Auctions with a multi-member bidder

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Abstract

I consider an auction in which one bidder is a team, consisting of symmetric individuals for whom the auctioned item is a public good: if the team wins it, they all enjoy it. Team members need to agree on a bid, and on splitting the burden of payment if they win; these decisions are taken through a mechanism. If the auction format is second-price, the game has a symmetric equilibrium. Under the first-price and all-pay formats, if there exists an equilibrium, then it extremely asymmetric: one member participates in the mechanism, and everybody else free ride.

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1 Introduction

Auction theory typically assumes that bidders are individuals. Namely, that "one bidder=one agent." In practice, however, bidding is often decided by groups of agents—firms' managements, consulting groups, and households, to name a few. If group members hold private information (e.g., about their preferences), the team needs to aggregate this information in order to operate. Somewhat surprisingly, the game theoretic literature has yet to provide an account on the matter. The present paper addresses this lacuna by studying an auction model in which one of the bidders is a team, who faces the aforementioned challenge.

One takeaway from the analysis is that in an environment with team bidders, there may be dramatic differences between the major auction formats. Another takeaway is that the second-price format exhibits appealing properties, and offers (under some conditions) a generalization of the results concerning the individual-bidders auction into the current setting. More specifically, the environment I consider and the results are as follows.

There is a single-item auction with two bidders: one who is a multi-member team, comprising n individuals (players 1 through n), and one who is a single individual (player $n+1$, the *regular bidder*). Players 1 though n are symmetric: each has a valuation for the item which is drawn from a distribution on $[0, 1]$, which is the same for all of them. The valuations of all $n+1$ individuals are private and independent. From the regular bidder's standpoint, the auction is standard; the novel angle is that of the team, whose perspective is as follows.

Given that team members' valuations are $(\theta_1, \cdots, \theta_n)$, if the team wins, the payoff of member i is $\theta_i - p_i$, where p_i is the payment he contributes to cover the item's cost. The team needs to decide what bid to submit, and how to split the burden of payment between its members if the auction is won. These decisions are taken by a mechanism to which team members, players 1 through n, send reports. The mechanism consists of a *bid aggregation rule*, A, and a *cost sharing rule*, s, and it works as follows: player i sends a report (or bid) $b_i \geq 0$, the reports are sent simultaneously, and the bid submitted on behalf of the team is $A(b_1, \cdots, b_n)$; if the auction is won, the cost is split between the players in proportions $(s_1(b_1, \cdots, b_n), \cdots, s_n(b_1, \cdots, b_n))$.[1] The interpretation is that by announcing b_i, player i reports his maximal willingness to pay for the item.[2]

I start the analysis by considering a second-price auction. Under the second-price format, it can be assumed that the regular bidder reports his valuation truthfully.[3] When the regular bidder's strategy is fixed at truthful reporting, the resulting game is a symmetric Bayesian game between the n team members, and under fairly general conditions it admits a symmetric (Bayes-Nash) equilibrium. After establishing this existence, I show that no equilibrium leads to an efficient allocation of the item. This is to be expected, because from the team's perspective, the auctioned item is a public good, hence free riding arises inevitably within the team.

Next, I turn to first-price and all-pay auctions. These are more challenging, because the regular bidder has no dominant strategy. The analysis of these auctions proceeds by considering two qualitatively different cases.

First, suppose that $n-1$ team members do not participate in the mechanism (equivalently, they mimic the zero type) and only one member actively participates. Then, the situation is that of a standard 2-player auction, played between the "team's representative" and the regular bidder. The challenge here is to provide the $n-1$ designated abstainers the incentives to report zero even if their type is high—incentives that can or cannot be provided, depending on the auction's format, the type distribu-

[1] Throughout, the word "bid" will be used in several senses. It will be used to denote an individual bid by player i—namely b_i, his report to the mechanism—and to denote the team's bid, $A(b_1, \cdots, b_n)$, that results from these reports. It will also be used to denote the regular bidder's bid (namely, the bid against which the team competes).

[2] A discussion about the definition of a mechanism—specifically, why I assume the aforementioned form rather than a "revelation mechanism"—appears in Appendix A.

[3] For that matter, one could assume an arbitrary number of out-of-team bidders, and identify the "regular bidder" with the first-order statistic of their type distributions.

tion, and the team's mechanism. For the case where these incentive *can* be provided, the resulting equilibrium is called an equilibrium with *complete free riding*.

Next, consider a candidate for an equilibrium without complete free riding. Then, at least two team members follow a non-zero reporting function. For each such member, the reporting function, while not identically zero, is still constantly zero on some interval of low types. The zero bidding of these low types induces low types of the regular bidder to bid zero as well, which is impossible: if all types below a cutoff report zero, any one of them would find it profitable to slightly increase his bid/report. Therefore, under the first-price and all-pay formats, in contrast to second-price auctions, if there exists an equilibrium, then it is an equilibrium with complete free riding. I show that both options—non-existence and existence of only complete free riding equilibria—are possible.

To get a closer look at team bidding in models where equilibria without complete free riding exist, I analyze a specific version of the second-price model—the *separable-proportional* one, which means that there exists a strictly increasing function ψ, with $\psi(0) = 0$, such that:

- Bid aggregation is separable: $A(b_1, \cdots, b_n) = \sum_{i=1}^{n} \psi(b_i)$; and
- Cost sharing is proportional: $\max\{b_1, \cdots, b_n\} > 0 \Rightarrow s_i(b_1, \cdots, b_n) = \frac{\psi(b_i)}{\sum_{j=1}^{n} \psi(b_j)}$ and $s_1(0, \cdots, 0) = \cdots = s_n(0, \cdots, 0) = \frac{1}{n}$.

All separable-proportional mechanisms are equivalent, therefore there is not loss of generality in assuming that ψ is the identity function.[4] I make this assumption. Under the second-price format and given that the regular bidder's type is distributed uniformly over a sufficiently large support, the separable-proportional model has a unique equilibrium. The equilibrium is symmetric, and is such that all team members send reports according to the following function:

[4]This point is explained in Appendix A.

$$\beta^{SPA}(\theta) = \max\{\theta - a, 0\}, \tag{1}$$

where the constant a is the (unique) solution to some equation. The equilibrium can be described as follows: Each team member computes an *adjusted valuation*, which is the maximum between zero and true valuation minus the constant a, and if the adjusted valuation is positive, the individual sends a report that increases the collective bid by the adjusted valuation; otherwise, the individual abstains. The constant a is equal to zero if and only if $n = 1$. Therefore, the bid function (1) generalizes the weak dominance equilibrium of the ordinary second-price action.[5]

The equilibrium expected utility of a type-θ team member is:

$$\pi^*(\theta) = \frac{1}{2M} \cdot [2\theta - \max\{\theta - a, 0\}] \cdot [2a + \max\{\theta - a, 0\}], \tag{2}$$

where $M > 0$ is the regular bidder's maximum type. Expression (2) generalize that equilibrium's payoffs for the case of uniform competition. By contrast, under the first-price format and with uniform type distributions, the above model has no equilibrium; and under the all-pay format, equilibria with complete free riding exist.

The paper is organized as follows. Subsection 1.1 reviews the literature. Section 2 provides definitions. Sections 3 is dedicated to second-price auctions, and Section 4 to first-price and all-pay auctions. Section 5 considers the separable-proportional model. Section 6 concludes. Appendix A is dedicated to remarks about modeling issues, and Appendix B collects proofs.

[5]When $n = 1$ the RHS of (1) is θ, therefore the bid submitted on behalf of the "team" is $A = \theta$.

