Procurement with supplier rationing rule for market clearing *†

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Abstract

Procurement auctions with multiple winners usually have a split award, with a pre-specified split percentage. However, there are auctions, like renewable energy capacity creation auctions in India and Brazil, where the auctioneer doesn't specify the split, but asks the bidders to report the quantity they can supply before the auction. Thereafter, in an English auction, bidders compete on price, and the highest price winner is given a residual quantity to clear the market, thereby splitting the award ex-post. The perfect Bayesian equilibrium of the English auction under IPV assumption, for given quantity bids, is characterized by lesser competitiveness and a strictly positive probability of bidding the reserve by the bidder with the highest quantity. PBE is unique with 2 players. The data from renewable energy auctions in India further provides some preliminary empirical evidence of bunching.

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1 Introduction

Large-scale procurement auctions usually have multiple winners, each of whom receives only a portion of the total quantity. Typically, the auctioneer devises a method to determine the quantity allocation of each winner. One such method, which has been thoroughly analysed in auction theory, involves a split-award rule. Under this rule, the auctioneer sets a split percentage of the total procurement quantity that each winner will provide beforehand. For example, the split can be set at $50 - 50$, and bidders are invited to bid for 100% and 50% of the quantity. The award quantity for each bidder is then determined based on the auctioneer's cost minimization (For examples, see Anton and Yao, 1989; Anton, Brusco, and Lopomo, 2010).

However, in practice, auctioneers don't always decide on the split beforehand. Instead, they may invite bidders to report the quantity they can supply before the auction begins, and use a market clearing rule to equate supply and demand during the auction. For example, in renewable energy auctions in India and Brazil (since 2018), bidders submit a quantity bid before the auction starts, which may depend on their capacity constraints (See SECI, 2017; Tolmasquim et al., 2021, respectively for each country). During the auction, this bid is frozen, and bidders compete on price. The award is decided by eliminating the high-price bidders until the point where the sum of the quantity bids of surviving bidders is weakly greater than the procurement demand. The market is cleared through a simple rationing rule of awarding a residual quantity to the highest-price winner (or the marginal bidder).[1] In a way, this procedure splits the award ex-post where exact split depends on bidding strategies and the quantity bids.

In this paper, I analyse the implications of the rule of rationing the marginal bidder's quantity award to clear the market on the price bids in the auction, assuming

[1]This market clearing rule is chosen due to its simplicity. In Brazil and in some auctions in India, bidders are allowed to reject the residual award. However, I don't provide that option in my framework to keep it simple. Another simple rule is to increase the demand, which was followed in Brazil earlier. Guatemala employs a more complex mechanism of using linear optimization to minimize cost. Other examples can be found in chapter 5 of IRENA's guidebook for designing renewable energy auctions (Rabia Ferroukhi and Nagpal, 2015).

the quantity bids to be exogenous. In particular, I characterise Perfect Bayesian Equilibrium in a procurement auction, where bidders see each other's quantity, and bidding is open and descending from a high reserve, like in a reverse English auction. I choose this auction format mainly because of its use in Indian renewable energy auctions, whose bidding data is publicly available and is used here to provide some empirical support for the PBE.

I model the game as a button auction. When the auction starts, each bidder holds and presses a button, and a screen displays the reserve bid, b^R, procurement target, and each bidder's frozen quantity bid (q_i for bidder B_i). As the auction proceeds, the bid on the screen reduces continually, and the bidders release the button when their bid appears on the screen and exit the auction. Upon exiting, a bidder may either get a residual award equal to the net of the procurement target and quantity of bidders who are yet to exit or get 0 if this residual award is negative. The auction ends when a bidder exits with a positive residual award. The bid shown at the screen at the time of exit of such a bidder is the price paid by the auctioneer to the bidders with positive quantity award.

The game can be seen as a softened war of attrition, with bidders having a cut-off bid strategy to determine the bid at which they exit and take the residual. Each bidder can decide to have a lower cut-off bid in order to avoid rationing (equivalent to waiting), or agree to get the residual award at a higher bid (equivalent to exiting). If the residual award is zero, the bidders don't pay anything, unlike in a traditional war of attrition where bid/time of wait is a sunk cost.[2] In the auction analysed in this article, we have winners, a marginal loser, and losers. The losers are the bidders who exit the game with an award of zero. The marginal loser is awarded the residual quantity, and the winners' award is the quantity they bid. The presence of a small positive award for one of the losers further softens the war of attrition. Consequently, for the most part of the paper, I refer to this auction game as a soft

[2]In a traditional war of attrition, two siblings each want a single piece of pie. Both of them bid for it, either with money or with time which they would spend waiting for the opponent to give up. The winner gets the whole pie, while the loser gets nothing and also has to pay their bid (For example, see Maynard Smith, 1974).

war of attrition.[3]

I assume a constant marginal cost of provision of the good.[4] I characterise the asymmetric Perfect Bayesian equilibrium (PBE) when bidders have independent private costs drawn identically from the same distribution. In any subgame, if a bidder's residual award is zero, she would find it weakly dominant to bid her cost. The exit of such a bidder starts a new subgame. With 2 players, each having a positive residual award, PBE is monotonic. The bidder with a higher quantity bid is less competitive vis-à-vis the one with a lower quantity bid, and a positive mass of her cost types exits at b^R, i.e., she bunches at the reserve. This is because the marginal cost of competition is higher for the bidder with a higher residual, who is also the one with a higher quantity bid. Thus, she is less reluctant to compete. Such an equilibrium is a direct consequence of the simple rule used for market clearing. The equilibrium is unique in a 2-bidder game. I use the publicly available data for renewable energy auctions conducted by Solar Energy Corporation of India (SECI), which employs such rationing rule, to provide a preliminary evidence of bunching by the highest quantity bidder.

If the rules are such that winners get all the quantity, and marginal loser is clubbed with losers and gets nothing, this game reduces strategically to a uniform price auction with unit demand, where it's weakly dominant to bid own type. This is more competitive. On the other hand, having an all-pay feature without any residual award, would make the game similar to (but not same as) the generalised symmetric war of attrition of Bulow and Klemperer (1999), where $N + K$ bidders, each with unit demand, compete for K awards. In that game, a set of $N - 1$ players exits immediately, because competing can potentially yield negative ex-post payoff. This situation is less competitive.

This paper contributes to the literature on auctions and war of attrition. Levin (2004) provides a simple overview of the theory of war of attrition, except for non-

[3]This terminology is due to Ran Speigler (TAU), who pointed out the softening aspect.

[4]This assumption should hold true for my main application of large-scale solar project development (usually above 100MW).

trivial asymmetric equilibrium. Latter can be found in Nalebuff and Riley (1985) and are characterised by bunching by one of the players. This study also showed that there is a continuum of such equilibria. The framework has been applied to study problems pertaining to firm exit from a declining market (Ghemawat and Nalebuff, 1985; Fudenberg and Tirole, 1986; Takahashi, 2015), public goods game (Bliss and Nalebuff, 1984; Li, 2019), and second price all-pay auctions (Krishna and Morgan, 1997). However, all of this literature focuses on the traditional war of attrition, which are characterised by sunk cost. Finally, as already pointed out, my work also contributes to the literature on split awards in procurement auctions.

The rest of the paper is organised as follows. Section 2 presents my game and notations used in the paper. In section 3, I give formal results on equilibrium. The proofs for results in section 3 are in the appendix. Section 4 provides a preliminary evidence in favour of bunching using data from renewable energy auctions in India. Section 5 concludes the paper.

2 Model and notations

Before the auction, government announces the procurement target M for that auction. Each bidder, B_i announces her quantity $q_i \leq M$, which is the capacity they can create and provide to the government. Set of all the bidders is denoted by $\mathcal{N}$. The auctioneer announces the reserve price ($=b^R$). The auction is conducted to discover the price at which good is provided.

Each bidder is assumed to have a constant marginal cost of supplying the product, denoted by c_i. For each bidder B_i, c_i is private information, revealed to her before the auction. $c_i \overset{i.i.d}{\sim} F(c)$ and $c_i \in [\underline{c}, \bar{c}]$. The distribution is atomless with $f(\underline{c}) > 0$ where $f(c) = F'(c)$, unless specified otherwise. I denote the reversed hazard rate of this distribution, $f(c)/F(c)$ by $\sigma(c)$ and assume that $\sigma'(c) < 0, \forall c > \underline{c}$.

Allocation of M is done via an open descending price auction. Bidders bid the per unit price they would ask the government for providing the good. Right before the start of the auction, each bidder's quantity, q_i is made public. The auction process can be thought of as a button auction. At the start of the auction, auctioneer displays bid b^R on a screen and all the bidders press and hold a button. As auction proceeds, the displayed bid reduces in a continuous manner. If a bidder wishes to exit at a bid b, she releases the button when screen displays $b \leq b^R$. When she releases the button, she gets a residual quantity award of $Max\{0, M - \sum_i q_i \mathbb{1}_{B_i \in \mathcal{I}(b)}\}$, where $\mathcal{I}(b)$ is the set of bidders holding the button at bid b. The auction stops when a bidder gets a positive award when she releases the button, or if $M - \sum_i q_i \mathbb{1}_{B_i \in \mathcal{I}(b)} = 0$. The bidders who are still holding the button at the end of auction are awarded their quantity at the bid displayed on the screen at the end of auction.

$\mathcal{L}(b) \subset \mathcal{I}(b)$ denotes the set of bidders who would get an award of 0 if they exited at b. Exit of bidders at some bid b starts a new subgame with reserve bid b. These bidders are said to have a status of fully rationed at bid b. $\mathcal{P}(b) = \mathcal{I}(b) \backslash \mathcal{L}(b)$ denotes the set of bidders whose status is partially rationed at bid b. They get a positive quantity award if they exit at b, and their exit ends the game.

These status can be better illustrated with a numerical example. Assume that the government wants $M = 350$MW of capacity generation and there are 5 bidders, $B_1, B_2, ..., B_5$. These bidders bid $40, 50, 60, 100$, and 250 Megawatts of capacity, respectively. One can notice that B_5 will get a positive award regardless of when she exits. $B_5 \in \mathcal{P}(b), \forall b$. However, B_1 will get 0 if she exits when $\mathcal{I}(b) \in \{\mathcal{N}, \{B_1, B_2, B_4, B_5\}, \{B_1, B_3, B_4, B_5\}, \{B_1, B_4, B_5\}\}$. If a fully rationed bidder exits at some bid b, a subgame starts with b acting as a reserve bid in that subgame. A partially rationed B_i, while deciding her bid, has to account for probabilities of all possible permutations of players who would continue.

In any subgame starting at b with $\mathcal{P}(b)$ partially rationed bidders and $\mathcal{L}(b)$ fully rationed bidders, a bidder B_i's bid is denoted by $b_{i,\mathcal{P}(b),\mathcal{L}(b)}$. This is the bid at which she would exit, if none of the opponents opponents exit before that bid is reached. If some $B_j \in \mathcal{L}(b)$ exits at bid more than $b_{i,\mathcal{P}(b),\mathcal{L}(b)}$, B_i revises her bid to $b_{i,\mathcal{P},\mathcal{L} \backslash B_j}$.

The equilibrium bid of type c_i of bidder B_i is given by function $\beta_{i,\mathcal{P},\mathcal{L}}(c_i)$ for each bidder B_i. The game ends when a partially rationed bidder exits and takes a residual quantity as award.

In the next section, I would analyse simple cases involving 2 to 3 bidders to explain the equilibrium. For each case, I provide formal statement and analysis of equilibrium, followed by the key economic insight it elicits about the problem. In each of these cases, $\pi^W_{i,\mathcal{P}(b),\mathcal{L}(b)}(b_i; c_i, \mathbf{q}, \mathbf{b}_{-i})$ and $\pi^L_{i,\mathcal{P}(b),\mathcal{L}(b)}(b_i; c_i, \mathbf{q}, \mathbf{b}_{-i})$ denote the ex-post payoffs of players conditional on winning and losing the war respectively when they bid b_i and their cost is c_i. $\mathbf{q}$ is the vector denoting quantities of all the players and $\mathbf{b}_{-i}$ is that of price bids of opponents of B_i. The cases become analytically complex as we move to more number of players. In the cases with $\mathcal{L}(b) = \emptyset$ or with just 2 players, I will suppress the notations by dropping subscripts $\mathcal{P}(b)$ and $\mathcal{L}(b)$. Finally, I will denote $\underset{x \to c^+}{lim} u(x)$ by $u(c^+)$, and $\underset{x \to c^-}{lim} u(x)$ by $u(c^-)$ for any continuous function $u(x)$ and any real number c, unless otherwise specified.

3 Equilibrium in some simple case

In this section, I characterise and show uniqueness and existence of equilibrium for 2 to 3 player cases. I focus only perfect bayesian equilibria (PBE), which enables me to filter out trivial equilibria where one of the player never exits. While I find the PBE, beliefs of players are not explicitly specified, and appear in the optimisation conditions. I specify the tie-breaking rule when I describe the cases being analysed. These cases together give insights into PBE of the game. Specifically I look at the following cases:

1. Case 1P1F: 2 bidders, 1 partially rationed and 1 fully rationed at b^R. For example, $M = q_1 = 100, q_2 = 50$.

2. Case 2P0F: 2 bidders, both partially rationed at b^R. For example, $M = 100, q_1 = 70, q_2 = 60$.

3. Case 2P1F: 3 bidders, 2 partially rationed and 1 fully rationed at b^R. For example, $M = 100, q_1 = 70, q_2 = 50, q_3 = 10$.

3.1 Case 1P1F: 1 partially rationed and 1 fully rationed bidder at b^R

Assume $M = q_1 > q_2$. Thus, B_2 is fully rationed at all the bids. In this case, exit of any player would end the game. Ex-post payoffs of a bidder B_i, conditional on winning and losing respectively, are:

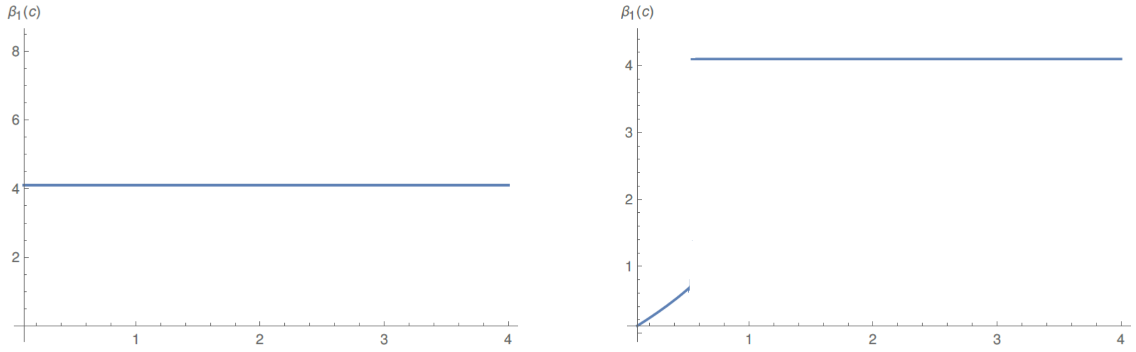
$$\pi^W_{i,B_1,B_2}(b_i; c_i, \mathbf{q}, b_{-i}) = q_i(b_{-i} - c_i)$$
$$\pi^L_{i,B_1,B_2}(b_i; c_i, \mathbf{q}, b_{-i}) = Max\{0, M - q_{-i}\}(b_i - c_i)$$

B_2 would find it weakly dominant to bid her cost. If she bids above and loses, she gets 0. If she wins, she still gets quantity award equal to her bid and a price equal to opponent's bid. Thus, she isn't really better off by bidding above her cost. Bidding lower than cost is dominated as that gives negative payoff. Thus, B_2's equilibrium bid function, $\beta_2(c) = c$.

