Dynamic One-Sided Matching*

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Abstract

We introduce a solution concept for dynamic one-sided matching models, called the Dynamic Core. The Dynamic Core is defined for a general class of dynamic markets where agents and objects arrive over time, and objects can be privately or collectively owned. We prove that the Dynamic Core is not empty and discuss the relation with the static notions of Core and Strong Core. We show that the output of a dynamic extension of the Top Trading Cycle algorithm, the Intertemporal Top Trading Cycle (ITTC), is in the Dynamic Core and the mechanism induced by the ITTC is efficient and group strategy-proof. In private economies the output of the ITTC can be supported as a dynamic competitive equilibrium.

Keywords: Dynamic Matching, One-sided Market, Stability, Dynamic Core, Intertemporal Top Trading Cycle.

JEL Codes: C71,C78,D47

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1 Introduction

In this paper we study the allocation and exchange of indivisible objects without monetary transfers in a dynamic model in which agents and objects arrive over time.

Our dynamic model entails several nuances that are absent in a static setting. First, agents face inter-temporal trade-offs. Second, consumption and exchange are different choices and, while consumption is irreversible, agents may choose to exchange repeatedly. Third, agents make conjectures about how the market may evolve in the future.

Our primary interest is to propose a notion of stability, the *Dynamic Core*, that is coherent with the inherently dynamic structure of the problem. The Dynamic Core is defined for a general model of one-sided matching in which objects enter the market owned by a single agent or collectively owned. Hence, our model can be interpreted as a dynamic version of the housing market with existing tenants (Abdulkadiroğlu and Sönmez, 1999) encompassing as special cases a dynamic version of the housing market (Shapley and Scarf, 1974) and of the house allocation model (Hylland and Zeckhauser, 1979).

The Dynamic Core builds over the notion of period-t blocking. An allocation in this dynamic economy specifies which objects agents exchange in each period and when they consume. An allocation is period-t blocked if there is no effective coalition of agents who strictly prefer the alternative of exchanging their endowments among themselves over the allocation. Agents in the blocking coalition can (i) either consume their new endowments or (ii) wait in the market for a better trade, with the prediction that only allocations that cannot be period-t' blocked at any $t' \geqslant t$ may realize in the future. The Dynamic Core is the set of allocations that cannot be period-t blocked at any period.

The Dynamic Core has a recursive structure that exhibits three properties. The Dynamic Core incorporates a *strong perfection* requirement[1] which ensures

[1] In dynamic matching, Kotowski (2019) attributes a "perfection requirement" to those stability notions which assess the "plausible" continuation within the solution concept. Examples are Corbae, Temzelides and Wright (2003); Kotowski (2019); Liu (2020); Doval (2022).

the credibility of the period-t blockings. Agents in the blocking coalition restrict the set of future allocations that are taken into consideration to those that will be unblocked. A major feature of our perfection requirement is that agents form rational conjectures about the set of future allocations that start at the period in which the blocking takes place. In SECTION 3 we argue that this is a main novelty with respect to previous models of dynamic exchange which exhibit a weaker perfection requirement.

Next, the Dynamic Core is a *farsighted*[2] solution: Agents can trade an object they own for one they like less in order to trade again in the future if this chain of exchanges entails a welfare improvement. Finally, the Dynamic Core is *conservative*[3]: only alternatives which give an improvement with certainty are eligible for the blocking coalition.

Our main result establishes that the Dynamic Core is not empty. We show that the output of a dynamic extension of the Top Trading Cycle algorithm, the Intertemporal Top Trading Cycle (ITTC), is in the Dynamic Core[4].

We then turn the attention to two polar cases in which we compare the Dynamic Core with standard Core notions. In private economies in which every object is owned by a single agent, the Strong Core is essentially unique and has a non-empty intersection with the Dynamic Core, which is in turn unrelated with the Core. In particular, we show that in a dynamic setting the Strong Core is too restrictive while the Core is too permissive. By contrast, the Dynamic Core offers consistent predictions. In public economies in which every object is collectively owned, we show that the Dynamic Core and the Strong Core are unrelated.

We also discuss the notion of Dynamic Strong Core by introducing a notion of weak period-t blocking. We prove that the Dynamic Strong Core is a subset,

[2]This assumption is pervasive in the coalition theory literature. Prominent examples are in Ray and Vohra (2015, 2019).

[3]Many economic models tend to take conservative views. This idea is manifested already in the seminal work of von Neumann and Morgenstern (1944) where the value of the coalition can be determined by playing a minimax game against the complementary coalition; Another classic example is Aumann and Peleg (1960) which explicitly assume that an acting coalition must expect the worst from the complementary coalition.

[4]The recursion used to define the Dynamic Core is different from the usual recursion based on the natural numbers, thus the proof of the non-emptiness of the Dynamic Core is based on a strong double induction argument, see SECTION 3.

possibly empty, of the Dynamic Core. In fact, the Dynamic Strong Core can be empty even in private economies with strict preferences. In public economies the Dynamic Strong Core is not empty and is unrelated to the Strong Core.

Finally, we show that the mechanism induced by the ITTC algorithm is Pareto efficient and (group) strategy-proof and, in private economies, its output can be supported as a dynamic competitive equilibrium.

Related Works

The literature on dynamic matching is rapidly growing in the last years. Most of the existing contributions in one-sided markets focus on the design of dynamic efficient mechanisms, rather than on their stability. Prominent examples are Gershkov and Moldovanou (2010); Unver (2010); Kurino (2014); Bloch and Cantala (2017); Andersson, Ehlers and Martinello (2018); Baccara and Yariv (2020); Leshno (2022). Notably, Abdulkadiroğlu, and Loertscher (2007) study a dynamic housing allocation problem where a mass of homogeneous objects must be allocated to a mass of agents over two periods. Authors design a dynamic mechanism where agents decide whenever to apply for the goods or to opt out. Those who apply for the goods gain priority in the first period while the other agents gain priority in the second period. They characterize the set of equilibria of this game.

To the best of our knowledge, we are the first to study a solution concept for a general class of dynamic one-sided matching with arrivals.

More prolific in this sense is a recent stream of literature in dynamic two-sided matching. Our solution concept, the Dynamic Core, has been inspired by the previous contributions of Kadam and Kotowski (2018); Kotowski (2019); Liu (2020) and Doval (2022) which provide a recursive notion of dynamic stability for two-sided markets where the "plausible" continuations of the market are addressed within the stability notion. This translates into a perfection requirement, a feature shared by our work. However, there are substantial differences. One above all, as argued in Section 3, our solution concept relies on stronger rationality assumptions which ensures the credibility of the blocking. In particular, we assume that agents in a blocking coalitions form rational expectation about how the market may evolve in the future already starting from the same period they block. This differs from previous models where agents form rational expectation only starting from the subsequent period they block. Moreover, differently from Liu (2020) and Doval (2022), we allow agents to match repeatedly and differently from Kadam and Kotowski (2018) and Kotowski (2019), that define agents' preferences over sequences of matching, we assume that agents only care about their final consumption. The latter feature makes our work widely related to the literature on farsighted one-sided matching (Kamijo and Kawasaki, 2009; Kawasaki, 2010; Klaus, Klijn and Walzl, 2010; Atay, Mauleon and Vannetelbosch, 2022) where agents decide to match or not, having in mind only the ultimate consequences of their choices.

Pereyra (2013) and Kennes, Daniel Monte and Tumennasan (2014) propose solution concepts for dynamic school choice models in which new cohorts of agents enter every period and study mechanisms that return fair and stable outcomes. In their settings agents do not face intertemporal tradeoffs and cannot exchange repeatedly, and therefore our concept of stability is distinct to ours.

Finally, our work contributes to the analysis of Core notions in dynamic exchange economies, a field that has always attracted the interest of economists. Gale (1978) introduces a notion of the Core for finite horizon Arrow-Debreu model with dated goods, and Becker and Chakrabarti (1995) propose a notion of the Core for an infinite horizon capital accumulation model. More recently, Kranich, Perea, Peters (2005); Habis and Herings (2010); Predtetchinski, Herings, Perea (2006) study Cores of sequences of characteristic function games. We complement the literature by providing a notion of the Core in dynamic markets without side payments.

Synopsis

The structure of the paper is as follows. Section 2 illustrates the general framework of dynamic one-sided matching. Section 3 introduces the Dynamic

Core, proves that it is not empty, and discusses how it differs from other existing notions of stability in dynamic models. A rifenement of the Dynamic Core, namely the Dynamic Strong Core, is discussed in SECTION 4. SECTION 5 discusses the relation between the Dynamic Core and the (strong) Core in private and public economies. SECTION 6 describes the Intertemporal Top Trading Cycle and shows that satisfies desirable normative properties. SECTION 7 concludes. All proofs are relegated to the APPENDIX A.

2 The Model

We consider a dynamic one-sided market with indivisible objects where agents and objects arrive over time. There are $n \in \mathbb{N}_+$ **periods**. For any period $t < n$, the set of **agents** and the set of **objects** entering at t are labelled $A_t \equiv \{a_1, ..., a_m\}$ and $H_t \equiv \{h_1, ..., h_\ell\}$, respectively. A **coalition** S is any non-empty subset of agents. We write $A_{\leqslant t} \equiv \bigcup_{k=1}^{t} A_k$ and $H_{\leqslant t} \equiv \bigcup_{k=1}^{t} H_k$, to denote all the agents and objects arrived in the market up to period t. Thus, $A \equiv A_{\leqslant n}$ and $H \equiv H_{\leqslant n}$ are the entire sets of agents and objects[5]. Abusing notation, we write $A_{>t} \equiv A \backslash A_{\leqslant t}$ and $H_{>t} \equiv H \backslash H_{\leqslant t}$ to denote the agents and the objects entering in the market from period $t + 1$ onward. We say that a pair (h, t) is **feasible for the agent** a if the agent a and the object h are in the market in period t. Each agent $a \in A$ has a **strict**[6] **preference relation** $\succ_a$over the feasible pairs (h, t) and an outside option $\emptyset$, which is the last choice for each agent. For all $a \in A$, we write $(h, t) \succ_a (h', t')$ if agent a strictly prefers object h in period t than object h' in period t'. We assume that agents' preferences exhibit **impatience**[7]: for all $a \in A$ and $h \in H$, $(h, t) \succ_a (h, t') \Longleftrightarrow t' > t$.

Objects can be owned either by a single agent or by the entire society. An object owned by a single agent is called **private** and **common** otherwise. An **ownership structure** establishes who are the owner(s) of an object when it

[5]We exclude the trivial cases $A = \varnothing \bigvee H = \varnothing$

[6]As usual, we denote by $\succeq$ the induced weak preference relation and we write $(h, t) \succsim_a (h', t') \Longleftrightarrow (h, t) \succ_a (h', t') \bigwedge (h, t) = (h', t')$.

[7]This assumption is not necessary and can be relaxed since in our model there is no cost to retain an object.

enters the market.

DEFINITION **1.** An **ownership structure** is a map $\omega : H \longrightarrow A \cup \{A\}$ satisfying two properties:

Single Private Object: for all $h, h' \in H$, $\omega(h) = \omega(h') \Longrightarrow \omega(h) = A$

Synchronous Entry: for all $t \leqslant n$, if $h \in H_t$ and $\omega(h) \neq A$ then $\omega(h) \in A_t$.

The first condition says that an agent can own at most a single private object. The second condition ensures that a private object and its initial owner enter in the economy at the same period. Note that this last condition implies that the pair (h, t) is always feasible for the owner of the private object h.

An **economy** is, thus, a tuple

$$\mathcal{E} = \left\langle (A_t, H_t)_{t=1}^n, \omega \right\rangle$$

consisting of a collection of agents and objects entering over time together with an ownership structure. An economy $\mathcal{E} = \langle (A_t, H_t,)_{t=1}^n, \omega \rangle$ is **private** if for every $h \in H$, $\omega(h) \neq A$ and **public** if for every $h \in H$, $\omega(h) = A$. The special cases of private and public economies are the dynamic extensions of the housing market (Shapley and Scarf, 1974) and the house allocation problem (Hylland and Zeckhauser, 1979) respectively.

Objects can be exchanged in more than one periods, but their consumption is irreversible and can be made only by an individual agent at a given period. To keep track of the exchanges made at any period t we introduce the notion of **period-t exchange**, a map specifying the owner of each object after that all the exchanges in period t took place.

DEFINITION **2.** A **period-t exchange** is a map $\sigma_t : H_{\leqslant t} \longrightarrow A_{\leqslant t} \cup \{A\}$ such that

- for all $h, h' \in H$, $\sigma_t(h) = \sigma_t(h') \Longrightarrow \sigma_t(h) = A$

The condition restates the *Single Private Object* property, that is, at the end of any period t, each agent owns at most one private object. Hence, $\sigma_t(h)$ represents either the agent or the society owning the object h in period t. Also, for convention, we write $\sigma_0 = \omega$ and for all $h \in H_{>t}$, $\sigma_t(h) = \omega(h)$. Let Σ_t be the set of all period-t exchanges.

We now define the consumption choice of an agent. If agent a is consuming object h in period t we write $\mu_t(a) = h$. If agent a does not consume any object in period t then we write $\mu_t(a) = h_0$.

Definition **3.** A **period-t consumption choice** is a map $\mu_t : A_{\leqslant t} \longrightarrow H_{\leqslant t} \cup \{h_0\}$ such that $\mu_t(a) = h_0$ for all $a \in A_{>t}$.

We also write $\mu_t(a) = h_0$ whether $t = 0$, or $a \in A_{>t}$. Let M_t be the set of all period-t consumption choices.

Next we define the notion of allocation for an economy.

Definition **4** (Allocation)**.** Given an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^{n}, \omega \rangle$, an **allocation** is a pair (σ, μ) where $\sigma \equiv (\sigma_1, ..., \sigma_n)$ is a list of period-t exchanges and $\mu \equiv (\mu_1, ..., \mu_n)$ a list of period-t consumption choices such that for all $a \in A_{\leqslant t}$, if $\mu_t(a) \neq h_0$ then the following conditions hold:

Consumption Rivalry: $\sigma_t \circ \mu_t(a) = a$;

Consumption Irreversibility: for all $t' > t$, $\sigma_{t'} \circ \mu_t(a) = a$ and $\mu_{t'}(a) = \mu_t(a)$.

The first condition requires that agents can only consume the private object they own, thus objects, private and common, are rivalrous. The second condition establishes that consumption is irreversible.

Given the allocation (σ, μ), we write $\mu(a)$ the object that agent a consumes and $t(a, \mu)$ the period when it is consumed by a.

Remark **1.** Note that, the property of *Consumption Irreversibility* implies that: $\mu(a) = \mu_n(a)$; and that for all $a, a' \in A$ with $a \neq a'$, $\mu(a) = \mu(a') \Longrightarrow \mu(a) = h_0$.