1.1 Literature

Studying auctions with a single team-bidder is both of its own merit,[6] and can be viewed as a first step towards the analysis of competition between multiple bidding teams. Such competition, from a game theoretic point of view, is an under-studied topic. A recent exception is a paper by Asker et al. (2021), who study teams-competition in auctions for patents. That paper's focus, however, is different from mine, since within each team there is no private information. Instead, the team's value is commonly known to all of its members, and since the auction format is second-price, bidding is straightforward—the entire team behaves as a single agent who has a well-defined value for winning, and the team's bid is equal to this value. Additionally, there are some earlier papers in the auctions literature about bidding in joint ventures, in particular for offshore drilling.[7] In these papers, too, each team's members' incentives are fully aligned.

At a general level, the paper is related to the literature about games that are played by teams. This literature is mostly experimental,[8] and the handful of theoretical exceptions it offers are substantially different from my model. For example, Kim et al. (2021) study a theoretical model of team play, but similarly to the papers mentioned in the previous paragraph, their model is such that there is no strategic interaction between members of the same team.[9]

[6]Bidding scenarios with only one multi-member bidder are actually encountered in the real world. For instance, the Israeli 5G spectrum auction that was carried out in 2020 is a case in point: it included several bidders, one of which was multi-member, comprising three telecommunication companies. A recent paper that describes this auction is Blumrosen and Solan (2022). Disclosure: I was involved in consulting to one of the operators in the 5G auction. The present paper is not related to this consulting.

[7]See, e.g., DeBrock and Smith (1983).

[8]See, e.g., Charness and Sutter (2012), Kugler et al. (2012). Experimental papers on team bidding include Cox and Hayne (2006) and Sutter et al. (2009).

[9]In subsection 2.1 of their paper Kim et al. write: *"Because it is a common value problem for the team, there is an implicit assumption of sincere reporting."* A similar approach was taken in an

From the existing literature, the model which is closest to the ones studied here is by Barbieri and Malueg (2010), who study private funding of public goods. This funding is decided via a game in which agents simultaneously make contributions, the public good is provided if the sum of contributions exceeds the good's cost, and otherwise it is not provided and the contributions are refunded. In that paper, the public good's provision-cost is stochastic, which is analogous to the stochastic bid (i.e., outside competition) against which the team competes in the second-price version of my model.

The literature on collusion can also be seen as somewhat relevant, to the extent that the team is viewed as a cartel. However, it is not a cartel in the ordinary sense of the word. Conceptually, the team in my model is an organic unit, not a fictitious construct the purpose of which is to advance the goals of its individual members. Operationally, collusion in auctions is typically based on transfers between the colluding agents, because the cartel allocates the won item to *one* of its members, and the others need to be compensated for leaving the auction empty-handed.[10,11] In the present paper, by contrast, transfers are not available, and losers-compensation is irrelevant, because when the team wins the auction, all of its members win together.[12]

Finally, another work which is related (albeit indirectly) to the present paper is by Haghpanah et al. (2021), who study selling a good to a group, where the transfer payed to the seller is incurred by each of the group's members.

earlier paper by Duggan (2001), where groups of players aggregate their actions via social choice rules, with no strategic issues involved in this aggregation.

[10]See, e.g., McAfee and McMillan (1992).

[11]Many papers on collusion consider repeated games. In such games, continuation play can substitute for monetary transfers, but the basic idea is the same—winners need to compensate losers. See, e.g., Athey and Bagwell (2001), Skrzypacz and Hopenhayn (2004), and Rachmilevitch (2013).

[12]Another context in which team members win together or lose together is that of group contests. See Eliaz and Wu (2018), Kobayashi and Konishi (2021), and the references therein.

2 Definitions

A mechanism, as described in the Introduction, consists of a bid aggregation rule A and cost sharing rule s. Both are continuous functions on $\mathbb{R}^n_+$, A takes values in $\mathbb{R}_+$ and s in the $(n-1)$-dimensional unit simplex. The function A satisfies $A(0, \cdots, 0) = 0$, it is weakly increasing in each coordinate, $A(b) > 0$ whenever there is at least one i with $b_i > 0$, and is such that for every $r > 0$ and b_{-i}, there exists a b_i such that $A(b_i, b_{-i}) \geq r$. Each player's payment-share s_i is weakly increasing in b_i. Since A and s are monotone in each argument, they are differentiable a.e. I assume that (at least) one of the following holds: either $\frac{\partial A}{\partial b_i} > 0$ or $\frac{\partial s_i}{\partial b_i} > 0$ (wherever the derivatives exist).

The valuation of each team member is distributed on $[0, 1]$ according to the cdf F, whose density is f. The regular bidder's valuation is distributed on an interval in $\mathbb{R}_+$ that contains zero, according to the cdf G, whose density is g.[13] All $n+1$ valuations (or types) are independent of one another.

A mechanism (A, s) gives rise to a Bayesian game played by the $n+1$ participants. The solution concept is Bayes-Nash equilibrium. When the auction format is second-price, it is assumed that the regular bidder reports his type truthfully and therefore, in this case, the game is actually considered as an n person game. In this game, a (pure) strategy for player i is a reporting (or bid) function $\beta_i \colon [0, 1] \to \mathbb{R}_+$. An equilibrium is *symmetric* if $\beta_1 = \cdots = \beta_n$. If the auction format is first-price or all-pay, a strategy for the regular bidder needs to be specified as well, hence a strategy profile takes the form $((\beta_1, \cdots, \beta_n), \beta_{n+1})$, where β_{n+1} is a bid function of the regular bidder, defined on his type-set. An equilibrium is with *complete free riding* if the profile of team reporting functions, $(\beta_1, \cdots, \beta_n)$, is such that $|\{i : \beta_i \equiv 0\}| = n - 1$.

A mechanism (A, s) *leads to an efficient allocation* if the corresponding game has an equilibrium under which the team wins the auction if $\sum_{i=1}^n \theta_i > \theta_{n+1}$, and the regular bidder wins if the reverse strict inequality holds.

[13] This interval need not be bounded, it can be $\mathbb{R}_+$.

3 Second-price auctions

Suppose that the auction format is second-price and let (A, s) be a mechanism. Denote the game by $\Gamma^{SPA}(A, s)$. Define the following functions on $\mathbb{R}^n_+$:

$$\Pi_i^{A,s}((b_1, \cdots, b_n)|\theta_i) \equiv$$
$$\equiv G(A(b_1, \cdots, b_n)) \times [\theta_i - s_i(b_1, \cdots, b_n) \cdot \mathbb{E}(\theta_{n+1} : \theta_{n+1} \leq A(b_1, \cdots, b_n))].$$

Since the regular bidder reports his type truthfully, $\Pi_i^{A,s}(.|\theta_i)$ is the objective that type-θ_i player maximizes in equilibrium, over b_i, given the reports of his teammates. I make the following assumption:

Assumption 1. *For every $i \in \{1, \cdots, n\}$, $\Pi_i^{A,s}(.|\theta_i)$ is a continuous function of the reports.*

It is not required that both A and s be continuous, just that $\Pi_i^{A,s}(.|\theta_i)$ is. In fact, it may be that one of these functions is not continuous; for example, in the separable-proportional model the cost sharing rule is not continuous,[14] but the functions $\Pi_i^{A,s}(.|\theta_i)$ are continuous.

Theorem 1. *Under Assumption 1, $\Gamma^{SPA}(A, s)$ has an equilibrium. Moreover, it has a symmetric equilibrium—a one in which all team members follow the same reporting function.*

Proof. Clearly, there exists an $K > 0$ such that no player will send a report in excess of K, hence the strategy sets can be constrained to $[0, K]$. Since the latter is compact and the objectives $\Pi_i^{A,s}(.|.)$ are continuous, it follows from Theorem 4.5 in Reny (2011) that a symmetric equilibrium exists. $\square$

As the following result shows, whereas an equilibrium exists, any equilibrium is inefficient. The result can be seen as related to Holmstrom (1982), who considered a

[14] $(0, \cdots, 0)$ is the (single) discontinuity point if the function $(b_1, \cdots, b_n) \mapsto (\frac{b_1}{\sum_j b_j}, \cdots, \frac{b_n}{\sum_j b_j})$.

model in which agents take non-observable actions that produce a monetary outcome, which the agents then share via a sharing rule. He showed that there does not exist a sharing rule such that the associated game has a Pareto efficient Nash equilibrium.