B_1's equilibrium bid function is obtained as her best response to $\beta_2(c)$. This is obtained by maximisation of B_1's expected payoff, which is given by:

$$\pi_1(b_1; c_1, \beta_2(c)) = (M - q_2)(b_1 - c_1) + q_1 \int_{b_1}^{b^R} (x - c_1)dF(x)$$

For B_1, this situation reduces, analytically, to a decision problem, rather than a game. $\beta_1(c_1)$ is attained by finding $b_1 \in \underset{b \leq b^R}{ArgMax} \pi_1(b; c_1, \beta_2(c))$ for each c_1. If $\beta_1(c_1) < b^R$, then $\sigma(\beta_1(c_1))(\beta_1(c_1) - c_1) = \frac{M-q_2}{q_2}$ which is the first order condition of optimisation at an interior point. If for some c_1 this equality doesn't hold $\forall b < b^R$, $\beta_1(c_1) = b^R$, i.e., B_1 exits immediately at b^R. Together $\beta_1(c), \beta_2(c)$ constitute the PBE strategies of this case.



(a) Complete pooling: $q_1 = 100$, $q_2 = 40$ (b) Partial pooling: $q_1 = 100$, $q_2 = 80$

Figure 1: Equilibrium bid function of B_1

Equilibrium bid function for B_1 when $M = 100$, $b^R = 4.1$, and $F : [0, 4] \to [0, 1]$ is constrained Log-Normal with $\mu = 1$, $\sigma = 1$. Note that the scales on x-axis and y-axis are different.

To illustrate the equilibrium, I assume $c_i \overset{iid}{\sim} U(0, 1)$ without an atom. For $c > 0.2\sqrt{31} - 0.8 \approx 0.313$, B_1 would bid b^R when $M = q_1 = 3, q_2 = 2$. For other values of c, $\beta_1(c) = 2c$. Notice that the bidding function is discontinuous. This discontinuity is further illustrated in Figure 1b where a truncated lognormal distribution is assumed.

Consider the situation where some type c_1 bids b^R because $\sigma(b)(b - c_1) < \frac{M-q_2}{q_2}$, $\forall b < b^R$. If M or q_1 rise, and/or q_2 declines, this inequality is likely to be satisfied for a wider range of c_1. Thus, the extent of bunching would increase. Intuitively, rise in M and decline in q_2 reduces the extent of rationing faced by B_1. This makes her reluctant to compete when her cost isn't low enough to defeat B_2 who bids truthfully.

3.2 Case 2P0F: 2 partially rationed bidders at b^R

Consider the same information assumptions as before. However, in this case, $M > q_1 > q_2$, and $q_1 + q_2 > M$. Ex-post win payoffs are same as before. Ex-post loss payoff, $\pi_i^L = (M - q_{-i})(b_i - c_i)$ for both i. The ties are broken in favour B_2.[5] Unlike,

[5]This tie-breaking rule is not without loss of generality. In fact, it is set in this way in order to have equilibrium existence. This is similar to the idea in Simon and Zame (1990) on endogenising the tie-breaking rule. They prove that in the game where indeterminacy can arise due to unspecified tie-breaking rule, one can always find a tie-breaking rule consistent with equilibrium existence.

1P1F and second price auction, none of the players would bid truthfully in this case. Thus, B_i's expected payoff from the auction when she bids b_i, conditional on opponent's bid, b_{-i} and quantities q_1, q_2, M is:

$$\pi_i(b_i; b_{-i}, c_i, \mathbf{q}, M) = (M - q_{-i})(b_i - c_i)Pr(b_i > b_{-i}) + q_i \mathbb{E}_F(b_i - c_i | b_i < b_{-i})Pr(b_i < b_{-i})$$

I find perfect Bayesian equilibrium. Equilibrium bid function of B_i is denoted by $\beta_i(c)$. I characterise this PBE in following lemma:

Lemma 1. *For each B_i, $\beta_i(c)$ constitutes a Perfect Bayesian Equilibrium strategy of the 2 player asymmetric soft war of attrition if and only if it satisfies following properties:*

(i) *$\beta_i(c)$ is non-decreasing in c.*

(ii) *$\beta_i(c)$ is continuous and atomless for $b < b^R$ for both i.*

(iii) $\beta_i(\underline{c}) = \underline{c}$, $\forall i$.

(iv) *For each player B_i, $\beta_i(c)$ solves:*

$$\sigma(\beta_{-i}^{-1}(\beta_i(c)))\beta_{-i}^{-1'}(\beta_i(c))(\beta_i(c) - c)(q_1 + q_2 - M) = (M - q_{-i}) \tag{1}$$

(v) *$\beta_2(\bar{c}) = b^R$, and $\exists c^*$ such that $\beta_1(c) = b^R$, $\forall c \in [c^*, \bar{c}]$.*

Proof. See Appendix A.1 ∎

Characteristic (i) can be show using single crossing property as defined in Athey (2001). (ii) can be shown through standard arguments. If there is an atom at some bid b, the opponent's type which bids b will deviate to a bid slightly lower than b, if latter's strategy is continuous. If there is a discontinuity in strategies, such that the type $\beta(c) = b$ and type $\beta(c^-) = b' < b$, than the opponent types bidding between b' and b would prefer to bid b. These deviations are shown in

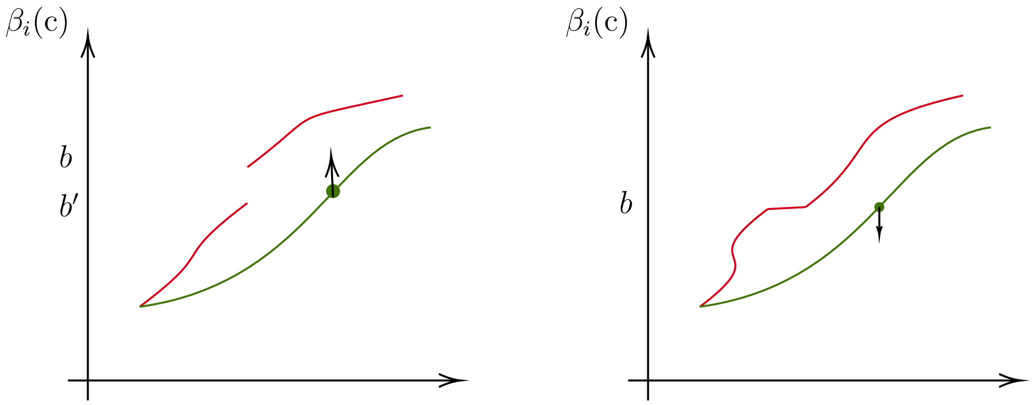


Figure 2: Possible deviations in case of discontinuity and presence of atom

Figure 2. Characteristic (iii) can be shown through arguments similar to Bertrand competition.[6] Characteristic (iv) can be obtained through first order conditions for optimum at an interior point. It requires invertibility of bid function, which is ensured by conditions (i) and (ii). Presence of $\sigma(\beta_i^{-1}(b))$ in FOCs exhibits the consistency in belief regarding opponent's exit, which is required for PBE.

Property (v) is the key characteristic of interest. It implies that a positive mass of high cost types of B_1 bid b^R, i.e., B_1 bunches at b^R. It relies on the relative marginal payoffs of two players at any point of intersection of the solution curves, which are such that $\frac{\beta_2'(c)}{\beta_1'(c)} = \frac{M-q_1}{M-q_2} < 1$ if $\beta_i(c)$s intersect at the cost c. The marginal payoffs are such that their solution curves intersect just once. Then, by continuity, strict monotonicity at $b < b^R$, and property (iii) and (iv), I show that even in the immediate neighbourhood of $\underline{c}$, $\beta_1(c) > \beta_2(c)$. Thus, the point of intersection can only be at $\underline{c}$. Therefore, the solution curves don't intersect at $b > \underline{c}$. Combined with the property that highest types of both players should bid b^R, it implies that $\beta_1(c) = b^R, \forall c \in [c^*, \bar{c}]$, while $\beta_2(c) = b^R$. This property also shows the importance of tie breaking rule in favor of B_2. In absence of this rule, whenever the two players bid b^R, B_2 has an incentive to reduce the bid slightly below b^R and avoid rationing with positive probability because B_1 is bunching at b^R. This tie-breaking rule makes

[6]Unlike here, in a traditional war of attrition, both the players would have a sunk cost of reducing the bid to $\underline{c}$. Thus, in absence of any award on losing, the player would prefer to unilaterally reducing their lowest to slightly below $\underline{c}$ which would lead to bids of $-\infty$ by the type $\underline{c}$, as in Nalebuff and Riley (1985).

B_2 indifferent between bidding b^R or slightly below b^R. Such an incentive doesn't exist for B_1 as possibility of tie for her is 0 because B_2 doesn't bunch.

Intuitively, B_1 is less aggressive and bunches because she has a higher marginal cost of competing (or reducing her bid) for any given cost type because she has a higher residual award. The gain in quantity conditional on winning is same for both the bidders ($=q_1 + q_2 - M$). Residual award is higher for B_1, which implies that competing is costlier for her. Thus, she is less aggressive, which gives her a higher markup ($= \beta_1(c) - c$) so that her overall marginal cost of competing is not as high. Thus, B_1's bid function is above B_2's until both of them have types in the immediate neighbourhood of $\underline{c}$. This also implies that for high cost types, B_1 has no incentive to compete at all, which leads to bunching.

Given the invertibility of $\beta_i(b)$; I can define the functions $\phi_i(b), \forall i$ as follows:

$$\phi_i(b) := \begin{cases} \beta_i^{-1}(b) & for\ b < b^R \\ Inf\{c : \beta_i(c) = b^R\} & for\ b = b^R \end{cases}$$

Hereafter, I refer to $\phi_i(b)$ as solution curve of B_i. Lemma 1 also implies that $\beta_1(c) > \beta_2(c), \forall c \in (\underline{c}, \bar{c}) \implies \phi_1(b) < \phi_2(b), \forall b > \underline{c}$. Figure 3 shows the equilibrium as characterised in Lemma 1, in terms of solution curves.

Till now, I haven't analysed the existence and uniqueness of PBE, which are not very obvious for the reasons similar to the ones for asymmetric first price auction (Lebrun, 2006). Any equilibrium is attained from the solution to Boundary value problem (BVP) given by FOCs (equations 1) and boundary conditions given by $\phi_2(b^R) = \bar{c}$, $\phi_1(b^R) = c^* < \bar{c}$ such that $\phi_1(\underline{c}) = \phi_2(\underline{c}) = \underline{c}$. The differential equations of this BVP have a division by 0 at the left boundary and hence, cauchy-lipschitz theorem is not applicable at $(\underline{c}, \underline{c})$. Thus, right boundary has to be used to establish existence, which is endogenously determined for $\phi_1(b)$. Using the FOCs, I can show existence of a c^* such that $\phi_1(b^R) = c^*$ and $\phi_1(\underline{c}) = \phi_2(\underline{c}) = \underline{c}$. Theorem 1 is formal statement of existence and uniqueness of PBE, which I prove in the appendix.

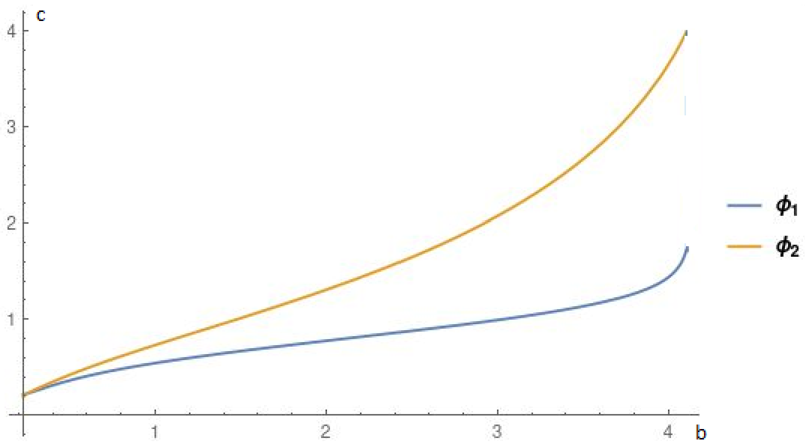


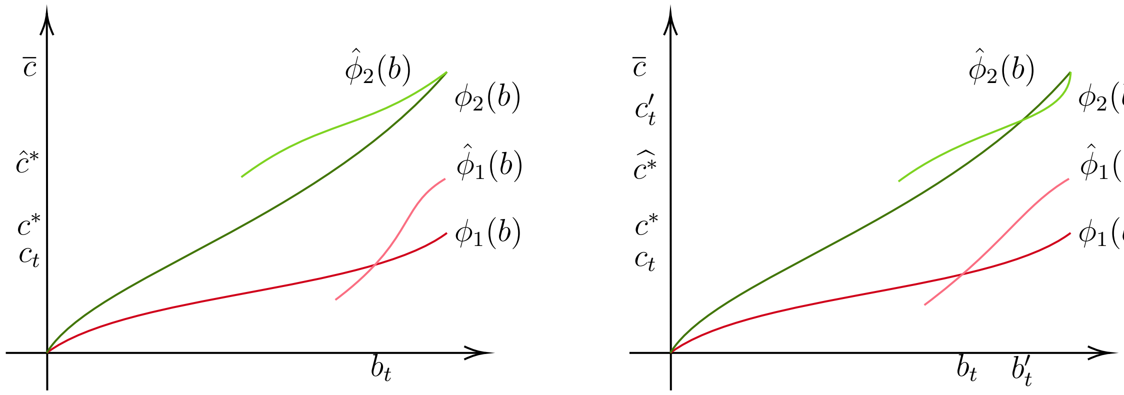
Figure 3: Assymetric equilibrium with 2 players

Equilibrium relation between bids and costs for B_1 and B_2 when $M = 200, q_1 = 175, q_2 = 150$ and $F : [0.2, 4] \to [0, 1]$ is constrained Log-Normal with $\mu = 1$, $\sigma = 1$. Y-axis has the costs and X-axis has the bid corresponding to those costs. All the cost types of B_1 above 1.73 are bunching at 4.1 and hence bid function for B_1 is not invertible in whole domain.

Theorem 1. *The PBE of 2P0F, as described in Lemma 1, exists and is unique.*

Proof. See Appendix A.2 ∎

Uniqueness can be understood through the argument similar to that of relative toughness in Lizzeri and Persico (2000). Consider two sets of solution curves $\phi_i(b)$ and $\hat{\phi}_i(b)$ such that $\phi_2(b^R) = \hat{\phi}_2(b^R) = \bar{c}$ and $\phi_1(b^R) = c^* < \hat{\phi}_1(b^R) = \hat{c}^*$, pertaining to "$\phi$" and "$\hat{\phi}$" situations respectively. As I show formally in appendix, this would imply that $\hat{\phi}_1(b) > \phi_1(b)$ and $\hat{\phi}_2(b) < \phi_2(b)$ for all $b \geq \underline{c}$. To understand this intuitively, consider the situation in the Figure 4a. At b^R, B_2 is bidding same in both equilibria, but is "marginally" more aggressive at b^R in $\hat{\phi}$ equilibrium (i.e., $\hat{\phi}_2'(b^R) < \phi_2'(b^R)$). As such, the probability of B_2's exit when B_1 bids in the immediate neighbourhood of b^R is lower. Thus, B_1 of type c^* should be less aggressive in $\hat{\phi}$ in order to compensate for this lower probability through a higher markup, as indicted by FOCs too. However, the figure suggests otherwise, and hence that situation can't happen. Thus, if $\hat{\phi}_1(b) > \phi_1(b), \hat{\phi}_2(b) < \phi_2(b)$ in the neighbourhood of b^R.



