New Approaches to Robust Inference on Market (Non-)Efficiency, Volatility Clustering and Nonlinear Dependence[1]

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[1]We thank W. Distaso, J.-M. Dufour, L. Giraitis, P. Kattuman, A. Min, T. Mikosch, U. K. Müller, J. L. M. Olea, A. Prokhorov, A. Rahbek, G. Székely, L. Trapani and Y. Zu for comments and discussions. Research of R. Ibragimov and A. Skrobotov was supported in part by a grant from the Russian Foundation for Basic Research (RFBR, Project No. 20-010-00960). R. S. Pedersen was supported by a grant from the Independent Research Fund Denmark (FSE 0133-00162B). Pedersen is a research fellow at the Danish Finance Institute.

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Abstract

We present novel, robust methods for inference on market (non-)efficiency, volatility clustering, and nonlinear dependence in financial return series. In contrast to existing methodology, our proposed methods are robust against well-known non-linear dynamics and tail-heaviness of returns. The methods are easy to implement and performs well in realistic settings. We revisit a recent study by Baltussen et al. (2019, Journal of Financial Economics, vol. 132, pp. 26-48) on autocorrelation in major stock indexes. Using our robust methods, we document that the evidence of presence of negative autocorrelation is weaker, compared to the conclusions of the original study.

Key words: robust inference, t-test, market efficiency, volatility clustering, nonlinear dependence, GARCH.

JEL Classification: C12, C14, G12, G14

1 Introduction

Many studies argue that time series of financial returns, R_t, and other key economic and financial variables and indicators like foreign exchange rates exhibit several common statistical properties, often referred to as stylized facts; see e.g. Campbell et al. (1997, Ch. 2), Cont (2001), Taylor (2008, Ch. 1-2), Tsay (2010, Ch. 1-3), Christoffersen (2012, Ch. 1), McNeil et al. (2015, Ch. 3), and references therein. The following three properties are the most important stylized facts that much of the empirical literature agrees upon, together with the standard mean-zero property, $E(R_t) = 0$, implying the absence of systematic gains or losses:

(i) Absence of linear dependence and linear autocorrelations: $\mathrm{Corr}(R_t, R_{t-h}) \approx 0$, even for small lags $h = 1, 2, ...$

(ii) The presence of nonlinear dependence and volatility clustering, captured by significant positive autocorrelation in simple nonlinear functions of the returns and different measures of volatility, such as squared returns: $\mathrm{Corr}(R_t^2, R_{t-h}^2) >> 0$, even for large lags $h > 0$.

(iii) Heavy tails: The (unconditional) returns distributions exhibit heavy power-law tails, $\lim_{x \to +\infty} x^{\zeta} P(|R_t| > x) = C$, with a constant $C > 0$ and the tail index $\zeta > 0$.

Note that property (i) is often cited as support for the *market efficiency hypothesis* (see, among others, Cont, 2001, and references therein). A standard way of testing properties (i)-(ii) for a given sample $(R_t)_{t=1,...,T}$ is to compute the sample counterparts of the correlations and rely on limiting Gaussian distributions of these (suitably scaled and standardized) statistics together with some "robust" (heteroskedasticity and autocorrelation consistent, HAC-type) estimates of their asymptotic variances; see, e.g., Baltussen et al. (2019) for a recent application in the analysis of linear autocorrelations in returns on major world stock indices. Such approaches may not be reliable under heavy-tailedness and nonlinear dependence (see, e.g., Granger and Orr, 1972, Davis and Mikosch, 1998, Mikosch and Stărică, 2000; Sec. 5.3 in Cont, 2001; Sec. 3.3.3 in Ibragimov et al., 2015, and references therein). Properties (i)-(ii) are commonly modelled using the much celebrated class of GARCH-type processes. Depending on the tail index ζ, characterizing the degree of heavy-tailedness in property (iii) of a given (GARCH-type) process, the correlations may not be defined and/or the sample correlations may have limiting distributions given by functions of multivariate non-Gaussian stable distributions, or may be inconsistent. For instance, $\text{Corr}(R_t, R_{t-h})$ ($\text{Corr}(R_t^2, R_{t-h}^2)$) is defined only if $\zeta > 2$ ($\zeta > 4$), and asymptotic normality of the sample counterpart requires that $\zeta > 4$ ($\zeta > 8$). The latter condition is hardly justified empirically, as it is typically found that $\zeta \in (2, 4)$ for financial returns in developed markets, whereas emerging markets' returns may even have $\zeta < 2$ and, hence, infinite variances (e.g., Loretan and Phillips, 1994, Cont, 2001, Sec. 1.2 and 3.2 in Ibragimov et al., 2015, and references therein). Note that the applicability of HAC inference approaches typically relies on moments of even higher order to be finite. Moreover, HAC-based inference methods often have poor finite sample properties, even in rather standard inference problems, especially with data with pronounced dependence and heterogeneity; see, among others, Andrews (1991), Andrews and Monahan (1992), den Haan and Levin (1997), and Ibragimov and Müller (2010) [IM (2010), henceforth].

In this paper, we present new robust approaches for dealing with the issue of heavy tails in testing for (non-)efficiency, volatility clustering and nonlinear dependence in financial return series. We exploit the property that if R_t has power-law tails with the index $\zeta > 0$, as in (iii), then for any $p > 0$, $|R_t|^p$ has the tail index ζ/p. This suggests that, even under pronounced heavy tails, correlations are well-defined and Gaussian limiting distributions for sample correlations can be obtained under suitable power transformations of the original return time series. Specifically, as a natural analogue to nonlinear dependence property (ii), we consider the property (ii') $\text{Corr}(|R_t|^p, |R_{t-h}|^p) >> 0$, even for large lags $h > 0$ for some $p > 0$.

Under heavy-tailedness property (iii), the correlations in (ii') are well-defined whenever $2p < \zeta$, and, as is shown in this paper, under general conditions, asymptotic normality of the corresponding sample correlations holds if $4p < \zeta$.

We further propose the correlations $\text{Corr}(R_t, |R_{t-h}|^s \text{sign}(R_{t-h}))$, $s > 0$, of 'signed' powers of absolute returns as measures of market (non-)efficiency. Similar to the power transformations in (ii'), these measures lead to formulation of natural analogues of property (i) under heavy-tailed and conditionally heteroskedastic time series:

(i') $\text{Corr}(R_t, |R_{t-h}|^s \text{sign}(R_{t-h})) \approx 0$, even for small lags $h = 1, 2, ...$ for some $s > 0$.

Similar to property (ii'), the correlations in (i') are well-defined for $\zeta > 2\max(1, s)$, and, under suitable conditions, asymptotic normality of the corresponding sample correlations holds if $\zeta > 2(1 + s)$.

To the best of our knowledge, property (ii') for powers of absolute returns was originally considered by Ding et al. (1993) in relation to detecting long-memory in returns. The purpose of power transformations in the present manuscript is different in the sense that power transformations of returns serve as a *necessary* step for making the corresponding correlations well-defined and for carrying out reliable inference in the presence of heavy-tailedness and conditional heteroskedasticity in returns.