For simplicity, I state the result under the assumption that the regular bidder's valuation is distributed on an interval that includes $[0, n]$. This assumption, however, is inessential.

Theorem 2. *Under the second-price format, no mechanism leads to an efficient allocation.*

Proof. Let (A, s) be a mechanism, and assume by contradiction that it leads to an efficient allocation. Then there exists an equilibrium, $(\beta_1, \cdots, \beta_n)$, such that for every profile of valuations, $(\theta_1, \cdots, \theta_n)$, it holds that $A(\beta_1(\theta_1), \cdots, \beta_n(\theta_n)) = \sum_{i=1}^{n} \theta_i$. W.l.o.g, it can be assumed that $A(b_1, \cdots, b_n) \equiv \sum_{i=1}^{n} b_i$ and each β_i is the identity. Consider player 1. When he reports r, his payoff, given that the others have types t_{-i}, is $G(r + \sum_{j \neq i} t_j) \cdot [\theta_1 - s_1(r, t_{-i}) \int_0^{r+\sum_{j\neq i} t_j} \frac{sh(s)}{G(r+\sum_{j\neq i} t_j)} ds]$. Integrating over the possible t_{-i}'s gives the expected payoff:

$$\theta_1 \underbrace{[\int_{t_{-i}} G(r + \sum_{j \neq i} t_j) f_{-i}(t_{-i}) dt_{-i}]}_{X} - \underbrace{[\int_{t_{-i}} s_1(r, t_{-i}) \int_0^{r+\sum_{j\neq i} t_j} sh(s) ds f_{-i}(t_{-i}) dt_{-i}]}_{Y}.$$

Since s_1 is weakly increasing in r, the term Y is differentiable (w.r.t r) almost everywhere, and it is clear that $Y' > 0$ whenever it exists. The derivative of the above expression is $\theta_1 \cdot X' - Y'$, which is negative for small enough θ_1's; thus, it is impossible for β_1 to be the identity—a contradiction. □

The above proof indicates what goes wrong if team bidding is required to mimic truthful individual bidding: the negativity of the aforementioned derivative not only implies that the reporting function cannot be the identity, but, moreover, that it is zero for all low enough θ_i's. Namely, low enough types mimic the minimum type, or, in other words, refrain from participation.

4 First-price and all-pay auctions

As indicated at the end of the previous section, the attempt to sustain "truthful team reporting" unravels, in the sense that all low enough types mimic the minimal type. Technically, the bid/reporting function starts flat, and assumes the value zero on an interval of low types. This pattern manifests itself in further ways, not only in the (failed) attempt to achieve efficiency in the second-price game. Under a first-price or an all-pay format, the implication of this pattern is this: if an equilibrium is not with complete free riding, then there are multiple team member who actively participate in the mechanism; between these members, there is free riding in the above sense—namely, low enough types send the same report as the minimal type—and this implies that the low types of the *regular bidder* also have an incentive to take the action of the minimal type. But this cannot be an equilibrium, because then each of these low types would have a profitable deviation, namely to slightly increase the bid/report.

This is the meaning of the following result, in the statement of which $\Gamma^{FPA}(A, s)$ is the game corresponding to a first-price format and the mechanism (A, s).

Theorem 3. *If $\Gamma^{FPA}(A, s)$ has an equilibrium, then it is an equilibrium with complete free-riding.*

To prove Theorem 3, I make use of the following lemma.

Lemma 1. *Fix an equilibrium of $\Gamma^{FPA}(A, s)$, and let $(\beta_1, \cdots, \beta_n)$ be the profile of team reporting functions in this equilibrium. Then there exists a non-empty subset of team members, I, such that the following holds for all $i \in I$: there exists an $a_i > 0$ such that β_i is identically zero on $[0, a_i]$.*

Proof. Fix an equilibrium. If it is an equilibrium with complete free-riding, then $|I| = n - 1$ and $a_i = 0$ for all $i \in I$. Suppose, then, that this is not the case. Consider a team member, i, with type θ_i, whose reporting function β_i is not identically zero. Given θ_{-i}, his expected utility from the report x is:

$$H(A(x, \beta_{-i}(\theta_{-i}))) \cdot [\theta_i - s_i(x, \beta_{-i}(\theta_{-i})) \cdot A(x, \beta_{-i}(\theta_{-i}))],$$

where H is the cdf of the regular bidder's equilibrium-bid. Therefore, his expected utility is:

$$U_i(x|\theta_i) = \mathbb{E}[H(A(x, \beta_{-i}(\theta_{-i}))) \cdot \theta_i] - \mathbb{E}[H(A(x, \beta_{-i}(\theta_{-i}))) \cdot s_i(x, \beta_{-i}(\theta_{-i})) \cdot A(x, \beta_{-i}(\theta_{-i}))].$$

The derivative of this objective is:

$$\frac{d}{dx} U_i(x|\theta_i) = \theta_i \mathbb{E}[h(A(x, \beta_{-i}(\theta_{-i})))A'] -$$
$$- \mathbb{E}\{h(A(x, \beta_{-i}(\theta_{-i})))A' \cdot s_i(x, \beta_{-i}(\theta_{-i})) \cdot A(x, \beta_{-i}(\theta_{-i})) -$$
$$-H(A(x, \beta_{-i}(\theta_{-i}))) \cdot s_i'(x, \beta_{-i}(\theta_{-i})) \cdot A(x, \beta_{-i}(\theta_{-i})) - H(A(x, \beta_{-i}(\theta_{-i}))) \cdot s_i(x, \beta_{-i}(\theta_{-i}))A'\},$$

where A' and s' is a shorthand for the appropriate partial derivative, and for brevity I omitted A's argument. Since the equilibrium is not an equilibrium with complete free-riding, there exists a $j \neq i$ such that β_j is no identically zero. Therefore $\frac{d}{dx} U_i(0|0) < 0$, which implies that $\beta_i(\theta_i) = 0$ for all small enough θ_i's. $\square$

Proof of Theorem 3: Assume by contradiction that an equilibrium without complete free riding exists. Fix such an equilibrium, and for each i let $a_i > 0$ be the constant described in Lemma 1. Let $p \equiv \Pi_{i=1}^n F(a_i)$. Now consider the regular bidder; suppose that his type is $\theta \in (0, p)$. I argue that it is optimal for this type of the regular bidder to bid zero. To see this, assume by contradiction that his optimal bid is $x > 0$.[15] Then the expected utility form x is at least as large as the expected utility from the bid zero:

$$p(\theta - x) + \int_0^x (\theta - x) dZ \geq p\theta,$$

where Z is the cdf of the team's bid. Therefore

$$\int_0^x (\theta - x) dZ \geq px,$$

[15] By assumption (by contradiction) an equilibrium exists, therefore, in particular, the aforementioned optimal bid exists.

which is impossible because $\int_0^x (\theta - x)dZ < \int_0^x (p-x)dZ < \int_0^x p dZ \le px$. Therefore both the team members and the regular bidder employ functions that are zero on $[0, a^*]$, where $a^* \equiv \min\{p, a_1, \cdots, a_n\}$. However, this cannot be an equilibrium, because then each participant with type $\theta \in (0, a^*)$—be it a team member or the regular bidder—has an incentive to slightly increase his report/bid. $\square$

Given a mechanism (A, s), let $\Gamma^{APA}(A, s)$ denote the associated game given that the auction format is all-pay. The result for all-pay auctions, whose proof is relegated to Appendix B, is analogous to the one about first-price auctions.