(a) (b)

Figure 4: Intersecting solution curves

Next, lets consider the points of intersection of $\hat{\phi}_1(b)$ and $\phi_1(b)$ and take the one with highest bid. Denote it by (b_t, c_t). B_1 is bidding same in both ϕ and $\hat{\phi}$ equilibria, but is less aggressive at the margin in the latter. As before, this will imply that B_2 should be more aggressive when her cost is c_t. This will suggest a situation show in the figure 4b. Finally, such an intersection in B_2's solution curves, by similar logic would imply that B_1 should be less aggressive when her type is the one who bids b_t' in ϕ, which is not in accordance to what we see in the figure. Thus, the solution curves in ϕ and $\hat{\phi}$ equilibrium should not intersect for all $b > \underline{c}$. Therefore, if $\hat{\phi}_1(b) > \phi_1(b), \hat{\phi}_2(b) < \phi_2(b)$.

At $\underline{c}$, given that the slope of the solution curves is infinite, we can't use the same logic. However, first order conditions require that the relative marginal payoff in the neighbourhood of a point of intersection should be such that $\phi_2'(\underline{c}^+)/\phi_2'(\underline{c}^+) = (M - q_2)/(M - q_1)$, which is a constant. This would imply that $\hat{\phi}_i(b) > \phi_i(b)$ for both i, which is not possible, as we already saw. Therefore, if $\hat{\phi}_1(b) > \phi_1(b), \hat{\phi}_2(b) < \phi_2(b), \forall b$. Therefore, only one possibility remains, in which either $\phi_1(\underline{c}) = \phi_2(\underline{c}) = \underline{c}$ or $\hat{\phi}_1(\underline{c}) = \hat{\phi}_2(\underline{c}) = \underline{c}$, but not both, as shown in Figure 5. Under certain regularity conditions, which I verify in the appendix, this implies uniqueness and existence of equilibrium.

While the result on existence and uniqueness is in line with the results on all-pay auctions without any residual reward for the losing bidder, there are some subtle

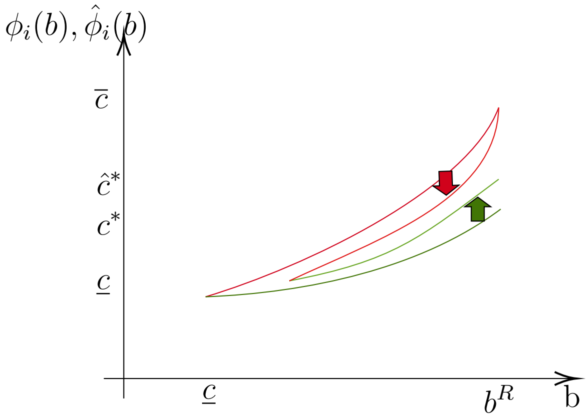


Figure 5: Co-movement of $\phi_1(b)$ and $\phi_2(b)$ in response to change in c^*

differences. For example, results in Lizzeri and Persico (2000) required loss payoff to be nonpositive. The result I have is attained even when the "loss" payoff is positive. Moreover, my result is in contrast with result on 2 player asymmetric war of attrition in Nalebuff and Riley (1985), which had a continuum of equilibria. In their case, many possible solutions to the FOCs satisfy the condition that player with highest type will wait for infinite time. While in this paper, the condition on slope ratio in the immediate neighbourhood of $(\underline{c}, \underline{c})$ prevents multiplicity.

The equilibrium characteristic that B_1 bunches depends crucially on the assumption that ex-post payoff are the only source of asymmetry and the cost distribution is same for both bidders. It should, however, be noted that payoff asymmetries can potentially arise from differences in cost distributions too. Till now, I have focused only on the former in order to clearly understand the effect of such an asymmetry. The insights developed here on the effect of quantity award heterogeneity also carry on to the situations where both sources of asymmetry are considered. However, the identity of bunching bidder depends on the net effect of dominance of cost distribution and ex-post award. I show this in Appendix A.3, where I provide a formal characterisation of the equilibrium and proofs for 2 cases of heterogeneity in cost distribution of the two players.

The first case is where $c_i \in [\underline{c}, \bar{c}_i]$, such that $\bar{c}_1 < \bar{c}_2$ and $c_i \overset{i.i.d}{\sim} F_i(c)$ such that $\sigma_1(c) = \sigma_2(c), \forall c \in [\underline{c}, min\{\bar{c}_1, \bar{c}_2\}]$. Intuitively speaking, B_2 is likely to have larger costs than B_1. This is a very specific case of first order stochastic dominance, but I analyse it because of its importance for the case with 2 partially and 1 fully rationed bidder. The proof of equilibrium characteristics is in Appendix A.3. In this case, B_1 is less aggressive than B_2. However, B_2 could bunch instead of B_1 if she is likely to have much higher cost than B_1, i.e., $\bar{c}_2 - \bar{c}_1 > \Delta > 0$. Given that the local incentives don't change much between this case, and the case with symmetric cost distribution, the equilibrium exists and is unique in this case. Note that if $\bar{c}_2 < \bar{c}_1$, then the equilibrium is same as the case with same cost distribution.

The second case is where $c_i \overset{i.i.d}{\sim} F_i(.)$ where each F_i has same support, $[\underline{c}, \bar{c}]$. Denote by $\sigma_i(c)$ the reversed hazard rate (RHR) of $F_i(c)$; $\sigma_i'(c) < 0$. I say that a distribution F_1 RHR dominates F_2 if $\sigma_1(c) \geq \sigma_2(c) c \in [\underline{c}, \bar{c}]$. Dominance can imply having higher probability of higher costs. It is possible that B_2 bunches if she is far more likely to have higher costs in comparison to B_1. If equilibrium exists in this case, then at any point of intersection of $\phi_1(b)$ and $\phi_2(b)$, $\phi_1'(b) < \phi_2(b)$ if $\frac{\sigma_1(c)}{\sigma_2(c)} > \frac{M-q_1}{M-q_2}, \forall c$. If $\sigma_1(c) > \sigma_2(c), \forall c$, this inequality is satisfied because $M - q_1 < M - q_2$ which implies that B_1 would be bunching. In this scenario, B_2 is more likely to have lesser costs and B_1 has a larger residual capacity, thus high cost types of latter would prefer to exit immediately. Furthermore, the effect of residual quantity is likely to dominate the effect of RHR dominance as long as the RHR are not very different when $\sigma_1(c) < \sigma_2(c), \forall c$. Theoretically, B_2 can bunch only if $\sigma_1(c) < \sigma_2(c)\frac{M-q_1}{M-q_2}, \forall c$, i.e., if B_2 is much more likely to have higher costs than B_1.

One can notice that the intuition attained on the effect of differences in ex-post quantity award in the case of same cost distributions for each bidder case is robust to differences in cost distributions, even though the net effect is different. What matters for the equilibrium structure, and specially for the identity of bunching bidder is the net effect of cost distribution dominance and quantity bids.

To conclude the analysis of 2 player setting, I should also provide the comparative statics with respect to M and q_i. For doing this in the simplest manner, I look at the special case of symmetric equilibrium attained when $q_1 = q_2 = q$. The symmetry assumption is just for the ease of analysis. The intuition obtained should, nonetheless, carry on to the asymmetric equilibrium too.

3.2.1 Special case: Symmetric 2 player equilibrium

Assume $q_1 = q_2 = q < M$. This gives rise to a symmetric equilibrium, where bidders do not bunch. For this section, suppose costs $c_i \overset{i.i.d}{\sim} U(0,1)$ and there is no atom at 0.[7] I can solve following differential equation to find equilibrium bid function:

$$\beta'(c) + \frac{M-2q}{M-q}\frac{\beta(c)}{c} = \frac{M-2q}{M-q}$$

The solution to above is:

$$\beta(c) = \begin{cases} c^{\frac{2q-M}{M-q}}\left(b^R + \frac{2q-M}{2M-3q}(1 - c^{\frac{2M-3q}{M-q}})\right) & ;\ M \neq 1.5q \\ c.b^R - c.ln(c) & ;\ M = 1.5q \end{cases}$$

This is the unique symmetric PBE. Equilibrium bid function is monotonically increasing in c because $\beta(c) \geq c$. This is in line with standard results on symmetric equilibrium in auctions.

Having a closed form solution eases the comparative statics with respect to q and M. We can see that any decrease in M or increase in q would be accompanied by decline in $\beta(c)$, and increase in $\beta'(c)$. In a way, bidders are more competitive, both absolutely and marginally. Decrease in M and increase in q leads to decline in the residual capacity, which reduces the marginal cost of competing. It also increases the quantity award in case of win, which increases the marginal benefit of competing. Thus, each bidder bids lesser for all the types. Morevover, the increase in $\beta'(c)$ means that the probability of opponent's exit $(= \sigma(\beta(c))/\beta'(c))$ also reduces.

[7] While it is possible to provide a solution for a general distribution $F(c)$, the expression there is not very clean. Hence, I use uniform distribution in order to provide an expression which is easy to follow and analyse.

3.3 Case 2P1F: 2 partially rationed 1 fully rationed bidder at b^R

Suppose 3 bidders B_1, B_2, and B_3 have quantities q_1, q_2, and q_3 respectively, such that,$q_1 > q_2 > q_3$, $q_1 + q_2 > M$ but $q_1 + q_3 < M$ and $q_2 + q_3 < M$. Thus, B_1 and B_2 can together cover the whole demand. Furthermore, assume a very small atom in the cost distribution at $\underline{c}$. In this scenario, exit of B_1 or B_2 will end the game, but exit of B_3 will start a subgame between the former 2. Denote the set of partially rationed bidders at any bid b by $\mathcal{P}(b)$, and the set of fully rationed bidders by $\mathcal{L}(b)$. In this example, $\mathcal{P}(b^R) = \{B_1, B_2\} = \mathcal{A}2$, $\mathcal{L}(b^R) = \{B_3\}$. Denote the equilibrium bid function of B_i by $\beta_{i,\mathcal{A}2,B_3}(c)$ in the subgame with all players, and $\beta_{i,\mathcal{A}2,\emptyset}(c)$ in the subgame started by B_3's exit.

Denote by $\mathbf{b}$, the vector of bids of all the players. If B_i is partially rationed and bids b_i, the other partially rationed bidder bids b_{-i}, and B_3 bids b_3, her payoff when her type is c_i is:

$$\begin{aligned}\pi_i(b_i; c_i, \mathbf{b}) =&(M - q_{-i} - q_3)(b_i - c_i)Pr(b_i = max_j\{b_j\})\\ &+ q_i\mathbb{E}(b_{-i} - c_i | b_{-i} > b_3, b_{-i} > b_i)Pr(b_{-i} = max_j\{b_j\})\\ &+ \mathbb{E}(\pi^*_{i,\mathcal{A}2,\emptyset}(b_3) | b_i < b_3, b_{-i} < b_3)Pr(b_3 = max_j\{b_j\})\end{aligned}$$

where $\pi^*_{i,\mathcal{A}2,\emptyset}(b_3)$ is the payoff for B_i in the subgame started by B_3's exit.

As in 1P1F, $\beta_{3,\mathcal{A}2,B_3}(c) = c$. B_1 and B_2 best respond to that and to each other in equilibrium. The equilibrium for 2P1F is formalised in the lemma below.

Lemma 2. *$\beta_{3,\mathcal{A}2,B_3}(c) = c$. $\beta_{i,\mathcal{A}2,B_3}(c)$ for $i \in \{1, 2\}$, gives a PBE if and only if:*

(i) $\beta_{i,\mathcal{A}2,B_3}(c)$ is non-decreasing in c.

(ii) $\beta_{i,\mathcal{A}2,B_3}(c)$ is continuous and atomless for $b < b^R$ for both i.

(iii) $\beta_{i,\mathcal{A}2,B_3}(\underline{c}) = \underline{c}$, $\forall i$.

(iv) $\forall i, \beta_{i,\mathcal{A}2,B_3}(c_i)$, *solve following differential equations:*

$$
\begin{aligned}
&(\pi^*_{i,\mathcal{A}2,\emptyset}(b;c_i) - (M - q_{-i} - q_3)(\beta_{i,\mathcal{A}2,B_3}(c_i) - c_i))\frac{f(\beta_{i,\mathcal{A}2,B_3}(c_i))}{F(\beta_{i,\mathcal{A}2,B_3}(c_i))}\mathbb{1}_{b\leq\bar{c}} \\
&+ (\beta_{i,\mathcal{A}2,B_3}(c_i) - c_i)(\sum_j q_j - M)\frac{f(\beta^{-1}_{-i,\mathcal{A}2,B_3}(\beta_{i,\mathcal{A}2,B_3}(c_i)))\beta^{-1'}_{-i,\mathcal{A}2,B_3}(\beta_{i,\mathcal{A}2,B_3}(c_i))}{F(\beta^{-1}_{-i,\mathcal{A}2,B_3}(\beta_{i,\mathcal{A}2,B_3}(c_i)))} = M - q_{-i} - q_3
\end{aligned}
\tag{2}
$$

where $\pi^*_{i,\mathcal{A}2,\emptyset}(b;c_i)$ *is the payoff of* B_i *in the subgame started with exit of* B_3.

(v) $\exists c^* \leq \bar{c}$ *such that* $\beta_{1,\mathcal{A}2,B_3}(c) = b^R, \forall c \in [c^*, \bar{c}]$, *and* $\beta_{2,\mathcal{A}2,B_3}(\bar{c}) = b^R$.

Proof. See Appendix A.4. ■

PBE described here looks the same that of 2P0F case, except that there is a kink at $b = \bar{c}$. The proof is also similar, except for some additional steps for (i) and (v). For (v), I show that there will be at most one point of intersection between $\beta_1(c)$ and $\beta_2(c)$. At any point of intersection, $\frac{\beta'_{1,\mathcal{A}2,B_3}(c)}{\beta'_{2,\mathcal{A}2,B_3}(c)} = \frac{M-q_2-q_3-(\pi^*_{1,\mathcal{A}2,\emptyset}(b,c)-(M-q_2-q_3)(b-c))\sigma(b)}{M-q_1-q_3-(\pi^*_{2,\mathcal{A}2,\emptyset}(b,c)-(M-q_1-q_3)(b-c))\sigma(b)}$ for $b \leq \bar{c}$. If B_3 were to exit at bid b pertaining to the point of intersection, then a subgame same as 2P0F starts with b as reserve. As we know from Lemma 1(v), B_1 of type c pertaining to this bid, will also exit at b in this subgame. This gives us the values for $\pi^*_{i,\mathcal{A}2,\emptyset}(b,c)$ for each i, which are such that the aforementioned slope ratio is above 1. Thus, there is only one possible point of intersection between $\beta_{1,\mathcal{A}2,B_3}$ and $\beta_{2,\mathcal{A}2,B_3}$, and that point is $(\underline{c}, \underline{c})$ for reasons same as in 2P0F.

