximum surplus is obtained when each buyer has the highest possible value, $v_j = 1$, and each seller has smallest possible cost, $c_i = 0$, so

$$S_{\max} = \max_{(c,v)} S(c, v) = n_b.$$

We apply Theorem 1 to find a market rule that satisfies Axioms 1–7.

Corollary 2. *If $\sum_j v_j > \sum_i c_i$, then the good is traded at the time $\tau^*(c, v)$ given by*

$$\delta^{\tau^*(c,v)} = \left(\frac{S(c, v)}{n_b}\right)^{n_s+n_b-1},$$

each seller i receives $p^*_{s,i}(c,v)$, and each buyer j pays $p^*_{b,j}(c,v)$ given by

$$p^*_{s,i}(c,v) = c_i + \frac{1}{n_s + n_b}\left(\sum\nolimits_j v_j - \sum\nolimits_i c_i\right), \quad i = 1, ..., n_s, \tag{18}$$

$$p^*_{b,j}(c,v) = v_j - \frac{1}{n_s + n_b}\left(\sum\nolimits_j v_j - \sum\nolimits_i c_i\right), \quad j = 1, ..., n_b. \tag{19}$$

If $\sum_j v_j \leq \sum_i c_i$, then the good is not traded.

This market rule is fair, in the sense that it provides equal net utility to each seller i and each buyer j. When there is a positive surplus, this utility is the equal share of the surplus, so for all $i = 1, ..., n_s$ and all $j = 1, ..., n_b$ it is given by

$$u_{s,i}(c,v|c_i) = u_{b,j}(c,v|v_j) = \frac{1}{n_s + n_b}\left(\sum\nolimits_{j'=1}^{n_b} v_{j'} - \sum\nolimits_{i'=1}^{n_s} c_{i'}\right)^n.$$

The total price paid by the buyers (and received by the sellers) is

$$P^*(c,v) = \sum\nolimits_{j=1}^{n_b} p^*_{b,j}(c,v) = \frac{n_s n_b}{n_s + n_b}\left(\frac{1}{n_s}\sum\nolimits_{i=1}^{n_s} c_i + \frac{1}{n_b}\sum\nolimits_{j=1}^{n_b} v_j\right).$$

5. Conclusion

Bargaining is the process of determining how to share a pie. Trade-offs need to be made as to who gets how much, as typically, if one player gets more then some other player gets less. The Nash bargaining solution is deemed to be the fair way to make these trade-offs under complete information. Fairness together with impartiality enters the solution via the Symmetry axiom of Nash (1950). Bargaining under incomplete information has the difficulty that it is hard to evaluate what is fair when individual wellbeing is influenced by some parameters that are only known to the player herself.

We set out to find a bargaining solution under incomplete information. To be able to unambiguously evaluate the consequences for the participants means to find a way to let players reveal their private information and to choose an approach where the outcome does not depend on any priors. We add surplus dependence as an impartiality condition. With these Axioms (1, 2 and 4) we obtain so much structure on the solution that only a mild additional axiom of Weak Efficiency (Axiom 3) results in a unique solution. The astonishing finding is that the shares are determined by the Nash bargaining solution that treats announcements as true values. Given this finding, we call our rule the fair bargaining rule. In particular, note that neither symmetry nor equitability was postulated, they both result from our axioms. Fairness is a result, not a postulate. Note also how an asymmetric solution can arise when players are heterogeneous, either in their discount factors or in their risk preferences.

A key to our paper is the axiom of Surplus Dependence. Without this axiom we would obtain many different possible rules, such as the trading rules found by Čopič and Ponsati (2016) and the public good provision rules in Mailath and Postlewaite (1990).

Another key to our approach is pragmatism. Probabilistic allocation is a useful mathematical technique to enrich the market rules and to find more ways to incentivize players to tell the truth. However it seems very unrealistic for real applications. Similarly, to post a price that has been randomly drawn from some distribution, as in Hagerty and Rogerson (1987) and Čopič and Ponsati (2016), does not seem to be a very realistic procedure. Thus we emphasize models with either a variable quantity or a strategic delay. The variable quantity model reveals new insights on the optimal relationship between quantity and surplus. The downside of the variable quantity model is that we assume constant marginal values and costs. An investigation of more general cost structures is on the agenda for future research. The model with strategic delays can be applied to any of the three settings: bargaining, trade, and public good provision. Exogenous deterioration of the output can be incorporated to model a finite horizon. Note that in the context of strategic delay the good need not be divisible.