We close the model by extending each preference $\succ_a$ to a preference over the set of allocations in the following way:

$$(\sigma, \mu) \succ_a (\tau, \nu) \Longleftrightarrow (\mu(a), t(a, \mu)) \succ_a (\nu(a), t(a, \nu)).$$

3 The Dynamic Core

Several aspects that are well-defined in a static setting, must be clarified when proposing a dynamic framework: (a) which coalitions of agents can block at a given period? (b) which objects a blocking coalition can redistribute among its members? and (c) which are the final consequences of blocking an allocation?

First, to answer question (a), it seems natural to require that only agents who are already present in the market in period t, and did not consume yet, can form a blocking coalition at this period. Therefore, at any period t a blocking coalition S must belong to $\mathcal{A}_t \equiv \{a \in A_{\leqslant t} | \mu_{t-1}(a) = h_0\}$.

Second, to answer question (b), we have to specify which objects are available for a blocking coalition $S \subseteq \mathcal{A}_t$ at any period t. Let $\mathcal{H}_t \equiv \{h \in H_{\leqslant t} | h \neq \mu_{t-1}(a),\ \forall a \in A_{\leqslant t}\}$ denote the set of objects that are in the market in period t and have not been previously consumed. The following notion of period-t endowment extends the usual notion of endowment to a dynamic setting, specifying the objects any blocking coalition can exchange among its members.

Definition 5. A **period-t endowment** $\omega_t : 2^{\mathcal{A}_t} \longrightarrow 2^{\mathcal{H}_t}$ is a map such that for all $S \in 2^{A_{\leqslant t}} \backslash \varnothing$,

$$\omega_t(S) \equiv \begin{cases} \mathcal{H}_t & \text{if } S = \mathcal{A}_t, \\ \{h \in \mathcal{H}_t | \sigma_{t-1}(h) \in S\} & \text{if } S \subsetneq \mathcal{A}_t. \end{cases}$$

The endowment of a coalition S in period t equals the set of all available objects at t if S contains all the available agents at t, otherwise it contains the objects obtained by the elements of S at previous period.

Then, at any period t, a coalition $S \subseteq \mathcal{A}_t$ can block an allocation (σ, μ) by proposing a period-t exchange τ_t and a period-t consumption choice ν_t. We impose that the set of admissible period-t exchanges for S consists of

$$\Sigma_t(S) \equiv \left\{ \tau_t \in \Sigma_t \;\middle|\; \begin{array}{ll} \tau_t(h) \in S \vee \tau_t(h) = A, & \forall h \in \omega_t(S), \\ \tau_t(h) = \sigma_{t-1}(h), & \forall h \in H_{\leqslant t} \backslash \omega_t(S). \end{array} \right\}$$

The first constraint requires that the blocking coalition can only redistribute their endowment among themselves; the second constraint requires that the blocking coalition cannot interfere with the exchange of the objects that do not belong to them. We also impose that the set of admissible period-t consumption choice for S is restricted to

$$\mathrm{M}_t(S) \equiv \{\nu_t \in \mathrm{M}_t | \nu_t(a) = \mu_{t-1}(a),\ \forall a \in A_{\leqslant t} \backslash S\}.$$

which states that agents outside the coalition who haven't consumed up to $t-1$ can freely choose their the consumption choices in period t.

Finally, we answer to question (c). In our setting, agents can form a blocking coalition without necessarily consuming the objects that they get at the period they block, in order to further exchange them in the future. It follows that blocking agents should form conjectures about which will be the final consequences of their block. This will depend not only on how contemporary agents outside the coalition will react, but also on the behavior of those agents who enter in the market in the following periods. The notion of continuation economy frames this idea. Let $S^w \equiv \{a \in s | \nu_t(a) = h_0\}$ be the agents in the blocking coalition S who stay in the market without consuming. A continuation economy $\mathcal{E}_{\geqslant t}(S, \tau_t, \nu_t)$ consists of

$$\left\langle (\mathcal{A}_t \backslash S, \mathcal{H}_t \backslash \omega_t(S)), (A_{t+1} \cup S^w, H_{t+1} \cup \omega_t(S) \backslash \nu_t(S)), ..., (A_n, H_n), \tau_t|_{H \backslash \nu_t(A_{\leqslant t})}) \right\rangle.$$

Agents in the period-t blocking coalition S form expectations on which allocations defined over the continuation economy will realize. The continuation economy starts in period t with the ownership structure $\tau_t|_{H_{\leqslant t} \backslash \nu_t(A_{\leqslant t})}$.

More precisely, agent in the blocking coalition only take under consideration the allocations in the continuation economy that are not blocked themselves. Moreover, every agent in the blocking coalition S should strictly improve. This defines our notion of **period-t blocking**.

Definition **6** (Period-t Blocking)**.** Let (σ, μ) be an allocation of the economy $\mathcal{E} = \langle (A_t, H_t,)_{t=1}^n, \omega \rangle$. A coalition $S \subseteq \mathcal{A}_t$ can **period-t block** (σ, μ) if there exists a pair $(\nu_t, \tau_t) \in \Sigma_t(S) \times \mathrm{M}_t(S)$ such that:

- $(\nu_t(a), t) \succ_a (\sigma, \mu)$ for all $a \in S \backslash S^w$
- $(\upsilon, \xi) \succ_a (\sigma, \mu)$ for all $a \in S^w$ and all allocations (υ, ξ) of the economy $\mathcal{E}_{\geqslant t}$ that cannot be period-t' blocked at any $t' \geqslant t$.

The **Dynamic Core** of an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ is the set of allocations that cannot be period-t blocked in any period, by any coalition.[8]

The following theorem states our main result.

THEOREM **1.** *For any economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ the Dynamic Core is not empty.*

REMARK **2.** The proof of THEOREM 1 is constructive and relies on an algorithm that is a modification of the Top Trading Cycle in our dynamic economy, and is described in SECTION 6. On a more technical level, the strategy of the proof of THEOREM 1 is based on a strong form of double induction over the number of periods and number of agents entering in the first period. The strong double induction employed in the proof is discussed in the APPENDIX A.

REMARK **3.** The Dynamic Core might contain more than one allocation. Example 2 in SECTION 4 makes this point.

EXAMPLE **1.** We want to illustrate our solution concept with an example. Consider a 2-period private economy with four agents, $a_1, ..., a_4$, and four objects, $h_1, ..., h_4$. Without loss of generality, assume that object h_i belongs to agent a_i. Agents a_1, a_2, a_3 enter in period 1 while agents a_4 enters in period 2. Agents' preferences are depicted in FIGURE 1. If $(h_\ell, 1)$ appears before than $(h_j, 2)$ in agent i's ranking, then she prefers being matched with object h_ℓ in period 1 than object h_j in period 2.

[8] The definition of the solution concept embeds a recursion. The recursion is well defined since, in the definition of period-t blocking, at any t the continuation economy $\mathcal{E}_{\geqslant t}$ is "smaller" than the economy $\mathcal{E}$. Indeed, either $\mathcal{E}_{\geqslant t}$ has less number of periods than $\mathcal{E}$ if the blocking occurs at $t > 1$, or it has the same number of periods than $\mathcal{E}$ but it has less agents in the first period because the agents in S enter in the following period. Note that the recursion employed here builds over number of periods and number of agents; This makes it different from the usual notion of recursion which is defined over the set of natural numbers.

$t=1$			$t=2$
a_1	a_2	a_3	a_4
$h_2, 1$	$h_3, 1$	$h_4, 2$	$h_2, 2$
$h_4, 2$	$h_1, 1$	$h_1, 1$	$h_4, 2$
$h_1, 1$	$h_2, 1$	$h_1, 2$	$\vdots$
$\vdots$	$\vdots$	$h_3, 1$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

FIGURE 1: Agents' preference profile.

To make the reader more familiar to the machinery of period-t blocking, we find convenient first to discuss an allocation which is not in the Dynamic Core. Consider the allocation (σ, μ) described in FIGURE 2. In period 1, a_1 and a_2 exchange their endowments and consume, agent a_3 consumes her endowment; in period 2, a_4 consumes her endowments.

$$(\sigma, \mu) = \left(\begin{array}{ll} a_1 = \sigma_1(h_2) & \mu_1(a_1) = h_2 \\ a_2 = \sigma_1(h_1) & \mu_1(a_2) = h_1 \\ a_3 = \sigma_1(h_3) & \mu_1(a_3) = h_3 \\ \hline a_4 = \sigma_2(h_4) & \mu_2(a_4) = h_4 \end{array} \right) \quad (\sigma', \mu') = \left(\begin{array}{ll} a_1 = \sigma'_1(h_1) & \mu'_1(a_1) = h_1 \\ a_2 = \sigma'_1(h_3) & \mu'_1(a_2) = h_3 \\ a_3 = \sigma'_1(h_2) & \mu'_1(a_3) = h_0 \\ \hline a_3 = \sigma'_1(h_4) & \mu'_1(a_3) = h_4 \\ a_4 = \sigma'_2(h_2) & \mu'_2(a_4) = h_2 \end{array} \right)$$

FIGURE 2: The allocation (σ, μ) is period-1 blocked by coalition $\{a_2, a_3\}$ via (τ_1, ν_1). The allocation (σ', μ') is in the Dynamic Core.

We claim that the allocation (σ, μ) is period-1 blocked by coalition $\{a_2, a_3\}$ via $(\tau_1, \nu_1) \in \Sigma_1(\{a_2, a_3\}) \times \mathrm{M}_1(\{a_2, a_3\})$ which is described below.

$$(\tau_1, \nu_1) = \left(\begin{array}{ll} a_1 = \tau_1(h_1) & \nu_1(a_1) = h_0 \\ a_2 = \tau_1(h_3) & \nu_1(a_2) = h_3 \\ a_3 = \tau_1(h_2) & \nu_1(a_3) = h_0 \end{array} \right)$$

According to (τ_1, ν_1), a_2 exchanges with a_3 and consumes h_3, while a_3 remains in the market with her new endowment h_2. Since $\tau_1 \in \Sigma_1(\{a_2, a_3\})$, by definition of $\Sigma_t(S)$, a_1, who does not belong to the blocking coalition, keeps its initial endowment h_1. The continuation economy $\mathcal{E}_{\geqslant t}(\{a_2, a_3\}, \tau_1, \nu_1)$ consists of

$$\left\langle (\{a_1\}, \{h_1\}), (\{a_3, a_4\}, \{h_2, h_4\}), \tau_1(h_1) = a_1, \tau_1(h_2) = a_3, \tau_1(h_4) = a_4, \right\rangle,$$

where a_1 enters in period-1 with her initial endowment h_1; a_3 enters in period-2 with her new endowment h_2; a_4 enters in period 2 with her initial endowment h_4.

The continuation economy $\mathcal{E}_{\geqslant t}(\{a_2, a_3\}, \tau_1, \nu_1)$ contains two allocations that are not blocked in any period $t' \geqslant 1$. An unblocked allocation is such that a_1 consumes her endowment in period 1, and a_3, with her new endowment h_2, exchanges with a_4 in period 2. The other unblocked allocation is such that a_1 remains in the market in period 1 and the cycle of exchanges (a_1, a_4, a_3) is formed in period 2. Those two alternatives are illustrated in FIGURE 3. Note that, agents a_2, a_3 are better off in (τ_1, ν_1). Indeed, a_2 consumes h_3 in period 1, her most preferred object, while a_3 consumes either h_1 or h_4 in period 2 and both options are strictly preferred to consuming h_3 in period 1.

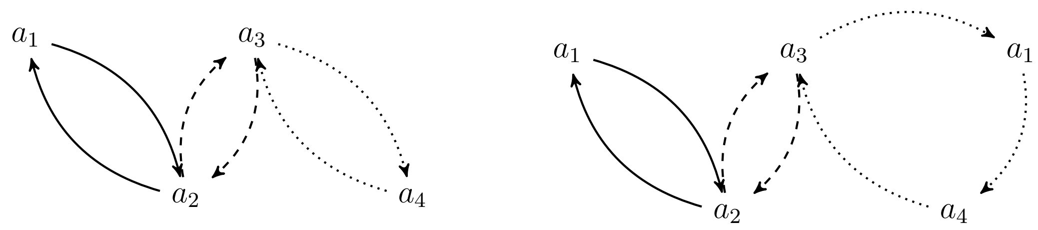

FIGURE 3: The figure illustrates the period-1 blocking (τ_1, ν_1) performed by coalition $\{a_2, a_3\}$; in bold the exchange between $\{a_1, a_2\}$ required by the allocation (σ, μ); in dashed the exchange performed by $\{a_2, a_3\}$ according to (τ_1, ν_1); in dotted the two unblocked exchanges that can take place in period 2.

Finally, we claim that the allocation (σ', μ') in the Dynamic Core. According to this allocation depicted in FIGURE 2, in period 1 a_1 consumes her endowment, a_2 and a_3 exchange, a_2 consumes and a_3 remains in the market; in period 2 a_3 exchanges with a_4 and they consume. In this allocation every agent consumes her most preferred object with the exception of a_1. It follows that a_1 can block by remaining in the market.[9] The continuation economy generated by the latter

[9]Agent a_1 can block (σ', μ') via $(\tau_1', \nu_1') = \begin{pmatrix} a_1 = \tau_1'(h_1) & \nu_1'(a_1) = h_0 \\ a_2 = \tau_1'(h_2) & \nu_1'(a_2) = h_0 \\ a_3 = \tau_1'(h_3) & \nu_1'(a_3) = h_0 \end{pmatrix}$.

blocking,[10] has two unblocked allocations: in both of them, a_2 and a_3 still exchange their endowments in period 1 and a_4 consumes h_2 in period 2. In the first one a_1 consumes h_4 and a_3 consumes h_1, in the second one the opposite occurs. For agent a_1 consuming h_1 in period 2 is a worst outcome than consuming h_1 in period 1, therefore remaining in the market is not a period-1 blocking for the coalition $\{a_1\}$.

Remark **4.** Our notion of stability is in the same spirit of other dynamic solutions defined in matching theory. In particular, our solution has similarities with the perfect α-Core (Kotowski, 2019) and dynamic stability (Doval, 2022) defined for two-sided matching. However, as we already pointed out, an essential characteristic of our stability notion is that the deviating agents form expectations already starting over the same period they block—the continuation economy starts from the same period of the blocking. Such a property, that we call strong perfection, is formally captured by a general form of recursion discussed in Footnote 8. This aspect distinguishes our stability notion from previous ones. For example, in Doval (2022) the deviating agents in period t may form non-rational expectation at t, and in Kotowski (2019) they do not form expectation at all about how the other agents react in period t.

The following example shows that our stronger requirement of rationality may return finer predictions. Consider a 2-period private economy with three agents. Agents 1 and 2 enter in period 1 with objects h_1 and h_2. Agent 3 enters in period 2 with h_3. Preferences are depicted in Section 3

$t=1$		$t=2$
a_1	a_2	a_3
$h_3, 2$	$h_2, 1$	$h_2, 2$
$h_1, 1$	$h_3, 2$	$h_1, 2$
$\vdots$	$h_2, 2$	$\vdots$

Figure 4: Agents' preference profile.