Theorem 4. *If $\Gamma^{APA}(A, s)$ has an equilibrium, then it is an equilibrium with complete free-riding.*

The following propositions show how Theorems 3 and 4 can be manifested both in the form of equilibrium existence, and in equilibrium non-existence.

Proposition 1. *If F and G are both uniform over $[0, 1]$ and the auction-format is all-pay, then the separable-proportional model has equilibria with complete free riding.*

Proposition 2. *If F and G are both uniform over $[0, 1]$ and the auction-format is first-price, then the separable-proportional model has no equilibrium with complete free riding. Therefore, it has no equilibrium.*

The intuition for the difference between the two propositions is the following. Under an equilibrium with complete free riding, one team member plays a standard 2-player auction against the regular bidder. The equilibrium of *this* auction is the same under any of the considered formats, due to payoff-equivalence. The issue, then, is whether the $n-1$ designated free riders have the incentives not to intervene (by increasing the team's bid) even if their type is high. Under the all-pay case, such incentives are provided by the harsh pay-your-bid rule; under the "softer" first-price format, they cannot.

5 The separable-proportional model

In the separable-proportional model, if the auction format is second-price and outside competition is uniform over a sufficiently large support, then the equilibrium is unique. The equilibrium, which is described below, generalizes the weak dominance equilibrium of the ordinary second-price auction.

Theorem 5. *Let the regular bidder's type be uniform on $[0, M]$, where $M \geq 2n$, and suppose that the auction-format is second-price. Then the separable-proportional model has a unique equilibrium, $(\beta_1, \cdots, \beta_n)$. The equilibrium is symmetric: $\beta_1 = \cdots = \beta_n = \beta^{SPA}$, where the common reporting function, β^{SPA}, is given by:*

$$\beta^{SPA}(\theta) = max\{\theta - a, 0\},$$

where a is the unique solution to:

$$a = \frac{n-1}{n+1} \cdot \left(\int_a^1 tf(t)dt + aF(a)\right). \tag{3}$$

In equilibrium, all non-abstaining types behave as in the weak dominance equilibrium of the ordinary second-price auction, in which their valuation is adjusted to $\theta - a$. Types below a free ride—they refrain from bidding, hoping that their partners' types be sufficiently high, in which case the auction will be won, but they will not be asked to contribute. Since a measure the free riders' segment, one would expect it to depend positively on n. The following result shows that this is indeed so. In its statement, a_n denotes the cutoff corresponding to a team of size n.

Proposition 3. *The cutoff a_n satisfies the following:*

1. *a_n is strictly increasing in n.*

2. *$lim_{n \to \infty} a_n = 1$.*

3. *$a_n \geq (\frac{n-1}{n+1})\mathbb{E}(\theta)$ for all $n \geq 1$.*

The intuition for *1* is that a greater number of team members implies stronger free-riding incentives for each member. Part *2* is a little delicate: whereas the assertion on the limit is mathematically correct, one need not forget that formula (3) applies when $M \geq 2n$; thus, the result should be interpreted as saying that if M is sufficiently large, then for a large enough n (but smaller than $\frac{M}{2}$) the equilibrium is close to a degenerate equilibrium, in which all types refrain from participation. The intuition behind part *3* is that if a team member believes that his partners' types are expected to be high, then he also believes that they are likely to place high bids and therefore there is a high probability that the auction will be won even if he does not contribute. By the same logic, one would expect the cutoff to increase under first-order stochastic dominance. This is indeed the case; the following result, in the statement of which $a(\cdot)$ denotes the cutoff as a function of the types distribution, formalizes of this fact.

Proposition 4. *If F^* first-order stochastically dominates F, then $a(F^*) \geq a(F)$.*

Proof. By integration by parts, equation (3) can be written as $a = k(1 - \int_a^1 F(t)dt)$, where $k \equiv \frac{n-1}{n+1}$. Let then F^* and F be as in the proposition's statement. Assume by contradiction that $a(F^*) < a(F)$. By the above argument, it holds that:

$$a(F) = k(1 - \int_{a(F)}^1 F(t)dt),$$

and

$$a(F^*) = k(1 - \int_{a(F^*)}^1 F^*(t)dt).$$

Equation-subtraction and invoking first-order dominance yields:

$$\begin{aligned} a(F) - a(F^*) &= k(\int_{a(F^*)}^1 F^*(t)dt - \int_{a(F)}^1 F(t)dt) \leq \\ &\leq k(\int_{a(F^*)}^1 F(t)dt - \int_{a(F)}^1 F(t)dt) = k\int_{a(F^*)}^{a(F)} F(t)dt \leq k(a(F) - a(F^*)), \end{aligned}$$

which is impossible because $k < 1$. $\square$

It is straightforward to see that if one sets $n = 1$, then Theorem 5's equilibrium boils down to the weak dominance equilibrium of the ordinary second-price auction. Correspondingly, the classical auctions' equilibrium payoffs are also obtained from the present payoffs, when one imposes $n = 1$.

Proposition 5. *Consider the separable-proportional model under the second-price format, and let the regular bidder's type be uniform on* $[0, M]$*, where* $M \geq 2n$*. Then the equilibrium-expected-utility of a team member with type* θ *is:*

$$\pi^*(\theta) = \frac{1}{2M} \cdot [2\theta - max\{\theta - a, 0\}] \cdot [2a + max\{\theta - a, 0\}].$$

Similarly to the bid function β^{SPA}, the type distribution F and the team size n affect equilibrium payoffs only through the cutoff a. Since $a = 0$ when $n = 1$, the above formula generalizes the standard auction's equilibrium payoffs formula.

It is easy to check that π^* is increasing in a, and therefore, by part *1* of Proposition 3, it is increasing in n. In particular, the model exhibits no group-size paradox (Olson 1965)—i.e., the free-riding that stems from collective bidding is not severe enough to create "decreasing returns to scale," namely to make smaller teams better off relative to larger ones.

Similarly to the team, the auctioneer also benefits from a large team size.

Proposition 6. *Consider the separable-proportional model under the second-price format, and let the regular bidder's type be uniform on* $[0, M]$*, where* $M \geq 2n$*. Then the team's equilibrium expected bid,* $n \times \mathbb{E}(\beta^{SPA})$*, is increasing in* n*.*

Proof. In Lemma 3 in Appendix B it is shown that $\mathbb{E}(\beta^{SPA}) = \frac{2a}{c(n-1)}$. Since $a = \frac{n-1}{n+1}(\int_a^1 tf(t)dt + aF(a))$, it follows that $n \cdot \mathbb{E}(\beta^{SPA}) = \frac{2na}{n-1} = \frac{2n}{n+1}(\int_a^1 tf(t)dt + aF(a)) = \frac{2n}{n+1}(1 - \int_a^1 F(t)dt)$. Monotonicity follows from the fact that the term in the parentheses is increasing in a. □

I end this section by considering, briefly, first-price auctions. As opposed to the second-price format, under the first-price format the linear-proportional model has no equilibrium, even when all type distributions (including that of the regular bidder) are uniform.

Proposition 7. *Let the regular bidder's type be uniform on* $[0, M]$*, where* $M \geq 1$*, and suppose that the auction-format is first-price. Then the separable-proportional model has no equilibrium.*

Note that Proposition 7 is a generalization of Proposition 2.