There are other closely related models that can be easily investigated with our methodology. For instance, one might wish to allocate a good to the player with the highest private value for this good. A related setting involves trade with multiple buyers and sellers where sellers produce the good independently and buyers do not jointly benefit from the sale. The analysis of these models is left for future research.

Appendix A. Proofs

A.1. **Proof of Theorem 1.** *Sufficiency.* It is straightforward to verify that every Nash bargaining solution with delay where the delay function is given by (4) satisfies Axioms 1–7.

Necessity. Let ψ satisfy Axioms 1–7. Let $V \in \mathcal{V}$ and let $(\kappa, \mu, \tau) = \psi(V)$. Let $U_i(v)$ be the payoff of player i under (κ, μ, τ) when all players announce their values truthfully. So, for each $v \in V$ and each $i \in \{1, ..., N\}$,

$$U_i(v) = u_i(v|v_i) = \delta^{\tau(v)}(v_{i,\kappa(v)} + \mu_i(v)). \tag{20}$$

Recall that $S(v) = \max_{k=0,1,...,K} \sum_{i=1}^N v_{i,k}$ for each $v \in V$. By Axiom 5 (Surplus Dependence), there exists a function $q : \mathbb{R} \to [0, 1]$ such that

$$\delta^{\tau(v)} = q(S(v)), \quad v \in \mathcal{V}. \tag{21}$$

Substituting (21) into (20) and summing it up across the players, by Axiom 2 (Renegotiation Proofness) and Axiom 6 (Budget Balance) we obtain

$$\sum\nolimits_{i=1}^N U_i(v) = \delta^{\tau(v)} \sum\nolimits_{i=1}^N (v_{i,\kappa(v)} + \mu_i(v)) = q(S(v))S(v). \tag{22}$$

The rest of the proof is divided into four steps.

Step 1. Using Axioms 1 and 2, we show that for each $i = 1, ..., N$, each $v_i, \hat{v}_i \in V_i$, and each $v_{-i} \in V_{-i}$,

$$U_i(\hat{v}_i, v_{-i}) = U_i(v_i, v_{-i}) + \int_{S(v_i,v_{-i})}^{S(\hat{v}_i,v_{-i})} q(s)\mathrm{d}s, \tag{23}$$

where $q(s)$ is weakly increasing.

Proof of Step 1. Consider $v_i, \hat{v}_i \in V_i$ and $v_{-i} \in V_{-i}$ such that for each reported profile (v_i, v_{-i}) and $(\hat{v}_i, v_{-i})$ there is a unique alternative that maximizes the surplus. Note that this holds for all $v_{-i}) \in V_{-i}$ and almost all $v_i, \hat{v}_i \in V_i$. Let

$$s_* = S(v_i, v_{-i}) \text{ and } s^* = S(\hat{v}_i, v_{-i}). \tag{24}$$

Let ϕ_i be a path in V_i that linearly transforms player i's report from $v_i = (v_{i,1}, ..., v_{i,K})$ to $\hat{v}_i = (\hat{v}_{i,1}, ..., \hat{v}_{i,K})$. So

$$\phi_i(z) = (1 - z)v_i + z\hat{v}_i.$$

So, $\phi_i(z)$ specifies player i's report as a linear combination of $\phi_i(0) = v_i$ and $\phi_i(1) = \hat{v}_i$. To simplify the notation, let

$$k_i(z) = \kappa(\phi_i(z), v_{-i}), \quad m_i(z) = \mu_i(\phi_i(z), v_{-i}), \quad \text{and} \quad s(z) = S(\phi_i(z), v_{-i}).$$