Consider the allocation such that every agent consumes her endowment as

[10]The continuation economy $\mathcal{E}_{\geqslant t}(\{a_1\}, \tau_1', \nu_1')$ induced by the blocking consists of $\left\langle (\{a_2, a_3\}, \{h_2, h_3\}), (\{a_1, a_4\}, \{h_1, h_4\}), \tau_1'(h_1) = a_1, \tau_1'(h_2) = a_2, \tau_3'(h_3) = a_3, \tau_1'(h_4) = a_4, \right\rangle$.

soon as she enters in the market, (agents 1 and 2 consume their endowments in period 1 and agent 3 in period 2). Such prediction is unsatisfactory since agent 1 could block by waiting the next period to exchange with agent 3. However, the blocking of agent 1 takes place only if she has rational expectations over agent 2's behavior in period 1. Suppose her expectations admit that agent 2 could wait for exchanging with agent 3 in period 2. These beliefs would prevent agent 1 from blocking by waiting without consuming in period 1. Since agent 2 prefers consuming her endowment in period 1 to consuming h_3 in period 2, these expectations are not rational. The only rational expectations that agent 1 may have over the continuation economy after the blocking are such that agent 2 consumes in period 1 and agent 3 agrees to exchange in period 2. Therefore the allocation such that agents 1 and 2 consume their endowments in period 1 is period-1 blocked by agent 1.

REMARK **5.** A key element in a theory of dynamic one-sided matching is to allow agents to exchange and remain in the market for a better trade in the future. Example 7 in the APPENDIX B shows that the Dynamic Core may be empty otherwise. Using the same example one can easily show that also the (strong) Core[11] may be empty in a dynamic economy with $t > 1$.

4 THE DYNAMIC STRONG CORE

The Dynamic Core is in the same spirit of the weak Core for housing matching market (Shapley and Scarf, 1974), where a coalition deviates from a given allocation only if every member of the coalition is better off. Indeed, it is straightforward to see that for $n = 1$ the Dynamic Core reduces to the Core.

An alternative solution concept in housing matching markets is the Strong Core (Roth and Postlewaite, 1977), which relies on a weaker notion of blocking: a coalition weakly blocks if all members are weakly better off, and at least one individual is strictly better off. Indeed, in housing markets when preferences are strict, the Strong Core is not empty, unique, and identified by the Gale's Top

[11] The notion of (strong) Core for a dynamic economy is introduced in SECTION 5.

Trading Cycle. In what follows, we present a natural extension of the Strong Core to a dynamic setting, and then we show why such an extension is problematic.

Definition **7** (Weak Period-t Blocking)**.** Let (σ, μ) be an allocation over the economy $\mathcal{E} = \langle (A_t, H_t,)_{t=1}^{n}, \omega \rangle$. A coalition $S \subseteq \mathcal{A}_t$ can **weakly period-t block** (σ, μ) if there exists a pair $(\tau_t, \nu_t) \in \Sigma_t(S) \times \mathrm{M}_t(S)$ such that:

- $(\nu_t(a), t) \succsim_a (\sigma, \mu)$ for all $a \in S \backslash S^w$
- $(\upsilon, \xi) \succsim_a (\sigma, \mu)$ for all $a \in S^w$ and for all allocations (υ, ξ) of the economy $\mathcal{E}_{\geqslant t}$ that cannot be weakly period-t' blocked at any $t' \geqslant t$,

and

- either $(\nu_t(a), t) \succ_a (\sigma, \mu)$ for some $a \in S \backslash S^w$,
- or for all allocations (υ, ξ) over the economy $\mathcal{E}_{\geqslant t}$ that cannot be weakly period-t' blocked at any $t' \geqslant t$, $(\upsilon, \xi) \succ_a (\sigma, \mu)$ for some $a \in S^w$.

The **Dynamic Strong Core** of an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^{n}, \omega \rangle$ is the set of allocations that cannot be weakly period-t blocked in any period by any coalition.

Theorem **2.** *For any economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^{n}, \omega \rangle$, the Dynamic Strong Core is a subset (possibly empty) of the Dynamic Core.*

Theorem 2 states that the Dynamic Strong Core is contained in the Dynamic Core allocations. However, differently than the Strong Core in static markets, the Dynamic Strong Core may be empty even in private economies when agents' preferences are strict. The following example makes the point.

Example **2.** There are four agents a_1, a_2, a_3, a_4, each agent endowed with an object. Agents a_1, a_2, a_3 enter in period 1, a_4 enters in period 2. Agents' preferences are depicted below.

The Dynamic Core consists of the following three allocations.

$t=1$			$t=2$
a_1	a_2	a_3	a_4
$h_3, 1$	$h_3, 1$	$h_4, 2$	$h_1, 2$
$h_1, 1$	$h_2, 1$	$\vdots$	$h_2, 2$
$\vdots$	$\vdots$	$\vdots$	$h_3, 2$
$\vdots$	$\vdots$	$\vdots$	$h_4, 2$

FIGURE 5: An example of a 2-period economy in which the Dynamic Strong Core is empty

$$(\sigma, \mu) = \left(\begin{array}{ll} a_1 = \sigma_1(h_3) & \mu_1(a_1) = h_3 \\ a_2 = \sigma_1(h_2) & \mu_1(a_2) = h_2 \\ a_3 = \sigma_1(h_1) & \mu_1(a_3) = h_0 \\ \hline a_3 = \sigma_2(h_4) & \mu_2(a_3) = h_4 \\ a_4 = \sigma_2(h_1) & \mu_2(a_4) = h_1 \end{array} \right) \qquad (\sigma', \mu') = \left(\begin{array}{ll} a_1 = \sigma_1(h_1) & \mu_1(a_1) = h_1 \\ a_2 = \sigma_1(h_3) & \mu_1(a_2) = h_3 \\ a_3 = \sigma_1(h_2) & \mu_1(a_3) = h_0 \\ \hline a_3 = \sigma_2(h_4) & \mu_2(a_3) = h_4 \\ a_4 = \sigma_2(h_2) & \mu_2(a_4) = h_2 \end{array} \right)$$

$$(\sigma'', \mu'') = \left(\begin{array}{ll} a_1 = \sigma''_1(h_1) & \mu''_1(a_1) = h_1 \\ a_2 = \sigma''_1(h_2) & \mu''_1(a_2) = h_2 \\ a_3 = \sigma''_1(h_3) & \mu''_1(a_3) = h_0 \\ \hline a_3 = \sigma''_2(h_4) & \mu''_2(a_3) = h_4 \\ a_4 = \sigma''_2(h_3) & \mu''_2(a_4) = h_3 \end{array} \right)$$

However, none of these is in the Dynamic Strong Core. To see this, note that whatever object agent a_3 owns at the end of period 1, she can exchange with agent a_4 in period 2, since agents 1 and 2 always consume in period 1. Therefore, a_3 is indifferent whether keeping her initial endowment, or exchanging it with agent a_1 or agent a_2. In both cases he consumes h_4 in period 2. This indifference impairs the existence of the Dynamic Strong Core. Consider, for example, the allocation (σ, μ). It is weakly period-1 blocked by coalition $\{a_2, a_3\}$ that exchanges their endowments in period 1. Applying the same logic, (σ', μ') and (σ'', μ'') are not in the Dynamic Strong Core either.

We have showed in the previous example that the Dynamic Strong Core of an economy can be empty. This fact has undesirable implications. Consider the following modification of the Example 3. Suppose that there is a time $t = 0$ in which agents 1,2 and 3 enter in the economy with their endowment. Suppose that consuming the endowment at time $t = 0$, $(h_i, 0)$, is the preferred alternative for each $i = 1, 2, 3$. We should expect that the allocation in which every agent consumes her endowment as soon as enters in the market is in the Dynamic Strong Core. However, this allocation is not in the Dynamic Strong Core because it can be blocked by the coalition of agents (a_1, a_2, a_3) who wait for, without consuming at time $t = 0$. Since the Dynamic Strong Core in the continuation economy (Example 3) is empty such coalition period-0 blocks the allocation. In fact the definition requires that all agents in S^w should prefer every allocation that cannnot be period-t' blocked at any $t' \geqslant 0$ to the original allocation. Since the set of such allocations is empty, the requirement is (vacuously) satisfied.[12]

We can conclude that the definition of Dynamic Strong Core is somehow too demanding and allocations that seem very natural outcomes of a dynamic economy are ruled out.

5 The Dynamic Core and the Core in Private and Public Economies

As we pointed out, the theory we present encompasses both private and public economy as special cases. This section studies the relation between the Dynamic Core and the existing solution concepts for housing market (Shapley and Scarf, 1974) and house allocation problem (Hylland and Zeckhauser, 1979).

The application of standard solution concepts in a dynamic matching environment with arrivals is not immediate. A way to approach the problem is to ignore the dynamics and assume a pre-stage game in which all the agents can make binding agreements about the allocations that will realize in the original

[12]A possible solution to this problem could be a modification of the definition such that a coalition cannot block if in the continuation economy the Dynamic Strong Core is empty. However, this definition would also generate odd predictions.

economy[13].

Suppose that (σ, μ) is an allocation in the pre-stage game. Then, (σ, μ) is **weakly blocked** if there is a coalition $S \subseteq A$ and an allocation (τ, ν) such that all agents in S prefers (τ, ν) over (σ, μ) with a strict improvement for at least one agent, i.e., $(\tau, \nu) \succeq_a (\sigma, \mu)$ for all $a \in S$ and $(\tau, \nu) \succ_a (\sigma, \mu)$ for some $a \in S$, and S is effective for the allocation (τ, ν), i.e., $\{\nu(a) | a \in S\} \subseteq \omega^{-1}(S)$.

The **Strong Core** of an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ consists of the set of all allocations that cannot be weakly blocked.

It can be show that in any Strong Core allocation the consumption choice is the same for every agents. Given an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ and allocation (σ, μ), we denote by $[\sigma, \mu] = \{\tau, \nu | \nu(a) = \mu(a), \ \forall a \in A\}$ the equivalent class of (σ, μ) with respect to μ. Let (σ, μ) be a Strong Core allocation of an economy $\mathcal{E}$. Then, the Strong Core of $\mathcal{E}$ is said to be essentially unique if every allocation in the Strong Core belongs to $[\sigma, \mu]$.

THEOREM **3.** *Let $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ be a private economy. Then, the Strong Core of $\mathcal{E}$ is essentially unique.*

In light of THEOREM 3, the following result "almost" characterizes the Strong Core as a refinement of the Dynamic Core, when economies are private.

THEOREM **4.** *In private economies, there exists a Strong Core allocation which is in the Dynamic Core.*

However even if for dynamic private economies the Strong Core always exists, it may be considered too restrictive. Recall Example 2. It can be easily shown that the Strong Core consists of the allocation (σ, μ). In fact, in (σ, μ) all agents except a_2 get their most preferred objects. However, such an allocation is not necessarily the only reliable prediction of the economy. Indeed, agent a_3 could decide either to exchange with agent a_2 in period 1 or to keep her endowment, as predicted by the Dynamic Core with the allocations (σ', μ') and (σ'', μ'') respectively.

[13]This approach is analogous to dynamic matching with contracts: a contract specifies which exchanges will realize, the time at which they occur and the consumption choice of each agent. See for example Dimakopoulos and Heller (2019).

Another prominent solution concept in static private market is the **(weak) Core**, which builds on a stronger notion of blocking by requiring a strict improvement for all members of the blocking coalition[14]. The following examples show that, in private economies, the Dynamic Core and the Core are unrelated. Example 3 shows that an allocation in the Dynamic Core may not be in the Core.

EXAMPLE **3.** Consider a private economy with 4 agents and 4 objects. Agents a_1 and a_2 arrive in period 1, agents a_3 and a_4 in period 2. Each agent is endowed with an object. Preferences are depicted in the table on the left-hand side, while the allocation (σ, μ) on the right-hand side is in the Dynamic Core.

$t = 1$		$t = 2$	
a_1	a_2	a_3	a_4
$h_3, 2$	$h_3, 2$	$h_4, 2$	$h_1, 2$
$h_2, 1$	$h_1, 1$	$\vdots$	$h_2, 2$
$\vdots$	$\vdots$	$\vdots$	$h_4, 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$(\sigma, \mu) = \left(\begin{array}{cc} a_1 = \sigma_1(h_2) & \mu_1(a_1) = h_2 \\ a_2 = \sigma_1(h_1) & \mu_1(a_2) = h_1 \\ \hline a_3 = \sigma_2(h_3) & \mu_2(a_3) = h_3 \\ a_4 = \sigma_2(h_4) & \mu_2(a_4) = h_4 \end{array} \right)$$

FIGURE 6: An example of an allocation in the Dynamic Core that is not in the Core.

To check that (σ, μ) is in the Dynamic Core consider any possible blocking coalition S. Suppose first that $S = \{a_1\}$ period-1 blocks (σ, μ) by waiting with her initial endowment. This deviation is deterred by the fact that, agent a_2 may react by waiting for the next period and form a cycle of exchanges (a_2, a_3, a_4). This allocation is in the Dynamic Core of the continuation economy (both a_2 and a_3 get their most preferred object) and is worse for agent 1 than the original allocation since agent a_1 consumes her initial endowment at $t = 2$ instead of at $t = 1$. Similar argument applies if a_2 alone deviates in period 1. However (σ, μ) is not in the Strong Core. Indeed, in the pre-stage game, coalition $\{a_1, a_3, a_4\}$ can agree to exchange their objects in period 2.

[14]An allocation (σ, μ) in the pre-stage game is **blocked** if there is a coalition $S \subseteq A$ and an allocation (τ, ν) such that all coalitional members strictly prefers (τ, ν) over (σ, μ), i.e., $(\tau, \nu) \succ_a (\sigma, \mu)$ for all $a \in S$, and S is effective for the allocation (τ, ν), i.e., $\{\nu(a) | a \in S\} \subseteq \omega^{-1}(S)$. The **Core** of an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ consist of the set of all allocations that cannot be blocked.

The next example shows an allocation that is in the Core but not in the Dynamic Core.

Example **4.** Consider a private economy with 3 agents and 3 objects. Agents a_1 arrives in period 1 together with its object h_1 and agents a_2 and a_3 arrive in period 2 together with their objects h_2 and h_3. Preferences are on the left-hand side of the figure. The allocation (σ, μ) is in the Core. To see that there are no blocking coalition in the pre-stage game, note that agents a_1 and a_2 alone cannot form a blocking coalition and in any blocking coalition agent containing a_3, she must exchange with a_2 otherwise she is worse off. However, such an allocation is not in the Dynamic Core. Indeed, agent a_1 can period-1 block by waiting with her endowment in period 1. This implies that in period 2 the grand coalition form and every agent gets her most preferred object.