Proof. Suppose, w.l.o.g, that $c = 1$ and that $M > 1$ ($M = 1$ is covered in Proposition 2). By Theorem 3, if there is an equilibrium, then it is one with complete free riding. Consider then such an equilibrium, and suppose that player 1 is the team's representative who competes against the regular bidder in the associated 2-player auction. Griesmer et al. (1967) proved that the following inverse bid-functions constitute an equilibrium:

$$\theta_1(b) = \frac{2bM^2}{M^2 - b^2(1 - M^2)},$$

and

$$\theta_{n+1}(b) = \frac{2bM^2}{M^2 + b^2(1 - M^2)}.$$ [16]

Therefore, the maximal bid of either player, and, in particular, of player 2 of type $\theta_2 = 1$ is $\bar{b} = \frac{\sqrt{M}}{1+M}$. Thus, by "jumping in" and sending the report $\bar{b}$ this player can secure the payoff $1 - \bar{b} = \frac{1+M-\sqrt{M}}{1+M}$. Since in the 2-player auction the weaker player (the team's representative) wins with probability no greater than half, the payoff of type $\theta_2 = 1$ in the putative equilibrium is bounded above by $\frac{1}{2}$. Therefore, it is enough to show that $\frac{1+M-\sqrt{M}}{1+M} > \frac{1}{2}$, or $1 + M > 2\sqrt{M}$, or $1 + 2M + M^2 > 4M$. This is equivalent $(M - 1)^2 > 0$, which clearly holds. $\square$

[16] Equilibrium uniqueness follows, for example, from Lebrun (2006). A comprehensive analysis of the uniform case can be found in Kaplan and Zamir (2012).

6 Conclusion

I have considered an auction in which one of the bidders is a multi-member team. Under the second-price format, a symmetric equilibrium exists, but team bidding leads to inefficiency, because of within-team free riding. For a specific version of the model, the separable-proportional one, the equilibrium is unique, and is a generalization of the weak dominance equilibrium. A sharp contrast to classical auction theory is encountered when replacing the auction-format to first-price or all-pay. Under any of these two formats, the only possible equilibria are extremely asymmetric ones, specifically equilibria with complete free riding. Such equilibria may exist or fail to exist, depending on the environment's characteristics—the auction format, the type distributions, and the team's mechanism.

Whereas the main motivation for the paper is the prevalent phenomenon of team bidding, the paper can also be viewed from a more theoretical angle, as a robustness check for classical auction theory, as this theory corresponds to the imposition $n = 1$ in the present framework. From this perspective, the paper delivers both positive and negative results. On the negative side, the equivalence between first-, second-price, and all-pay auctions is not robust, since it breaks down when $n > 1$; on the positive side, the weak dominance equilibrium of the second-price auction is robust, in the sense that, when outside competition is uniform, the separable-proportional model's equilibrium generalizes it to the n-person-team case.

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Appendix A: Remarks about modeling

The definition of a "mechanism"

Throughout the paper, a mechanism is defined to be a pair, (A, s), of bid aggregation and cost sharing rules, and the players' reports to the mechanism are bids (real numbers). Whereas the word "natural" should be used with caution in the contexts of theoretical economic models, I believe that the above form is the natural way to model the phenomenon at hand. The reason outside competition, namely the presence of the regular bidder in the model.

First of all, the regular bidder is an explicitly-modeled entity that bids in the auction, and it is with respect to this entity that the team's operation needs to be described. It seems suitable to have the team members' description be given in the same language as the one describing the regular bidders; that is, all $n + 1$ individuals do the same thing—they all choose bids.

Secondly, the seemingly more conventional approach of setting up a revelation mechanism, to which the players report their types, is not suitable here. The reason is that such a mechanism has the players reported types as input and, as output, it produces allocation probabilities and expected payments. In a typical mechanism design problem, these probabilities and payments can be chosen independently of one another, but here they cannot—because of the result bidder. To take a simple example, it is impossible to allocate the item to the team with certainty, and at the same time charge zero payments from its members.

In general, the description of the feasible set of probabilities-and-transfers may be complicated. In the simplest case, where the format is second-price, the set of feasible probabilities is $Q \equiv \{G(b) : b \geq 0\}$ and the set of pairs of feasible probability-and-transfer is $\{(q, \mathbb{E}(\theta_{n+1} : \theta_{n+1} \leq G^{-1}(q))) : q \in Q\}$. If the auction format is not second-price, and hence the regular bidder need not report truthfully, it is not clear what the counterparts of the above sets are.

Finally, under the revelation game approach one needs to to specify the incentive constraints. These constraints depend on the way the burden of payments is split within the team, the description of which is missing from the description of the revelation mechanism, as given in the previous paragraphs. The explicit description of (A, s) takes care of all these shortcomings.

The functional form in the separable-proportional model

In the general formulation of the separable-proportional model, the bid aggregation rule is $A(b_1, \cdots, b_n) = \sum_{i=1}^{n} \psi(b_i)$ and player i's share of the payment (when not all reports are zero) is $\frac{\psi(b_i)}{\sum_{i=1}^{n} \psi(b_i)}$, where ψ is some strictly increasing function that satisfies $\psi(0) = 0$. There is no loss of generality in assuming that ψ is the identity, namely that $\psi(x) \equiv x$.

To see this, let M be a version of the model when the above function is some ψ, and let M^* be the same model, except that ψ is the identity. I will show that every equilibrium of M, $(\beta_1, \cdots, \beta_n)$, can be mapped into an equilibrium of M^* in which, conditional on every type-realization, the team's behavior is the same as in M; and, similarly, one can may every equilibrium of M^* to an equivalent on in M.

Let then $(\beta_1, \cdots, \beta_n)$ be an equilibrium of M. For each i, define $\gamma_i \equiv \psi \circ \beta_i$. I argue that $(\gamma_1, \cdots, \gamma_n)$ is an equilibrium of M. To see this, assume by contradiction that, given the behavior of the others fixed at γ_{-i}, player i of some type θ_i has a profitable deviation $\gamma_i(\theta_i) \mapsto x$. Given θ_{-i}, the deviation changes the team's bid by $\delta_i = x - \gamma_i(\theta_i) = x - \psi(\beta_i)$; and it changes player i's payment-share by $\frac{\delta_i}{\delta_i + \sum j \neq i \sum \gamma_i(\theta_i)} = \frac{\delta_i}{\delta_i + \sum_{j \neq i} \sum_{\psi}(\beta_i(\theta_i))} = \frac{x - \psi(\beta_i)}{x - \psi(\beta_i) + \sum_{j \neq i} \sum \psi(\beta_i(\theta_i))}$. But this means that the deviation $\beta_i(\theta_i) \mapsto \psi^{-1}(x)$ is profitable in M—in contradiction to equilibrium. In the same way, one can map equilibria of M^* to equilibria of M.

Therefore, all specifications of ψ-functions are equivalent, because all are equivalent to the case where ψ is the identity. Let then the bid aggregation and cost sharing rules be:

- $A(b_1, \cdots, b_n) = \sum_{i=1}^{n} b_i$; and
- $\max\{b_1, \cdots, b_n\} > 0 \Rightarrow s_i(b_1, \cdots, b_n) = \frac{b_i}{\sum_{j=1}^{n} b_j)}$ and $s_1(0, \cdots, 0) = \cdots = s_n(0, \cdots, 0) = \frac{1}{n}$.

Now, let $c > 0$. It is easy to see that $(\beta_1, \cdots, \beta_n)$ is an equilibrium of the above game if and only if $(\tilde{\beta}_1, \cdots, \tilde{\beta}_n)$ of the game in which A is replaced by $c \cdot A$, where $\tilde{\beta}_i \equiv \frac{1}{c} \cdot \beta_i$ for all $i = 1, \cdots, n$.