The intuition behind a similar equilibrium as in 2P0F is that B_3's presence affects both B_1 and B_2 in the same way. It reduces their residual capacity by the same amount and the marginal probability of B_3's exit at any bid is same for both the bidders. Thus, B_1 is still less reluctant to compete vis-a-vis B_2.

Since the equilibrium characteristics are similar to that of 2P0F, the existence and uniqueness results remain unchanged. The two key conditions leading to uniqueness and existence in 2P0F are: (i) solutions to the boundary value problem for different boundaries is such that $\phi_2(b)$ is lower if $\phi_1(b)$ is higher for a given boundary, and (ii) if $\phi_1(b)$ and $\phi_2(b)$ intersect at $\underline{c}$, ratio of their slopes at $\underline{c}$ is fixed. These conditions are unaffected in 2P1F. Hence, as I prove in the appendix, the equilibrium exists and is unique in 2P1F.

Theorem 2. *The equilibrium described by Lemma 2 exists and is unique.*

Proof. See Appendix A.5. ■

Here I showed how the equilibrium from 2P0F extends to 2P1F. Barring some important nuances, the characteristics of PBE are same.

In the next section, I take my framework to data. In particular, I first provide the rules of renewable energy auctions in India, and comment on the applicability of my model in that setting. Then, I provide some preliminary empirical evidence in favor of bunching.

4 Preliminary empirical evidence for bunching

In this section, I provide empirical evidence in favour of PBE. For this purpose, I use the data from renewable energy auctions in India, where the rationing rule as described in section 2 is employed for market clearing. I will first provide a synopsis of institutional background to explicitly provide link between those auctions and the soft WoA presented in the previous section. After that, I provide data and preliminary evidence.

4.1 Relevant Institutional background: Renewable energy auctions in India

A common feature of SECI auctions for renewable energy capacity creation is a pre-announced procurement target, 2-round bidding procedure, and supply side rationing rule for market clearing same as the one analysed in section 3. The auctions provide 25 year contracts to bidders to sell the electricity to SECI. There are 2 types of auctions. In one of them, the bidders price bids refer to the amount of subsidy (VGF) they would need per MW of capacity created, for making their project financially viable at a reserve tariff set by SECI. In others, bidders' bids are the tariffs they would charge per KiloWatt-hour (KWh) of electricity produced from the capacity they create. Bidders in VGF auctions can bid with tariffs, provided the tariff is less than the reserve tariff. Tariff bids are ranked better than VGF ones. Using a capacity utilisation factor (CUF), one can convert VGF to tariff and vice versa. Furthermore, this CUF depends on weather conditions and can be assumed to be same for all bidders in same location, and over 25 years period (once adjusted for seasonality). Thus, the model presented in previous section can be used for such auctions.

Since the focus here is on finding some preliminary evidence supporting the equilibrium results derived in the soft WoA in section 3, I focus only on the second round of auction. I describe this round properly and abstract away from all the other institutional details. While interesting in themselves, these details affect the PBE of soft WoA only mildly, and hence can be skipped for the sake of brevity.

Before the auction, the auctioneer announces the total capacity it wants winning bidders to develop, denoted by M. With the knowledge of M, each bidder declares her capacity bid q_i and gives a price bid which would act as her own reserve bid in the auction. If $\sum_i q_i < M$, auctioneer reduces M in a predefined manner. The auction is open and happens online with each bidder getting a pseudo-identity. Right at the start of the auction, each bidder's q_i, and first round bid (which can't be increased) is publicly displayed.[8] Each bidder decides whether she wants to reduce her bid from first round bid or not. If no bidder reduces their bid after 52$^{\text{nd}}$ minute, the auction ends in 1 hour. If some bidder changes their bid after 52$^{\text{nd}}$ minute, then

[8]The first round is a sealed bid auction, where each bidder submits the capacity and tariff bid. The bidders are ordered as per tariff bid. Starting from the top, a bidder is selected if the cumulative capacity till that bidder is below the demand. Once the cumulative supply exceeds demand, the top half of the remaining bidders is also selected. The price bids in first round form the maximum bid in second round, and quantity bids are frozen.

everyone else gets 8 minutes to change their bid. Auction ends when no bidder has changed their bid for 8 minutes. At every moment during the auction, bidders can see each others' current bids.

As the auction proceeds, fully rationed bidders start reducing their bids. After exit of some such bidders, we reach a point in the auction where a group of bidders, at least one of whom is partially rationed, have same bid. One can consider this bid as the reserve bid, b^R for the auction game with these bidders. Competing further implies reducing this bid. For my analysis of soft WoA, I only consider these bidders. There is another caveat I must mention. There are some bidders who have first round bid much lesser than b^R. This doesn't affect the applicability of the theoretical framework, as the war is only for the capacity left after such bidders are allotted their quantity bid. Unless we are in a subgame where the reserve is equal to or smaller than their bids, these bidders are tentative winners and can't change their bid. They can't be strategic and the only impact they have is in terms of quantity award. Thus, they are not important in bidding strategy of other bidders. Furthermore, it's convenient for modeling to assume same reserve bid ($=b^R$) for all the WoA players.

One can see how this auction is easily modeled by the button auction described in previous section. At any given point after some bidders have a bid of b^R, these bidders will reduce their bid by the minimum reduction allowed. A change in bid by one bidder is analogous to change in the displayed bid. Decision of a bidder to not compete by reducing own bid to match this bid is equivalent to releasing the button and exiting. If this bidder gets 0 quantity, the game continues, until no one changes their bid for another 8 minutes. Once auction ends, the top W ranked bidders would get some quantity award. Among these, W^{th} ranked bidder would only get a residual of M and quantity awards of top $W-1$ ranking bidders. In analogy of button auction, it is as if the top $W-1$ bidders are holding the button, while everyone else has released it, including the partially rationed W^{th} ranked bidder.

Finally, one can wonder that in the second round, bidders might be able to learn something about opponents' costs from their first round bids. Given that there is some time between the 2 rounds, bidders' private information on cost is subject to some shock. As a result, the learning from the first round would manifest itself in the posterior distribution over opponents' cost in the second round. The posteriors can be different. As I show in section 3.2.2 and appendix A3, this heterogeneity can be easily incorporated in the soft WoA framework with same cost distributions for each bidder. Hence, the equilibrium found in the previous section, and the intuitive understanding developed there regarding the effect of different quantity awards, are unaffected by such learning.

4.2 Data and preliminary evidence

I collect data from two sets of documents in the public domain. The first set of documents are named Request for Submission (RfS) released by SECI to the bidders. These documents provide auction specific characteristics like the capacity targets, reserve price, location of project, and so on. The other set of documents provide the bids and awards of all bidders for both rounds.

I observe data from 43 auctions by SECI, which provide a total of 265 bids for the second round where soft WoA emerges. Among these 123 bids are of fully rationed bidders, which are same as their respective costs.

From the data, we can make some observations about equilibrium behaviour in soft war of attrition. Among the tariff auctions with large procurement targets (above 200MW), I observe that in 11 out of 28 auctions, the marginal loser immediately exits, i.e;, has bid within Rupee 0.01 of the lowest of bids of all the losing bidders. In 9 of such 11 auctions, it is the bidder with highest quantity bid who exits immediately and agrees to be rationed. Three of these auctions have no partially rationed bidders. In 2 auctions, the bidders exercise the option to reject the residual capacity allotted to them, which is provided to them only in some recent auctions, if the award is less than half of capacity bid. In these auctions, we do not observe any competition.

Among the 14 VGF auctions analysed (with $M \geq 40$), 6 have no partial rationing as each bidder's quantity bid equal M. Among the remaining, there was immediate exit in 3 auctions, all of them by the bidder with highest quantity bid. In 2 auctions, we do not see immediate exit. In 3 auctions, the winner had a very low first round bid and quantity bid equal to M, which led to absence of competition in second round. Overall, I can say that in the auctions where soft war of attrition occurred, there was immediate exit in almost half of the occurrences by the highest quantity bidder. This indicates that theoretical equilibrium should exhibit partial pooling by the bidder with highest quantity, if it has to explain the observed data.

To further explore the relation between quantity bid and immediate exit, I estimate a simple linear probability model and a probit model. These models are in no way causal, and are estimated just to capture the correlation between extent of rationing and decision to exit immediately. I use only the tariff auctions data for this purpose.

The immediate exit is captured by indicator variable $concede_{it}$ which equals 1 if in auction t, the partially rationed bidder B_i, bids same or 0.01 less than the fully rationed bidder with lowest bid. I don't use the bids from auctions without partial rationing, and auctions where some bidder exercised the right to reject the residual award. As we noticed earlier, whenever a fully rationed bidder exits, a subgame is created among remaining bidders. If I observe such a situation in a particular auction, I consider the subgame generated by exit of a bidder as a separate auction. In each subgame where the partially rationed bidder doesn't exit immediately, the indicator for conceding is set at 0. In the terminal subgame where a partially rationed bidder exits, this indicator is set at 1. Treating these subgames as independent of each other is a limitation. As such, the model here measures just a correlation, and not the exact effect of rationing on decision to exit immediately.

To capture rationing, I calculate a potential residual award for all the winning bidders who were awarded their desired capacity, if they had chosen to concede. To this end, I subtract the quantity bids of bidders whose first and second round bids are same (if any) from M as they are not yet in the relevant soft war of attrition.

This gives me adjusted M. Moreover, I also remove these bidders from analysis. The potential residual award is then difference between adjusted M and capacity of all other bidders in WoA. The potential residual is then floored at 0. I take its ratio with respect to the desired capacity to measure the extent of rationing. Another measure of interest, motivated by first order conditions, is the ratio of residual to the excess of cumulative capacity of WoA players($= \sum_i q_i$) over M. As noted in previous section, this ratio of rationed quantity vis-a-vis the level of rationing in auction expresses relative benefit of immediate exit. I also use number of fully rationed and number of partially rationed bidders as determinant to account for impact of competition. I model following regression specifications:

$$concede_{it} = \beta_0 + \beta_1(residual_{it}/bid_{it}) + \beta_2 SP_{it} + \beta_3 nPR_t + \beta_4 nFR_t + \epsilon_{it}$$
$$concede_{it} = \beta_0 + \beta_1(residual_{it}/excess_t) + \beta_2 SP_{it} + \beta_3 nPR_t + \beta_4 nFR_t + \epsilon_{it}$$

The results are provided in Table 1. I must add a caveat that I haven't used any variable to capture the cost of the bidders, which is an important determinant of exit decisions. I am working on this at the moment. Regardless, we can notice that the measure of rationing is an important determinant of probability of immediate exit by a bidder. Moreover, there is a positive relation between both of the variables, which implies that the bidder is more likely to exit immediately if she is not being rationed a lot. The relation also carries on to Probit regression. Another important determinant is the number of partially rationed bidders, which reflects the number of those players in the WoA, whose exit ends the game. More the number of such players, less costly is to compete, and hence, the bidders are less likely to exit immediately.

The histogram of ratio of residual and quantity bid in Figure 6 can help visualise the relationship between the extent rationing and decision to exit immediately. For this histogram, I filtered out the observations where rationed quantity was zero. One can see that when residual quantity is less compared to the quantity bid, the players are less like to exit immediately.

	Dependent variable:			
	Immediate exit			
	OLS		*probit*	
	(1)	(2)	(3)	(4)
res/own	0.736***		3.247***	
	(0.119)		(0.777)	
res/excess		0.047**		0.188**
		(0.019)		(0.096)
SP	0.006	−0.079	0.033	−0.371
	(0.114)	(0.125)	(0.618)	(0.548)
VGF	0.067	0.068	0.460	0.347
	(0.082)	(0.091)	(0.491)	(0.450)
nFR	−0.012	−0.059***	−0.080	−0.294***
	(0.019)	(0.019)	(0.123)	(0.101)
nPR	−0.056***	−0.053***	−0.268***	−0.263**
	(0.015)	(0.017)	(0.098)	(0.103)
Constant	0.155	0.403***	−1.270**	0.070
	(0.095)	(0.093)	(0.591)	(0.425)
Observations	147	147	147	147
R^2	0.299	0.144		
Adjusted R^2	0.274	0.114		
Log Likelihood			−39.983	−49.964
Residual Std. Error (df = 141)	0.299	0.331		
F Statistic (df = 5; 141)	12.044***	4.749***		
Note:			*p<0.1; **p<0.05; ***p<0.01	

Table 1: Relation between decision of immediate exit and rationing

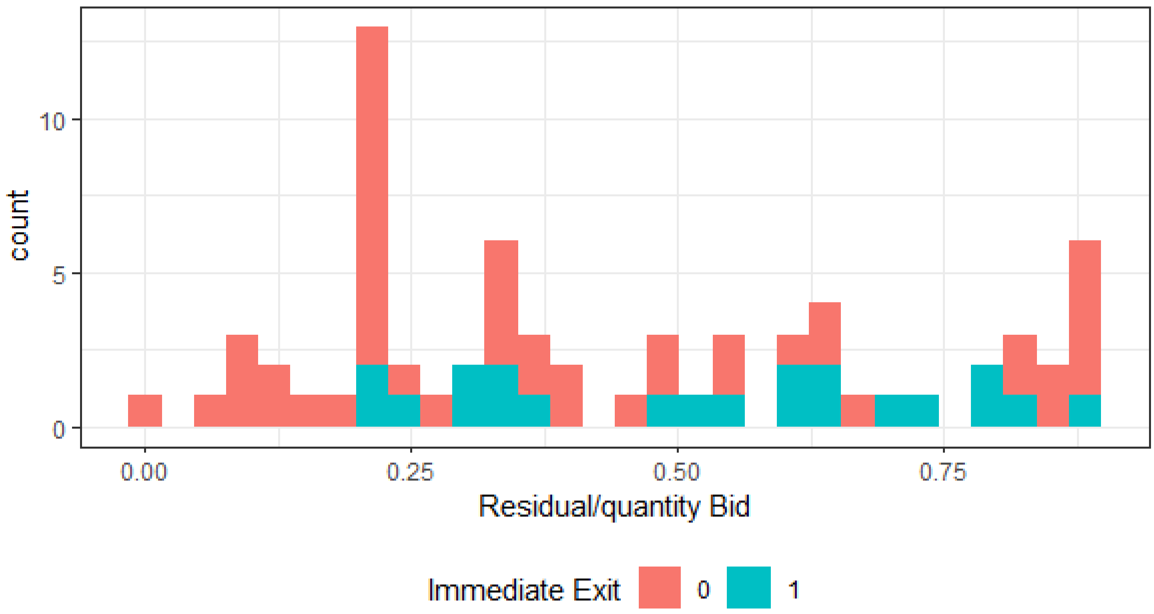


Figure 6: Histogram depicting Potential residual capacity/Quantity bid and decision to exit

These reduced form empirical models present correlation between residual quantity and the decision to immediately exit. While the models here are not at all causal, the presence of such correlations and immediate exit by highest quantity bidders can be seen as a first evidence in the favor of bidders' strategies being consistent with PBE of a soft WoA.

5 Conclusion

In this article, I have analysed a simple supply side rationing rule used by procurement agencies in order to clear the market. This rationing rule enables auctioneer to forego the need to decide a split-award beforehand, and instead, lets the bidding strategies unveil the split ex-post. The rule manifests itself in the form of a softened war of attrition, which is different from traditional WoA because of absence of sunk costs. I provide the perfect Bayesian equilibrium of this game, which has bunching at the reserve price by player with highest quantity bid. I provide results on its uniqueness and existence of PBE in some simple settings. The equilibrium is such that any mechanism which uses soft WoA produces an inefficient selection. This can also have implications in the field of IO, where WoA has been used to analyse exit decision of firms.