$t=1$	$t=2$	
a_1	a_2	a_3
$h_3, 2$	$h_1, 2$	$h_2, 2$
$h_1, 1$	$h_3, 2$	$h_1, 2$
$h_2, 2$	$h_2, 2$	$h_3, 2$
$h_1, 2$	$\vdots$	$\vdots$

$$(\sigma, \mu) = \left(\begin{array}{cc} a_1 = \sigma_1(h_1) & \mu_1(a_1) = h_1 \\ \hline a_2 = \sigma_2(h_3) & \mu_2(a_2) = h_3 \\ a_3 = \sigma_2(h_2) & \mu_2(a_3) = h_2 \end{array} \right)$$

Figure 7: An example of an allocation in the Core that is not in the Dynamic Core.

The antipodes of a private economy is a public economy, where all objects are collectively owned (recall that an economy is **public** if for every $h \in H$, $\omega(h) = A$).

Section 5 illustrates how, in public economies, a Strong Core allocation might be not in the Dynamic Core. This, together with the established fact that for $t = 1$ the Core is not contained in the Strong Core, proves that, for public economies, the Dynamic Core and Strong Core are unrelated.

Example **5.** There are 2 agents and 1 public object. Agents a_1 arrives in period 1 together with the common object h_1 and agent a_2 arrive in period 2. Preferences are on the left-hand side of the figure. The allocation (σ, μ) such that the object

$t=1$	$t=2$
a_1	a_2
$h_1, 1$	$h_1, 2$
$h_1, 2$	$\vdots$
$\vdots$	$\vdots$

$$(\sigma, \mu) = \left(\begin{array}{cc} A = \sigma_1(h_1) & \mu_1(a_1) = h_0 \\ \hline A = \sigma_2(h_1) & \mu_2(a_1) = h_0 \\ & \mu_2(a_2) = h_1 \end{array} \right)$$

FIGURE 8: An example of an allocation in the Strong Core that is not in the Dynamic Core.

is consumed by a_2 is in the Strong Core but not in the Dynamic Core. Indeed, agent a_1 can period-1 block (σ, μ) by consuming h_1 in period 1.

In the previous section, we argued that the Dynamic Strong Core can be empty. However, for the special case of public economies, it is not empty.

THEOREM **5.** *In public economies, the Dynamic Strong Core is not empty.*

We conclude this section by showing that in public economies, the Dynamic Strong Core is unrelated to the Strong Core. Note that since the Core allocation in is not in the Dynamic Core, then it is not in the Dynamic Strong Core. It remains to show that an allocation in the Dynamic Strong Core might lie outside the Strong Core. The following example fills the gap.

EXAMPLE **6.** There are 3 agents and 2 public objects. Agents a_1 arrives in period 1 together with the public object h_1 and agents a_2, a_3 arrive in period 2 with the public object h_2. Preferences are on the left-hand side of the figure. The allocation (σ, μ) such that h_1 is consumed by a_1 in period 1 and h_2 is consumed by a_2 in period 2 is in the Dynamic Strong Core. Indeed, only agent a_1 could weakly block by waiting for until period 2. However, her deviation is deterred by the fact that there is a continuation economy in which all the objects are assigned to the other agents. Clearly, (σ, μ) is not in the Strong Core

6 THE INTERTEMPORAL TOP-TRADING CYCLE

The Top Trading Cycle, first described by Shapley and Scarf (1974) and attributed to David Gale, is one of the most influential algorithms in matching

$t=1$	$t=2$	
a_1	a_2	a_3
$h_2, 2$	$h_1, 2$	$h_2, 2$
$h_1, 1$	$h_2, 2$	$h_1, 2$
$h_1, 2$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$

$$(\sigma, \mu) = \left(\frac{A = \sigma_1(h_1) \quad \mu_1(a_1) = h_1}{A = \sigma_2(h_2) \quad \mu_2(a_2) = h_2} \right)$$

FIGURE 9: An example of an allocation in the strong Dynamic Core which is not in the Strong Core.

Private Economies

- SC is essentially unique
- sc $\in$ Dynamic Core
- C $\neq$ Dynamic Core

Public Economies

- SC $\neq$ Dynamic Core
- SC $\neq$ Dynamic Strong Core
- Dynamic Strong Core is non-empty

FIGURE 10: We summarize the comparison results with the Core (C) and the Strong Core (SC) for private and public economies. For private economies, SC is essentially unique and an SC allocation (sc) always belongs to the Dynamic Core. C and Dynamic Core are unrelated. For public economies, SC is unrelated with the Dynamic Core and the Dynamic Strong Core. The latter is non-empty.

theory. The Top-Trading Cycle is declined in several matching environments with the scope to identify a stable allocation. A Top-Trading Cycle consists of an iterative procedure which constructs a directed graph as follows: each agent points to her most preferred object among those available and each object points to either her owner or the agent with the highest priority. As there are a finite number of agents, there must exist a cycle and objects are allocated accordingly.

In this section, we provide an extension of the Top-Trading Cycle, called **Intertemporal Top Trading Cycle**, hereafter ITTC, to our dynamic setting that identifies an allocation in the Dynamic Core.

To provide an intuition on how the ITTC algorithm works consider a 2-period private economy.

Pointing: Each agent points to the object belonging to her preferred feasible pair and each object points to its owner.

Clearing: Consider any cycle composed only by agents and objects that are present at time $t = 1$, and perform the exchanges accordingly; every agent consumes at time $t = 1$ the object she points. Remove agents and objects.

Trimming and Clearing: Consider any cycle involving some agents (and objects) that enter in period 1 and some that enter in period 2. Identify each chain starting from an object entered in period 1 that ends with an agent who points to an object that enters in period 2. Perform the exchanges along the chain accordingly, and assign the first object of the chain to the last agent in the chain. All agents in the chain except the last one consume at $t = 1$ the object they point. Remove all agents who consume and objects that are consumed.

Repeat this procedure until it exhausts all exchanges performed in period 1, that is until every agent entered in period 1 either consumes at $t = 1$ or points to an object that enters in period 2.

Every agent who is still present in period 2 points to her preferred remaining object, and each object points to its current owner. Any time a cycle is formed, an agent consumes at time $t = 2$ the object is pointing. At this stage the ITTC works as the TTC.

We prove that the outcome of the ITTC algorithm is an allocation in the Dynamic Core.

Now, we formally define the ITTC. A **preference profile** $\succ \equiv (\succ_a)_{a \in A}$ is a collection of preferences, one for each agent. Let $\mathcal{L}$ be the domain of all preference profiles $\succ$. Given an economy $\mathcal{E} = \langle (A_k, H_k)_{k=1}^n, \omega \rangle$ and a preference profile $\succ \in \mathcal{L}$, a **direct mechanism** ϕ is a map which associates an allocation (σ, μ) to each preference profile $\succ \in \mathcal{L}$. The ITTC mechanism is a direct mechanism whose selected allocation is identified by the Intertemporal Top-Trading Cycle algorithm, hereafter ITTC algorithm. In what follows, we introduce the main ingredients of the algorithm.

We write $a \rhd_h b$ to say that agent a has a priority over b for the object h. A priority for an object h describes pre-existing social, legal, or economic relationships among agents. The priority ordering for each object respects the order according to which agents enter in the economy: an agent who enters in period t has priority on any objects that is present in period t over any agent who enters at any period

$t' > t$. Next definition formalizes the notion of priority ordering.

Definition 8 (Priority). A **priority** $\rhd_h$ is a strict linear order over the set of agents such that for all $h \in H$ and $t' < t''$ if $a \in A_{t'}$ and $b \in A_{t''}$ then $a \rhd_h b$. A priority structure $\rhd \equiv (\rhd_h)_{h \in H}$ is a profile of priority orders.

A **cycle** $C = (h_i, a_i)_{i=1}^m$ with $m \in \mathbb{N}_+$ is an ordered list of objects and agents such that object h_1 points to agent a_1, agent a_1 points to object h_2,....,object h_m points to agent a_m, and agent a_m points to object h_1. A **subcycle** C' of C is a cycle whose elements are contained in C and listed in the same order as in C.

We say that a cycle is **intertemporal at time** t if there exists at least one element that is not present at time t.

A **period-t chain** $C^t = (h_i, a_i)_{i=1}^{\ell}$ with $\ell \in \mathbb{N}_+$ is an ordered list of agents and objects that are present at time t such that object h_1 points to agent a_1, agent a_1 points to object h_2,....,object h_ℓ points to agent a_ℓ, and agent a_ℓ points to an object h that is not available in period t. With abuse of notation, let C^t_A and C^t_H the agents and the objects involved in C^t, respectively.

A **period-t chain** is a subset of consecutive elements of a cycle that are all present at time t. A **maximal period-t chain** is a period-t chain that is maximal with respect to set inclusion.

Definition 9 (IITC). An **ITTC algorithm** constructs a pair (σ_t, μ_t) in any period $t = 1, ..., n$ via $k \geqslant 1$ iterations. Let $A^{t,1} \equiv \mathcal{A}_t \cup A_{>t}$ and $H^{t,1} \equiv \mathcal{H}_t \cup H_{>t}$. The priorities over objects may change during time. Let $\rhd_h^1 = \rhd_h$. In Step (t,k) proceeds as follows with inputs $A^{t,k}$, $H^{t,k}$, and $\rhd^t$.

Step (t,k). Let $A^{t,k}$ and $H^{t,k}$ be the sets of available agents and objects at **Step** (t,k). Each agent a points to the object belonging to her favorite feasible pair among those composed of an object in $H^{t,k}$ and a period greater or equal than t. Each object h points to its owner if she belongs to $A^{t,k}$, to the $\rhd_h^t$-maximal element of $A^{t,k}$ otherwise. Since the numbers of agents and objects are finite, there is at least one cycle, namely $C^{t,k}$. [**Pointing**]

Case 1. If the cycle $C^{t,k}$ is not intertemporal then each agent $a_i \in C_A^{t,k}$ is assigned to and consumes the object $h_{i+1} \in C_H^{t,k}$. [**Clearing**]

Case 2. If the cycle $C^{t,k}$ is intertemporal then, for every maximal period-t chain, each agent, with the exception of the last one in the chain, is assigned to and consumes the subsequent object [**Trimming and Clearing**], and

2.1 If the chain starts with an agent a_i, the last agent in the chain becomes the $\rhd_{h_i}^{t+1}$-maximal agent and does not consume at time t.

2.2 If the chain starts with an object h_i and there exists at least one agent in the chain, the last agent of the chain a_j gets the ownership of object h_i and does not consume at time t.

2.3 If the chain is formed by a single object h_i, the object h_i is made common ($\sigma_t(h_i) = A$) with a_i as the $\rhd_{h_i}^{t+1}$-maximal element.

Remove all agents and objects in the cycle from $A^{t,k}$ and $H^{t,k}$ to get $A^{t,k+1}$ and $H^{t,k+1}$, respectively. If $A^{t,k+1} = \varnothing$ or $H^{t,k+1} = \varnothing$ proceed to **Step** $(t+1, 1)$, otherwise proceed to **Step** $(t, k+1)$. At **Step** $(n+1, 1)$ the algorithm ends.

At the end of time t, when no more cycle can be created, every agent a which does not belong to any cycle at t waits for the next period ($\mu_t(a) = h_0$) and every object which does not belong to any cycle at t, is made public ($\sigma_t(h) = A$).

Remark **6.** It is possible to show that if a cycle C is formed at some step t and if an agent remains in the market for some of the following steps, then all the cycles to which he/she belongs are contained in C. Moreover, in the ITTC algorithm, the cycles formed at **Step** $(1, k)$ drive the entire procedure. Indeed, each agent, in all cycles which she belongs to, points to the object she pointed to in the cycle formed at **Step** $(1, k)$. These observations implies that agents consume their most preferred object among those available at Step $(1, k)$. In the technical Appendix A, Lemma 2, Lemma 3, Lemma 4, Corollary 1, Corollary 2, and Corollary 3 make this point.

Suppose that $\phi_{\rhd}(\succ, \mathcal{E})$ is the output of the ITTC at some profile $\succ\in \mathcal{L}$. Theorem 6 proves that $\phi_{\rhd}(\succ, \mathcal{E})$ consists of an allocation in the Dynamic Core of $\mathcal{E}$.

Theorem **6.** *Let $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega\rangle$ be an economy. Then, the output of any ITTC is an allocation in the Dynamic Core of the economy $\mathcal{E}$.*

We conclude this section illustrating how the IITC algorithm works in a 2-period economy with four agents and four objects: h_{c1} and h_{c2} are collectively owned objects, h_2 and h_4 are private and they are owned by a_2 and a_4, respectively. Agents a_1 a_2 and a_3 enter in period 1, a_4 in period 2; private objects enter with their owner, common object h_{c1} enters in period 1 while h_{c2} in period 2. Agents' preferences and objects' priorities are described in Figure 11.

$t=1$			$t=2$
a_1	a_2	a_3	a_4
$h_{c1}, 1$	$h_4, 1$	$h_4, 2$	$h_{c2}, 2$
$\vdots$	$\vdots$	$h_2, 1$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$t=1$		$t=2$	
$\rhd_{h_2}$	$\rhd_{h_{c1}}$	$\rhd_{h_{c2}}$	$\rhd_{h_4}$
a_2	a_2	a_1	a_4
a_3	a_3	a_3	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 11: Agents' preferences in the motivating example of Section 2.

The ITTC mechanism for this economy consists of four Steps: $(1,1), (1,2), (2,1)$. **Step** $(1,1)$ constructs a graph cycle where each private object points to its owner, each common object points to its $\rhd^1$-maximal element (where $\rhd_h^1 = \rhd_h$ for each h), and each agents points her most preferred object (see Figure 12). There exists a cycle $C^{1,1} = (h_{c2}, a_1, h_{c1}, a_2, h_4, a_4)$. This is an intertemporal cycle since a_1, a_2, a_3, h_{c1} enter in period 1, while a_4, h_{c2}, h_4 enter in period 2. Then, **Case 2** of the algorithm applies. Trim and clear the cycle $C^{1,1} = (h_{c2}, a_1, h_{c1}, a_2, h_4, a_4)$ by assigning h_{c1} to a_1 and by making a_2 the $\rhd_{h_{c2}}^2$-maximal element—from now on a_2 has the highest priority over the common object h_{c2}.