The role of outside competition in the second-price separable-proportional model

The requirement that M be sufficiently large is imposed in Theorem 5 in order to guarantee that each bid-increase by a team member increases the probability of winning, no matter the partners' types. When that is not the case, the objective faced by a team member changes significantly relative to the large-M case. To illustrate, consider $n = 2$ and suppose that $M > 0$ is small enough, so that if player 1 reports x and θ_2 exceeds some value—call this value $\theta(x)$—the team wins for sure. Then, the objective faced by type θ_1 of player 1, given that player 2 follows some monotonic reporting function $\tilde{\beta}$, is to maximize the following expression over x:

$$\frac{1}{M}\int_0^{\theta(x)} (x+\tilde{\beta}(t))\cdot(\theta_1 - \frac{x}{x+\tilde{\beta}(t)} \cdot \frac{x+\tilde{\beta}(t)}{2})f(t)dt + \int_{\theta(x)}^{1} [\theta_1 - \frac{xM}{2(x+\tilde{\beta}(t))}]f(t)dt. \quad (4)$$

The first-order condition associated with this objective is a differential equation that depends on F non-trivially, and for which I have no closed-form solution. Similar to the equilibrium bid function β^{SPA}, the function $\tilde{\beta}$ is identically zero on an interval that starts at the origin and ends at some cutoff, but to the right of this cutoff $\tilde{\beta}$ is non-linear. The function β^{SPA} stems from a more tractable optimization because the largeness of M implies that $\theta(x) \geq 1$ for every x a player may report in equilibrium, hence the second term in (4) disappears.

The requirement $M \geq 2n$ is sufficient to guarantee that, no matter how the equilibrium looks like, it necessarily has the property that any bid-increase by a team member increases the winning probability. Therefore, $M \geq 2n$ is sufficient for equilibrium uniqueness. However, for the equilibrium from Theorem 4 to exit (but without making a claim for uniqueness), it is sufficient and necessary that $n(1-a) \leq M$, where a is given by (3). This is because a sufficient and necessary condition for the existence of this equilibrium is that $A(\beta^{SPA}(1), \cdots, \beta^{SPA}(1)) \leq M$, and $A(\beta^{SPA}(1), \cdots, \beta^{SPA}(1)) = n(1-a)$.

For the parameter-range $M \in ((1-a)n, 2n)$, I believe that the equilibrium need not be unique. I end the section with a sketch (without a proof) why I believe this is so. Consider a team with $N = n+1$ members, and suppose that $M = n(1-a) + \epsilon$, where $\epsilon > 0$ is arbitrarily small and $a = a_n$ is the cutoff from the n-member version of Theorem 5's equilibrium. Now, consider the profile under which team members 1 through n play the n-version of Theorem 5's equilibrium, and member N reports zero independent of his valuation. Clearly, members 1 through n are playing best-responses. As for N, his payoff equals θ_N times the probability of winning. I argue that if n and $\mathbb{E}(\theta)$ are large, then this probability is approximately one. One can show that the team's winning probability is the probability is $\frac{2na}{(n-1)M}$.[17] By assumption on M, $\frac{2na}{(n-1)M} \sim \frac{2na}{(n-1)n(1-a)} = \frac{2a}{(n-1)(1-a)} \equiv g(a)$. This expression is increasing in a, and by part *3* of Proposition 3, if $\mathbb{E}(\theta) \sim 1$ then a is bounded from below by a number which is approximately $\frac{n-1}{n+1}$. Plugging $a = \frac{n-1}{n+1}$ in the expression for the approximated probability yields $g(\frac{n-1}{n+1}) = 1$.

[17]The calculation appears in an earlier version of this paper. The details are available from the author upon request.

Appendix B: Proofs

Proof of Proposition 1: Let team member 1 follow the function $\beta(\theta_1) = \frac{\theta_1^2}{2}$, let the regular bidder follow $\beta(\theta_{n+1}) = \frac{\theta_{n+1}^2}{2}$, and let all other team member report/bid zero independent of their type. In other words, team member 1 is the team's representative, and he plays, against the regular bidder, the equilibrium of the 2-bidder all-pay IPV auction with uniform types;[18] all other team member abstain. It remains to show that no member $i > 1$ has an incentive to submit a non-zero report.

Consider such a member. When his type is θ and he bids $x > 0$, the probability that the team wins, conditional on $\theta_1 = t$, is:

$$P(x,t) \equiv Pr(x + \frac{t^2}{2} \geq \frac{\theta_{n+1}^2}{2}) = Pr(\theta_{n+1} \leq \sqrt{2x + t^2}) = \sqrt{2x + t^2}.$$

The expected winning probability is therefore $P(x) \equiv \int_0^1 P(x,t)dt = \int_0^1 \sqrt{2x + t^2}dt$. Player i's expected utility from (the deviation to the report) x is:

$$V(\theta_1, x) \equiv P(x) \cdot \theta_1 - x.$$

To establish the equilibrium, it suffices to show that $\frac{dV(\theta_1,x)}{dx} < 0$ at all $x > 0$. For this purpose, it is enough to show that $P'(x) \leq 1$, which is indeed the case: $P'(x) = \sqrt{2x+1} - \sqrt{2x} \leq 1$ is equivalent to $\sqrt{2x+1} \leq 1 + \sqrt{2x}$, which clearly holds. $\square$

Proof of Proposition 2: Consider a designated free rider whose type is $\theta_i = 1$. By sending a report of $\frac{1}{2}$ in the first-price auction, player i obtains the utility $1 - \frac{1}{2} \cdot \int_0^1 \frac{1}{1+t}dt$, which exceeds what he is getting under complete free riding, namely $\frac{1}{2}$. $\square$

Lemma 2. *Fix an equilibrium of $\Gamma^{FPA}(A,s)$, and let $(\beta_1, \cdots, \beta_n)$ be the profile of team reporting functions in this equilibrium. Then for each i there exists an $a_i > 0$ such that β_i is identically zero on* $[0, a_i]$.

[18] See, e.g., Krishna 2002, p.31-32.

Proof. Fix an equilibrium as above. Following the notation of Lemma 1's proof, the objective that team member i maximizes is:

$$\mathbb{E}[H(A(x, \beta_{-i}(\theta_{-i}))) \cdot \theta_i - s_i(x, \beta_{-i}(\theta_{-i})) \cdot A(x, \beta_{-i}(\theta_{-i}))].$$

The derivative w.r.t x is $\mathbb{E}[\frac{d}{dx}H(A(x, \beta_{-i}(\theta_{-i})))\cdot\theta_i] - \mathbb{E}[s_i' A + s_i A']$, which is negative for small enough θ_i's. $\square$

Proof of Theorem 4: Assume by contradiction that a symmetric equilibrium exists. Fix such an equilibrium, and let $a_i > 0$ be the constant described in Lemma 1. Let $p \equiv \Pi_{i=1}^{n} F(a_i)$. Now consider the regular bidder; suppose that his type is $\theta \in (0, p)$. I argue that it is optimal for this type of the regular bidder to submit the bid zero. To see this, assume by contradiction that his optimal bid is $x > 0$. Then the following holds for every $\epsilon \in (0, x)$:

$$p\theta + \int_0^x \theta dZ - x \geq p\theta + \int_0^{x-\epsilon} \theta dZ - (x - \epsilon),$$

where Z is the cdf of the team's bid. Rearranging this inequality yields $\int_{x-\epsilon}^{x} \theta dZ \geq \epsilon$, which is impossible. Therefore both the team members and the regular bidder employ functions that are zero on an interval of low enough types, which, as explained in Theorem 3's proof, is impossible. $\square$

Proof of Proposition 3:

1. Equation (3) can be written as $\phi \equiv \int_a^1 tf(t)dt - a - (\frac{2a}{n-1}) + aF(a) = 0$. Let us view n as a continuous, rather than discrete, variable. By the implicit function theorem, he sign of the derivative $\frac{\partial a}{\partial n}$ is the same as that of $-[\frac{\partial \phi}{\partial n}]/[\frac{\partial \phi}{\partial a}]$. Since $\frac{\partial \phi}{\partial n} = -2a(\frac{1}{n-1})' > 0$ and $\frac{\partial \phi}{\partial a} = -1 - \frac{2}{n-1} + F(a) < 0$, the result follows.