The main contribution of this paper is the development of robust approaches to inference on measures of market (non-)efficiency, nonlinear dependence, and volatility clustering, such as the correlations in (i') and (ii'). Firstly, we establish asymptotic normality of sample auto(cross)correlations of arbitrary transformations of a time series under general mixing conditions for the data-generating process (DGP). Further, in order to avoid (HAC-based) estimation of limiting variances of the statistics, we propose robust $t-$statistic inference approaches in the spirit of IM (2010, 2016), and prove their asymptotic validity for general classes of DGPs, including GARCH-type time series. A similar approach was recently considered in Pedersen (2020) in relation to inference about an autoregressive coefficient in linear autoregressive models in the presence of heavy-tailed *symmetric* GARCH-type errors. In contrast, the robust inference approaches proposed in this paper do not impose any symmetry restriction on the DGPs. This is a desirable feature, as it quite is common for financial time series to have skewed marginal distributions (gain/loss asymmetry) as well as leverage effects (e.g., Cont, 2001).

We provide a numerical analysis that demonstrates appealing finite sample properties of the robust $t-$statistic inference approaches. Lastly, we revisit the aforementioned study by Baltussen et al. (2019) and illustrate the applicability of the approaches in relation to inference on properties (i') and (ii') in major stock market indices. Importantly, we document that all the associated return series are likely to have tail indices $\zeta < 4$, i.e. infinite fourth moments. Moreover, when applying our robust approaches, taking into account return heavy-tailedness, we find weak evidence of negative serial dependence in returns, in contrast to the conclusions made by Baltussen et al. (2019).

The $t-$statistic approaches to robust inference in IM (2010, 2016) and those proposed in this paper complement and are related to inference approaches based on self-normalization (see the review in de la Peña et al., 2009, Shao, 2015; Remark 2.5 and references therein) and fixed-smoothing (fixed-b) heteroskedasticity and autocorrelation robust (HAR) methods that do not rely on consistency of limiting variance estimators and use nonstandard "fixed-b" asymptotics or Student-t or F distributional approximations (see Kiefer et al., 2000, Kiefer and Vogelsang, 2002, 2005, Jansson, 2004, Müller, 2007, 2014, Sun et al., 2008, Sun, 2013, 2014a; 2014b; Lazarus et al., 2018, Lazarus et al., 2021, and Remark 2.6 in this paper).

The paper is organized as follows. Section 2 introduces and discusses measures of serial and nonlinear dependence based on autocorrelations of powers of absolute returns, and provides asymptotic theory for estimators of these measures. We further propose robust

$t-$statistic approaches for reliable inference on the measures, and show that the approaches are asymptotically valid under general conditions. Section 3 investigates the finite-sample properties of the inference methods. Section 4 provides an empirical illustration of the inference methods. Section 5 concludes and discusses suggestions for future research. Proofs and additional simulation results can be found in the Online Appendix.

2 Inference on measures of market (non-)efficiency, nonlinear dependence and volatility clustering

2.1 Autocovariances and -correlations for transformed returns

The results in this paper hold for a wide class of stationary time series processes that satisfy mixing and moment conditions stated below. (Throughout, "stationarity" refers to the notion of strict stationarity.) This includes heavy-tailed GARCH(p, q) time series, generalized GARCH processes (e.g., Pedersen, 2020), and heavy-tailed stochastic volatility processes (e.g., Davis and Mikosch, 2001), among others. To present the main ideas, we focus on the GARCH(1,1) process as an ongoing example.

Let $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. A GARCH$(1, 1)$ process, $(R_t)_{t\in\mathbb{Z}}$, is given by

$$R_t = \sigma_t Z_t,\ t \in \mathbb{Z}, \tag{1}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. random variables (r.v.'s) with mean zero and unit variance, $E(Z_t) = 0$ and $\mathrm{Var}(Z_t) = 1$, and $(\sigma_t^2)_{t\in\mathbb{Z}}$ is a conditional volatility process,

$$\sigma_t^2 = \omega + \alpha R_{t-1}^2 + \beta\sigma_{t-1}^2, \quad \omega > 0, \quad \alpha, \beta \geq 0. \tag{2}$$

As is well-known, the process in (1)-(2) has a stationary and ergodic version if and only if $E[\log(\alpha Z_t^2 + \beta)] < 0$ (e.g., Nelson, 1990). In addition, under mild conditions on the distribution of Z_t, e.g., if it has a Lebesgue density, the GARCH process is β-mixing with geometric rate; e.g., Francq and Zakoïan (2006, Thm. 3). This implies that the process is also α-mixing with geometric rate (see, e.g., Rio, 2017, for additional details on mixing processes). Under, essentially, the conditions listed above, the stationary solution to (1)-(2) satisfies Kesten's theorem (e.g., Mikosch and Stărică, 2000). Specifically, the unconditional distribution of R_t has power-law tails as in (iii) with the tail index $\zeta > 0$ given by the unique positive solution to the equation

$$E[(\alpha Z_t^2 + \beta)^{\zeta/2}] = 1. \tag{3}$$

For instance, $\zeta \in (2,4)$ if $1 - (\kappa_Z - 1)\alpha^2 < (\alpha + \beta)^2 < 1$, where $\kappa_Z \equiv E[Z_t^4]$.

Given a stationary process, $(R_t)_{t\in\mathbb{Z}}$, we consider the following population autocovariance and autocorrelation functions of order h for measuring nonlinear dependence and volatility clustering in the process. We emphasize that the measures are non-zero if the process is conditionally heteroskedastic. For $p > 0$ and $E[|R_t|^{2p}] < \infty$, let

$$\gamma_{|R|^p}(h) = \mathrm{Cov}(|R_t|^p, |R_{t-h}|^p), \quad h = 0, 1, \ldots, \tag{4}$$

$$\rho_{|R|^p}(h) = \mathrm{Corr}(|R_t|^p, |R_{t-h}|^p) = \frac{\gamma_{|R|^p}(h)}{\gamma_{|R|^p}(0)}, \quad h = 1, 2 \ldots \tag{5}$$

To quantify the degree of efficiency, i.e. if R_t is predictable with respect to its lagged values, we define, for $s > 0$ and $E[|R_t|^{1+s}] < \infty$,

$$\gamma'_{R,|R|^s\mathrm{sign}(R)}(h) = \mathrm{Cov}(R_t, |R_{t-h}|^s \mathrm{sign}(R_{t-h})), \quad h = 0, 1, \ldots, \tag{6}$$

and for $\max\{E[|R_t|^{2s}], E[|R_t|^2]\} < \infty$, denote

$$\rho'_{R,|R|^s\mathrm{sign}(R)}(h) = \mathrm{Corr}(R_t, |R_{t-h}|^s \mathrm{sign}(R_{t-h})) = \frac{\gamma'_{R,|R|^s\mathrm{sign}(R)}(h)}{\sqrt{\gamma_R(0)\gamma_{|R|^s\mathrm{sign}(R)}(0)}}, \quad h = 1, 2, \ldots, \tag{7}$$

where $\gamma_R(0) = \mathrm{Var}(R_t)$ and $\gamma_{|R|^s\mathrm{sign}(R)}(0) = \mathrm{Var}(|R|^s \mathrm{sign}(R))$.

Example 2.1 *As indicated in the introduction, in the presence of heavy tails, e.g., when*

$(R_t)_{t\in\mathbb{Z}}$ *follows a GARCH(1,1) process with the tail index* $\zeta > 0$*, the quantities in* (4) *and* (5) *are defined if* $\zeta > 2p$*. Likewise, the covariances in* (6) *are defined if* $\zeta > 1 + s$*, and the correlations in* (7) *are defined if* $\zeta > 2\max\{1, s\}$*.*

Remark 2.1 *For* $s = 1$*,* (6) *and* (7) *are identical to the usual linear autocovariances and autocorrelations, respectively. For* $s \neq 1$*,* (6) *and* (7) *are also able to detect market (non-)efficiency. In particular, if* $(R_t)_{t\in\mathbb{Z}}$ *is a martingale difference sequence, e.g. if it is a GARCH process, the quantities in* (6) *and* (7) *are equal to zero, exactly like the usual linear autocovariances and autocorrelations.*

In the next section, we consider estimation of the dependence measures and provide large sample theory for their estimators.