Observe that by Axiom 2 (Renegotiation Proofness), there exists a threshold $z^* \in (0,1)$ such that along the path of reports $(\phi_i(z), v_{-i})$ the selected alternative $k_i(z)$ satisfies $k_i(z) = k_i(0)$ for $z < z^*$ and $k_i(z) = k_i(1)$ for $z > z^*$. Thus

$$\kappa(\phi_i(z), v_{-i}) \text{ is almost everywhere constant w.r.t. } z \in [0,1]. \tag{25}$$

Let $z', z'' \in [0,1]$ such that $z' < z''$ and $k_i(z') = k_i(z'')$. By (25) the chosen alternative is constant on $[z', z'']$. For each $z \in \{z', z''\}$, by Axiom 2 (Renegotiation Proofness) we have $v_{i,k_i(z)} = s(z) - \sum_{j \neq i} v_{j,k_i(z)}$. Therefore, we have by (20) and (21)

$$U_i(\phi_i(z), v_{-i}) = q(s(z)) \left(s(z) - \sum\nolimits_{j \neq i} v_{j,k_i(z)} + m_i(z) \right), \quad z \in \{z', z''\}.$$

Axiom 1 (Dominant Strategy Incentive Compatibility) then implies

$$\begin{aligned} U_i(\phi_i(z'), v_{-i}) &= q(s(z')) \left(s(z') - \sum\nolimits_{j \neq i} v_{j,k_i(z')} + m_i(z') \right) \\ &\geq q(s(z'')) \left(s(z') - \sum\nolimits_{j \neq i} v_{j,k_i(z'')} + m_i(z'') \right), \text{ and} \\ U_i(\phi_i(z''), v_{-i}) &= q(s(z'')) \left(s(z'') - \sum\nolimits_{j \neq i} v_{j,k_i(z'')} + m_i(z'') \right) \\ &\geq q(s(z')) \left(s(z'') - \sum\nolimits_{j \neq i} v_{j,k_i(z')} + m_i(z') \right), \end{aligned}$$

Using the above inequalities and the assumption that $k_i(z') = k_i(z'')$, we obtain

$$q(s(z'))(s(z'') - s(z')) \leq U_i(\phi_i(z''), v_{-i}) - U_i(\phi_i(z'), v_{-i}) \leq q(s(z''))(s(z'') - s(z')),$$

which can only hold if $q(s)$ is weakly increasing. Dividing all terms by $(z'' - z')$ and taking the limit of $z'' \to z'$, we obtain

$$\mathrm{d}U_i(\phi_i(z'), v^r_{-i}) = q(s(z'))\mathrm{d}s(z').$$

By (25), the above holds for almost all $z' \in [0,1]$, specifically, where $k_i(z')$ is locally constant. Therefore, after changing the variable $s = s(z)$, we obtain

$$U_i(\hat{v}_i, v_{-i}) - U_i(v_i, v_{-i}) = \int_0^1 q(s(z))\mathrm{d}s(z) = \int_{s_*}^{s^*} q(s)\mathrm{d}s \tag{26}$$

for almost all $v_i, \hat{v}_i \in V_i$. Because the incentive compatibility implies that $U_i(v)$ is continuous, (26) holds for all $v_i, \hat{v}_i \in V_i$. By (24), (26) is equivalent to (23). This completes the proof of Step 1. $\square$

Step 2. Using Step 1 and Axioms 3 and 7, we show that for each $i, j = 1, ..., N$ and each $v \in V$,

$$U_i(v) = U_j(v). \tag{27}$$

Proof of Step 2. A set $\tilde{V}$ is a *comprehensive extension of* V if it contains all profiles that are weakly smaller than v for each $v \in V$, so

$$\tilde{V} = \left\{ \tilde{v} \in \mathbb{R}^{N \times K} : \tilde{v} \leq v \text{ for some } v \in V \right\}.$$

Let $(\tilde{\kappa}, \tilde{\mu}, \tilde{\tau}) = \psi(\tilde{V})$, and let $\tilde{U}_i(v_i, v_{-i})$ be the payoff of player i under $(\tilde{\kappa}, \tilde{\mu}, \tilde{\tau})$ given by (20). We first show that (27) holds under the mechanism $(\tilde{\kappa}, \tilde{\mu}, \tilde{\tau})$ for the set $\tilde{V}$ of value profiles.