2. Let $a^* \equiv \lim_{n\to\infty} a_n$. Equation (3) implies $a^*(1 - F(a^*)) = \int_{a^*}^1 tf(t)dt$, or $a^* = \mathbb{E}(\theta : \theta \geq a^*)$. Therefore $a^* = 1$.

3. This is clear for $n = 1$, so suppose $n \geq 2$. It follows from (3) that:

$$a \cdot (\frac{n+1}{n-1}) = \int_a^1 tf(t)dt + aF(a) =$$
$$= \int_a^1 tf(t)dt + a\int_0^a f(t)dt > \int_a^1 tf(t)dt + \int_0^a tf(t)dt = \mathbb{E}(\theta). \quad \square$$

Proof of Proposition 5: Let b denote the team's collective bid. I will calculate the payoff for player i conditional on a given b, and then integrate over b.

The above mentioned payoff is $Pr(u \leq b) \cdot (\theta - \frac{c\beta^{SPA}(\theta_i)}{b} \cdot \frac{b}{2})$, where u is the uniform random variable on $[0, M]$. This expression is equal to $\frac{b}{M} \cdot (\theta - \frac{\max\{\theta_i - a, 0\}}{2})$. Therefore, the expected payoff is:

$$\frac{1}{M} \cdot (\theta - \frac{\max\{\theta - a, 0\}}{2}) \cdot \int_0^M b \cdot Pr(c\beta^{SPA}(\theta_i) + c\sum_{j \neq i} \beta^{SPA}(\theta_j) = b)db =$$
$$= \frac{1}{M} \cdot (\theta_i - \frac{\max\{\theta_i - a, 0\}}{2}) \cdot \{\max\{\theta_i - a, 0\} + \mathbb{E}[c(n-1)\beta(\theta)]\} =$$
$$= \frac{1}{2M} \cdot [2\theta_i - \max\{\theta_i - a, 0\}] \cdot [\max\{\theta_i - a, 0\} + 2a]. \quad \square$$

Lemma 3. *Let $(\beta_1, \cdots, \beta_n)$ be an equilibrium of the linear-proportional model. Then each β_i is weakly increasing.*

Proof. Let p (resp. p') and t (resp. t') be the wining probability and expected payment of types θ_i (resp. θ_i') when they send their equilibrium bids, where $\theta_i > \theta_i'$. If $b_i(\theta_i') > b_i(\theta_i)$, then $p' > p$ and $t' > t$. Incentive compatibility implies:

$$p\theta_i - t \geq p'\theta_i - t',$$

and

$$p'\theta_i' - t' \geq p\theta_i' - t.$$

Rearranging these inequalities yields $t' - t \geq \theta_i(p' - p)$ and $t - t' \geq \theta_i'(p - p')$. Summing the rearranged inequalities yields $0 \geq (p' - p) \cdot (\theta_i - \theta_i') > 0$—a contradiction. □

Lemma 4. *Suppose that $M \geq 2n$. If $(\beta_1, \cdots, \beta_n)$ is an equilibrium of the separable-proportional model, then $\beta_1 = \cdots = \beta_n = \beta$, where β satisfies* (1) – (3).

Proof. Let $(\beta_1, \cdots, \beta_n)$ be an equilibrium. Let I be the set of players who follow a non-null bidding strategy. That is, $I \equiv \{i : \beta_i(\theta_i) > 0 \text{ for some } \theta_i\}$. Obviously, $I \neq \emptyset$. By Lemma 2, for each $i \in I$ the function β_i is positive-valued on $(a_i, 1]$, where $a_i \equiv \inf\{\theta_i : \beta_i(\theta_i) \geq 0\}$.

I argue that $|I| > 1$. To see this, assume by contradiction that I is a singleton, and, w.l.o.g, that $1 \notin I$. When type θ_1 reports r, his payoff, conditional on $(\theta_2, \cdots, \theta_n) = (t_2, \cdots, t_n)$, is[19]:

$$\frac{(r + \sum_{i\in I, i\neq 1} \beta_i(t_i))}{M} \cdot (\theta_1 - \frac{r}{r + \sum_{i\in I, i\neq 1} \beta_i(t_i)} \cdot \frac{(r + \sum_{i\in I, i\neq 1} \beta_i(t_i))}{2}) =$$
$$= \frac{1}{M} \cdot [(r + \sum_{i\in I, i\neq 1} \beta_i(t_i))\theta_1 - \frac{r^2}{2} - \frac{r\sum_{i\neq 1} \beta_i(t_i)}{2}].$$

Clearly, the objective is independent of M, so to ease the notation I assume, in what follows, that $M = 1$. Therefore, when type θ_1 reports r, his expected payoff is:

$$(r + \sum_{i\in I, i\neq 1} \mathbb{E}(\beta_i))\theta_1 - \frac{r^2}{2} - \frac{r\sum_{i\in I, i\neq 1} \mathbb{E}(\beta_i)}{2}. \tag{5}$$

The derivative of this expression w.r.t r is:

$$\theta_1 - r - \frac{\sum_{i\in I, i\neq 1} \mathbb{E}(\beta_i)}{2}. \tag{6}$$

If $I = \{j^*\}$ for some $j^* \neq 1$, then this j^* plays an ordinary SPA against the regular bidder, and therefore sends the report θ_{j^*}. Therefore, at $\theta_1 = 1$ and $r = 0$ the above

[19] This payoff formula applies regardless of the cardinality of I.

derivative is equal to $1 - \frac{\mathbb{E}(\theta)}{2} > 0$. Therefore, $|I| > 0$.

Suppose that $1 \in I$. For $\theta_1 > a_1$, the first-order condition is:

$$\theta_1 - r - \frac{\sum_{i \in I, i \neq 1} \mathbb{E}(\beta_i)}{2} = 0.$$

The condition is satisfied at $r = \beta_1(\theta_1)$, hence:

$$\beta_1(\theta_1) = \theta_1 - \frac{1}{2} \cdot \sum_{i \in I, i \neq 1} \mathbb{E}(\beta_i). \tag{7}$$

The analogous formula holds for any other $i \in I$. Therefore, the following holds for each $i \in I$:

$$\beta_i(\theta_i) = \begin{cases} 0 & \text{if } \theta_i < a_i \\ \theta_1 - \frac{1}{2} \cdot \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j) & \text{if } \theta_1 > a_1 \end{cases}$$

Consider type a_i. This type is indifferent between bidding zero and bidding $\beta_i(a_i)$. This type's expected payoff from bidding zero is a_i times the probability of winning: $a_i \cdot \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)$. The expected utility from bidding $\beta_i(a_i)$ is $(\beta_i(a_i) + \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)) \cdot (a_i - \frac{\beta_i(a_i)}{2})$. The indifference condition is:

$$a_i \cdot \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j) = (\beta_i(a_i) + \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)) \cdot (a_i - \frac{\beta_i(a_i)}{2}),$$

or:

$$\frac{\beta_i(a_i)}{2} \cdot (\beta_i(a_i) + \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)) = a_i \beta_i(a_i).$$

I argue that $\beta_i(a_i) = 0$. To see this, note that if $\beta_i(a_i) > 0$ then the above equation implies $a_i - \frac{\beta_i(a_i)}{2} = \frac{\sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)}{2}$, which implies $\beta_i(a_i) = 2\beta_i(a_i)$, hence $\beta_i(a_i) = 0$.

It therefore follows that the following holds for all $i \in I$:

$$\beta_i(\theta_i) = \begin{cases} 0 & \text{if } \theta_i < a_i \\ \theta_i - a_i & \text{if } \theta_i \geq a_i, \end{cases}$$

where $a_i = \frac{1}{2} \cdot \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)$.