2.2 Limit theory for sample dependence measures

Let $(R_t)_{t=1,\dots,T}$ be a sample of observations. Denote by $\hat{\mu}_R$, $\hat{\mu}_{|R|^p}$ and $\hat{\mu}_{|R|^s\text{sign}(R)}$, respectively, the sample means of R_t, $|R_t|^p$, and $|R_t|^s\text{sign}(R_t)$, for $p, s > 0$, i.e.

$$\hat{\mu}_R = \frac{1}{T}\sum_{t=1}^{T} R_t, \quad \hat{\mu}_{|R|^p} = \frac{1}{T}\sum_{t=1}^{T} |R_t|^p, \quad \hat{\mu}_{|R|^s\text{sign}(R)} = \frac{1}{T}\sum_{t=1}^{T} |R_t|^s\text{sign}(R_t). \tag{8}$$

The sample versions of (4) and (5) are given, respectively, by

$$\hat{\gamma}_{|R|^p}(h) = \frac{1}{T}\sum_{t=h+1}^{T} (|R_t|^p - \hat{\mu}_{|R|^p})(|R_{t-h}|^p - \hat{\mu}_{|R|^p}), \tag{9}$$

$$\hat{\rho}_{|R|^p}(h) = \frac{\hat{\gamma}_{|R|^p}(h)}{\hat{\gamma}_{|R|^p}(0)}. \tag{10}$$

Likewise, the sample versions of (6) and (7) are

$$\hat{\gamma}'_{R,|R|^s\text{sign}(R)}(h) = \frac{1}{T}\sum_{t=h+1}^{T} (R_t - \hat{\mu}_R)(|R_{t-h}|^s\text{sign}(R_{t-h}) - \hat{\mu}_{|R|^s\text{sign}(R)}), \tag{11}$$

$$\hat{\rho}'_{R,|R|^s\text{sign}(R)}(h) = \frac{\hat{\gamma}'_{R,|R|^s\text{sign}(R)}(h)}{\sqrt{\hat{\gamma}_R(0)\hat{\gamma}_{|R|^s\text{sign}(R)}(0)}}, \tag{12}$$

where $\hat{\gamma}_R(0)$ and $\hat{\gamma}_{|R|^s\text{sign}(R)}(0)$ denote the sample variances of R_t and $|R_t|^s\text{sign}(R_t)$ defined in the usual way similar to $\hat{\gamma}_{|R|^p}(0)$ in (9).

The following Lemmas 2.1 and 2.2 provide a basis for asymptotic inference on the properties (ii') and (i'), respectively. The lemmas follow from the general results in the Online Appendix for sample autocovariances and autocorrelations of *arbitrary* functions of α-mixing processes.

Lemma 2.1 *Let $(R_t)_{t\in\mathbb{Z}}$ be a stationary α-mixing process. For $p > 0$, assume that there exists a value $\delta > 0$ such that $E[|R_t|^{4p+\delta}] < \infty$, and such that the mixing coefficients $\alpha(n)$ satisfy $\sum_{n=1}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty$. Then, with $\hat{\gamma}_{|R|^p}(h)$ and $\hat{\rho}_{|R|^p}(h)$ defined in (9) and (10), respectively, for a fixed integer m, one has*

$$\sqrt{T}(\hat{\gamma}_{|R|^p}(h) - \gamma_{|R|^p}(h))_{h=0,1,\ldots,m} \to_d (G_{h,p})_{h=0,\ldots,m}, \tag{13}$$

$$\sqrt{T}(\hat{\rho}_{|R|^p}(h) - \rho_{|R|^p}(h))_{h=1,\ldots,m} \to_d (H_{h,p})_{h=1,\ldots,m}, \tag{14}$$

where the limits are multivariate Gaussian with mean zero.

Lemma 2.2 *Let $(R_t)_{t\in\mathbb{Z}}$ be a stationary α-mixing process. For $s > 0$, assume that there exists a value $\delta > 0$ such that $E[|R_t|^{2(1+s)+\delta}] < \infty$, and such that the mixing coefficients $\alpha(n)$ satisfy $\sum_{n=1}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty$. Then, with $\hat{\gamma'}_{R,|R|^s\mathrm{sign}(R)}(h)$ defined in (11), one has*

$$\sqrt{T}(\hat{\gamma'}_{R,|R|^s sign(R)}(h) - \gamma'_{R,|R|^s\mathrm{sign}(R)}(h))_{h=0,1,\ldots,m} \to_d (G'_{h,s})_{h=0,1,\ldots,m}, \tag{15}$$

where $(G'_{h,s})_{h=0,1,\ldots,m}$ is multivariate Gaussian with mean zero.

If $\gamma'_{R,|R|^s\mathrm{sign}(R)}(h))_{h=1,\ldots,m} = (0,\ldots,0)$, with $\hat{\rho'}_{R,|R|^s\mathrm{sign}(R)}(h)$ defined in (12), one has

$$\sqrt{T}(\hat{\rho'}_{R,|R|^s\mathrm{sign}(R)}(h))_{h=1,\ldots,m} \to_d ((\gamma_R(0)\gamma_{|R|^s\mathrm{sign}(R)}(0))^{-1/2} G'_{h,s})_{h=1,\ldots,m}. \tag{16}$$

If $\max\{E[|R_t|^{4+\delta}], E|R_t|^{4s+\delta}]\} < \infty$, then

$$\sqrt{T}(\hat{\rho'}_{R,|R|^s\mathrm{sign}(R)}(h) - \rho'_{R,|R|^s\mathrm{sign}(R)}(h))_{h=1,\ldots,m} \to_d (H'_{h,s})_{h=1,\ldots,m}, \tag{17}$$

where $(H'_{h,s})_{h=1,\ldots,m}$ is multivariate Gaussian with mean zero.

In Lemma 2.2, the moment conditions for asymptotic normality of sample covariances in (15) are weaker than those of sample correlations in (17). The reason is that asymptotic normality of sample correlations relies on joint asymptotic normality of sample covariances and variances of R_t and $|R_t|^s \text{sign}(R_t)$, if the true correlations are non-zero. If the true correlations are zero, as in (16), the convergence of the sample correlations only relies on asymptotic normality of the sample covariances and consistency of the sample variances, which in turn requires the same moment conditions as for convergence of sample covariances.