Because $\tilde{V}$ is unbounded from below, there exists $v \in \tilde{V}$ such that $S(v) = 0$. For any such v, by (22) we have $\sum_i \tilde{U}_i(v) = 0$, and by Axiom 3 (Individual Rationality) we have $\tilde{U}_i(v) \geq 0$ for each i. We thus obtain

$$S(v) = 0 \;\text{ implies }\; \tilde{U}_i(v) = 0 \;\text{ for each } i = 1, ..., N. \tag{28}$$

If $S(v) = 0$ for all $v \in \tilde{V}$, then the proof of Step 2 is complete.

Suppose that there exists $v \in \tilde{V}$ such that $S(v) > 0$. Consider any such v. Because $\tilde{V}_i$ is unbounded from below, there exists a deviation $\tilde{v}_i$ of player i such that $S(\tilde{v}_i, v_{-i}) = 0$. Then, by (20), (28), and Step 1 we obtain for each $i = 1, ..., N$ and each $v \in \tilde{V}$

$$\tilde{U}_i(v_i, v_{-i}) = \delta^{\tilde{\tau}(v)}(v_{i,\tilde{\kappa}(v)} + \tilde{\mu}_i(v)) = \int_0^{S(v)} q(s)\mathrm{d}s.$$

Next, by Axiom 7 (Independence of Irrelevant Information), for each $v \in V \subset \tilde{V}$ we have $\kappa(v) = \tilde{\kappa}(v)$ and $\mu(v) = \tilde{\mu}(v)$. Thus, for each $i = 1, ..., N$ and each $v \in V$ we obtain

$$U_i(v_i, v_{-i}) = \delta^{\tau(v)}(v_{i,\kappa(v)} + \mu_i(v)) = \frac{\delta^{\tau(v)}}{\delta^{\tilde{\tau}(v)}} \int_0^{S(v)} q(s)\mathrm{d}s,$$

which is the same for each i. This immediately implies (27). □

Step 3. Using Steps 1 and 2, and Axiom 4, we show that for each $v \in V$ and each $i = 1, ..., N$,

$$U_i(v) = \frac{S_{\max}}{N} - \int_{S(v)}^{S_{\max}} q(s)\mathrm{d}s, \tag{29}$$

where

$$S_{\max} = \sup_{v \in V} S(v).$$

Proof. Let $v \in V$. First consider $v \in V$ such that $S(v) = S_{\max}$. By (22) and Step 2 we have

$$U_i(v) = q(S_{\max})\frac{S_{\max}}{N} \text{ for each } i = 1, ..., n.$$

By Axiom 4 (Weak Nonwastefulness), $\tau(v) = 0$, so $q(S(v)) = 1$ for some $v \in V$. By Step 1, $q(s)$ is weakly increasing. It follows that $q(S_{\max}) = 1$. We thus obtain

$$S(v) = S_{\max} \text{ implies } U_i(v) = \frac{S_{\max}}{N} \text{ for each } i = 1, ..., n. \tag{30}$$

Now consider $v \in V$ such that $S(v) < S_{\max}$. Because $V = \times_{i=1}^N V_i$, there exists a finite sequence $\hat{v}^0, \hat{v}^1, ..., \hat{v}^R$ of value profiles in V such that $\hat{v}^0 = v$,

$$S(v) = S(\hat{v}^0) < S(\hat{v}^1) < ... < S(\hat{v}^R) = S_{\max},$$

and for each $r = 1, ..., R$, the profiles $\hat{v}^r$ and $\hat{v}^{r-1}$ differ by a unilateral deviation of a single player.