Finally, I use the data from renewable energy auctions in India which incorporate soft WoA. The positive correlation between a bidder's decision to exit immediately and the residual award they get provides some evidence in favor of bunching. However, these empirical results are not at all causal, and require further investigation.

At present, I am working on extending the results obtained here to the more general setting like those with 3 partially rationed bidders. Besides strengthening empirical result, an important avenue for future research would be to compare the inefficiencies generated in soft WoA with an optimal mechanism, and some other mechanisms for market clearing. This would provide policy implications. In this paper, I took q_i as exogenously provided. Future work can involve endogenising the report of q_i, to see if there is any incentive of false reporting. For this purpose, one can think of a stage game, where first stage is player's choice of q_i report, depending on the capacity they have. Soft WoA, then becomes second stage of this game. One can also think of incorporating more institutional details from the first round of the renewable energy auctions, and see the effect of that on the selection efficiency.

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A Proofs

A.1 Proof of Lemma 1

Proof. I continue with the assumption that $q_1 > q_2$. Throughout the proofs I denote $\underset{x \to x^-}{lim} u(x)$ by $u(x^-)$ and $\underset{x \to x^+}{lim} u(x)$ by $u(x^+)$ for any function $u(x)$.

To prove condition (i), it is sufficient to show that payoff of a player satisfies Single crossing property of incremental returns (SCP IR) as defined in Athey (2001), when opponents play a non-decreasing strategy. Consider any 2 arbitrary cost types of B_i, c_i and c_i' such that $c_i < c_i'$ and 2 bids b_i, b_i' such that $b_i < b_i'$. The the property is satisfied if $\pi_i(b_i', c_i) - \pi_i(b_i, c_i) > 0$ implies $\pi_i(b_i', c_i') - \pi_1(b_i, c_i') > 0$ when B_{-i} bids with a non-decreasing strategy. Without loss of generality, assume $i = 1$.

$$\begin{aligned} \pi_1(b_1', c_1; b_2) &= (M - q_2)(b_1' - c_1)Pr(b_2 < b_1') + q_1\mathbb{E}(b_2 - c_1|b_2 > b_1')Pr(b_2 > b_1') \\ \pi_1(b_1, c_1; b_2) &= (M - q_2)(b_1 - c_1)Pr(b_2 < b_1) + q_1\mathbb{E}(b_2 - c_1|b_2 > b_1)Pr(b_2 > b_1) \end{aligned} \tag{3}$$

where b_2 is the random variable denoting B_2's possible bid.

$$\begin{aligned} \therefore & A(b_1', b_1, c_1, b_2) \equiv \pi_1(b_1', c_1; b_2) - \pi_1(b_1, c_1; b_2) \\ =& (M - q_2)[(b_1' - c_1)Pr(b_2 < b_1') - (b_1 - c_1)Pr(b_2 < b_1)] \\ & + q_1[\mathbb{E}(b_2 - c_1|b_2 > b_1')Pr(b_2 > b_1') - \mathbb{E}(b_2 - c_1|b_2 > b_1)Pr(b_2 > b_1)] \end{aligned} \tag{4}$$

Similarly,

$$
\begin{aligned}
&\pi_1(b_1', c_1'; b_2) - \pi_1(b_1, c_1'; b_2) \\
=&(M - q_2)[(b_1' - c_1')Pr(b_2 < b_1') - (b_1 - c_1')Pr(b_2 < b_1)] \\
&+ q_1[\mathbb{E}(b_2 - c_1'|b_2 > b_1')Pr(b_2 > b_1') - \mathbb{E}(b_2 - c_1'|b_2 > b_1)Pr(b_2 > b_1)] \\
=&(M - q_2)[(b_1' - c_1 + c_1 - c_1')Pr(b_2 < b_1') - (b_1 - c_1 + c_1 - c_1')Pr(b_2 < b_1)] \\
&+ q_1[\mathbb{E}(b_2 - c_1 + c_1 - c_1'|b_2 > b_1')Pr(b_2 > b_1') - \mathbb{E}(b_2 - c_1 + c_1 - c_1'|b_2 > b_1)Pr(b_2 > b_1)] \\
=&A(b_1', b_1, c_1, b_2) + (M - q_2)(c_1 - c_1')[Pr(b_2 < b_1') - Pr(b_2 < b_1)] + q_1(c_1 - c_1')[Pr(b_2 > b_1') - Pr(b_2 > b_1)] \\
=&A(b_1', b_1, c_1, b_2) + (M - q_2)(c_1 - c_1')[Pr(b_2 < b_1') - Pr(b_2 < b_1)] + q_1(c_1 - c_1')[-Pr(b_2 < b_1') + Pr(b_2 < b_1)] \\
=&A(b_1', b_1, c_1, b_2) + (M - q_2 - q_1)(c_1 - c_1')[Pr(b_2 < b_1') - Pr(b_2 < b_1)]
\end{aligned}
\tag{5}
$$

As $b_1' > b_1$, $Pr(b_1' = max\{b_1', b_2\}) - Pr(b_1 = max\{b_1, b_2\}) > 0$ because opponent's cost type bidding b_1' is weakly lesser than the type bidding b_1. This along with $A(b', b, c_1, b_2) > 0$, $c_1 < c_1'$, $M < q_1 + q_2$, ensures that above expression above is positive. Thus, $\pi_1(b_1', c_1'; b_2) - \pi_1(b_1, c_1'; b_2) > 0$, which proves the property.

(ii) property establishes atomlessness at $b < b^R$ and continuity of bidding strategies.

Continuity: Given the monotonicity of equilibrium, the only type of discontinuity is the one where for some type c_1 of B_1, $\beta_1(c_1^-) = b' < \beta_1(c_1) = b$. Assume that other player plays a continuous strategy.[9] Then, $\exists \tilde{c}_2 \beta_2(\tilde{c}_2) \in [b', b]$. The payoff to this type of B_2 is $\pi_2(\beta_2(\tilde{c}_2), \tilde{c}_2) = (\beta_2(\tilde{c}_2) - \tilde{c}_2)(M - q_1)Pr(b_1 < \beta_2(\tilde{c}_2)) + q_2\mathbb{E}(b_1 - \tilde{c}_2|b_1 > \beta_2(\tilde{c}_2))Pr(b_1 > \beta_2(\tilde{c}_2))$. If she bids b, her payoff is $\pi_2(b, \tilde{c}_2) = (b - \tilde{c}_2)(M - q_1)Pr(b_1 < b) + q_2\mathbb{E}(b_1 - \tilde{c}_2|b_1 > b)Pr(b_1 > b)$. Note that, given the monotonicity of B_1's strategy and a hole in her bid distribution on (b', b), $Pr(b_1 > b) = Pr(b_1 > \beta_2(\tilde{c}_2))$ and $Pr(b_1 < b) = Pr(b_1 < \beta_2(\tilde{c}_2))$. Thus, $\pi_2(b, \tilde{c}_2) - \pi_2(\beta_2(\tilde{c}_2), \tilde{c}_2) = (b - \beta_2(\tilde{c}_2))(M - q_1)Pr(b_1 < b) + q_1\mathbb{E}(b_1 - \tilde{c}_2|b_1 > b)Pr(b_1 > b) > 0$.

No atom at bids below b^R: In any equilibrium, a cost type of a bidder has to be locally indifferent between the bid suggested by PBE and a bid slightly lower or higher. Suppose that in equilibrium, B_1 has an atom of probability mass $\varepsilon > 0$ at some bid $b_1 < b^R$. If opponent bids continuously. Thus, she has a type c_2^+ which

[9] The proof would extend to the case where opponent also plays a discontinuous strategy.

bids b_1^+. If this type decides to reduce her bid to b_1^-, then her marginal cost is almost zero, but marginal benefit is $(q_1+q_2-M)\varepsilon(b_1-c_2)$. Thus, B_2 of this type (c_2^+) can profit by bidding slightly lower than b_1.

From (i) and (ii), we know that $\beta_i(c)$ is invertible for all c as long as $\beta_i(c) \neq b^R$. Thus, I can define the functions $\phi_i(b), \forall i$ as follows:

$$\phi_i(b) := \begin{cases} \beta_i^{-1}(b) & for\ b < b^R \\ Inf\{c : \beta_i(c) = b^R\} & for\ b = b^R \end{cases}$$

Condition (iii) can be argued as follows. Consider a candidate equilibrium where $\beta_1(\underline{c}) = \underline{b}$ but $\beta_2(c_*) = \underline{b}$ for some $c_* > \underline{c}$ (assumed without loss of generality) and $\underline{b} > \underline{c}$. Given the monotonicity, the type $c_* + \epsilon, \epsilon \to 0$ of B_2 would bid some $\underline{b} + \delta(\epsilon)$, $\delta(\epsilon) \to 0$. It's payoff is:

$$\begin{aligned}
&\pi_2(\underline{b}+\delta(\epsilon), c_*+\epsilon) \\
=&(M-q_1)F(\phi_1(\underline{b}+\delta(\epsilon)))(\underline{b}+\delta(\epsilon)-c_*-\epsilon) + q_2\int_{\underline{b}+\delta(\epsilon)}^{b^R}(x-c_*-\epsilon)dF(\phi_1(x)) \\
\approx&q_2\int_{\underline{b}}^{b^R}(x-c_*-\epsilon)dF(\phi_1(x)) - \delta(\epsilon)(q_1+q_2-M)f(\phi_1(\underline{b}))\phi_1'(\underline{b})(\underline{b}-c_*-\epsilon) \\
&+(M-q_1)F(\phi_1(\underline{b}))(\underline{b}+\delta(\epsilon)-c_*-\epsilon) \\
<&q_2\int_{\underline{b}}^{b^R}(x-c_*-\epsilon)dF(\phi_1(x))(1-F(\phi_1(\underline{b}))+(M-q_1)F(\phi_1(\underline{b}))(\underline{b}+\delta(\epsilon)-c_*-\epsilon) \\
<&q_2\int_{\underline{b}}^{b^R}(x-c_*-\epsilon)dF(\phi_1(x))(1-F(\phi_1(\underline{b}))+q_2F(\phi_1(\underline{b}))(\underline{b}+\delta(\epsilon)-c_*-\epsilon) \\
=&\pi_2(\underline{b}-\gamma, c_*+\epsilon)\ ,\forall\gamma>0
\end{aligned}$$

Thus, there is a positive deviation for the type $c_* + \epsilon$ which implies that we can't have such an equilibrium. Therefore, in equilibrium $\beta_i(\underline{c})$ is same for both i. Similar deviation can be shown if $\underline{b} > \underline{c}$. Thus, $\underline{b} = \underline{c}$ in equilibrium.

(iv) is attained as the first order condition for payoff opitimisation of bidder B_i when

B_{-i} is playing as per $\phi_{-i}(b)$. To see this, note that the payoff of B_i of type c_i when she bids b_i while opponent bids according to $\phi_{-i}(b)$ is:

$$\pi_i(b_i; c_i, \phi_{-i}(b)) = F(\phi_{-i}(b_i))(b_i)(b_i - c_i)(M - \sum_{j \neq i} q_j) + q_i \int_{b_i}^{b^R} (x - c_i) dF(\phi_{-i}(x)) \tag{6}$$

On differentiating with respect to b_i and equating it to zero, we will find the first order condition, satisfied by (1).

Finally I prove (v), which states that B_1 partially pools at b^R in equilibrium. If B_2 has to pool, $\phi_2(b^R) < \phi_1(b^R)$. At any point of intersection of $\phi_1(b)$ and $\phi_2(b)$, one can see from (1) that $\frac{\phi_2'(b)}{\phi_1'(b)} = \frac{M - q_2}{M - q_1} > 1$. This would imply that $\phi_2(b))$ should intersect that $\phi_1(b)$ just once and from below and left of it on a graph. This is because the inequality $\phi_2'(b) > \phi_1'(b)$ will not be satisfied at the second point of intersection. Note that if $\phi_1(b) < \phi_2(b)$ for some $b < b^R$, there will be no intersection between the two functions for bids above this b.

Next, suppose that $\exists b_t \leq b^R$, such that $\phi_1(b) \geq \phi_2(b), \forall b \leq b_t$ with equality only at $b = b_t$. Since $\phi_2(b)$ can intersect $\phi_1(b)$ only from left and below, all other cases are ruled out.

Given $(i), (ii), (iii), (iv)$, as $c \to \underline{c}^+$, $\beta_1(c) \to \underline{c}^+, \beta_2(c) \to \underline{c}^+$. This implies that $\beta_1(c) \to \beta_2(c)$ as $c \to \underline{c}^+$. From (i) and (ii), $\beta_i(c)$, where $c \to \underline{c}^+$, is strictly monotonic. Given its strict monotonicity, $\beta_i(c)$ would be invertible when $c \to \underline{c}^+$.

Consider some $c = \underline{c} + \delta, \delta \to 0$. Then $\phi_1(\underline{c} + \delta) - \phi_1(\underline{c}) \approx \delta \phi_1'(\underline{c} + \delta)$ and $\phi_2(\underline{c} + \delta) - \phi_2(\underline{c}) \approx \delta \phi_2'(\underline{c} + \delta)$. Therefore,

$$\frac{\phi_2'(\underline{c} + \delta)}{\phi_1'(\underline{c} + \delta)} \approx \frac{\phi_2(\underline{c} + \delta) - \phi_2(\underline{c})}{\phi_1(\underline{c} + \delta) - \phi_1(\underline{c})} \tag{7}$$

From FOCs (equations 1), $\frac{\phi_2'(\underline{c} + \delta)}{\phi_1'(\underline{c} + \delta)} = \frac{M - q_2}{M - q_1} \frac{\sigma(\phi_1(\underline{c} + \delta))}{\sigma(\phi_2(\underline{c} + \delta))} \frac{\underline{c} + \delta - \phi_2(\underline{c} + \delta)}{\underline{c} + \delta - \phi_1(\underline{c} + \delta)}$. Suppose $\phi_i(\underline{c} + \delta) = \underline{c} + \epsilon_i(\delta)$, for where $\epsilon_i(\delta) \to 0$ by continuity $\phi_i(b)$. Thus, $\frac{\phi_2'(\underline{c} + \delta)}{\phi_1'(\underline{c} + \delta)} =$

$\frac{M-q_2}{M-q_1}\frac{\sigma(\underline{c})+\epsilon_1(\delta)\sigma'(\underline{c})}{\sigma(\underline{c})+\epsilon_2(\delta)\sigma'(\underline{c})}\frac{\delta-\epsilon_2(\delta)}{\delta-\epsilon_1(\delta)}$. Now $\frac{\sigma(c)}{\sigma'(c)}=\frac{f(c)}{f'(c)-f^2(c)/F(c)}$. At $c=\underline{c}$, $\frac{\sigma(c)}{\sigma'(c)}=0$. Thus $\frac{\phi_2'(\underline{c}+\delta)}{\phi_1'(\underline{c}+\delta)}=\frac{M-q_2}{M-q_1}\frac{\sigma'(\underline{c})(\sigma(\underline{c})/\sigma'(\underline{c})+\epsilon_1(\delta))}{\sigma'(\underline{c})(\sigma(\underline{c})/\sigma'(\underline{c})+\epsilon_2(\delta))}\frac{\delta-\epsilon_2(\delta)}{\delta-\epsilon_1(\delta)}=\frac{M-q_2}{M-q_1}\frac{\epsilon_1(\delta)}{\epsilon_2(\delta)}\frac{\delta-\epsilon_2(\delta)}{\delta-\epsilon_1(\delta)}$. Alongwith Equation 7, this implies $\frac{\phi_2(\underline{c}+\delta)-\phi_2(\underline{c})}{\phi_1(\underline{c}+\delta)-\phi_1(\underline{c})}\approx\frac{\epsilon_2(\delta)}{\epsilon_1(\delta)}=\frac{M-q_2}{M-q_1}\frac{\epsilon_1(\delta)}{\epsilon_2(\delta)}\frac{\delta-\epsilon_2(\delta)}{\delta-\epsilon_1(\delta)}$ which implies $\frac{\epsilon_2^2(\delta)}{\epsilon_1^2(\delta)}\approx\frac{M-q_2}{M-q_1}\frac{\delta-\epsilon_2(\delta)}{\delta-\epsilon_1(\delta)}$. If $\epsilon_2(\delta)<\epsilon_1(\delta)$, LHS<1, but RHS >1. Thus, $\epsilon_2(\delta)>\epsilon_1(\delta)$.