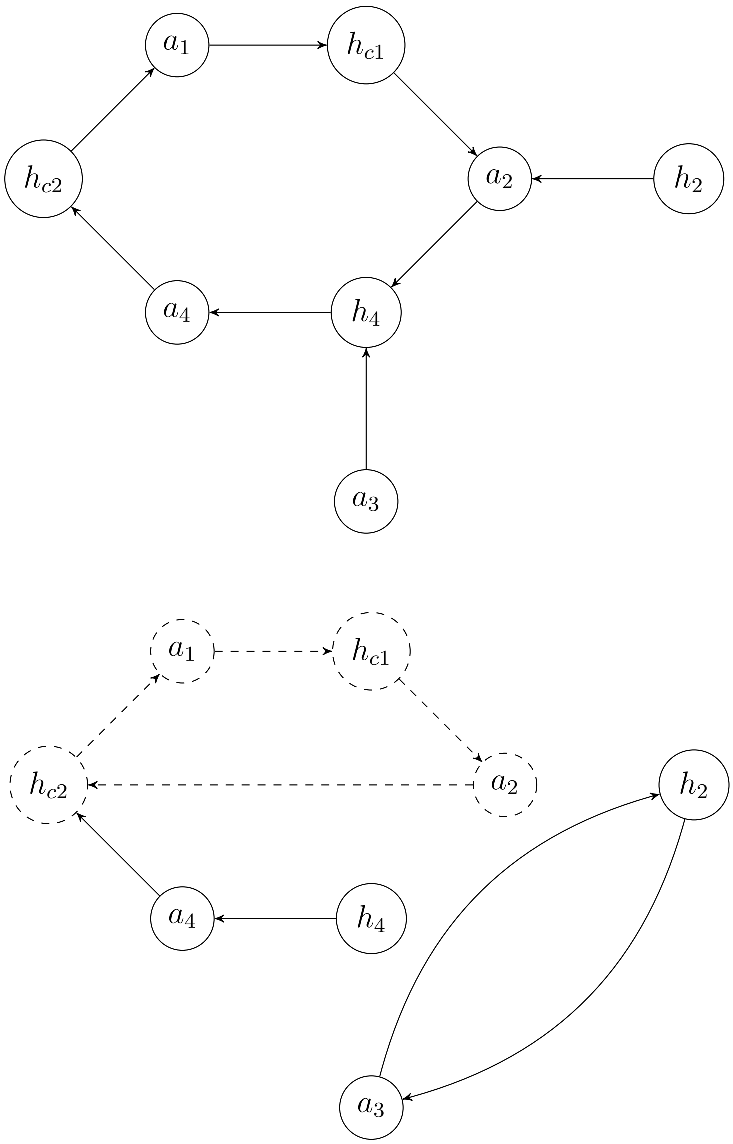

FIGURE 12: Upper part of the figure illustrates the pointing procedure of **Step** (1,1): Each private object points to its owner and each common object points to the agent having the highest priority. The cycle $C^{1,1}(h_{c2}, a_1, h_{c1}, a_2, h_4, a_4)$ forms. Since $C^{1,1}$ is intertemporal at $t = 1$ then the cycle is trimmed. This is illustrated by the bottom part of the figure: agent a_1 consumes the assigned objects and is removed. Agent a_2 receives priority for object h_{c2} that will arrive in period 2—a_2 is the $\rhd^2_{h_{c2}}$ maximal element. In **Step** (1,2), agents outside $C^{1,1}$ forms a cycle $C^{1,2} = (h_2, a_3)$. Indeed, according to priority $\rhd_{h_2}$, the object h_2 points now to a_3, since its owner is in an another cycle.

Step $(1, 2)$ constructs a direct graph using the set of agents and objects not in $C^{1,1}$ that is, $A^{1,2} = \{a_3\}$ and $H^{1,2} = \{h_2\}$. Note that the initial owner of h_2 does not belong to $A^{1,2}$, thus, according the the ITTC algorithm, it points to its $\rhd_{h_2}$-maximal element in $A^{1,2}$, that is a_3. Then, the cycle $C^{1,2} = (h_2, a_3)$ forms.

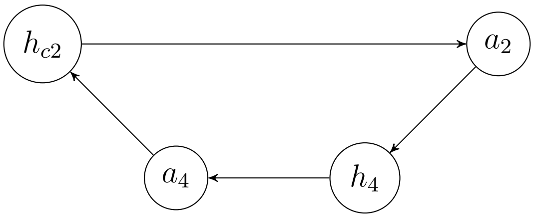

FIGURE 13: The figure illustrates **Step** (2,1): the cycle $C^{2,1} = (h_{c2}, a_2, h_4, a_4)$ is created. Since it is not intertemporal at $t = 2$, it is cleared accordingly.

Since both h_2 and a_3 are present in period 1, h_2 is assigned to a_3 who consumes. **Steps** (1,1) and (1,2) are illustrated in FIGURE 12.

This results in the following "partial assignment"

$$(\sigma_1, \mu_1) = \begin{pmatrix} a_1 = \sigma_1(h_{c1}) & \mu_1(a_1) = h_{c1} \\ & \mu_1(a_2) = h_0 \\ a_3 = \sigma_1(h_2) & \mu_1(a_3) = h_2 \end{pmatrix}$$

Since there are no more cycles that can be formed, the algorithm proceeds to **Step** $(2, 1)$ which generates the cycle $C^{2,1} = (h_{c2}, a_2, h_4, a_4)$. Since $C^{2,1}$ is not intertemporal at $t = 2$, then **Case 1** of the algorithm applies and the cycle is cleared accordingly. This results in the following "partial assignment":

$$(\sigma_2, \mu_2) = \begin{pmatrix} a_2 = \sigma_2(h_4) & \mu_2(a_2) = h_4 \\ a_4 = \sigma_1(h_{c2}) & \mu_1(a_4) = h_{c2} \end{pmatrix}$$

This final step is illustrated in FIGURE 13. The output of the algorithm is the allocation $(\sigma_1, \sigma_2, \mu_1, \mu_2)$.

Note that for private economies, the algorithm can be run without the use of objects' priorities. For an example, we refer the reader to the APPENDIX B.

PROPERTIES OF THE ITTC

We have shown that the ITTC mechanism identifies an allocation in the Dynamic Core at every preference profile. In this section we show that the ITTC mechanism also satisfies desirable normative properties.

Definition 10 (Pareto efficiency). An allocation (σ, μ) is **Pareto efficient** if there is no allocation (τ, ν) such that $(\tau, \nu) \succsim_a (\sigma, \mu)$ for all $a \in A$ with strict preference for at least one agent. A mechanism is Pareto efficient if its output is Pareto efficient at every preference profile.

Theorem 7. *The ITTC mechanism is Pareto efficient.*

An additional important property of a mechanism for implementation purpose is that agents do not have incentive to misreport their preferences (Roth, 1982), even when they can coordinate their misrepresentation.

Definition 11 (Group strategy-proofness). Given the economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$, the mechanism $\phi_\triangleright$ is **group strategy-proof** if for all $\succ \in \mathcal{L}$ there is no $S \subseteq A$ and $\succ'$ such that

$$\phi_\triangleright(\succ'_S, \succ_{A \setminus S}, \mathcal{E}) \succ_a \phi_\triangleright(\succ, \mathcal{E}) \quad \forall a \in S$$

Theorem 8. *The ITTC mechanism is group strategy-proof.*

We conclude with another important observation when we restrict the attention to private economies. The allocation induced by the ITTC can be supported by a dynamic competitive equilibrium. Let $p \in \mathbb{R}_+^{|H|}$ denote a price vector and p_h the price of the object h with $p_\varnothing = 0$.

Definition 12. Given a private economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ an allocation (σ, μ) can be supported as dynamic competitive equilibrium for a profile $\succ \in \mathcal{L}$ if there exists a price vector p such that for all $a \in A$ the following conditions hold:

1. $p_{\sigma_t^{-1}(a)} \leqslant p_{\sigma_{t-1}^{-1}(a)}$ for all $t \in \{1, ..., n\}$
2. if $(\tau, \nu) \succ_a (\sigma, \mu)$ then $p_{\tau_t^{-1}(a)} > p_{\tau_{t-1}^{-1}(a)}$ for some $t \in \{1, ..., n\}$

Theorem 9. *In private economies, the output of the ITTC can be supported as a dynamic competitive equilibrium.*

7 Final Discussions

We provide a novel solution concept, the Dynamic Core, for dynamic one-sided matching models in which agents and object arrive over time, and objects are either privately or collectively owned. We prove that the Dynamic Core is always non-empty. We present a dynamic version of the Gale's TTC mechanism, named the Intertemporal Top-Trading Cycle (ITTC). The ITTC identifies an allocation in the Dynamic Core at every preference profile, it is Pareto efficient, and group strategy proof. For private economies, its outcome can be supported as a dynamic competitive equilibrium.

A real-life problem in which our model can be applied is kidney transplantation. The problem has an inherently dynamic structure and the ownership structure resembles the one of the model. New patients and organs continuously arrive over time, and often patients waiting for a transplant face intertemporal tradeoffs having to decide whether to accept the kidney for transplantation, or decline the offer and rejoin the candidate pool for a future reassignment (Agarwal et al., 2021). Organs available for transplantation are either collectively owned, deceased donor organs and organs from Samaritan donors[15], or privately owned, living donors of incompatible pairs who participate to kidney exchange programs[16].

The ITTC algorithm can be used to incorporate recent proposals to merge allocation programs that allocate deceased donor organs with kidney exchange programs (Sonmez, Unver and Yemez, 2020; Furian et al., 2019, 2020).

Consider the following example. Suppose that Ann and Bob are in dialysis. Ann immediately needs a kidney transplant because her health conditions are deteriorating, while Bob's health conditions are still good. They both have an incompatible living donors, and Bob's donor is compatible with Ann, while Anna's donor is not compatible with Bob. Carla has a chronic kidney disease and she will need a kidney transplant in a year. Carla is compatible with Ann's donor and Bob

[15]Samaritan donors are living donors who anonymously donate a kidney to a patient waiting for a transplant.

[16]Kidney exchange programs allow patients with incompatible living donors to swap their living donors so each recipient receives a compatible transplant (see Roth, Sömnez and Ünver, 2003).

is compatible with Carla's donor. An intertemporal paired exchange program that uses the ITTC algorithm is such that Ann immediately receives Bob donor's kidney and Ann's donor becomes Bob's new "endowment". In a year Bob will perform a kidney paired exchange with Carla, so Carla will eventually receive original Ann donor's kidney and Bob receives Carla donor's kidney. Since Ann's donor could renege after she received the organ or Carla could become not anymore eligible for a transplant, Bob could be insured against this risk with an increased priority in the deceased donor waiting list in case any of these events would occur (Sonmez, Unver and Yemez, 2020). In the algorithm this corresponds to assigning Bob the highest priority over a deceased donor organ that enters the market in the following period.

This discussion suggests that an important direction towards which the analysis of allocations that are dynamically stable should be extended, is the inclusion of uncertainty in the model. Uncertainty could regard which agents and objects will enter the market in the future, as also which preferences future agents may have.

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Appendix A

The following notation will be used thereafter. Denote by $C^{t,k}$ the cycle generated by the algorithm at **Step** (t, k), and by $m(t)$ the number of iterations produced at any period-t.

For any economy $\mathcal{E}$, we denote by $q(\mathcal{E})$ the number of agents entering in the first period of the economy. To maintain the notation simple, we only write q. Then, we refer to (n, q)[17] as the *size* of an economy. Sizes are ordered lexicographically. Thus, (n', q') is ranked below (n, q) if either $n' < n$ or $n' = n$ and $q' < q$. We say that for any two economies $\mathcal{E}$ and $\mathcal{E}'$ with sizes (n, q) and (n', q'), $\mathcal{E}'$ is "smaller" than $\mathcal{E}$ if (n', q') is ranked below (n, q).

Next, since our main results rely on a double induction argument, we find convenience to introduce such a mathematical instrument in details.

Theorem **10** (Strong Double Induction). *Let $P(n, q)$ a statement over every economy $\mathcal{E}$ having size (n, q).*

If

- *For all $(n, q) \in \mathbb{N}^+ \times \mathbb{N}$, if for all $(n', q') \in \mathbb{N}^+ \times \mathbb{N}$ ranked below (n, q) $P(n', q')$ holds, then $P(n, q)$ holds*

then $P(n, q)$ is true for every $(n, q) \in \mathbb{N}^+ \times \mathbb{N}$.

That is, if the statement holds for every economy of size $(1, 1)$ and if we can prove it for every economy of size (n, q), given that the statement holds for every economy of smaller size, then the statement holds for every economy of any size.

This version of strong double induction can be found in Hrbacek and Jech (1978). Its proof is left as an exercise. For the sake of completeness, we include

[17] Remember that n is the total number of periods of the economy.

here a proof of its validity.

Proof of Theorem 10. Let $\preceq_\ell$ be the lexicographic order over $\mathbb{N}^+ \times \mathbb{N}$. First, note that $(\mathbb{N}^+ \times \mathbb{N}, \preceq_\ell)$ is a well-ordered set. The ordered set $(\mathbb{N}^+ \times \mathbb{N}, \preceq_\ell)$ is well-ordered since it is total ordered and it is such that every non-empty set $C \subseteq \mathbb{N}^+ \times \mathbb{N}$ has a minimal element[18] of C according to $\preceq_\ell$. Indeed, given any $C \subseteq \mathbb{N}^+ \times \mathbb{N}$, a minimal element of C is the element with the minimal first component, and, in case of a tie, the one with the minimal second component among the elements with a minimal first component.

Next, suppose toward a contradiction, that the set

$$D := \{(n, q) \in \mathbb{N}^+ \bigtimes \mathbb{N} | P(n, q) \text{ does not hold } \}$$

is non-empty. Let (n^*, q^*) be its minimal element. Since (n^*, q^*) is a minimal element of D, then for all $(n', q') \in \mathbb{N}^+ \times \mathbb{N}$ ranked below (n^*, q^*), we have $(n', q') \notin D$. Therefore, $P(n', q')$ holds and, by inductive hypothesis, $P(n^*, q^*)$ holds as well, which led to a contradiction. ■

Remark **7.** The reader who is interested in this type of exercise can now verify that the strong double induction is equivalent to a strong transfinite induction. Indeed, there is a natural order isomorphism ϕ between $(\mathbb{N}^+ \times \mathbb{N}, \preceq_\ell)$ and the ordinal $\boldsymbol{\omega}^{\mathbf{2}}$, that is $\phi(n, q) := (n-1)\boldsymbol{\omega} + q$.

The following lemma will be used thereafter.