Fix $r = 1, ..., R$ and let i be the player whose report is different between $\hat{v}^r$ and $\hat{v}^{r-1}$, so $v_i^r \neq v_i^{r-1}$ and $v_{-i}^r = v_{-i}^{r-1}$. By Step 1,

$$U_i(\hat{v}^{r-1}) = U_i(\hat{v}^r) - \int_{S(\hat{v}^r)}^{S(\hat{v}^{r-1})} q(s)\mathrm{d}s. \tag{31}$$

By Step 2, the utilities are the same for all players, so $U_j(\hat{v}^{r-1}) = U_i(\hat{v}^{r-1})$ for each $j \neq i$. Using (30) to determine $U_i(\hat{v}^R)$ and applying equation (31) recursively for each $r = R, R-2, ..., 1$, we obtain

$$U_i(\hat{v}^{r-1}) = \frac{S_{\max}}{N} - \int_{S(\hat{v}^{r-1})}^{S_{\max}} q(s)\mathrm{d}s.$$

In particular, we obtain $U_i(v) = U_i(\hat{v}^0)$ is given by (29). □

Step 4. Suppose that $S_{\max} > 0$. Using Step 3, we show that

$$q(s) = \left(\frac{s}{S_{\max}}\right)^{N-1} \text{ for each } s \leq S_{\max}.$$

Proof of Step 4. Denote

$$Q(s) = \frac{S_{\max}}{N} - \int_s^{S_{\max}} q(x)\mathrm{d}x \text{ and } Q'(s) = q(s).$$

Let $v \in \mathcal{V}$ and let $s = S(v)$. By Step 3, we have $U_i(v) = Q(s)$, so

$$\sum_{i=1}^N U_i(v) = NQ(s).$$

By (22) we have

$$\sum_{i=1}^{N} U_i(v) = q(s)s = Q'(s)s.$$

We thus obtain a differential equation

$$NQ(s) = Q'(s)s \ \text{ s.t. } Q(S_{\max}) = \frac{S_{max}}{N} \text{ and } Q'(S_{\max}) = 1,$$

where condition $Q'(1) = q(1) = 1$ is by Axiom 4 (Weak Non-Wastfulness) and the monotonicity of $q(s)$ as shown in Step 1. This is an ODE initial value problem. Because $Q(s)$ is continuous and bounded on $[0, S_{max}]$, by Picard-Lindelöf Theorem there is a unique solution. This solution is

$$Q(s) = \frac{s^N}{NS_{\max}^{N-1}}.$$

Thus $q(s) = Q'(s) = (s/S_{\max})^{N-1}$. □

To complete the proof of Theorem 1, we use Steps 3 and 4 to show that (κ, μ, τ) satisfy the conditions of Theorem 1.

Let $V \in \mathcal{V}$. If $S_{\max} = 0$, then τ trivially satisfies (4). If $S_{\max} > 0$, then by Step 4, $q(S(v)) = (S(v)/S_{\max})^{N-1}$. Substituting this into (21) determines τ as in (4).

Next, by (20), (21), and Steps 3 and 4 we obtain for each $i = 1, ..., N$ and each $v \in V$

$$U_i(v) = \delta^{\tau(v)}(v_{i,\kappa(v)} + \mu_i(v)) = \frac{S_{\max}}{N} - \int_{S(v)}^{S_{\max}} q(s)\mathrm{d}s = q(S(v))\frac{S(v)}{N} = \delta^{\tau(v)}\frac{S(v)}{N}.$$

Thus, (κ, μ) must satisfy the conditions (1)–(2) of the Nash bargaining solution with delay. This completes the proof. □

A.2. **Indispensability of the Axioms.** Let us point out that each of the seven axioms is indispensable. For each axiom we present a mechanism, or a class of mechanisms, that violates this axiom but satisfies the other six axioms.

Consider the *first-best mechanism* that splits the surplus equally and without delay whenever this surplus is positive. Specifically, for each $v \in V$, let $\tau(v) = 0$, and let $\kappa(v)$ and $\mu(v)$ be given by Nash bargaining solution, (1) and (2). This mechanism satisfies Axioms 2–7, but violates Axiom 1 (Dominant Strategy Incentive Compatibility).

Consider the same first-best mechanism, but now let the monetary transfers be the VCG transfers. This mechanism satisfies Axioms 1–5 and 7, but violates Axiom 6 (Budget Balance).