Note that $\mathbb{E}(\beta_i) = \int_{a_i}^{1} (t - a_i) f(t) dt = \int_{a_i}^{1} t f(t) dt - a_i(1 - F(a_i))$. Therefore:

$$\sum_{j \in I, j \neq 1} \mathbb{E}(\beta_j) = \sum_{j \in I, j \neq 1} \{ \int_{a_j}^{1} t f(t) dt - a_j (1 - F(a_j)) \} = 2a_1.$$

Since $|I| > 1$, assume, w.l.o.g, that $2 \in I$. Then:

$$\sum_{j \in I, j \neq 2} \{ \int_{a_j}^{1} t f(t) dt - a_j (1 - F(a_j)) \} = 2a_2.$$

I argue that $a_1 = a_2$. To see this, assume by contradiction, w.l.o.g, that $a_1 > a_2$. Therefore, the above equations imply:

$$\int_{a_2}^{a_1} t f(t) dt - a_2(1 - F(a_2)) + a_1(1 - F(a_1)) = 2(a_1 - a_2). \tag{8}$$

At $a_1 = a_2$ both sides are equal to zero; the derivative of the LHS w.r.t. a_1 is $1 - F(a_1) \leq 1$ and that of the RHS is 2, hence $a_1 = a_2$ is the unique solution, in contradiction to $a_1 > a_2$. It follows that there exists an a such that $a_i = a$ for all $i \in I$, and therefore all bid functions coincide; the equilibrium bid function, β, is given by:

$$\beta(\theta) = \begin{cases} 0 & \text{if } \theta < a \\ \theta - a & \text{if } \theta \geq a, \end{cases}$$

where $a = \frac{(n-1)}{2} \cdot \mathbb{E}(\beta)$, or $\mathbb{E}(\beta) = \frac{2a}{(n-1)}$. This condition can be written as:

$$\mathbb{E}(\beta) = \int_{a}^{1} (t - a) f(t) dt = \int_{a}^{1} t f(t) dt - a(1 - F(a)) = \frac{2a}{(n-1)},$$

or:

$$\int_{a}^{1} t f(t) dt = a \cdot (\frac{n+1}{n-1}) - aF(a).$$

Note that $a = 0$ the LHS exceeds the RHS, and at $a = 1$ the converse holds; therefore, a solution, a, exists. To see that it is unique, assume by contradiction that there exists a $b \neq a$ such that:

$$\int_b^1 tf(t)dt = b \cdot (\frac{n+1}{n-1}) - bF(b).$$

Suppose, w.l.o.g, that $b > a$. The above equations imply $\int_a^b tf(t)dt = (a-b) \cdot (\frac{n+1}{n-1}) - aF(a) + bF(b)$. At $b = a$ both sides are equal to zero, the derivative of the LHS w.r.t b is $bf(b)$ and that of the RHS is $-(\frac{n+1}{n-1}) + F(b) + bf(b) < bf(b)$, hence the solution is unique.

Finally, it remains to show that $|I| = n$. To see this, assume by contradiction that $|I| = k < n$ and let i be such that $i \notin I$. For $\theta_i = 1$ and $r = 0$, the (counterpart of the) derivative (6) is $1 - \frac{k\mathbb{E}(\beta)}{2} = 1 - \frac{k\frac{2a}{(n-1)}}{2} = 1 - \frac{k\frac{2a}{(n-1)}}{2} \geq 1 - \frac{k\frac{2a}{(n-1)}}{2} \geq 1 - a > 0$. $\square$

Proof of Theorem 5: By Lemma 4, if $(\beta_1, \cdots, \beta_n)$ is an equilibrium, then $\beta_i = \beta^{SPA}$ for all i and (1) and (3) hold. Conversely, consider β such that (1) and (3) hold. The arguments from Lemma 2's proof establish that this is an equilibrium; for types $\theta > x$ the FOC is satisfied at $r = \beta^{SPA}(\theta)$ and for types $\theta < x$ not participating is optimal, because the derivative of their objective function w.r.t. r is negative. $\square$

References

[1] Asker, J., Baccara, M. and Lee, S., 2021. Patent auctions and bidding coalitions: structuring the sale of club goods. The RAND Journal of Economics, 52(3), pp.662-690.

[2] Athey, S. and Bagwell, K., 2001. Optimal collusion with private information. RAND Journal of Economics, pp.428-465.

[3] Barbieri, S. and Malueg, D.A., 2010. Threshold uncertainty in the private-information subscription game. Journal of Public Economics, 94(11-12), pp.848-861.

[4] Blumrosen, L., and Solan, E., 2022. Selling Spectrum in the Presence of Shared Networks: The Case of the Israeli 5G Auction. Working Paper.

[5] Charness, G. and Sutter, M., 2012. Groups make better self-interested decisions. Journal of Economic Perspectives, 26(3), pp.157-76.

[6] Cox, J.C. and Hayne, S.C., 2006. Barking up the right tree: Are small groups rational agents?. Experimental Economics, 9(3), pp.209-222.

[7] DeBrock, L.M. and Smith, J.L., 1983. Joint bidding, information pooling, and the performance of petroleum lease auctions. The Bell Journal of Economics, pp.395-404.

[8] Duggan, J., 2001. Non-cooperative games among groups. Working Paper.

[9] Eliaz, K. and Wu, Q., 2018. A simple model of competition between teams. Journal of Economic Theory, 176, pp.372-392.

[10] Griesmer, J.H., Levitan, R.E. and Shubik, M., 2000. (1967),Toward a Study of Bidding Processes Part IV-Games with Unknown Costs. INTERNATIONAL LIBRARY OF CRITICAL WRITINGS IN ECONOMICS, 113, pp.108-125.

[11] Haghpanah, N., Kuvalekar, A. and Lipnowski, E., 2021, July. Selling to a Group. In Proceedings of the 22nd ACM Conference on Economics and Computation (pp. 583-584).

[12] Holmstrom, B., 1982. Moral hazard in teams. The Bell Journal of Economics, pp.324-340.

[13] Kaplan, T.R. and Zamir, S., 2012. Asymmetric first-price auctions with uniform distributions: analytic solutions to the general case. Economic Theory, 50(2), pp.269-302.

[14] Kim, J., Palfrey, T.R. and Zeidel, J.R., 2021. Games Played by Teams of Players. American Economic Journal: Microeconomics, forthcoming.

[15] Kobayashi, K. and Konishi, H., 2021. Effort complementarity and sharing rules in group contests. Social Choice and Welfare, 56(2), pp.205-221.

[16] Krishna, V., 2002. Auction theory. Academic press.

[17] Kugler, T., Kausel, E.E. and Kocher, M.G., 2012. Are groups more rational than individuals? A review of interactive decision making in groups. Wiley Interdisciplinary Reviews: Cognitive Science, 3(4), pp.471-482.

[18] Lebrun, B., 2006. Uniqueness of the equilibrium in first-price auctions. Games and Economic Behavior, 55(1), pp.131-151.

[19] McAfee, R.P. and McMillan, J., 1992. Bidding rings. The American Economic Review, pp.579-599.

[20] Olson, M., 1965. The logic of collective action: public goods and the theory of groups.

[21] Rachmilevitch, S., 2013. Endogenous bid rotation in repeated auctions. Journal of Economic Theory, 148(4), pp.1714-1725.

[22] Reny, P.J., 2011. On the Existence of Monotone PureStrategy Equilibria in Bayesian Games. Econometrica, 79(2), pp.499-553.

[23] Skrzypacz, A. and Hopenhayn, H., 2004. Tacit collusion in repeated auctions. Journal of Economic Theory, 114(1), pp.153-169.