Lemma **1.** *For any economy $\mathcal{E}$ and any $t \leqslant n$, a continuation economy $\mathcal{E}_{\geqslant t}$ is smaller than $\mathcal{E}$*

Proof of Lemma 1. Let $\mathcal{E}$ be an economy of size (n, q), (σ, μ) be an allocation over $\mathcal{E}$, and S be a coalition which period-t blocks (σ, μ) via (τ_t, ν_t) for some $t \leqslant n$. We have that either $t = 1$ or $t > 1$. If $t > 1$, then the continuation economy $\mathcal{E}_{\geqslant t}$,

[18] A minimal element of a set A with respect to $\preceq_\ell$ is an element $m \in A$ such that if $s \in A$, satisfies $s > m$ then necessarily $m \leqslant s$.

has size $(n - t + 1, q')$, thus it is smaller than $\mathcal{E}$. If $t = 1$, we claim that the continuation economy $\mathcal{E}_{\geqslant t}$ has size (n, q') with $q' < q$. Indeed, the agents in the coalition S, that are present in $t = 1$ of $\mathcal{E}$, are absent in $t = 1$ of $\mathcal{E}_{\geqslant t}$. To see this, note that, by definition of period-t blocking, agents in S either leave the economy to consume or they wait in the continuation economy from period 2. Moreover, no other agents enter in the continuation economy until period 2. ■

PROOF OF THEOREM 1. The result is implied by THEOREM 6 ■

PROOF OF THEOREM 2. We proceed by strong double induction. Define $P(n, q)$ as "the Dynamic Strong Core is a subset of the Dynamic Core for every economy of size (n, q)". To prove the inductive step, fix any size (n, q) and suppose, by inductive hypothesis, that the statement P holds for every economy of smaller size. We must show that $P(n, q)$ is true. Take any economy $\mathcal{E}$ of size (n, q) and consider any allocation in $\mathcal{E}$, namely (σ, μ), which is period-t blocked for some t by a coalition S via (τ_t, ν_t). We proceed by showing that the same coalition S can weakly period-t block (σ, μ) via (τ_t, ν_t). First, note that by definition of period-t blocking we have that $(\nu_t(a), t) \succ_a (\sigma, \mu)$ for all $a \in S \backslash S^w$. This implies that $(\nu_t(a), t) \gtrsim_a (\sigma, \mu)$ for all $a \in S \backslash S^w$ with at least one strict inequality for some agent in $S \backslash S^w$. Moreover, by same argument we have that $(\upsilon, \xi) \succ_a (\sigma, \mu)$ for all $a \in S^w$ and all allocations (υ, ξ) in the Dynamic Core of the economy $\mathcal{E}_{\geqslant t}$. Thus $(\upsilon, \xi) \gtrsim_a (\sigma, \mu)$ for all $a \in S^w$ and all allocations (υ, ξ) in the Dynamic Core of the economy $\mathcal{E}_{\geqslant t}$ with at least one strict inequality for some agent in S^w.

By LEMMA 1, the continuation economy $\mathcal{E}_{\geqslant t}$ is smaller than $\mathcal{E}$. Then, by the inductive hypothesis, the Dynamic Strong Core is a subset of the Dynamic Core in the continuation economy $\mathcal{E}_{\geqslant t}$. It follows that $(\upsilon, \xi) \gtrsim_a (\sigma, \mu)$ for all $a \in S^w$ with at least one strict inequality for some agent in S^w and all allocations (υ, ξ) in the Dynamic Strong Core of the economy $\mathcal{E}_{\geqslant t}$.

Therefore, since any allocation that is period-t blocked is also weakly period-t blocked, it must be that the set of allocations that is not in the Dynamic Core is neither in the Dynamic Strong Core. By strong double induction, this must hold for every economy of any size. ■

The proof of THEOREM 6 builds on the following lemmas and corollaries

LEMMA **2.** *For any* $\succ \in \mathcal{L}$*, the output* $\phi_{\triangleright}(\succ, \mathcal{E})$ *is an allocation.*

PROOF OF LEMMA 2. Fix any economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^{n}, \omega \rangle$ with preference profile $\succ \in \mathcal{L}$ and let $\phi_{\triangleright}(\succ, \mathcal{E}) \equiv (\sigma, \mu)$ be the outcome of ITTC at $\succ$ for some priority structure $\triangleright$. Note that, by construction, σ is an n-list of period-t exchange and μ an n-list of period-t consumption choices. We only need to show that for every $t \leqslant n$ the properties of consumption rivalry and irreversibility hold for (σ_t, μ_t). Since at any **Step** (t, k), agents consume only the object they are assigned to, then the property of consumption rivalry is satisfied for any σ_t, μ_t. Moreover, by construction of the ITTC, at any time t agents can consume only once and at the end of any time, every agent that consumed is removed from the procedure. Therefore, the property of consumption irreversibility is satisfied for any (σ_t, μ_t). ■

LEMMA **3.** *If a maximal period-t chain ends with an object, then it is composed by that object alone.*

PROOF OF LEMMA 3. We will prove the lemma by contradiction. Let's consider a maximal period-t chain that ends with the object h_i and that is not composed by h_i alone. This means that a_{i-1} is present in period t and that a_i is not present in period t. It follows that a_i is not the owner of h_i but the $\triangleright^t_{h_i}$-maximal agent. Therefore, agent a_i has higher priority over h_i than a_{i-1} who is already present in period t, This contradicts the definition of priority structure (Definition 8). ■

LEMMA **4.** *Let* (t, k) *and* (t', k') *with* $t < t'$ *be two steps of the ITTC algorithms. If there exists an agent* $a \in A$ *such that* $a \in C_A^{t,k} \cap C_A^{t',k'}$*, then* $C^{t',k'}$ *is the only subcycle of* $C^{t,k}$ *among the cycles formed at* t'*.*

Proof of Lemma 4. Let (t, k) and (t', k') with $t < t'$ be two steps of the ITTC algorithms. Suppose that there exists an agent $a \in A$ such that $a \in C_A^{t,k} \cap C_A^{t',k'}$. We show that $C^{t',k'}$ is the only subcycle of $C^{t,k}$ among the cycles formed at t'.

First we show that the statement hold for $t' = t + 1$ and $k = 1$

Since, by assumption, agent a belongs to $C^{t,1}$ and to $C^{t+1,k'}$ then $C^{t,1}$ must be intertemporal, otherwise no agent would belong to $C^{t+1,k'}$ since they would consume in period t.

To prove that $C^{t+1,1}$ is the only subcycle of $C^{t,1}$ is sufficient to show that at **Step** $(t + 1, 1)$ the following conditions hold:

[a] If an agent $a_i \in C_A^{t,1}$ is still present in period $t+1$, then he/she points to h_{i+1},

[b] If an object $h_i \in C_H^{t,1}$ is present in period $t+1$, then it points to the first agent in $C^{t,1}$ after h_i who is still present in period $t + 1$.

To prove $[a]$, suppose that an agent a_i in $C_A^{t,1}$ is present in period $t + 1$ and let (h_{i+1}, t'') be her preferred feasible pair. First note that (h_{i+1}, t'') is feasible for a_i. Indeed, it cannot be that $t'' = t$, otherwise agent a_i would have consumed in period t and she would be not present in period $t + 1$. Since the set of feasible pairs at **Step** $(t+1, 1)$ is a subset of the set of feasible pairs at step $(t, 1)$, then we have that (h_{i+1}, t'') is still the preferred feasible pair for agent a_i at **Step** $(t+1, 1)$. To prove $[b]$, consider any object $h_i \in C_H^{t,1}$ which is present in period $t + 1$. There are two cases to consider: either a_i is present at Step $(t + 1, 1)$ or he is not.

Case 1: Agent a_i is still present at **Step** $(t + 1, 1)$. In this case, either h_i is private and a_i is its owner or h_i is public and a_i is the agent with the highest priority. In both, h_i still points to a_i at **Step** $(t + 1, 1)$.

Case 2: Agent a_i is not present at **Step** $(t + 1, 1)$. This happens when a_i consumes at time t object h_{i+1}. Thus, a_i is the first agent of a chain at **Step** $(t, 1)$. If $h_i \in H_{\leqslant t}$, the owner of h_i at **Step** $t + 1, 1$ is the last agent a_ℓ of the chain initiated by agent a_i. If $h_i \notin H_{\leqslant t}$, the algorithm makes a_ℓ the agent with the highest priority for h_i at **Step** $(t + 1, 1)$. In both cases h_i points to a_ℓ at **Step** $(t+1,1)$ who is the first agent in $C^{t,1}$ after h_i still present in period $t+1$.

Next, fix $t' = t+1$ and $k' = 2$. The proof of the statement follows the same logic as in the previous argument. Although there are two caveats:

- $C^{t+1,2}$ is formed either at **Step** $(t+1,1)$, if all agents and objects in $C^{t,1}$ are present in period t, or at **Step** $(t+1,2)$ after the cycle $C^{t+1,1}$ is formed at **Step** $(t+1,1)$. In the latter case we proved that $C^{t+1,1}$ is a subcycle of $C^{t,1}$ comprising all agents and objects in $C^{t,1}$ still present in period $t+1$. Thus, for every agent in $C^{t,2}$ who does not consume in **Step** $(t,2)$ the set of feasible pairs at **Step** $(t+1,2)$ is a subset of the set of feasible pairs at **Step** $(t,2)$.
- There might exist some objects in $C^{t,2}$ that are private and their owners belong to $C_A^{t,1}$. These objects behave exactly as public objects in this case.

By iterating the same argument we can prove the statement for all cycles $C^{t,k}$ until they are cleared. ■

The following COROLLARY 1 states that cycles formed at step 1 drive the entire procedure. Namely, each agent, in all cycles which she belongs to, points to the object she pointed to in the cycle formed at **Step** $(1,k)$.

COROLLARY **1.** *Let* $(1,k)$ *and* (t',k') *with* $1 < t'$ *be two steps of the ITTC algorithms. If* $a_i, h_{i+1} \in C^{1,k}$ *and* $a_i \in C_A^{1,k} \cap C_A^{t',k'}$*, then* $a_i, h_{i+1} \in C^{t',k'}$

Moreover, the output of the ITTC mechanism is such that each agent consumes the object she points. This observation is a further corollary to LEMMA 4.

COROLLARY **2.** *For all cycle* $C^{t',k'}$*, if* $a_i, h_{i+1} \in C^{t',k'}$ *then* $\mu_t(a_i) = h_{i+1}$ *where* t *is the first period when* a_i *and* h_{i+1} *are both present in the economy.*

It follows that, according to the ITTC, each agent consumes the object she points in the cycle formed at **Step** $(1,k)$.

Corollary **3.** *For any cycle* $C^{1,k'}$*, If* $a_i, h_{i+1} \in C^{1,k'}$*, then* $\mu_t(a_i) = h_{i+1}$ *where* t *is the first period when* a_i *and* h_{i+1} *are both present in the economy.*

Proof of Theorem 6. We proceed by strong double induction. Define $P(n,q)$ as "In every economy of size (n,q) the outcome of the ITTC is in the Dynamic Core".

Fix an economy $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ of size (n,q), a preference profile $\succ \in \mathcal{L}$ and let $\phi_\triangleright(\succ, \mathcal{E}) \equiv (\sigma, \mu)$ be an allocation induced by the ITTC for some priority structure $\triangleright$.

By inductive hypothesis, suppose that for all economies $\mathcal{E}' = \langle (A'_t, H'_t)_{t=1}^{n'}, \omega' \rangle$ of smaller size, each allocation (σ', μ') induced by an ITTC is in the Dynamic Core of $\mathcal{E}'$.

We prove that the outcome (σ, μ) is in the Dynamic Core of $\mathcal{E}$. Suppose, toward a contradiction, that there exists a coalition $S \subseteq \mathcal{A}_t$ that period-t blocks (σ, μ) via $(\tau_t, \nu_t) \in \Sigma_t(S) \times \mathrm{M}_t(S)$. Then, by definition of period-t blocking

- $(\nu_t(a), t) \succ_a (\sigma, \mu)$ for all $a \in S \backslash S^w$
- $(\upsilon, \xi) \succ_a (\sigma, \mu)$ for all $a \in S^w$ for all allocations (υ, ξ) over the economy $\mathcal{E}_{\geqslant t}$ that are in the Dynamic Core

Note that, the property Consumption Rivarly (Definition 4) together with the fact that $\tau_t \in \Sigma_t(S)$, implies that $\nu_t(a) \in \omega_t(S) \cup \{h_0\}$.

Let k be the smaller number such that $S \cap C_A^{t,k} \neq \varnothing$ if it exists. Otherwise $k = m(t) + 1$[19]. Consider any agent $a \in S \cap C_A^{t,k}$. There are two cases to consider, both of them led to a contradiction.

Case 1: $a \in S \backslash S^w$ and $(\nu_t(a), t) \succ_a (\sigma, \mu)$. Note that agent a points to the object that belongs to the most preferred feasible pair, that is $(\mu(a), t(\mu, a))$, among those composed of object in $H^{t,k}$ in accordance with the ITTC.[20] By Corollary 2, agent a consumes object $\mu(a)$ as soon it arrives in the market, that is at $t(\mu, a)$.

[19]This is the case in which there are more agents than objects and all the members of the coalition do not belong to any cycle and they are not receiving any object in the end.

[20]If $k = m(t) + 1$, then $H = t, h$ is empty and any object is better for a.

Since $(\nu_t(a), t) \succ_a (\mu(a), t(\mu, a))$ and $(\nu_t(a), t)$ is feasible, it must be that $\nu_t(a)$ is not in $H^{t,k}$. This implies that $\nu_t(a)$ must belong to one of the previous cycles. Let $C^{t,j}$ be the cycle such that $\nu_t(a) \in C_H^{t,j}$ where $j < k$ and let a' be the agent toward which $\nu_t(a)$ points in $C^{t,j}$. Note that there are three cases under which an object points to an agent: either the object is private and the agent is its owner, or the object is private and the agents is the one with the highest priority among those still unassigned, or the object is public. We show that is all such a cases it holds that $S \neq \mathcal{A}_t$ and $\nu_t(a) \notin \omega_t(S)$, a contradiction.

- $\nu_t(a)$ is private in period t and a' is its owner in $C^{t,j}$. Note that a' is not in S but she is present in period t (private objects enter in the economy with their owner and objects cannot be left to agents not yet in the market). Therefore, $S \neq \mathcal{A}_t$ and $\nu_t(a) \notin \omega_t(S)$.
- $\nu_t(a)$ is private but its owner does not belong to $A^{t,j}$ and a' is the agent with the highest priority among those in $A^{t,j}$. This means that the owner belongs to a previous cycle than $C^{t,j}$, and thus she is not in S, but she is present in period t (private objects enter in the economy with their owner and objects cannot be left to agents not yet in the market). Thus $S \neq \mathcal{A}_t$ and $\nu_t(a) \notin \omega_t(S)$.
- $\nu_t(a)$ is public in period t and a' is the agent with the highest priority among those in $A^{t,k}$. In this case a' cannot enter the market after a, otherwise a would have a higher priority than a' according to $\nu_t(a)$. Thus $S \neq \mathcal{A}_t$ and, since $\nu_t(a)$ is public, it does not belong to $\omega_t(S)$.

Case 2: $a \in S^w$. The contradiction arises from the fact that there exists an allocation (υ, ξ) in the economy $\mathcal{E}_{\geqslant t}(S, \tau_t, \nu_t)$ which is in the Dynamic Core and such that not $(\xi(a), t(a, \xi)) \succ_a (\sigma, \mu)$. To see this, let (υ, ξ) be the outcome of the mechanism $\phi_{\rhd^t}(\succ, \mathcal{E}_{\geqslant t})$. Note by LEMMA 1, the continuation economy $\mathcal{E}_{\geqslant t}(S, \tau_t, \nu_t)$ has a smaller size than the economy $\mathcal{E}$. Then, by the inductive hypothesis, (υ, ξ) is in the Dynamic Core of $\mathcal{E}_{\geqslant t}(S, \tau_t, \nu_t)$. Since a

improves in the continuation economy with respect to (σ, μ), then $\xi(a)$ must belong to a previous cycle, $C^{t,j}$ with $j < k$, otherwise a would have pointed toward $\xi(a)$ at **Step** (t, k).