Next, consider the *status quo* mechanism that always implements alternative 0 (status quo) without delay, so $\tau(v) = 0$, $\kappa(v) = 0$, and $\mu(v) = (0, ..., 0)$ for all $v \in V$. This mechanism satisfies Axioms 1 and 5–7, but violates Axiom 2 (Renegotiation Proofness).

Next, consider a *biased split mechanism* that implements the delayed Nash bargaining solution as in Theorem 1, but, in addition, player 2 makes an extra payment of ε to player 1, where ε is a positive constant. Specifically, for each $v \in V$, let $\kappa(v)$ and $\tau(v)$ be given by (1) and (4). For players 1 and 2 let

$$\mu_1(v) = \frac{S(v)}{N} - v_{1,\kappa(v)} + \varepsilon \quad \text{and} \quad \mu_2(c) = \frac{S(v)}{N} - v_{1,\kappa(v)} - \varepsilon.$$

If there are more than two players, then for each player $i > 2$ let $\mu_i(v)$ be given by (2). This mechanism satisfies Axioms 1–2 and 4–7, but violates Axiom 3 (Individual Rationality).

Relaxing Axiom 4 (Weak Non-Wastefulness) while keeping the rest of the axioms leads to a family of bargaining mechanisms $(\kappa, \mu, \tau_\lambda)$ parameterized by $\lambda \in [0, 1]$, where $\kappa(v)$ and $\mu(v)$ are given by (1) and (2), and $\tau_\lambda(v)$ is given by

$$\delta^{\tau_\lambda(v)} = \lambda \left(S(v)\right)^{N-1} \quad \text{for each } v \in V.$$

Each of these bargaining mechanisms differs from the mechanism (κ, μ, τ) in Theorem 1 only in that the time of the agreement is increased by $-\ln\lambda$. This follows easily from the proof of Theorem 1. Notice that $(\kappa, \mu, \tau_\lambda)$ is Pareto inferior to (κ, μ, τ_1) for each $\lambda < 1$. In particular, this means that (κ, μ, τ_1) is the unique ex-post undominated bargaining mechanism that satisfies Axioms 1–3 and 5–7 (for the formal definition of ex-post dominance see Čopič and Ponsati, 2016).

Relaxing Axiom 5 (Surplus Dependence) while keeping Axioms 1–4 and 6–7 leads to a large family of bargaining mechanisms with no closed form characterization. A particular example in case of a single alternative, $K = 1$, is the family of *posted sharing mechanisms* (also referred to as mechanisms in threshold form by Kuzmics and Steg, 2017). Any such mechanism is described by a fixed profile of shares $\bar{s} = (\bar{s}_1, ..., \bar{s}_n) \in \Delta^n$. It is as if a designer proposes to share the revenue from the alternative $k = 1$ according to $\bar{s}$, and players simultaneously accept or reject this proposal.

Finally, we present a solution that satisfies Axioms 1–6 but violates Axiom 7 (Independence of Irrelevant Information). Axiom 7 plays no role when $N = 2$, so we present a counterexample for $N = 3$ and $K = 1$.

Let $\tilde{V}_i = [0, 1]$ for each $i = 1, 2, 3$, so $\tilde{V} = [0, 1]^3$. Let v_i denote the value of the alternative $k = 1$ for player i. Note that the maximum surplus is $\max_{v\in[0,1]^3} v_1 + v_2 + v_3 = 3$. For each $s \in [0, 3]$ let

$$q(s) = \frac{s^2}{9} \quad \text{and} \quad Q(s) = \frac{s^3}{27}.$$

Consider the following solution $\tilde{\psi}$. For each $V \in \mathcal{V}$ let the bargaining mechanism $\tilde{\psi}(V)$ be identical to one that satisfies the conditions of Theorem 1, with a single difference. When $V = \tilde{V}$, the transfer rule $\tilde{\mu}$ is as follows. For $v = (0, 0, 0)$ let $\tilde{\mu}_1(v) = \tilde{\mu}_2(v) = \tilde{\mu}_3(v) = 0$. For each $v \in \tilde{V}\backslash\{(0, 0, 0)\}$ let

$$\tilde{\mu}_1(v) = \frac{Q(v_1 + v_2 + v_3) + Q(v_2) + Q(v_3)}{q(v_1 + v_2 + v_3)} - v_1,$$

$$\tilde{\mu}_2(v) = \frac{Q(v_1 + v_2 + v_3) - Q(v_3)}{q(v_1 + v_2 + v_3)} - v_2,$$

$$\tilde{\mu}_3(v) = \frac{Q(v_1 + v_2 + v_3) - Q(v_2)}{q(v_1 + v_2 + v_3)} - v_3.$$