Example 2.2 *As discussed in Section 2.1, under suitable conditions, GARCH(1,1) processes are α-mixing with geometric decay, and hence satisfy the conditions on the mixing coefficients in Lemmas 2.1 and 2.2. In particular, Lemma 2.1 holds if the tail index $\zeta > 4p$. Whenever $\zeta > 2(1+s)$, the normal asymptotics in (15) and (16) hold for the GARCH(1, 1) processes. If the moment conditions in Lemmas 2.1 and 2.2 are not satisfied, the rates of convergence of the sample covariances and correlations are slower than $\sqrt{T}$ and the limits are given by functions of r.v.'s with non-Gaussian (in general, asymmetric) stable distributions. Importantly, the rates of convergence and the limiting distributions depend on the (unknown) tail index ζ and the powers p and s. E.g., from the results in Davis and Mikosch (1998) and Mikosch and Stărică (2000) (see also Davis and Mikosch, 2009) it follows that, in the case $p = 1$ and $\zeta \in (2,4)$, $T^{1-2/\zeta}(\hat{\gamma}_{|R|}(h) - \gamma_{|R|}(h))$ has an infinite variance asymmetric stable limiting distribution with the index of stability given by $\zeta/2$. A similar result applies to $T^{1-2/\zeta}(\hat{\gamma}_{|R|^2}(h) - \gamma_{|R|^2}(h))$ when $\zeta \in (4,8)$, where the index of stability of the limiting asymmetric stable distribution is $\zeta/4$. Moreover, for the cases $\{p = 1$ and $\zeta \in (0,2)\}$ and $\{p = 2$ and $\zeta \in (0,4)\}$, $\hat{\gamma}_{|R|^p}(h)$ is inconsistent and has a non-Gaussian asymmetric stable limit. As the rate of convergence and the limiting distributions depend on the unknown value of the tail index, ζ, the results on convergence of full-sample covariance and correlation estimators are not directly applicable in terms of hypothesis testing and other inference problems. However, as discussed in Remark 2.3 and Example 2.3 below, the stable limit theory may be used (under suitable conditions) in inference using the robust t-statistic approaches considered in the next section.*

The formulas for the covariance matrices of the limiting Gaussian variables in Lemmas 2.1 and 2.2 are provided in the Online Appendix for the case of covariances and correlations, $\text{Cov}(f(R_t), g(R_{t-h}))$ and $\text{Corr}(f(R_t), g(R_{t-h}))$, for general functions f and g. The asymptotic covariance matrices have a complicated structure: For instance, for the case of (13), the asymptotic covariance matrix depends on autocovariances of any order of the time series of the products $(|R_t|^p |R_{t-h}|^p)_{h=0,\dots,m}$. Under suitable conditions, including more restrictive moment conditions, the limiting covariance matrices may be estimated by HAC-type estimators, as discussed in the introduction. For instance, following Newey and West (1987, Theorem 2), one has to assume that $E[|R_t|^{2p(4+\epsilon)}] < \infty$ for some $\epsilon > 0$ for Newey-West-type standard errors to be applicable. Such conditions are restrictive for financial applications as, e.g., letting $p = 1$ requires that R_t has finite eighth-order moments, i.e., $\zeta > 8$ in terms of the tail index (see also Remark 2.6 on self-normalization and HAR approaches to time series inference that typically may be used under more relaxed moment conditions as compared to HAC). Importantly, HAC-based inference methods often have poor finite sample properties, even in rather standard inference problems; see the discussion and references in the introduction.

In the next section, we propose robust approaches to inference on covariances and correlations of the form (4)-(7) using $t-$statistics in estimates of these quantities computed over groups of time series observations. The main advantage of these approaches is that no estimation of limiting covariance matrices is needed. We establish asymptotic validity of the robust inference approaches by relying on the large-sample results in Lemmas 2.1 and 2.2 and new results on asymptotic independence of the group-based estimators.

2.3 Robust inference on market (non-)efficiency, volatility clustering and nonlinear dependence

Following IM (2010, 2016), we consider robust $t-$statistic inference on a parameter β of a general stationary process $(R_t)_{t\in\mathbb{Z}}$. In the following, the parameter β of interest may be the population covariance $\beta = \gamma_{|R|^p}(h), \gamma'_{R,|R|^s\text{sign}(R)}(h)$ or correlation $\beta = \rho_{|R|^p}(h), \rho'_{R,|R|^s\text{sign}(R)}(h)$. Let $(R_t)_{t=1,\dots,T}$ be a sample of observations. Consider a partition of the sample into a fixed number $q \geq 2$ of (approximately) equal sized groups of consecutive observations, i.e. the observations in group $j = 1, \dots, q$ have time indexes $(j-1)\lfloor T/q \rfloor < t \leq j\lfloor T/q \rfloor$, where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$. The robust $t-$statistic inference on β is conducted using its group estimators, $(\hat{\beta}_j)_{j=1,\dots,q}$, given by the sample covariances/correlations in (4)-(7) based on the observations in group j, e.g., $\hat{\beta}_j$ may equal

$$\hat{\gamma}_{j,|R|^p}(h) = \frac{1}{\lfloor T/q \rfloor} \sum_{t=(j-1)\lfloor T/q \rfloor + h + 1}^{j\lfloor T/q \rfloor} (|R_t|^p - \hat{\mu}_{j,|R|^p})(|R_{t-h}|^p - \hat{\mu}_{j,|R|^p}), \tag{18}$$

or

$$\hat{\gamma}'_{j,R,|R|^s \mathrm{sign}(R)}(h) = \frac{1}{\lfloor T/q \rfloor} \sum_{t=(j-1)\lfloor T/q \rfloor + h + 1}^{j\lfloor T/q \rfloor} (R_t - \hat{\mu}_{j,R})(|R_{t-h}|^s \mathrm{sign}(R_{t-h}) - \hat{\mu}_{j,|R|^s \mathrm{sign}(R)}), \tag{19}$$

where $\hat{\mu}_{j,|R|^p}$ is the group-based version of $\hat{\mu}_{|R|^p}$ in (8) based on the observations in group j, and similar for $\hat{\mu}_{j,R}$ and $\hat{\mu}_{j,|R|^s \mathrm{sign}(R)}$.

Suppose that one seeks to test the null hypothesis $H_0 : \beta = \beta_0$, e.g. that the covariance $\gamma'_{R,|R|^s \mathrm{sign}(R)}(h) = 0$, against the two-sided alternative $H_a : \beta \neq \beta_0$. Let t_β denote the t-statistic in the group estimators, $(\hat{\beta}_j)_{j=1,\ldots,q}$, i.e.

$$t_\beta = \sqrt{q} \frac{\overline{\hat{\beta}} - \beta_0}{s_{\hat{\beta}}}, \tag{20}$$

with $\overline{\hat{\beta}} = q^{-1} \sum_{j=1}^{q} \hat{\beta}_j$ and $s^2_{\hat{\beta}} = (q-1)^{-1} \sum_{j=1}^{q} (\hat{\beta}_j - \overline{\hat{\beta}})^2$. The robust t-statistic approaches rely on rejecting the null hypothesis H_0 in favor of the two-sided alternative H_a at level $\tilde{\alpha} \leq 0.083\ldots$, if the absolute value of the $t-$statistic, $|t_\beta|$, exceeds the $(1 - \tilde{\alpha}/2)$ quantile of a Student's t-distribution with $q - 1$ degrees of freedom. The test of H_0 against H_a at level $\tilde{\alpha} \leq 0.1$ is conducted in the same way if $2 \leq q \leq 14$. Using the results in Bakirov and Székely (2005) and IM (2010), one can further calculate the $p-$values of the above $t-$statistic robust tests in the case of an arbitrary number q of groups thus enabling conducting $t-$statistic robust tests of an arbitrary level (see Theorem 2.1 below and the empirical applications in Section 4).

According to Theorem 2.1 below, the t-statistic approaches to inference on properties (i') and (ii') are asymptotically valid and have asymptotically correct size. The theorem follows from, firstly, asymptotic normality and asymptotic independence of the group estimators

$(\hat{\beta}_j)_{j=1,...,q}$, implied by Lemmas 2.1 and 2.2 in the previous section and Lemma 2.3 below, and, secondly, a small sample result on the conservativeness property of the t-statistic in heterogeneous normal r.v.'s originally proved by Bakirov and Székely (2005); see also IM (2010, Thm. 1).