First, we show that objects belonging to previous $k-1$ cycles do not belong to $\omega_t(S)$. Indeed, for private objects, their owners belong to $C_A^{<k}$ and thus not to S. Then, private objects cannot belong to $\omega_t(S)$. For public object, let us suppose there exists a public object h_p in the first $k-1$ cycles that is part of a cycle at **Step** (t, j) with $j < k$. At **Step** (t, j), h_p points toward an agent a' who is part of the cycle $C^{t,j}$. Recall that a is part of a cycle in a later step (t, k) and she belongs to $A^{t,k}$. Since by construction of the algorithm is true that $A^{t,k} \subset A^{t,j}$, then a belongs to $A^{t,j}$ as well. Hence, a must have a lower priority than a' on h_p. This implies that a' enters in the economy before or at same period than aTherefore, $a' \in \mathcal{A}_t \backslash S$, S is not composed by all the agents in $\mathcal{A}_t$ and public objects as h_p does not belong to $\omega_t(S)$.

Next, we show that the outcome of the ITTC $\phi_{\rhd^t}(\succ, \mathcal{E}_{\geqslant t}) = (\nu, \xi)$ equals (σ, μ) for the agents in the first $k-1$ cycles. Indeed, all agents and objects in the first $k-1$ cycles are present in the first period of the continuation economy, that is at t. In particular, all the agents and objects in $C^{t,1}$ are present at **Step** $(t, 1)$ of the ITTC algorithm applied to the continuation economy. Since we are using the same priority $\rhd^t$ and initial ownership σ_{t-1} and agents have the same preferences $\succ$[21] then those agents and objects point to the same object or agent. This implies that the same cycle $C^{t,1}$ is formed and the same agents and objects are removed. For the same reason, if $k > 2$ at **Step** $(t, 2)$ the same cycle is formed. Iterating the argument, the same holds until step $(t, k-1)$.

Recall that, by COROLLARY 3, the final consumption of these agents are the objects they point to in the first period. Then, the agent that precedes the object $\xi(a)$ in $C^{t,j}$, a', consumes it. Thus, by the property of *Consumption*

[21] τ_t coincides with σ_{t-1} outside $\omega_t(S)$.

Rivalry it must be that $a' = a$, a contradiction.

Proof of Theorem 4. Let $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ be an economy. Fix any $\succ \in \mathcal{L}$ and let (σ, μ) be the output of the ITTC. By Theorem 6, (σ, μ) is in the Dynamic Core of the economy. To prove the claim, we show that (σ, μ) is also in the Strong Core. We list the first $m(1)$ cycles: $C^{1,1}, C^{1,2}..., C^{1,m(1)}$. Let $A(1), ..., A(m(1))$ be the corresponding set of agents, where $A(1)$ is the set of agents involved in the cycle $C^{1,1}$, $A(2)$ is the set of agents involved in the cycle $C^{1,2}$, and so on. Suppose, toward a contradiction, that (σ, μ) is weakly blocked by a coalition S. Moreover, let us assume that S is minimal, i.e., there is no blocking coalition S' strictly contained in S. Note that, since A is finite, if there exists a blocking coalition, the there exists a minimal one. Note also that, since Theorem 7 implies that (σ, μ) is Pareto efficient, then S must differ from A and hence any exchanges among agents in S must involve private objects only. So $S \subset A$, $(\tau, \nu) \succeq_a (\sigma, \mu)$ for all $a \in S$, $(\tau, \nu) \succ_a (\sigma, \mu)$ for some $a \in S$, and $\{\nu(a) | a \in S\} \subseteq \omega^{-1}(S)$.

Consider the first $j \leqslant m(1)$ such that $A(j) \cap S \neq \varnothing$ if exists. Otherwise $j = m(1) + 1$. We have that either there exists an agent $a \in A(j) \cap S$ such that $(\tau, \nu) \succ_a (\sigma, \mu)$ or not.

Suppose the former case. Since, by Corollary 2, agent a is already consuming in (σ, μ) the most preferred object that is feasible to him among those in $H^{1,j}$, it must be that $\nu(a)$ have been traded in some cycle $C^{1,j'}$ with $j' < j$. Since $\nu(a)$ is private, it belongs to $\omega^{-1}(A(j'))$. However S is effective for (τ, ν), so $\nu(a) \in \omega^{-1}(S)$. It follows that $A(j') \cap S \neq \varnothing$. This contradicts the assumption that j was the smallest integer such that with $A(j) \cap S \neq \varnothing$.

Suppose now that such an agent a does not belong to $A(j)$. This means that for all $a \in A(j) \cap S$, $(\sigma, \mu) \sim_a (\tau, \nu)$ and since no agents are indifferent between two goods, we have that $\mu(a) = \nu(a)$ for all $a \in A(j) \cap S$. Take any $a \in A(j) \cap S$. Since $\nu(a) = \mu(a)$, it holds that $\mu(a) \in \omega^{-1}(S)$ ($\mu(a)$ is a private object, thus its owner belongs to S. There are two possibilities to consider. (1) The owner of $\mu(a)$ belongs to a previous cycle $j' < j$. Then, $S \cap A(j') \neq \varnothing$, and j is not the smallest integer such that $S \cap A(j) \neq \varnothing$, a contradiction. (2) The owner of

$\mu(a)$ does not belong to a previous cycle, and thus she is pointed by $\mu(a)$ in $C^{1,j}$. Since $\mu(a)$ is pointed by a in $C^{1,j}$, the agents following a in $C^{1,j}$ belongs to S and this holds for every $a \in S \cap A(j)$, and thus $S \cap A(j)$ is closed under successor in $A(j)$. There are only two subsets $A'(j)$ of $A(j)$ closed under successor: $A(j)$ itself and the empty set. Since $A(j) \cap S \neq \emptyset$, then $A(j) \cap S = A(j)$ and $A(j) \subseteq S$. Thus $\nu(A(j)) = \mu(A(j)) = \omega^{-1}(A(j))$, so $\nu(S \backslash A(j)) \subseteq \omega^{-1}(S \backslash A(j))$ which means that $S \backslash A(j)$ is effective for (τ, ν). This together with the fact that there exists at least one agent in $S \backslash A(j)$ that strictly improve in (τ, ν) implies that $S \backslash A(j)$ is a blocking coalition, which contradicts the assumption that S is minimal.

■

PROOF OF THEOREM 5. Let $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ be a public economy. We show that a Dynamic Strong Core allocation exists. We proceed constructively. Let us consider a serial dictatorship type of rule. We impose a priority order $\rhd$ over the set of agents A which we require to be period consistent, i.e., agents that enter in the market in previous periods always have a higher priority than agents that enter later. The algorithm works as the usual serial dictatorship: every agent, in priority order, chooses the best object still unchosen and, as soon as they are both in the market, she consumes it. First, note that the output of this algorithm is an allocation of the economy $\mathcal{E}$. Let (σ, μ) be such an allocation. We claim that (σ, μ) is in the Dynamic Strong Core. We prove the claim by strong double induction. We prove the inductive step by contradiction. Suppose that there exists an economy $\mathcal{E}$ of size (n, q) and a coalition S which weakly period-t blocks (σ, μ) with (ξ_t, ν_t). There are two possibilities. There is at least one agent in S who improves in period t or not. Suppose the former case. Let a be the agent with the highest priority among those who strictly improves in period t. Note that $\nu_t(a)$ is public object that was assigned to some agent a' with a higher priority than a (otherwise a cannot improve), i.e., $\mu(a') = \nu_t(a)$. Since $\nu_t(a)$ is public and $\nu_t(a) \in \omega_t(S)$, then $S = \mathcal{A}_t$. Since $\rhd$ is period consistent then a' enters in the economy at the same period as or before a, and she has not consumed yet in period t—otherwise $\nu_t(a)$ would not be in the economy. This implies that $a' \in \mathcal{A}_t = S$. Let $a_1, a_2, ..., a_m$ be the agents in S in priority order. Note that agent a_1 consumes $\mu(a_1)$ either according to ν_t or in every allocation in the Dynamic Strong Core of the continuation economy. Indeed, every other preferred object has been previously consumed by an agent with an highest priority. Similar argument applies to a_2. In addition, agent a_2 cannot consume $\mu(a_1)$ nor according to ν_t nor in any allocation in the Dynamic Strong Core of the continuation economy since it is consumed by a_1, by previous argument. Thus, a_2 consumes $\mu(a_2)$ either according to ν_t or in every allocation in the Dynamic Strong Core of the continuation economy. By iterating the argument, we have that a' consumes $\mu(a')$ either according to ν_t or in every allocation in the Dynamic Strong Core of the continuation economy. A contradiction, since we stated that $\mu(a') = \nu_t(a)$.

Suppose now that no such agent $a \in S$ exists. By inductive hypothesis, the allocation on the continuation economy induced by the serial dictatorship is in the Dynamic Strong Core. Without loss of generality let us consider S composed only by agents who change their consumption with respect to μ, that is $S = \{a \in \mathcal{A}_t | \mu_t(a) \neq \nu_t(a)\}$. Take the agent a in S with the highest priority. By definition of weak blocking, she is weakly better off in every allocation in the Dynamic Strong Core of the continuation economy. However, since $\nu_t(a) \neq \mu_t(a)$ and the fact that she was consuming in μ_t, she cannot be indifferent in the continuation economy $\mathcal{E}_{\geqslant t}(\tau_t, \nu_t)$. Thus, in every allocation in the Dynamic Strong Core of the continuation economy, she consumes an object $\mu(a')$ for some a' with a higher priority. However, by applying the serial dictatorship algorithm to the continuation economy using the same priority $\triangleright$, we obtain an allocation in the Dynamic Strong Core where all the agents with a priority higher than a consume the same object as in (σ, μ), a contradiction. ■

Proof of Theorem 7. Let $\mathcal{E} = \langle (A_t, H_t)_{t=1}^n, \omega \rangle$ be an economy. Fix any $\succ \in \mathcal{L}$ and let (σ, μ) be the output of the ITTC.

We list the first $m(1)$ cycles: $C^{1,1}, C^{1,2} ..., C^{1,m(1)}$. Let $A(1), ..., A(m(1))$ be the corresponding set of agents, where $A(1)$ is the set of agents involved in the cycle $C^{1,1}$, $A(2)$ is the set of agents involved in the cycle $C^{1,2}$, and so on.

Suppose toward a contradiction that (σ, μ) is not Pareto efficient. Then there is an allocation (τ, ν) such that $(\nu(a), t(a, \nu)) \gtrsim_a (\mu(a), t(a, \mu))$ for all $a \in A$ with strict improvement for at least one agent.

We claim that for any $k \leqslant m(1)$, $\mu(a) = \nu(a)$ and $t(a, \mu) = t(a, \nu)$ for all $a \in A(k)$. To prove the claim we proceed by induction. First note that for all $a \in A(1)$ it must be that $\mu(a) = \nu(a)$ and $t(a, \mu) = t(a, \nu)$. Indeed, by Corollary 3 each $a \in A(1)$ consumes his first choice at (σ, μ) the same has to be under (τ, ν).

Next, fix any positive integer $k < m(1)$. Suppose $\mu(a) = \nu(a)$ for all $a \in \bigcup_{i=1}^{k} A(i)$. We show that $\mu(a) = \nu(a)$ for all $a \in A(k+1)$. To see this, note that by inductive hypothesis whether under (σ, μ) or under (τ, ν) every agent in $A(k+1)$ can obtain their favorite object among those that are not consumed by agents in $\bigcup_{i=1}^{k} A(i)$. Since, by Corollary 3 each agent $a \in A(k+1)$ consumes his first choice among the available objects under (σ, μ), the same has to be under (τ, ν). Then, $\mu(a) = \nu(a)$ and $t(a, \mu) = t(a, \nu)$ for all $a \in A(k+1)$.

Since the number of cycle is finite, the inductive reasoning proves our claim. Finally, note that, our claim together with the fact that $A = C_1 \cup ... \cup C_m$ implies that $\mu_T(a) = \upsilon_T(a)$ and $t(a, \mu) = t(a, \nu)$ for all $a \in A$, a contradiction. ■

Proof of Theorem 8. Let $\mathcal{E} = \langle (A_t, H_t)_{t=1}^{n}, \omega \rangle$ be an economy. Fix any $\succ \in \mathcal{L}$ and let (σ, μ) be the output of the ITTC for some $\rhd$.

As usual, we list the first $m(1)$ cycles: $C^{1,1}, C^{1,2} ..., C^{1,m(1)}$ and let $A(1), ..., A(m(1))$ be the corresponding set of agents.

We show that there is no $S \subseteq A$ and $\succ' \in \mathcal{L}$ with $\succ' = (\succ'_S, \succ_{A \setminus S})$ such that $\phi_{\rhd}(\succ', \mathcal{E}) \succ_a \phi_{\rhd}(\succ, \mathcal{E})$ for all $a \in S$. We prove the claim inductively by showing that for all $k \in \{1, ..., m(1)\}$ it holds that $A(k) \cap S = \varnothing$ and that $\phi_{\rhd}(\succ', \mathcal{E})$ generates the same cycles as in $\phi_{\rhd}(\succ, \mathcal{E})$.

First note that, by Corollary 3, for all $a \in A(1)$ it holds that all agents in the cycle $C^{1,1}$ get already their top alternatives. Therefore, $S \cap A(1) = \varnothing$ and $C^{1,1}$ can be the first cycle generated by $\phi_{\triangleright}(\succ', \mathcal{E})$.

Next, fix any positive integer $k < m(1)$. Suppose that for all $\ell \in \{1, ..., k\}$, that $A(\ell) \cap S = \varnothing$ and $C^{1,\ell}$ is generated by $\phi_{\triangleright}(\succ', \mathcal{E})$ also. We show that $C^{1,k+1} \cap S = \varnothing$ and $C^{1,k+1}$ is generated by $\phi_{\triangleright}(\succ', \mathcal{E})$.