It is easy to verify that the mechanism $\tilde{\psi}(\tilde{V}) = (\tilde{\kappa}, \tilde{\mu}, \tilde{\tau})$ satisfies the conditions of Axioms 1–6. But Axiom 7 is violated. For example, let $V = (-\infty, 1]^3$ and denote by (κ, μ, τ) the bargaining mechanism induced by $\tilde{\psi}$ for V, so $\tilde{\psi}(V) = (\kappa, \mu, \tau)$. Then for each $v \in \tilde{V}\backslash\{(0, 0, 0)\} \subset V$ we have $\mu_i(v) \neq \tilde{\mu}_i(v)$ for $i = 1, 2, 3$, which is a violation of Axiom 7.

A.3. **Proof of Theorem 2.** Suppose that the players play $(\gamma^*, \sigma^*, \alpha^*)$ and have beliefs B^* with arbitrary (possibly, degenerate) priors.

First, consider the punishment phase. In this phase, the best share that everyone except the deviant can be offered is $a + (1 - na)/(n - 1)$, and the deviant can only be offered a. Any different offer would make at least one of the players strictly worse off. So making a different offer cannot be a profitable deviation. For any player i whose value satisfies that $w_i \leq a + (1 - na)/(n - 1)$, rejecting the equilibrium offer is not profitable either, because no player can expect a better offer in the future. For any player i whose value satisfies that $w_i > a + (1 - na)/(n - 1)$, accepting $a+(1-na)/(n-1)$ is strictly inferior to rejecting it, and offering to split the quantity

$q = 1$ where she gets $a + (1 - na)/(n - 1)$ is strictly inferior to offering $q = 0$. Finally, the deviant cannot get more than zero payoff, because any offer where she gets more than $w_i \geq a$ will be rejected by at least one other player. So, the play during the punishment phase satisfies the conditions of a perfect Bayesian equilibrium.

We now consider the cooperative phase. Any player who makes an inconsistent offer in this phase triggers the punishment phase, in which she gets zero payoff. But she gets at least zero in equilibrium. So, deviations to inconsistent offers cannot be profitable.

By making a consistent offer, the player reveals to the others a value $\hat{w}_i$, which then becomes the assessment value. Notice that a player can reject offers until her turn to move comes, through which she makes an "announcement" of her value. So, an offer is accepted by all if no player wishes to make another consistent offer, thereby changing her announcement. Whenever an offer is accepted by all, it is equal to the outcome of the fair bargaining rule, where each player prefers to reveal her value truthfully, so deviations from truthful announcements are not profitable. Finally, acceptance of an offer that should be rejected in equilibrium leads to a strictly smaller payoff. Rejection of an offer that should be accepted in equilibrium cannot lead to a better payoff, because in equilibrium the other players' assessments never decrease, and one's payoff is decreasing in the assessments of the others. It follows that strategy profile $(\gamma^*, \sigma^*, \alpha^*)$ satisfies the conditions of a perfect Bayesian equilibrium.

It remains to verify that the beliefs B^* are obtained by the Bayes rule whenever possible. Observe that the equilibrium strategies are separating. When player i moves for the first time, her consistent offer induces a degenerate belief about player i's value. If this value is in the support of the prior, then the posterior must be degenerate with the unit mass on that value. Alternatively, if this value is not in the support of the prior, then the posterior can be arbitrary, in particular, it can be degenerate with the unit mass on that value. Players move at most one in equilibrium. It is easy to verify that on the out-of-equilibrium paths, no inconsistencies with the Bayes rule arise. □

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