As discussed in IM (2010), asymptotic validity of the robust $t-$statistic approaches further implies that the confidence intervals

$$\overline{\hat{\beta}} \pm \mathrm{cv} s_{\hat{\beta}}, \tag{21}$$

where cv is the usual $(1+C)/2 \times 100$ percentile of the Student-t distribution with $q-1$ degrees of freedom, have asymptotic coverage of at least C for all $C \geq 0.917\ldots$. Further, for $2 \leq q \leq 14$, confidence intervals (21) with cv being the usual 95% percentile of the Student-t distribution with $q-1$ degrees of freedom, have asymptotic coverage of at least 0.9.

According to the following lemma, the group estimators, $(\hat{\beta}_j)_{j=1,...,q}$, are asymptotically independent under the assumptions in Lemmas 2.1 and 2.2.

Lemma 2.3 *Suppose that $(R_t)_{t\in\mathbb{Z}}$ is stationary and β-mixing. Under the assumptions of Lemma 2.1, the centered and scaled group sample covariances $\sqrt{\lfloor T/q \rfloor}(\hat{\gamma}_{i,|R|^p}(h) - \gamma_{|R|^p}(h))$ and $\sqrt{\lfloor T/q \rfloor}(\hat{\gamma}_{j,|R|^p}(h) - \gamma_{|R|^p}(h))$, are asymptotically independent for $i, j = 1, 2, ..., q$, with $i \neq j$. Likewise, under the conditions of Lemma 2.2, $\sqrt{\lfloor T/q \rfloor}(\hat{\gamma'}_{i,R,|R|^s sign(R)}(h) - \gamma'_{R,|R|^s \mathrm{sign}(R)})$ and $\sqrt{\lfloor T/q \rfloor}(\hat{\gamma'}_{j,R,|R|^s sign(R)}(h) - \gamma'_{R,|R|^s sign(R)})$ are asymptotically independent for $i, j = 1, 2, ..., q$, with $i \neq j$. The asymptotic independence property also holds for the group sample correlations $\hat{\rho}_{j,|R|^p}(h)$ and $\hat{\rho'}_{j,R,|R|^s sign(R)}$, $j = 1, 2, ..., q$, under the assumptions in Lemmas 2.1 and 2.2, respectively.*

Remark 2.2 *The proof of Lemma 2.3 in the case of covariances and correlations of general functions of the process (R_t) is given in the Online Appendix and relies on exact coupling properties for β-mixing processes. A similar argument was recently used in Pedersen (2020) in the context of inference on the autoregressive coefficient in linear autoregressive models in the presence of heavy-tailed GARCH-type errors under symmetry. The proof is general also in the sense that the arguments hold for arbitrary limiting distributions of the group estimators. The limiting distributions may be non-Gaussian, e.g., stable (with the index of stability less than 2), as discussed in Example 2.2.*

The following main result follows by Lemmas 2.1-2.3 and IM (2010, Thm. 1).

Theorem 2.1 *Consider, as above, testing the hypothesis* $H_0 : \beta = \beta_0$ *against* $H_a : \beta \neq \beta_0$ *for covariances/correlations* $\beta = \gamma_{|R|^p}(h)$, $\gamma'_{R,|R|^s \mathrm{sign}(R)}(h)$, $\rho_{|R|^p}(h)$, $\rho'_{R,|R|^s \mathrm{sign}(R)}(h)$. *Suppose that the assumptions of Lemma 2.3 are satisfied. Let* T_{q-1} *denote a r.v. with a Student's t-distribution with* $q-1$ *degrees of freedom, and let* $\mathrm{cv}(q, \tilde{\alpha})$ *satisfy* $P(T_{q-1} > \mathrm{cv}(q, \tilde{\alpha})) = \tilde{\alpha}/2$. *With* t_β *defined in (20) using group estimators,* $\hat{\beta}_j$, *of* β *one has, under the null hypothesis* H_0,

$$\limsup_{T\to\infty} P(|t_\beta| > \mathrm{cv}(q, \tilde{\alpha})|H_0) \leq \tilde{\alpha}, \tag{22}$$

for any $\tilde{\alpha} \leq 2\Phi(-\sqrt{3}) = 0.08326\ldots$, *where* $\Phi(\cdot)$ *is the standard normal cdf. Inequality (22) also holds for* $2 \leq q \leq 14$ *if* $\tilde{\alpha} \leq 0.1$. *Moreover, for any* $x \geq 0$ *and* $q \geq 2$,

$$\limsup_{T\to\infty} P(|t_\beta| > x|H_0) \leq \max_{R<k\leq q} P\Big(|T_{k-1}| > \sqrt{\frac{R(k-1)}{k-R}}\Big), \tag{23}$$

where $R = R(x) = \frac{qx^2}{x^2+q-1}$.

Theorem 2.1 demonstrates the asymptotic validity of robust $t-$statistic approaches for inference on the properties (i')-(ii') under appropriate conditions on the degree of heavy-tailedness (the tail index ζ) for the return process (R_t) and powers p and s in (i')-(ii') like $p < \zeta/4$, or $s < \zeta/2 - 1$. In contrast to HAC-based inference approaches, with estimates of limiting covariance matrices that depend on the particular values of powers p and s appearing in $\gamma_{|R|^p}(h)$ and $\gamma'_{R,|R|^s \mathrm{sign}(R)}(h)$, the robust $t-$statistic approaches are applicable irrespective of the values of p and s provided they satisfy the above conditions/inequalities.

Remark 2.3 *As pointed out by IM (2010), the t-statistics approaches are applicable whenever the group-based estimators weakly converge to scale mixtures of normal distributions and are asymptotically independent (Example 2.2 and Remark 2.2). Hence, the limiting distributions are, in principle, not required to be Gaussian, as further discussed in Pedersen (2020) in the symmetric case as well as in Example 2.3 below.*

Example 2.3 *As discussed in Section 2.1, under suitable conditions, the GARCH(1,1) process is* β*-mixing (with geometric rate) and, hence, satisfies the assumptions of Lemmas 2.1-2.3 under the conditions on the tail index* ζ *and powers* p, s *discussed in Example 2.2. Hence, Theorem 2.1 holds. Thus, for instance, the (centered and scaled) group estimators*

$\hat{\gamma}_{j,|R|^p}(h)$ *and* $\hat{\rho}_{j,|R|^p}(h)$ *are asymptotically independent and normal if* $4p < \zeta$*. Likewise, the group estimators* $\hat{\gamma'}_{j,R,|R|^s \text{sign}(R)}(h)$ *are asymptotically independent and normal if* $2(1+s) < \zeta$*. Similar conclusions hold for the group estimators* $\hat{\rho'}_{j,R,|R|^s \text{sign}(R)}(h)$*. Suppose instead that* $2(1+s) > \zeta > 1+s$*. Then the group estimators* $\hat{\gamma'}_{j,R,|R|^s \text{sign}(R)}(h)$ *are asymptotically independent and stable; see also Remark 2.2. If the innovations,* Z_t*, are symmetric, then the limiting stable distributions are symmetric, and given by scale mixtures of normal distributions. In such situations, as pointed out in Remark 2.3, the* $t-$*statistic approaches are applicable for inference on property (i'). These aspects of the* t*-statistic approaches are further investigated in the simulation experiments in the next section.*