However, this implies that $\phi_2(b)>\phi_1(b)$ even in the immediate neighbourhood of $\underline{c}$. This implies that there is no point of intersection between $\phi_2(b)$ and $\phi_1(b)$ for $b>\underline{c}$. Thus, for any $b\in(\underline{c},b^R]$, $\phi_2(b)>\phi_1(b)$ and, in particular, $\exists c^*<\bar{c},\ s.t.\ \phi_1(b^R)=c^*$. For $c>c^*$, $\beta_1(c)=b^R$ due to non-decreasing nature of β_is. ■

A.2 Proof of Theorem 1

Proof. Assume $q_1>q_2$. If there is an equilibrium, it will be such that any solution to boundary value problem (BVP) given by (iv) and (v) will satisfy condition (iii). Note that the Cauchy-Lipschitz theorem is satisfied at all points except at $(\underline{c},\underline{c})$. Rewriting the differential equations as below, one can see that RHS is continuous, and hence the BVP will have a unique solution for every c^*. Let's think of one such solution $\phi_i(b),\forall i$ of this boundary value problem. I will now show that if the solution is such that $\phi_1(b)$ and $\phi_2(b)$ intersect at $\underline{c}$, then, $\phi_1(\underline{c})=\phi_2(\underline{c})=\underline{c}$. Since ϕ_is solve the equations 1, I can write:

$$\begin{aligned}\phi_2'(b)&=\frac{M-q_2}{q_1+q_2-M}\frac{1}{\sigma(\phi_2(b))(b-\phi_1(b))}\\ \phi_1'(b)&=\frac{M-q_1}{q_1+q_2-M}\frac{1}{\sigma(\phi_1(b))(b-\phi_2(b))}\end{aligned}\tag{8}$$

Define functions $u(b):=\sigma(\phi_2(b))(b-\phi_1(b))$ and $v(b):=\sigma(\phi_2(b))(b-\phi_1(b))$. Using the Taylor series to first order, one can approximate $u(b)$ and $v(b)$ as:

$$u(b) \approx \sigma(\phi_2(a))(a-\phi_1(a)) + (b-a)(\sigma'(\phi_2(a))\phi_2'(a)(a-\phi_1(a)) + \sigma(\phi_2(a))(1-\phi_1'(a)))$$
$$v(b) \approx \sigma(\phi_1(a))(a-\phi_2(a)) + (b-a)(\sigma'(\phi_1(a))\phi_1'(a)(a-\phi_2(a)) + \sigma(\phi_1(a))(1-\phi_2'(a)))$$

in the neighbourhood of a. Inputting these approximation in 8, we get:

$$\begin{aligned}
\phi_2'(b) &\approx \frac{M-q_2}{q_1+q_2-M}\left(\frac{1}{\sigma(\phi_2(a))(a-\phi_1(a)) + (b-a)(\sigma'(\phi_2(a))\phi_2'(a)(a-\phi_1(a)) + \sigma(\phi_2(a))(1-\phi_1'(a)))}\right)\\
\phi_1'(b) &\approx \frac{M-q_1}{q_1+q_2-M}\left(\frac{1}{\sigma(\phi_1(a))(a-\phi_2(a)) + (b-a)(\sigma'(\phi_1(a))\phi_1'(a)(a-\phi_2(a)) + \sigma(\phi_1(a))(1-\phi_2'(a)))}\right)
\end{aligned} \tag{9}$$

Suppose that we choose c^* such that $\phi_1(\underline{c}) = \phi_2(\underline{c}) = \underline{b} \geq \underline{c}$. Take a as something just higher than $\underline{b}$ (denoted $\underline{b}^+$) and input it in 9. Then, for some $b \to \underline{b}^+$,

$$\begin{aligned}
\phi_2'(b) &\approx \frac{M-q_2}{q_1+q_2-M}\left(\frac{1}{\sigma(\underline{c}^+)(\underline{b}^+ - \underline{c}^+) + (b-\underline{b}^+)(\sigma'(\underline{c}^+)\phi_2'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+) + \sigma(\underline{c}^+)(1-\phi_1'(\underline{b}^+)))}\right)\\
\phi_1'(b) &\approx \frac{M-q_1}{q_1+q_2-M}\left(\frac{1}{\sigma(\underline{c}^+)(\underline{b}^+ - \underline{c}^+) + (b-\underline{b}^+)(\sigma'(\underline{c}^+)\phi_1'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+) + \sigma(\underline{c}^+)(1-\phi_2'(\underline{b}^+)))}\right)
\end{aligned} \tag{10}$$

which implies

$$\frac{\phi_2'(b)}{\phi_1'(b)} \approx \frac{M-q_2}{M-q_1}\left(\frac{\sigma(\underline{c}^+)(\underline{b}^+ - \underline{c}^+) + (b-\underline{b}^+)(\sigma'(\underline{c}^+)\phi_1'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+) + \sigma(\underline{c}^+)(1-\phi_2'(\underline{b}^+)))}{\sigma(\underline{c}^+)(\underline{b}^+ - \underline{c}^+) + (b-\underline{b}^+)(\sigma'(\underline{c}^+)\phi_2'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+) + \sigma(\underline{c}^+)(1-\phi_1'(\underline{b}^+)))}\right) \tag{11}$$

From the proof of Lemma 1, we know that at any point of intersection $\frac{\phi_2'(b)}{\phi_1'(b)} = \frac{M-q_2}{M-q_1}$. Differentiability of $\phi_i(b)$s $\forall b \neq \underline{c}$ implies that $\frac{\phi_2'(b)}{\phi_1'(b)} = \frac{M-q_2}{M-q_1}$ in the neighbourhood of the point of intersection. For this to be valid in the neighbourhood of $\underline{b}$, one needs:

$$\begin{aligned}
&\sigma'(\underline{c}^+)\phi_1'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+) - \sigma(\underline{c}^+)\phi_2'(\underline{b}^+) \approx \sigma'(\underline{c}^+)\phi_2'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+) - \sigma(\underline{c}^+)\phi_1'(\underline{b}^+)\\
\implies &\sigma'(\underline{c}^+)(\underline{b}^+ - \underline{c}^+)(\phi_1'(\underline{b}^+) - \phi_2'(\underline{b}^+)) \approx \sigma(\underline{c}^+)(\phi_2'(\underline{b}^+) - \phi_1'(\underline{b}^+))\\
\implies &-\sigma'(\underline{c}^+)(\underline{b}^+ - \underline{c}^+) \approx \sigma(\underline{c}^+)\\
\implies &\frac{\sigma(\underline{c}^+)}{\sigma'(\underline{c}^+)} \approx -(\underline{b}^+ - \underline{c}^+)
\end{aligned}$$

LHS in above approximation is almost 0 while RHS is positive if $\underline{b} > \underline{c}$. Thus above approximation holds only when $\underline{b} \approx \underline{c}$ which implies that $\frac{\phi_2'(b)}{\phi_1'(b)} \to \frac{M-q_2}{M-q_1}$ only if $\underline{b} = \underline{c}$.

Thus $\underline{b} = \underline{c}$ in the solution of the boundary value problem. When the ordinate of the point of intersection of the 2 solution curves is $\underline{c}$, the corresponding abscissa is also $\underline{c}$.

Next I will show that there will be only one set of solutions to the boundary value problem which will intersect at $(\underline{c}, \underline{c})$.

Suppose there are two equilibria called ϕ and $\hat{\phi}$ equilibrium. Assume that $\hat{\phi}_1(b^R) = \hat{c}^* > \phi_1(b^R) = c^*$, while $\hat{\phi}_2(b^R) = \phi_2(b^R) = \bar{c}$. Note that $\hat{\phi}_1(b)$ and $\phi_1(b)$ can't intersect. Suppose they intersect at some point (b_1^t, c_1^t), $c_1^t > \underline{c}$. Then, there are two solutions to the boundary value problem defined by Equations 1 and $\phi_1(b_1^t) = c^t$, $\phi_2(b^R) = \bar{c}$. This violates the cauchy-lipschitz theorem. Thus, $\hat{\phi}_1(b) > \phi_1(b), \forall b > \underline{c}$. By the same logic $\hat{\phi}_2(b)$ and $\phi_2(b)$ can't intersect.

From FOC for B_1, we can write $\sigma(\hat{\phi}_2(b))\hat{\phi}_2{}'(b)(b - \hat{\phi}_1(b)) = \sigma(\phi_2(b))\phi_2{}'(b)(b - \phi_1(b))$. $\forall b > \underline{c}, \hat{\phi}_1(b) > \phi_1(b) \implies b - \hat{\phi}_1(b) < b - \phi_1(b) \implies \sigma(\hat{\phi}_2(b))\hat{\phi}_2{}'(b) > \sigma(\phi_2(b))\phi_2{}'(b) \implies \sigma(\hat{\phi}_2(b^R))\hat{\phi}_2{}'(b^R) > \sigma(\phi_2(b^R))\phi_2{}'(b^R)$. Since $\hat{\phi}_2(b^R) = \phi_2(b^R) = \bar{c}, \hat{\phi}_2{}'(b^R) > \phi_2{}'(b^R)$. Thus, $\hat{\phi}_2(b^R-) < \phi_2(b^{R-})$. Since the two can't intersect for any $c > \underline{c}$, $\hat{\phi}_2(b) < \phi_2(b), \forall b > \underline{c}$.

Finally, I show by contradiction that either $\hat{\phi}_2(\underline{c}) = \hat{\phi}_1(\underline{c}) = \underline{c}$ or $\phi_2(\underline{c}) = \phi_1(\underline{c}) = \underline{c}$ but not both. Suppose that $\hat{\phi}_2(\underline{c}) = \hat{\phi}_1(\underline{c}) = \phi_2(\underline{c}) = \phi_1(\underline{c}) = \underline{c}$. As in the proof of lemma 1, consider a type $c = \underline{c} + \delta$, $\delta \to 0$. Suppose that $\phi_i(\underline{c} + \delta) = \underline{c} + \epsilon_i(\delta), \epsilon_i(\delta) \to 0$, and $\hat{\phi}_i(\underline{c} + \delta) = \underline{c} + \hat{\epsilon}_i(\delta), \hat{\epsilon}_i(\delta) \to 0$. As $\hat{\phi}_1(b) > \phi_1(b)$, $\hat{\epsilon}_1(\delta) > \epsilon_1(\delta)$. Using the same arguments as before, I can write, $\dfrac{\epsilon_2^2}{\epsilon_1^2} = \dfrac{M-q_2}{M-q_1}\dfrac{\delta - \epsilon_2(\delta)}{\delta - \epsilon_1(\delta)}$ and $\dfrac{\hat{\epsilon}_2^2}{\hat{\epsilon}_1^2} = \dfrac{M-q_2}{M-q_1}\dfrac{\delta - \hat{\epsilon}_2(\delta)}{\delta - \hat{\epsilon}_1(\delta)}$. Thus, $\dfrac{\epsilon_2^2(\delta)}{\epsilon_1^2(\delta)}\dfrac{\hat{\epsilon}_1^2(\delta)}{\hat{\epsilon}_2^2(\delta)} = \dfrac{\delta - \epsilon_2(\delta)}{\delta - \epsilon_1(\delta)}\dfrac{\delta - \hat{\epsilon}_1(\delta)}{\delta - \hat{\epsilon}_2(\delta)} \implies \dfrac{\hat{\epsilon}_1^2(\delta)}{\epsilon_1^2(\delta)} = \dfrac{\delta - \epsilon_2(\delta)}{\delta - \epsilon_1(\delta)}\dfrac{\delta - \hat{\epsilon}_1(\delta)}{\delta - \hat{\epsilon}_2(\delta)}\dfrac{\hat{\epsilon}_2^2(\delta)}{\epsilon_2^2(\delta)}$. LHS>1, while $\dfrac{\delta - \hat{\epsilon}_1(\delta)}{\delta - \epsilon_1(\delta)} < 1$. Thus, for the equality to hold, we need $\dfrac{\delta - \epsilon_2(\delta)}{\delta - \hat{\epsilon}_2(\delta)}\dfrac{\hat{\epsilon}_2^2(\delta)}{\epsilon_2^2(\delta)} > 1$, which happens when $\hat{\epsilon}_2(\delta) > \epsilon_2(\delta)$. However, we already showed that $\hat{\phi}_2(b) < \phi_2(b), \forall b > \underline{c}$, which implies that $\hat{\epsilon}_2(\delta) < \epsilon_2(\delta)$, which is a contradiction. Thus, either $\hat{\phi}_2(\underline{c}) = \hat{\phi}_1(\underline{c}) = \underline{c}$ or $\phi_2(\underline{c}) = \phi_1(\underline{c}) = \underline{c}$, but not both.

From these arguments, we can infer that if $\hat{\phi}_1(b) > \phi_1(b)$, $\hat{\phi}_2(b) < \phi_2(b), \forall b$ including $\underline{c}$. Now, consider a function, $H(c)$ which maps the ordinate at the right boundary of $\phi_1(b)$ at (b^R, c^*) to the ordinate of point of intersection of $\phi_1(b)$ and $\phi_2(b)$. By construction, it has a maxima at $\bar{c}$ (because I can always set $c^* = \bar{c}$). Since the choice of ϕ and $\hat{\phi}$ equilibria was arbitrary and since the changes in $\phi_1(b)$ and $\phi_2(b)$ in response to change in c^* are as shown above, $H(c)$ is strictly positively monotonic.

Now I argue that $H(c)$ is continuous too. To see this, notice that the RHS of equations 8 are continuous and hence, the solution to IVP defined by them and $\phi_1(b^R) = c^*, \phi_2(b^R) = b^R$ is continuous with respect to the boundary point.[10] If $H(c_t^*) = c_t$ for some c_t^*, then $\phi_2(b) - \phi_1(b) < 0$ at $b < b_t$, where $\phi_i(b_t) = c_t, \forall i$ when $\phi_1(b^R) = c_t^*$. Given the strict monotonicity of $H(c)$, the continuity of IVP solution with respect to initial value implies that if $\phi_1(b^R) = c_t^* - \omega, \omega \to 0$ $\phi_2(b) - \phi_1(b) < 0$ for $b < b_t - \delta(\omega), \delta(\omega) \to 0$, because $\phi_2(b_t) = c_t + \epsilon_2(\delta(\omega)), \phi_1(b_t) = c_t - \epsilon_1(\delta(\omega)), \epsilon_i(\delta(\omega)) \to 0, \forall i \implies \phi_i(b_t - \delta(\omega)) = c_t - \epsilon(\delta(\omega)), \epsilon(\delta(\omega)) \to 0, \forall i$. Thus, $H(c_t^* - \omega) = c_t - \epsilon(\delta(\omega)), \epsilon(\delta(\omega)) \to 0$, thereby establishing continuity of $H(c)$.