By the fact that for all $\ell \leqslant k$ $C^{1,\ell}$ is generated by $\phi_{\triangleright}(\succ', \mathcal{E})$, it descends that $H \backslash H_{k+1}$ are consumed by the agents in $\bigcup_{i \leqslant k} A(i)$. Then, COROLLARY 3 applies and for all $a \in A(k)$ it holds that $\phi_{\triangleright}(\succ, \mathcal{E}) \succsim_a \phi_{\triangleright}(\succ', \mathcal{E})$ for all $a \in S$ and for any $\succ' \in \mathcal{L}$, that is all agent in $A(k+1)$ get their top items among those that are not exchanged by agent in $A(\ell)$ for some $\ell \leqslant k$. Therefore, $S \cap A(k+1) = \varnothing$ and $C^{k+1,1}$ is generated by $\phi_{\triangleright}(\succ', \mathcal{E})$. Note that all agents who do not belong to any cycles under $\phi_{\triangleright}(\succ, \mathcal{E})$ continue to do not belong to any cycles also under $\phi_{\triangleright}(\succ', \mathcal{E})$ thus they have no incentives to misreport their preferences. Therefore, the inductive reasoning proves that such a coalition S does not exist. ■

PROOF OF THEOREM 9. Let $\mathcal{E} = \langle (A_t, H_t)_{t=1}^{n} \omega \rangle$ be a private economy and fix any preference profile $\succ \in \mathcal{L}$. Let $(\sigma, \mu) \equiv \phi_{\triangleright}(\succ, \mathcal{E})$ be the allocation induced by the ITTC algorithm. As in the proof of THEOREM 4, denote by $A(1), ..., A(m(1))$ an ordered list corresponding to the set of agents involved in the algorithm' cycles formed in period 1. By similar argument, denote by $H(1), ..., H(m(1))$ an ordered list corresponding to the set of objects involved in the algorithm' cycles in period 1. Pick a decreasing sequence $p_1, ..., p_{m(1)}$ and let p_i be the price of all objects in $H(i)$, for all $i = 1, ..., m(1)$.

We claim that such a $p_1, ..., p_{m(1)}$ supports (σ, μ) as a dynamic competitive equilibrium. First, we prove that in any period the exchanges tracked by σ are feasible. Fix any $j \leqslant m(1)$ and any agent $a \in A(j)$. Then, a owns an object that is worth p_j and, in period 1, it can only afford objects in $H(j) \cup ... \cup H(m(1))$. By LEMMA 4, at any period-t, agent a can only trade objects that are contained in $H(j)$ with agents in $A(j)$. Therefore, any exchange for a will occur at price p_j. Finally we show that for agent a, $\mu(a)$ is the most preferred object among those with an affordable price. To see this, let $h \in H(j)$ be the object that he points in the cycle, that is the a'most preferred object among the feasible ones. By COROLLARY 1, agents continue to point h in any cycle he might belong to. Therefore, it must be that $p_{\mu(a)} \leqslant p_j$. Finally, according to COROLLARY 3, at (σ, μ) agent a consumes h at the time he prefers to. Since the choice of j and a was arbitrarily, we conclude that the same holds for any other agent in the economy. ■

APPENDIX B

EXAMPLE **7.** Let us consider a private economy which lasts two periods. There are five agents, $a_1, ..., a_5$, and five objects, $h_1, ..., h_5$. Without loss of generality, assume that object h_i belongs to agent a_i. Agents a_1, a_2, a_3 enter in period 1 while agents a_4, a_5 enter in period 2. Agents preferences are depicted in FIGURE 14 below. If $(h_\ell, 1)$ appears before than $(h_j, 2)$ in agent i's ranking, then she prefers being matched with object h_ℓ in period 1 than object h_j in period 2. Suppose that agents can wait only worth their own endowment, that is when they exchange they also consume.

$t = 1$			$t = 2$	
a_1	a_2	a_3	a_4	a_5
$h_2, 1$	$h_3, 1$	$h_5, 2$	$h_3, 2$	$h_1, 2$
$h_4, 2$	$h_1, 1$	$h_2, 1$	$h_4, 2$	$h_5, 2$
$h_1, 1$	$h_2, 1$	$h_3, 1$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

FIGURE 14: Preference list

We claim that no Dynamic Core allocations exist. Consider the allocation in which, in period 1, no agent exchanges and agent a_2 consumes her endowment; in period 2, a_1 gets object h_4, agent a_4 gets object h_3, agent a_3 gets object h_5, agent a_5 gets object h_1. This cycle is depicted in FIGURE 15.

The allocation obtained through these exchanges is period-1 blocked by agents $\{a_1, a_2\}$ who can exchange and consume their endowments already in period 1. By

$t=1$			$t=2$	
a_1	a_2	a_3	a_4	a_5
$h_2, 1$	$h_3, 1$	$\boldsymbol{h_5, 2}$	$\boldsymbol{h_3, 2}$	$\boldsymbol{h_1, 2}$
$\boldsymbol{h_4, 2}$	$h_1, 1$	$h_2, 1$	$h_4, 2$	$h_5, 2$
$h_1, 1$	$\boldsymbol{h_2, 1}$	$h_3, 1$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

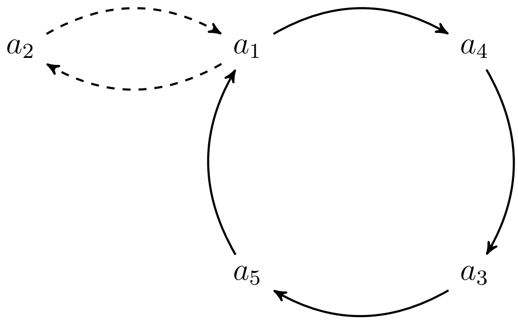

FIGURE 15: The allocation (in bold) obtained when all exchanges occur in period 2 and the blocking coalition (dashed).The allocation where a_1, a_3, a_4, a_5 exchange their objects is period-1 blocked by the couple a_1, a_2.

a similar argument, in any other allocation in which no exchange occurs in period 1 there is a period-1 blocking.

Next, consider any allocation involving an exchange in period 1. Notice that there is not an allocation in which agents a_1, a_2, a_3 are all involved in a cycle in period 1 and get an object better than their endowment. Consider then any allocation with a pairwise exchange among agents in period 1. Note that, under the assumption that agents can exchange only once, only exchanges between a_1 and a_2 and between a_2 and a_3 are profitable (a_1 and a_3 cannot perform any mutually advantageous trade since they prefers their endowments to exchange each others). Suppose that agents a_1 and a_2 exchange their endowments in period 1. Given that a_1 and a_2 are now out of the market, the other agents are not able to perform any further trade and remain with their endowments. A careful reader can note that this allocation can be period-1 blocked by agents a_2 and a_3 who can both gain by exchanging and consume their endowment in period 1. FIGURE 16 is an illustration.

Suppose now that agents a_2 and a_3 exchange their endowments in period 1. Given that a_2 and a_3 are now out of the market, the others agents are not able to perform any further trade and remain with their endowments. This allocation is challenged by agents $\{a_1, a_3\}$, who can wait together to exchange with a_4 and a_5 in period 2. Agents 1 and 3 form rational expectations on the allocations they could get in the continuation economy. In any allocation of the continuation economy

$t=1$			$t=2$	
a_1	a_2	a_3	a_4	a_5
$\boldsymbol{h_2, 1}$	$h_3, 1$	$h_5, 2$	$h_3, 2$	$h_1, 2$
$h_4, 2$	$\boldsymbol{h_1, 1}$	$h_2, 1$	$\boldsymbol{h_4, 2}$	$\boldsymbol{h_5, 2}$
$h_1, 1$	$h_2, 1$	$\boldsymbol{h_3, 1}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

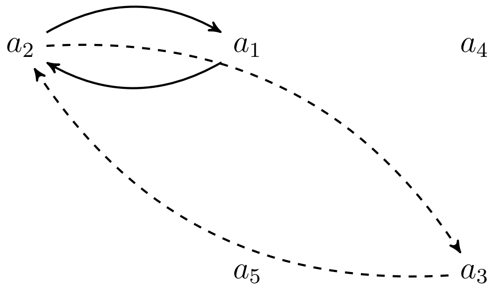

FIGURE 16: The allocation where a_1, a_2 exchange their objects is period-1 blocked by the couple a_2, a_3.

that cannot be period-t' blocked for $t' \geqslant 1$, agent 2 consumes her endowment in period 1, because she prefers consuming her endowment in period 1 than consuming any other object in period 2. In period 2 the only allocation that is not period-2 blocked by any coalition is the one in which the cycle of exchanges (a_1, a_4, a_3, a_5) is performed. It follows that the coalition $\{a_1, a_3\}$ can period-1 block the allocation in which agents 2 and 3 exchange their endowment in period 1.

The emptiness of the Dynamic Core in this example is driven by the assumption that agents can exchange only once. By relaxing this assumption, it is possible to restore the non-emptyness of the Dynamic Core. Suppose that agents a_1, a_2, a_3 form a cycle and exchange their goods; agents a_1 and a_2 consume and leave the market, while agent a_3 does not consume the object she received and remains in the market together with her new endowment h_1. Agent a_3 can now exchange with a_5 in period 2. In this allocation everyone except a_4 gets her most preferred object and agent a_4 consumes her endowment. Therefore no coalition of agents can improve upon it.

We find illustrative to apply the ITTC algorithm to the simple private economy discussed in Example 7. The ITTC mechanism for this economy consists of four Steps: $(1,1), (1,2), (2,1), (2,2)$. **Step** $(1,1)$ constructs a cycle $C^{1,1} = (a_1, h_2, a_2, h_3, a_3, a_5, h_1)$ where each private object points to its owner and each agents points her most preferred object (see FIGURE 17).

This is an intertemporal cycle since a_1, a_2, a_3 enter in period 1, while a_5 enters in period 2.

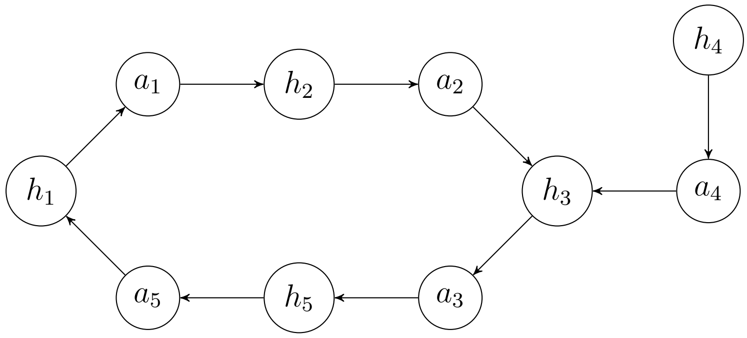

FIGURE 17: **Step (1,1)** The cycle $C^{1,1} = (a_1, h_2, a_2, h_3, a_3, a_5, h_1)$ forms.

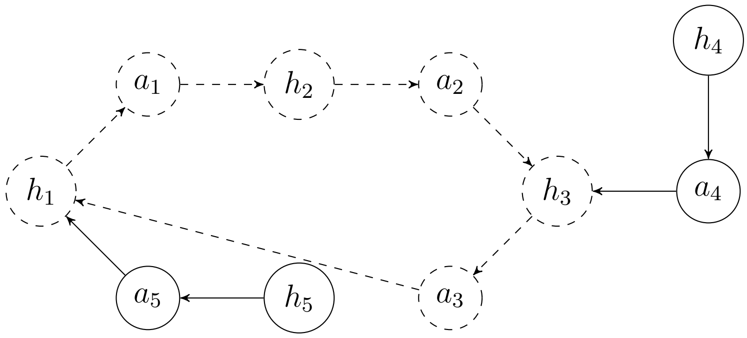

FIGURE 18: **Step (1,1)**. Since $C^{1,1}$ is intertemporal at $t = 1$ then the cycle is trimmed and agents a_1 and a_2 consume the assigned objects and are removed. Agent a_3 receives object h_1 and remains until period 2.

Then **Case 2** of the algorithm applies. Trim and clear the cycle $C^{1,1} = (a_1, h_2, a_2.h_3, a_3, a_5, h_1)$ by assigning h_2 to a_1, h_3 to a_2 and h_1 to a_3. Agents a_1 and a_2 consume the object assigned to them and are removed, agent a_3 does not consume h_1 that becomes her endowment in period 2. All other agents remain with their endowment. This results in the following "partial assignment"

$$(\sigma_1, \mu_1) = \begin{pmatrix} a_1 = \sigma_1(h_2) & \mu_1(a_1) = h_2 \\ a_2 = \sigma_1(h_3) & \mu_1(a_2) = h_3 \\ a_3 = \sigma_1(h_1) & \mu_1(a_3) = h_0 \end{pmatrix}$$

This step is illustrated in FIGURE 18.

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Step $(1, 2)$ constructs a direct graph using the set of agents and objects not in $C^{1,1}$ that is, in this case, only $A^{1,2} = \{a_4\}$ and $H^{1,2} = \{h_4\}$. The cycle $C^{1,2} = (h_4, a_4)$ is intertemporal at $t = 1$ since a_4 is not in the market at time 1. Then

Case 2 of the algorithm applies. Since there are no maximal open chain, there is no change in priorities, ownership nor consumption. Since there are no more cycles that can be formed, the algorithm proceeds with **Step** $(2,1)$ which generates one cycle between (h_5, a_5, h_1, a_3) and (h_4, a_4). Let $C^{2,1} = (h_5, a_5, h_1, a_3)$. Since $C^{2,1}$ is not intertemporal at $t = 2$ then **Case 1** of the algorithm applies and the cycle is cleared accordingly. This results in the following "partial assignment":

$$(\sigma_2, \mu_2) = \begin{pmatrix} a_3 = \sigma_2(h_5) & \mu_2(a_3) = h_5 \\ a_4 = \sigma_1(h_4) & \mu_1(a_4) = h_4 \\ a_5 = \sigma_1(h_1) & \mu_1(a_5) = h_1 \end{pmatrix}$$

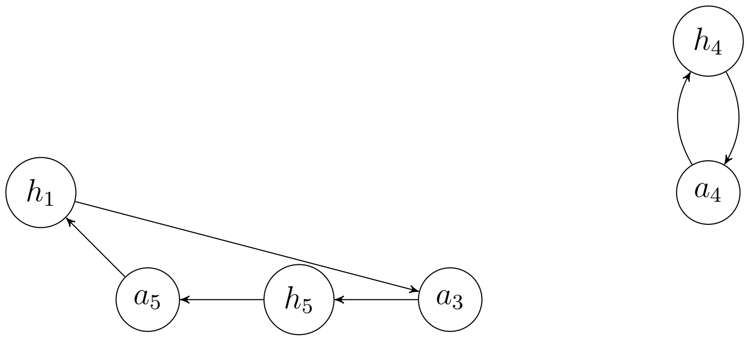

FIGURE 19: **Step (2,1) and Step (2,2).** Cycles $C^{2,1} = (h_5, a_5, h_1, a_3)$ and $C^{2,2} = (h_4, a_4)$ are created. Since both are not intertemporal at $t = 2$ then they are cleared accordingly.

Finally, in **Step** $(2,2)$ the cycle $C^{2,2} = (h_4, a_4)$ forms. The cycle is not intertemporal at $t = 2$ and is cleared according to **Case 1** of the algorithm. This final step is illustrated in FIGURE 19. The output of the algorithm is the allocation $(\sigma_1, \sigma_2, \mu_1, \mu_2)$.