Remark 2.4 *As discussed in Ibragimov and Müller (2010), the* $t-$*statistic approach provides a number of important advantages over the existing methods. In particular, it can be employed when data are potentially heterogeneous and correlated in a largely unknown way. In addition, the approach is simple to implement and does not need new tables of critical values. The assumptions of asymptotic normality for group estimators in the approach are explicit and easy to interpret, in contrast to conditions that imply validity of alternative procedures.* [3]

The numerical results in Ibragimov and Müller (2010) demonstrate that, for many dependence and heterogeneity settings considered in the literature and typically encountered in practice for time series, panel, clustered and spatially correlated data, the choice $q = 8$ *or* $q = 16$ *leads to robust tests with attractive finite sample performance. The asymptotic efficiency results for* $t-$*statistic based robust inference further imply that it is not possible to use data dependent methods to determine the optimal number of groups* q *to be used in the approach when the only assumption imposed on the data generating process is that of asymptotic normality for the group estimators* $\hat{\beta}_j$*.*

Remark 2.5 *Inference using* $t-$*statistics, essentially in any context, is related to using self-normalized sums and statistics. E.g., as is well-known, for the* $t-$*statistic* $t_q = \sqrt{q}\overline{X}_q/s_q$ *in any r.v.'s* $X_1, ..., X_q$*,* $q \geq 2$ *(e.g., group estimators* $\hat{\beta}_j$*,* $j = 1, ..., q$*, of a parameter of interest* β*, as in robust* $t-$*statistic inference approaches proposed in IM (2010, 2016) and this paper), where, as usual,* $\overline{X}_q = \frac{1}{q}\sum_{j=1}^q X_j$*,* $s_q^2 = \frac{1}{q-1}\sum_{j=1}^q (X_j - \overline{X}_q)^2$*, and the self-normalized sum* $SN_q = \frac{\sum_{j=1}^q X_j}{\sqrt{\sum_{j=1}^q X_j^2}}$ *of* X_j'*s one has* $SN_q = t_q/\sqrt{1 + (t_q^2 - 1)/q}$ *(see, among others, Efron, 1969, Edelman, 1990, Pinelis, 1994, Dufour and Hallin, 1993, de la Peña and Ibragimov, 2017,*

[3]As is shown in Ibragimov and Müller (2010), the $t-$statistic based approach to robust inference efficiently exploits the information contained in these regularity assumptions, both in the small sample settings (uniformly most powerful scale invariant test against a benchmark alternative with equal variances) and also in the asymptotic frameworks.

and references therein). Therefore, for any $x > 0$,

$$P(|t_q| > x) = P(|SN_q| > x/\sqrt{1 + (x^2 - 1)/q}). \quad (24)$$

As discussed in the above and other works in the literature (see, among other, de la Peña et al., 2009, and references therein), self-normalization allows one to conduct inference under heavy tails and relaxed moment conditions. For instance, in the case of an arbitrary sample size $q \geq 2$, *and independent r.v.'s* $X_1, ..., X_q$ *symmetric about a common median* μ, *the test of* $H_0 : \mu = 0$ *against* $H_a : \mu \neq 0$ *can be based on bounds for the tail probabilities of linear combinations of i.i.d. symmetric Bernoulli r.v.'s and the implied bounds for* $t-$*statistics* t_q *and self-normalized sums* SN_q *of* $X_j's$ *in* (24) *established in the above papers. In particular, from Edelman (1990) it follows that, under* H_0, *for any* $x > 0$,

$$P(|t_q| > x) = P(|SN_q| > x/\sqrt{1 + (x^2 - 1)/q}) <$$
$$1 - \Phi\Big[\frac{x}{\sqrt{1 + (x^2 - 1)/q}} - 1.5\frac{\sqrt{1 + (x^2 - 1)/q}}{x}\Big] = G(x). \quad (25)$$

Thus, $H_0 : \mu = \mu_0$ *is rejected in favor of* $H_a : \mu \neq \mu_0$ *at level* $\tilde{\alpha}$ *if* $G(|t_q|) \leq \tilde{\alpha}$. *Dufour and Hallin (1993) show that similar bounds can be used in tests on regression and autocorrelation coefficients under independence and symmetry in observations, and de la Peña and Ibragimov (2017) consider the case of random polynomials and generalized cross-correlations. In the context of robust* $t-$*statistic inference approaches using group estimators* $\hat{\beta}_j$, $j = 1, ..., q$, *of a parameter of interest* β, *i.e. the autocovariances/autocorrelations of functions of a GARCH time series as considered in this paper, the bounds on the* $t-$*statistics* t_q *and self-normalized sums* SN *in (24) under independence and symmetry can be used to construct tests on* β *that are valid under asymptotically independent group estimators* $\hat{\beta}_j$, $j = 1, ..., q$, *with* any *(not necessarily identical) limiting distributions that are symmetric about* β. *E.g., under these conditions, using bounds* (25) *in Edelman (1990), the hypothesis* $H_0 : \beta = \beta_0$ *is rejected in favor of* $H_a : \beta \neq \beta_0$ *at level* $\tilde{\alpha}$ *if the* $t-$*statistic* t_β *in group estimators* $\hat{\beta}_j$, $j = 1, ..., q$, *in* (20) *if* $G(|t_\beta|) \leq \tilde{\alpha}$. *The small-sample results in Bakirov and Székely (2005) and* $t-$*statistic robust inference approaches in IM (2010,2016) and this paper may be viewed as improvements on bounds* (25) *for independent and symmetric observations and the implied (conservative) testing procedures using the observations or asymptotically independent and symmetric group estimators of a parameter of interest. The improvements hold under the assumptions that the group estimators are asymptotically independent and have asymptotic distributions that are scale mixtures of normal ones.*

Remark 2.6 *Several works in the literature have focused on HAR inference using self-*

normalized statistics with non-standard asymptotics, including testing uncorrelatedness of a dependent time series (see Lobato, 2001, and the review in Shao, 2015). Politis (2011) (see also the working paper version, Politis, 2009) has proposed a new class of higher-order accurate large-sample covariance and spectral density matrix estimators based on flat-top kernels. The above inference approaches may be used under more relaxed moment assumptions as compared to HAC methods. However, direct application of HAC/HAR and self-normalization approaches, including the aforementioned ones, in the context of inference on, e.g., linear autocovariances and autocorrelations of GARCH-type time series, requires asymptotic normality of respective sample autocovariances and autocorrelations. On the other hand, according to Lemmas 2.1 and 2.2 (see also the discussion in the introduction), asymptotic normality of the above sample linear autocovariances and autocorrelations requires finite fourth moments of the GARCH processes (we note, in particular, that GARCH processes dealt with in the numerical analysis of inference on linear autocorrelations in Lobato, 2001, have coefficients implying finite fourth moments).

3 Finite-sample properties

In this section, we present numerical results on finite-sample properties of the t-statistic approaches and compare them with those of HAC-based approaches in inference on properties (i')-(ii'). We consider the AR-ARCH DGPs given by

$$R_t = \phi R_{t-1} + \varepsilon_t, \quad t = 1, \dots, T, \tag{26}$$

$$\varepsilon_t = \sigma_t Z_t, \tag{27}$$

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2, \tag{28}$$

where $\omega = 0.1$, $0 < \alpha < 1$, $0 \leq \phi < 1$, and $(Z_t)_{t=1,\dots,T}$ are i.i.d. r.v.'s with $E(Z_t) = 0$ and $\text{Var}(Z_t) = 1$. In terms of the distribution of the innovations Z_t, we consider:

(a) *Symmetric light-tailed distribution*: Z_t is standard normal, $Z_t \sim \mathcal{N}(0, 1)$.