Given its continuity and strict monotonicity, $H(c)$ has minima at $\underline{c}$ which is also the left boundary of its co-domain. By Weierstrass extreme value theorem, there will be a point c^* as described by Lemma $1(v)$ such that this minima is attained and is unique given the strictly positive monotonicity of $\mathcal{H}(c)$. Thus, there will only one value of c^* such that both the curves intersect at $c = \underline{c}$. As I have already shown, $\phi_1(\underline{c}) = \phi_2(\underline{c}) = \underline{c}$. This establishes uniqueness and existence. ■

A.3 2P0F extensions

In this section, I present two extensions with asymmetric cost information. In the first extension the 2 bidders have cost distributions which can ordered as per their

[10]See Buchauer, Hiltmann, and Kiehl (1994) for an example of sensitivity analysis of IVP. It states that the solution to an IVP $\dot{y} = u(t, y), y(t_0) = y_0$ is continuous in y_0, i.e., $y(t, t_0, y_0)$ is continuous in y_0, if u is continuous in y.

Reversed Hazard Rates. In the second extension, I assume that the distribution of one of the bidders is truncated version of that of another bidder. While both cases enables me to extend the results from the main text, the second is important for the formalisation of 2P1F equilibrium.

A.3.1 Different reversed hazard rates

Suppose $c_i \overset{i.i.d}{\sim} F_i(c)$, however, $c_i \in [\underline{c}, \bar{c}]$ for each i. Denote reverserd hazard of $F_i(c)$ by $\sigma_i(c)$. Suppose that they can be ordered in terms of their reversed hazard rate, i.e $\sigma_i(c) < \sigma_{-i}(c)$. Then, as before, I can characterise the equilibrium in following lemma:

Lemma 3. *For each B_i, $\beta_i(c)$ constitutes a Perfect Bayesian Equilibrium of the 2 player asymmetric soft war of attrition if and only if it satisfies following properties:*

(i) $\beta_i(c)$ is non-decreasing in c.

(ii) $\beta_i(c)$ is continuous and atomless for $b < b^R$ for both i.

(iii) $\beta_i(\underline{c}) = \underline{c}$, $\forall i$.

(iv) For each player B_i, $\beta_i(c)$ solves:

$$\sigma_{-i}(\beta_{-i}^{-1}(\beta_i(c)))\beta_{-i}^{-1'}(\beta_i(c))(\beta_i(c) - c)(q_1 + q_2 - M) = (M - q_{-i}) \tag{12}$$

(v) $\exists c^*$ such that $\beta_i(c^*) = b^R$, $\forall c \in [c^*, \bar{c}]$ for at most one i, and $\beta_{-i}(\bar{c}) = b^R$.

Proof. Proof of $(i), (ii), (iii), (iv)$, are same as in case with same cost distributions for each bidder. For (v), I can proceed in the same way as before. Define ϕ_i as

$$\phi_i(b) := \begin{cases} \beta_i^{-1}(b) & for\ b < b^R \\ Inf\{c : \beta_i(c) = b^R\} & for\ b = b^R \end{cases}$$

At any point of intersection of $\phi_1(b)$ and $\phi_2(b)$, I can write $\dfrac{\phi_2'(b)}{\phi_1'(b)} = \dfrac{(M-q_2)\sigma_1(\phi_1(b))}{(M-q_1)\sigma_2(\phi_2(b))}$. If $\sigma_1(c) > \sigma_2(c), \forall c$, $\phi_2'(b) > \phi_1'(b)$ at point of intersection. Then, by same arguments as in proof of Lemma 1, B_1 will bunch. Furthermore, B_1 will bunch as long as $\frac{\sigma_1(c)}{\sigma_2(c)} > \frac{M-q_1}{M-q_2}, \forall c$.

If $\frac{\sigma_1(c)}{\sigma_2(c)} < \frac{M-q_1}{M-q_2}, \forall c$, then B_2 bunches at b^R. ■

However, note that this lemma doesn't allow for 2 equilibria, one where B_1 bunches and the other where B_2 bunches. Given the parameters of the model, only one of B_1 or B_2 can bunch.

A.3.2 Asymmetric support, same RHR

For each B_i, $c_i \in [\underline{c}, \bar{c}_i]$. However, $\sigma(c)$ is same for both i for $c \in [\underline{c}, min_i\{\bar{c}_i\}]$. Equilibrium is characterised by the lemma below:

Lemma 4. *For each B_i, $\beta_i(c)$ constitutes a Perfect Bayesian Equilibrium of the 2 player asymmetric soft war of attrition only if it satisfies following properties:*

(i) $\beta_i(c)$ *is non-decreasing in* c.

(ii) $\beta_i(c)$ *is continuous and atomless for* $b < b^R$ *for both* i.

(iii) $\beta_i(\underline{c}) = \underline{c}$, $\forall i$.

(iv) For each player B_i, $\beta_i(c)$ *solves:*

$$\sigma_i(\beta_{-i}^{-1}(\beta_i(c)))\beta_{-i}^{-1'}(\beta_i(c))(\beta_i(c) - c)(q_1 + q_2 - M) = (M - q_{-i}) \tag{13}$$

(v) $\exists \Delta$ *such that if* $\bar{c}_2 - \bar{c}_1 < \Delta$, $\exists c_1^*$ *such that* $\beta_1(c) = b^R$, $\forall c \in [c_1^*, \bar{c}_1]$ *and* $\beta_2(\bar{c}_2) = b^R$, *else,* $\exists c_2^*$ *such that* $\beta_2(c) = b^R$, $\forall c \in [c_2^*, \bar{c}_2]$ *and* $\beta_1(\bar{c}_1) = b^R$

Proof. Proof of $(i), (ii), (iii), (iv)$ are same as in case with same cost distributions for each bidder. As before, define $\phi_i(b)$ as inverse of $\beta_i(c)$. For (v), it can be seen in the same way as in proof of Lemma 1 that $\phi_2(b) > \phi_1(b), \forall b > \underline{c}$ for a given set of least upper bounds (LUBs) of support of cost distribution, $\{\bar{c}_1, \bar{c}_2\}$. If $\bar{c}_1 > \bar{c}_2$, B_1 would bunch because $\phi_2(b^R) = \bar{c}_2$ which needs to be higher than $\phi_1(b^R)$. This would imply that $\phi_1(b^R) < \bar{c}_2 < \bar{c}_1$.

Consider the case where $\bar{c}_1 \leq \bar{c}_2$. Consider two sets of LUBs, $[\bar{c}_1, \bar{c}_1]$ and $[\bar{c}_1, \hat{\bar{c}}_2]$ such that $\hat{\bar{c}}_2 > \bar{c}_1$. Denote the corresponding equilibrium inverse bid functions generated from these LUBs as $\phi_i(b)$ and $\hat{\phi}_i(b)$ respectively. From Lemma 1, we know that $\phi_1(b^R) = c^* < \bar{c}_1$ and $\phi_2(b^R) = \bar{c}_1$ and that $\phi_1(\underline{c}) = \phi_2(\underline{c}) = \underline{c}$. Given the characteristics of equilibrium, $\exists c \in [\underline{c}, \bar{c}_1)$ where $\phi_2(c)$ and $\hat{\phi}_2(c)$ intersect. Use c^t to denote the supremum of such c and b^t to denote the bid at that c^t. Suppose $c^t > \underline{c}$. Then, such an intersection implies two solutions to IVP characterised by FOCs (13) and $\phi_1(b^R) = c^*, \phi_2(b^t) = c^t$. Thus, the only possibility is that $\phi_2(c)$ and $\hat{\phi}_2(c)$ intersect at $(\underline{c}, \underline{c})$.

Suppose $\hat{\phi}_2'(\underline{c}^+) > \phi_2'(\underline{c}^+)$, then $\hat{\phi}_1'(\underline{c}^+) > \phi_1'(\underline{c}^+)$ because $\frac{\hat{\phi}_2'(\underline{c}^+)}{\hat{\phi}_1'(\underline{c}^+)} \approx \frac{\phi_2'(\underline{c}^+)}{\phi_1'(\underline{c}^+)} \approx \frac{M-q_2}{M-q_1}$. This implies $\hat{\phi}_1(\underline{c}^+) > \phi_1(\underline{c}^+)$ which implies $\hat{\phi}_1(b) > \phi_1(b) \forall b$ otherwise we would have similar violation of cauchy-lipschitz theorem.

By the same logic, I can say that if $\hat{\phi}_2'(\underline{c}^+) < \phi_2'(\underline{c}^+)$, then $\hat{\phi}_1(b) < \phi_1(b) \forall b$ and $\hat{\phi}_2(b) < \phi_2(b) \forall b$. However, that implies both players having an atom at b^R, which would give B_1 an incentive to deviate as tie-breaking rule favors B_2. Thus, this would not happen

Define a function $M(\bar{c}_2) : [\bar{c}_1, \infty) \to \mathbb{R}^+$ such that $M(\bar{c}_2)$ maps LUB of support of c_2 to $\phi_1(b^R)$, where $\bar{c}_1$ is LUB of an arbitrary support of c_1. Since the choice of $\hat{\bar{c}}_2$ above is arbitrary, we can say that $M'(\bar{c}_2) > 0$. Continuity can be argued in the same way as in proof of Theorem 1 in Appendix A.2. Thus, for a given $\bar{c}_1$, as $\bar{c}_2$ increases from $\bar{c}_1$, c^* increases, and the size of B_1's atom at b^R reduces. The maximum value of c^* can be $\bar{c}_1$, which corresponds to atom size of 0. Due to monotonicity and continuity of $M(\bar{c}_2)$, $\exists \bar{c}_2^T$ such that $M(\bar{c}_2^T) = \bar{c}_1$. Then for $\bar{c}_2 \in [\bar{c}_1, \bar{c}_2^T)$, B_1 bunches at b^R and for $\bar{c}_2 > \bar{c}_2^T$, B_2 would bunch. This holds true regardless of the value of $\underline{c}_1$. I can thus define $\Delta \equiv \underline{c}_2^T - \underline{c}_1$, such that $B_1(B_2)$ bunches if $\underline{c}_2 < (>) \underline{c}_1 + \Delta$. This proves (v). ■

This result here has similar intuition as in previous extension. B_2 would bunch at b^R only if it is likely to have costs much higher than that of B_1. This extension is important not only for robustness checks, but also for formalising equilibrium in case with 2 partially rationed and 1 fully rationed player.

Finally, I establish existence and uniqueness of this PBE in order to have characterisation of equilibrium of 2P1F case.

Theorem 3. *Equilibrium defined by Lemma 4 exists and is unique.*

Proof. From Lemma 4, it can be inferred that for some given values of $\bar{c}_1, \bar{c}_2$, only one of the bidders, B_1 or B_2 will be bunching.

The boundary value problem which gives equilibrium bid function is characterised by the differential equation 13, and boundaries given by $\phi_1(\underline{c}) = \phi_3(\underline{c})$, and $\phi_2(b^R) = \bar{c}_2$ when $\bar{c}_2 > \bar{c}_1 + \Delta$, and $\phi_1(b^R) = \bar{c}_1$ otherwise. Comparing to the boundary value problem for 2P0F case, it can be noticed that the differential equation and left boundary are the same, while right boundary can be different.

From the proof of Theorem 1, we already know that equilibrium exists and is unique if the right boundary is $\phi_2(b^R) = \bar{c}_2$. Moreover, same arguments can be applied to the case where the right boundary is $\phi_1(b^R) = \bar{c}_1$ because the differential equation given by FOC of B_2 is analogous to the one given by that of B_1 and the local incentives at $\underline{c}$ remain unchanged. ■

A.4 Proof of Lemma 2

Proof. For fully rationed bidder B_3, it is weakly dominant to bid her cost. The reason is same as for 1P1F case.

As in Section A.1, I show (i) condition by proving that payoff follows increasing differences property. As before, I will show it for B_1. Consider any two types c_1, c_1' of B_1, such that $c_1 < c_1'$, and any two arbitrary bids b_1, b_1', where $b_1 < b_1'$. To show monotonicity, all I need to show is that when B_2 follows a non-decreasing strategy, if $\pi_1(b_1', c_1; b_2, c_3) - \pi_1(b_1, c_1; b_2, c_3) > 0$, then $\pi_1(b_1', c_1'; b_2, c_3) - \pi_1(b_1, c_1'; b_2, c_3) > 0$, where b_2 is random variable (RV) denoting B_2's bid, and c_3 is RV for B_3's cost type (and equivalently, her bid).

$$\begin{aligned}
\pi_1(b_1', c_1; b_2, c_3) =&(M - q_2 - q_3)(b_1' - c_1)Pr(b_1' = max\{b_1', b_2, c_3\}) \\
&+ q_1\mathbb{E}(b_2 - c_1 | b_2 = max\{b_1', b_2, c_3\})Pr(b_2 = max\{b_1', b_2, c_3\}) \\
&+ \mathbb{E}(\pi_{1,\mathcal{A}2,\emptyset}(c_3, c_1) | c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
\pi_1(b_1, c_1; b_2, c_3) =&(M - q_2 - q_3)(b_1 - c_1)Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1\mathbb{E}(b_2 - c_1 | b_2 = max\{b_1, b_2, c_3\})Pr(b_2 = max\{b_1, b_2, c_3\}) \\
&+ \mathbb{E}(\pi_{1,\mathcal{A}2,\emptyset}(c_3, c_1) | c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\})
\end{aligned} \tag{14}$$

Denote $\pi_1(b_1', c_1; b_2, c_3) - \pi_1(b_1, c_1; b_2, c_3)$ by $A(b_1', b_1, c_1, b_2, c_3)$, or simply, A. Suppose that $A > 0$ always. Furthermore,

$$\begin{aligned}
\pi_1(b_1', c_1'; b_2, c_3) =&(M - q_2 - q_3)(b_1' - c_1')Pr(b_1' = max\{b_1', b_2, c_3\}) \\
&+ q_1\mathbb{E}(b_2 - c_1'|b_2 = max\{b_1', b_2, c_3\})Pr(b_2 = max\{b_1', b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1')|c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
\pi_1(b_1, c_1'; b_2, c_3) =&(M - q_2 - q_3)(b_1 - c_1')Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1\mathbb{E}(b_2 - c_1'|b_2 = max\{b_1, b_2, c_3\})Pr(b_2 = max\{b_1, b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1')|c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\})
\end{aligned} \tag{15}$$

which implies,

$$\begin{aligned}
\pi_1(b_1', c_1'; b_2, c_3) =&(M - q_2 - q_3)(b_1' - c_1' + c_1 - c_1)Pr(b_1' = max\{b_1', b_2, c_3\}) \\
&+ q_1\mathbb{E}(b_2 - c_1' + c_1 - c_1|b_2 = max\{b_1', b_2, c_3\})Pr(b_2 = max\{b_1', b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1')|c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1)|c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
&- \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1)|c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
\pi_1(b_1, c_1'; b_2, c_3) =&(M - q_2 - q_3)(b_1 - c_1' + c_1 - c_1)Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1\mathbb{E}(b_2 - c_1' + c_1 - c_1|b_2 = max\{b_1, b_2, c_3\})Pr(b_2 = max\{b_1, b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1')|c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1)|c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\}) \\
&- \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1)|c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\})
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\therefore \pi_1(b_1', c_1'; b_2, c_3) - \pi_1(b_1, c_1'; b_2, c_3) \\
=&A + (M - q_2 - q_3)(c_1 - c_1')Pr(b_1' = max\{b_1', b_2, c_3\}) - (M - q_2 - q_3)(c_1 - c_1')Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1\mathbb{E}(c_1 - c_1'|b_2 = max\{b_1', b_2, c_3\})Pr(b_2 = max\{b_1', b_2, c_3\}) - q_1\mathbb{E}(c_1 - c_1'|b_2 = max\{b_1, b_2, c_3\})Pr(b_2 = max\{b_1, b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1')|c_3 = max\{b_1', b_2, c_3\} - \pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1)|c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
&- \mathbb{E}(\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1')|c_3 = max\{b_1, b_2, c_3\} - \pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1)|c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\})
\end{aligned} \tag{17}$$