(b) *Asymmetric light-tailed distribution*: Z_t has an asymmetric (skewed) t-distribution with 50 degrees of freedom and the skewness parameter of 0.5: $Z_t \sim t(50, 0.5)$.

(c) *Asymmetric heavy-tailed distribution*: Z_t has an asymmetric (skewed) t-distribution with 3 degrees of freedom and the skewness parameter of 0.5: $Z_t \sim t(3, 0.5)$.

The densities of the asymmetric $t-$distributions in (b) and (c) are given in (10)-(13) in Hansen (1994) with $\lambda = 0.5$ and $\eta = 50, 3$, respectively.

The conclusions from the numerical results are similar for other AR-GARCH-type processes, including processes with asymmetric GJR-GARCH dynamics. All tests considered have a 5% nominal level. We use a sample size of 5,000 observations and 10,000 Monte Carlo replications. All computations were done in MATLAB (v. 2020a).

3.1 Testing for linear (in)dependence and market (non-)efficiency

For investigating the finite-sample properties of inference methods, we consider the processes in (26)-(28) with $\alpha = \pi^{1/3}/2$ and innovations Z_t with distributions in (a)-(c). By Kesten's equation (3), when $Z_t \sim \mathcal{N}(0, 1)$ as in (a), the processes R_t and ε_t in (26)-(28) have heavy-tailed power-law distributions as in (iii), with the tail index $\zeta = 3$. Likewise, solving Kesten's equation (3) numerically, we obtain that R_t and ε_t have heavy-tailed power-law distributions with the tail indices $\zeta \approx 2.89$ and $\zeta \approx 2.24$, respectively, in cases (b) and (c). Therefore, under distributions (a)-(c) for innovations Z_t, the tail indices ζ of the processes R_t and ε_t lie in the interval $(2, 4)$, as is typically the case for financial returns in developed markets (see the discussion in the introduction). Further, as $\zeta \in (2, 3]$, the absolute third moments of the processes are infinite.

We consider the finite-sample size and power properties of tests of the null hypotheses $H_0 : \beta = 0$ against the two-sided alternative $H_a : \beta \neq 0$ for $\beta = \rho'_{R,|R|^s \text{sign}(R)}(1) = \text{Corr}(R_t, |R_{t-1}|^s \text{sign}(R_{t-1}))$ in (i') with the lag $h = 1$ and different powers $s > 0$. Note that, under H_0, the autoregressive coefficient in (26) is zero, $\phi = 0$. For $s = 1$, H_0 corresponds to the standard property of absence of linear autocorrelations, $\text{Corr}(R_t, R_{t-1}) = 0$, as in (i). In simulations below, we consider the powers $s = 0.1, 0.25, 0.5$.

Based on the stated values of the tail index, $\zeta = 4, 2.89, 2.24$, Example 2.2 implies asymptotic normality of the full-sample estimator $\hat{\beta} = \hat{\rho}'_{R,|R|^s \text{sign}(R)}(1)$ of $\beta = \rho'_{R,|R|^s \text{sign}(R)}(1)$ in (16) whenever $s < 0.5$, $s < 0.445$, and $s < 0.12$, respectively, for the cases (a), (b) and (c). Likewise, from Example 2.3 it follows that, under H_0 and the same conditions on powers s, asymptotic normality and asymptotic independence hold for the group estimators $\hat{\beta}_j = \hat{\rho}'_{j,R,|R|^s \text{sign}(R)}(1)$, implying, by Theorem 2.1, asymptotic validity of robust $t-$statistic inference approaches.

The first class of tests we consider is based on the HAC-based $t-$statistic of (full-)sample correlations $\hat{\beta} = \hat{\rho'}_{R,|R|^s \text{sign}(R)}(1)$, with a long-run variance estimator based on a QS kernel with automatic bandwidth selection (Andrews, 1991; see also Section 2.2), and the critical values based on the standard normal distribution. The second class of tests is based on $t-$statistics in the group estimators for $q = 4, 8, 12$ and 16 groups.

3.1.1 Size properties

The results on size properties of HAC-based and robust $t-$statistic approaches for testing $H_0 : \beta = \rho'_{R,|R|^s \text{sign}(R)}(1) = 0$, as in property (i'), against $H_a : \beta \neq 0$ are provided in Table 1. We note that the standard HAC-based tests are oversized. In particular, in the case of asymmetric heavy-tailed innovations Z_t (case (c)), the size distortions are severe in the case where $s = 1$, i.e., when one carries out the usual test for the absence of linear autocorrelations as in property (i). For the robust $t-$statistic approaches, the size control is good even for the case $s = 1$, except for the case of asymmetric heavy-tailed innovations. For the latter case, the limit of the full-sample and group estimators $\hat{\beta} = \hat{\rho'}_{R,|R|^s \text{sign}(R)}(1)$ and $\hat{\beta}_j = \hat{\rho'}_{j,R,|R|^s \text{sign}(R)}(1)$ is asymmetric stable, invalidating the use of both the HAC and robust $t-$statistic approaches (Example 2.2). In contrast, in the case where $s = 1$ and the distribution of Z_t is standard normal (case (a)), one has that the group estimators $\hat{\beta}_j = \hat{\rho'}_{j,R,|R|^s \text{sign}(R)}(1)$ are asymptotically independent and *symmetric* stable (Example 2.3), implying that the $t-$statistic robust tests of H_0 are asymptotically valid. This is reflected in the attractive size properties of the robust $t-$statistic approaches, e.g., in contrast to those of the standard HAC-approaches. The same conclusions hold in the case $s = 0.5$. The robust $t-$statistic approaches to testing the hypotheses H_0 under (a), (b) with $s = 0.1, 0.25$ and under (c) with $s = 0.1$ – where estimators are asymptotically normal – are slightly over-sized, with reasonable size control for $q = 4$ and $q = 8$. Quite remarkably, the tests with the most desirable size properties are the ones for testing H_0 with $s = 0.1$ based on the $t-$statistic approaches with $q = 4$ or $q = 8$ number of groups. The reason might be that the asymptotic (Gaussian) distributions in (16) in Lemma 2.2 provide relatively good approximations to the distributions of the group estimators for small choices of powers s, whereas the quality of the approximations worsens for larger values of s.

Remark 3.1 *We note that for some of the DGPs, and for some powers s, the size distortions of the HAC-based approach are not overly severe, although the approach is not theoretically justified, in contrast to $t-$statistic inference. The reason may be two-fold. Firstly, finite sample distributions of the HAC-based t-statistics may be fairly well approximated by a standard Gaussian distribution, if the tails of the DGPs are not too heavy. Secondly, kindly pointed out by the Associate Editor, for any fixed bandwidth, the HAC-based t-statistic may be viewed as having a self-normalized structure. Further, as discussed in Remarks 2.5 and Remark 2.6, time series inference approaches based on self-normalization typically exhibit robustness to infinite higher-order moments.*

[Table 1 about here.]