From Lemma 4, I can write continuation value in the subgame following B_3's exit, $\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1)$, as:

$$\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1) = \underset{b_1'' \leq c_3}{Max}\left[(M - q_2)(b_1'' - c_1)\frac{F(\phi_2(b_1''))}{a(c_3)} + q_1 \int_{b_1''}^{c_3} (x - c_1)\frac{dF(\phi_2(x))}{a(c_3)}\right]$$

where $\phi_2(b)$ is characterised by Lemma 4 and $a(c_3)$ denotes the probability that B_2's cost type is from that subset of $[\underline{c}, \bar{c}]$ which bids less than c_3 in the subgame with preceding B_3's exit. I can further write,

$$\begin{aligned}
\pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1) =& \underset{b_1'' \leq c_3}{Max}\left[(M - q_2)(b_1'' - c_1 + c_1' - c_1')\frac{F(\phi_2(b_1''))}{a(c_3)} + q_1 \int_{b_1''}^{c_3} (x - c_1 + c_1' - c_1')\frac{dF(\phi_2(x))}{a(c_3)}\right] \\
\implies \pi^*_{1,\mathcal{A}2,\emptyset}(c_3, c_1) \leq& \underset{b_1'' \leq c_3}{Max}\left[(M - q_2)(x - c_1')\frac{F(\phi_2(b_1''))}{a(c_3)} + q_1 \int_{b_1''}^{c} 3(x - c_1')\frac{dF(\phi_2(x))}{a(c_3)}\right] \\
&+ \underset{b_1'' \leq c_3}{Max}\left[(M - q_2)(c_1' - c_1)\frac{F(\phi_2(b_1''))}{a(c_3)} + q_1 \int_{b_1''}^{c_3} (c_1' - c_1)\frac{dF(\phi_2(x))}{a(c_3)}\right] \\
\implies \pi_1(c_3, c_1') - \pi_1(c_3, c_1) \geq& - \underset{b_1'' \leq c_3}{Max}\left[(M - q_2)(c_1' - c_1)\frac{F(\phi_2(b_1''))}{a(c_3)} + q_1 \int_{b_1''}^{c_3} (c_1' - c_1)\frac{dF(\phi_2(x))}{a(c_3)}\right]
\end{aligned} \tag{18}$$

Since we have supposed that B_2 has non-decreasing strategies in the subgame before B_3's exit, and Lemma 4(i) states that $\phi_2(x)$ is an increasing function, (18) implies

$$\begin{aligned}
&\pi_1(c_3, c_1') - \pi_1(c_3, c_1) \geq -\underset{b_1'' \leq c_3}{Max}\left[(M - q_2)(c_1' - c_1)\frac{F(\phi_2(b_1''))}{a(c_3)} + q_1(c_1' - c_1)\frac{a(c_3) - \phi_2(b_1'')}{a(c_3)}\right] \\
\implies &\pi_1(c_3, c_1') - \pi_1(c_3, c_1) \geq -q_1(c_1' - c_1)
\end{aligned} \tag{19}$$

where the last line follows from the idea that this objective function will be maximised when $b_1'' = \underline{c}$.

$$
\begin{aligned}
&\pi_1(b_1', c_1'; b_2, c_3) - \pi_1(b_1, c_1'; b_2, c_3) \\
&\geq A + (M - q_2 - q_3)(c_1 - c_1')Pr(b_1' = max\{b_1', b_2, c_3\}) - (M - q_2 - q_3)(c_1 - c_1')Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1(c_1 - c_1')Pr(b_2 = max\{b_1', b_2, c_3\}) - q_1(c - c_1)Pr(b_2 = max\{b_1, b_2, c_3\}) \\
&+ q_1(c_1 - c_1')Pr(c_3 = max\{b_1', b_2, c_3\}) - q_1(c - c_1)Pr(c_3 = max\{b_1, b_2, c_3\}) \\
&= A + (M - q_2 - q_3)(c_1 - c_1')Pr(b_1' = max\{b_1', b_2, c_3\}) - (M - q_2 - q_3)(c_1 - c_1')Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1(c_1 - c_1')Pr(b_1' \neq max\{b_1', b_2, c_3\}) - q_1(c_1 - c_1')Pr(b_1 \neq max\{b_1, b_2, c_3\}) \\
&= A + (M - q_2 - q_3)(c_1 - c_1')Pr(b_1' = max\{b_1', b_2, c_3\}) - (M - q_2 - q_3)(c_1 - c_1')Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1(c_1 - c_1')(1 - Pr(b_1' = max\{b_1', b_2, c_3\})) - q_1(c_1 - c_1')(1 - Pr(b_1 = max\{b_1, b_2, c_3\})) \\
&= A + (M - q_2 - q_3 - q_1)(c_1 - c_1')(Pr(b_1' = max\{b_1', b_2, c_3\}) - Pr(b_1 = max\{b_1, b_2, c_3\}))
\end{aligned} \tag{20}
$$

$Pr(b_1' = max\{b_1', b_2, b_3\}) - Pr(b_1 = max\{b_1, b_2, b_3\}) > 0$ because opponents' cost type bidding b_1' is weakly lesser than the type bidding b_1. This along with $A > 0$, $c_1 < c_1'$, $M < q_1 + q_2 + q_3$, $b_1' > b_1$, ensures that above expression is positive. This proves condition (i).

Proof of $(ii), (iii)$ is same as 2P0F. (iv) can be shown from first order conditions of optimisation of B_i's payoff.

For (v), consider a point of intersection (b, c) of $\phi_{1,\mathcal{A}2,B_3}$ and $\phi_{2,\mathcal{A}2,B_3}$ for some $b < \bar{c}$. At this point,

$$
\frac{\phi'_{2,\mathcal{A}2,B_3}(b)}{\phi'_{1,\mathcal{A}2,B_3}(b)} = \frac{M - q_2 - q_3 - (\pi^*_{1,\mathcal{A}2,\emptyset}(b, c) - (M - q_2 - q_3)(b - c))\sigma(b)}{M - q_1 - q_3 - (\pi^*_{2,\mathcal{A}2,\emptyset}(b, c) - (M - q_1 - q_3)(b - c))\sigma(b)} \tag{21}
$$

Note that $\pi^*_{1,\mathcal{A}2,\emptyset}(b, c)$ is the payoff if B_3 exits at b. Since this is also a point of intersection, the subgame started by B_3's exit is same as 2P0F, with $c_i \in [\underline{c}, c]$. Moreover, at this point, both players have type c and the reserve bid for 2P0F is b. Thus, from Lemma 1 and tie breaking rule in favor of non-bunching player, B_1 will bunch and B_2 will also bid b. Their continuation value at this point are

$\pi^*_{1,\mathcal{A}2,\emptyset}(b,c) = (M-q_2)(b-c)$, $\pi^*_{2,\mathcal{A}2,\emptyset}(b,c) = q_1(b-c)$. Thus, we can write

$$\frac{\phi'_{2,\mathcal{A}2,B_3}(b)}{\phi'_{1,\mathcal{A}2,B_3}(b)} = \frac{(M-q_2-q_3)-q_3(b-c)\sigma(b)}{(M-q_1-q_3)-(\sum_{j=1}^{3} q_j - M)(b-c)\sigma(b)} > 1$$

where inequality arises because $M-q_1-q_3 < M-q_2-q_3$ while $\sum_j q_j - M > q_3$. Following the same arguments as in the proof of Lemma 1(v), I can now say that $\phi_2(b) > \phi_1(b), \forall b \in (\underline{c}, \bar{c}]$. Furthermore, if $\phi_1(b)$ was to intersect $\phi_2(b)$ for $b > \bar{c}$, it would require that $\phi'_2(b) > \phi'_1(b)$. This implies that $\phi_1(b)$ doesn't intersect with $\phi_1(b)$ for $b > \bar{c}$, as the intersection would necessarily lead to $\phi'_1(b)$ more than $\phi'_2(b)$ given the continuity and positive monotonicity of these functions. This would imply that $\phi_2(b^R) = \bar{c} > \phi_1(b^R) = c_1^*$. ■

A.5 Proof of Theorem 2

Proof. The subgame started by exit of B_3 is as described in Appendix A.3.2, where I also show the existence of uniqueness of equilibrium of this subgame. Thus, to show existence and uniqueness of the equilibrum in whole, I need to show the same for the subgame before B_3 exits. That proof follows the same steps as in 2P0F case.

The BVP of interest is defined by Lemma 1(iv), (v). For this proof, I will suppress, $\mathcal{A}2$ and B_3 from the notation and refer to solutions as $\phi_i(b)$, simply. I first show that if the solution to this BVP intersect at the point with ordinate at $\underline{c}$ has abscissa at $\underline{c}$ too.

To see this, suppose that $\phi_1(\underline{c}) = \phi_2(\underline{c}) = \underline{b} \geq \underline{c}$. From the differential equations, I can write:

$$\frac{(b-\phi_1(b))\sigma(\phi_2(b))\phi'_2(b)}{(b-\phi_2(b))\sigma(\phi_1(b))\phi'_1(b)} = \frac{M-q_2-q_3-(\pi^*_{1,\mathcal{A}2,\emptyset}(b,\phi_1(b))-(M-q_2-q_3)(b-\phi_1(b)))\sigma(b)}{M-q_1-q_3-(\pi^*_{2,\mathcal{A}2,\emptyset}(b,\phi_1(b))-(M-q_1-q_3)(b-\phi_2(b)))\sigma(b)}$$

Applying first order taylor series expansion to $u(b) := (b-\phi_1(b))\sigma(\phi_2(b))$ and $v(b) := (b-\phi_2(b))\sigma(\phi_1(b))$ at $\underline{b}^+$ which is infinitesimally larger that $\underline{b}$, I get:

$$\frac{\phi_2'(\underline{b}^+)((\underline{b}^+ - \underline{c}^+)\sigma(\underline{c}^+) + (b - \underline{b}^+)(\sigma(\underline{c}^+)(1 - \phi_1'(\underline{b}^+)) + \sigma'(\underline{c}^+)\phi_2'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+)))}{\phi_1'(\underline{b}^+)((\underline{b}^+ - \underline{c}^+)\sigma(\underline{c}^+) + (b - \underline{b}^+)(\sigma(\underline{c}^+)(1 - \phi_2'(\underline{b}^+)) + \sigma'(\underline{c}^+)\phi_1'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+)))}$$
$$\approx \frac{M - q_2 - q_3 - (\pi^*_{1,\mathcal{A}2,\emptyset}(\underline{b}^+, \underline{c}^+) - (M - q_2 - q_3)(\underline{b}^+ - \underline{c}^+))\sigma(\underline{b}^+)}{M - q_1 - q_3 - (\pi^*_{2,\mathcal{A}2,\emptyset}(\underline{b}^+, \underline{c}^+) - (M - q_1 - q_3)(\underline{b}^+ - \underline{c}^+))\sigma(\underline{b}^+)}$$

The property that slope ratio at the point of intersection is given by (21) has to be satisfied when $\phi_i(b)$s intersect at $(\underline{b}, \underline{c})$ for $\underline{b} \geq \underline{c}$. Since the functions $\phi_i(b)$s are differentiable, the slope ratio in the immediate neighbourhood of $\underline{b}$ is be approximately satisfied. Thus, we would have:

$$\sigma(\underline{c}^+)(1 - \phi_1'(\underline{b}^+)) + \sigma'(\underline{c}^+)\phi_2'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+) \approx \sigma(\underline{c}^+)(1 - \phi_2'(\underline{b}^+)) + \sigma'(\underline{c}^+)\phi_1'(\underline{b}^+)(\underline{b}^+ - \underline{c}^+)$$
$$\implies \sigma(\underline{c}^+)(\phi_2'(\underline{b}^+) - \phi_1'(\underline{b}^+)) \approx (\underline{b}^+ - \underline{c}^+)\sigma'(\underline{c}^+)(\phi_1'(\underline{b}^+) - \phi_2'(\underline{b}^+))$$
$$\implies \frac{\sigma(\underline{c}^+)}{\sigma'(\underline{c}^+)} \approx -(\underline{b}^+ - \underline{c}^+)$$

The LHS of this approximation is almost 0 under the assumption that there is a vanishingly small atom at $\underline{c}$, while RHS is 0 only if $\underline{b}^+ \approx \underline{c}^+$.

Given the continuity of $\phi_1'(b)$ and $\phi_2'(b)$, solution to BVP exists and is unique for a given set of boundary conditions. Thus, multiple equilibria would arise only if there are 2 or more sets of boundary conditions for which ϕ_is intersect at $\underline{c}$. I now show that there is at most one value of c^* for which ϕ_is intersect at $(\underline{c}, \underline{c})$. Consider two different initial values, $\phi_1(b^R) = c^*$ giving a solution $\phi_1(b), \phi_2(b)$, and $\hat{c}^*$ giving solution $\hat{\phi}_1(b), \hat{\phi}_2(b)$. Further suppose that $\hat{c}^* > c^*$. Also, suppose that $\phi_i(\bar{c}) = c_{ik}$, $\hat{\phi}_i(\bar{c}) = \hat{c}_{ik}$, for $i = 1, 2$.

Now, as before, $\hat{\phi}_1(b) > \phi_1(b), \forall b > \bar{c}$, else we would have two solutions to the differential equations given by (2) and boundary condition given by point of intersection and $\phi_2(b^R) = \bar{c}$. For the same reasons $\hat{\phi}_1(b) > \phi_1(b), \forall b \in (\underline{c}, \bar{c}]$. Since B_3 isn't active for $b \in (\bar{c}, b^R]$, the differential equations (2) are same as (1) except for an adjustment due to q_3. Thus, we will also have $\hat{\phi}_2(b) < \phi_2(b), \forall b \in (\bar{c}, b^R]$ and subsequently for $b \in (\underline{c}, \bar{c}]$.

Finally consider an intersection at $\underline{c}$. Note that $\pi_{i,\mathcal{A}2,\emptyset}(\underline{c},\underline{c}) = 0$ for $i = 1, 2$. Thus, $\frac{\phi_2'(\underline{c})}{\phi_1'(\underline{c})} = \frac{M-q_2-q_3}{M-q_1-q_3}$.[11] Hereafter, I can use the same argument as in 2P0F case to establish that there is at most one solution. Finally, as before, I can show the conditions for application of Weierstrass extreme value theorem to the mapping between c^* and point of intersection $\phi_i(b)$s, to show that there is exactly one equilibrium. ■

[11] $\sigma(\underline{c}) \neq \infty$ because of the assumption of a small atom at $\underline{c}$.