3.1.2 Power properties

To investigate the power properties of the HAC-based and robust $t-$statistics tests of $H_0 : \beta = 0$ against $H_a : \beta \neq 0$ for $\beta = \rho'_{R,|R|^s \text{sign}(R)}(1) = \text{Corr}(R_t, |R_{t-1}|^s \text{sign}(R_{t-1}))$ as in (i'), we consider the alternatives where the autoregressive parameter ϕ in (26) ranges from 0 to 0.5. Figure 1 provides the size-adjusted rejection frequencies for the HAC-based and robust $t-$statistic (with $q = 8$ groups) tests of H_0 for the case of normal innovations (case (a)) and asymmetric heavy-tailed innovations (case (c)). When Z_t are standard normal, the power curves for the HAC and the robust $t-$statistic tests are very close to each other. We note that the rejection frequencies are generally lower when $s = 1$, i.e., when testing the classical hypothesis of no linear dependence/absence of linear autocorrelations as in (i). As in the previous section, we note that, under $s = 1$ and standard normal innovations, the robust $t-$statistic tests remain asymptotically valid, in contrast to the HAC-based tests. Similar conclusions hold in the case of asymmetric heavy-tailed innovations (case (c)), with the only exception that the robust $t-$statistic tests have much better power properties than the HAC-based tests for the case of $s = 1$, although, as discussed in the previous section, the use of HAC-based and the robust $t-$statistic tests are not theoretically justified for this case. For the sake of brevity, we do not report the results for the case of light-tailed asymmetric innovations (case (b)), where the power properties of the tests are similar to those for the Gaussian case in (a). Overall, the tests with $s = 0.1, 0.25$ are the most powerful.

[Figure 1 about here.]

Figure 2 contains rejection frequencies for the tests under standard normal innovations (case (a)) for $s = 1$ and different number q of groups in robust $t-$statistic approaches. In this case, as mentioned, the robust $t-$statistic approaches are asymptotically valid, in contrast to HAC-based tests. Note that the rejection frequencies are increasing in q, and that the tests based on $q = 8, 12, 16$ groups appear to be more powerful than the HAC-based tests. Similar conclusions hold for other powers, s, and other distributions of the innovations Z_t (see Online Appendix).

[Figure 2 about here.]

To conclude, the numerical results indicate that in order to conduct reliable inference on property (i'), one may choose $s > 0$ small and rely on the robust t-statistic approaches with $q = 4$ or $q = 8$ groups. This ensures very reasonable size control as well as quite attractive power properties, e.g., in comparison with the widely used HAC-based approaches. The robust $t-$statistic approaches to inference may thus be viewed as useful complements to the traditional HAC-based methods.

3.2 Testing for nonlinear dependence and volatility clustering

Next, we turn to the finite sample properties of the HAC-based and robust $t-$statistic approaches to inference on property (ii') with the lag order of $h = 1$ and powers $p \in \{0.1, 0.25, 1, 2\}$. The DGPs considered follow (26)-(28) with $\phi = 0$ and the ARCH parameter $\alpha \in (0, 1)$. For the sake of brevity, we present the results for the case of standard normal innovations Z_t in (a). The results in the Online Appendix for asymmetric/heavy-tailed cases (b) and (c) reveal the same qualitative conclusions.

We investigate the relative performance of the t-statistic and HAC-based approaches by comparing the coverage levels of the corresponding t-statistic and HAC-based confidence intervals for the unknown population correlation $\beta = \rho_{|R|^p}(1)$. Specifically, the confidence intervals based on the t-statistic approaches are constructed as in (21), and the HAC-based confidence intervals are computed by standard methods, relying on asymptotic normality of the full-sample estimator $\hat{\beta} = \hat{\rho}_{|R|^p}(1)$. Note that with $\alpha \in (0, 1)$, the processes R_t and ε_t have heavy-tailed power-law distributions as in (iii), with the tail index $\zeta > 2$. From Example 2.2, it follows that the sample correlation $\hat{\beta} = \hat{\rho}_{|R|^p}(1)$, is asymptotically normal if $\zeta > 4p$, and hence whenever $p \leq 0.5$. From Example 2.3, under the same conditions on powers p, it holds that the group-based estimators are asymptotically normal and asymptotically independent, implying asymptotic validity of the robust $t-$statistic inference approaches.

Figure 3 contains coverage levels of the confidence intervals for the HAC-based and $t-$statistic inference approaches for different choices of powers p and different number of groups q in the t-statistic approaches. As expected, the coverage is very unstable for the tests based on the powers $p = 2$ and $p = 1$, due to the loss of asymptotic normality for the full-sample and group estimators $\hat{\beta} = \hat{\rho}_{|R|^p}$ and $\hat{\beta}_j = \hat{\rho}_{j,|R|^p}$ and their convergence to functions of r.v.'s with asymmetric stable distributions for sufficiently large values of α (Example 2.2). Specifically, this holds if $\zeta \leq 4p$, and, hence, by Kesten's equation (3), whenever $\alpha \geq 3^{-1/2} \approx 0.5574$ ($\alpha \geq 105^{-1/4} \approx 0.3124$) for $p = 1$ ($p = 2$).

The coverage improves for smaller powers, in particular for $p = 0.1, 0.25$. The best coverage across all values of α is observed for the robust $t-$statistic tests with $p = 0.1$ and $q = 4, 8$ number of groups. For $p \leq 0.5$, the coverage levels of the HAC-based approaches are comparable to those of the $t-$statistic approaches with $q = 8$ groups, although the coverage for $q = 4$ groups is always better and closer to the correct 95%.

We emphasize that for the case where $p = 0.5$, the HAC-based methods are not theoretically justified due to infinite moments (as this requires $\zeta > 8p$). This is in contrast to the robust $t-$statistic approaches that are asymptotically valid under $p = 0.5$. Again, the reasonable performance of the HAC-based approach may be due to self-normalized structure of the HAC-based t-statistic, as discussed in Remark 3.1.

Similar to the findings in Section 3.1, to make reliable inference on property (ii'), one may choose $p > 0$ small in the robust t-statistic approaches with $q = 4$ or $q = 8$ groups. Similar to the discussion in Section 3.1, the latter approaches may be viewed as useful complements to HAC-based inference methods in the analysis of nonlinear dependence/volatility clustering property (ii').

[Figure 3 about here.]

4 Illustration: Revisiting Baltussen et al. (2019)

In this section, we revisit a recent study by Baltussen et al. (2019) that (among other contributions) tests for linear dependence in returns on the world's major stock market indices. Specifically, relying on HAC-based inference applied to the first-order autocorrelations, $\rho'_{R,R}(1)$, Baltussen et al. (2019) state that serial dependence in daily returns on 20 major market indices covering 15 countries in North America, Europe, and Asia was significantly positive until the end of the 1990s, and switched to being significantly negative since the early 2000s. In light of the discussion in the introduction and Examples 2.2 and 2.3 with $s = 1$, asymptotic normality of sample linear autocorrelations requires finite fourth-order moments of the (GARCH-type) return process. In the case where such moment conditions are not satisfied, the sample linear autocorrelations weakly converge (under suitable conditions) to functions of non-Gaussian stable variables, invalidating the HAC-based inference approaches based on asymptotic normality.

We consider daily percentage returns on the major stock indexes from March 3, 1999 to December 31, 2016 as in Baltussen et al. (2019). The second and third columns of Table 2 provide, respectively, the (bias-corrected) log-log rank-size regression estimates of the tail indices for the return time series and their 95% confidence intervals (see Gabaix and Ibragimov, 2011). Importantly, for 19 out of 20 series, the estimates of the tail index are smaller than 4. The left end-points of the confidence intervals vary from 2.49 to 3.41 across the return series, and several of the intervals lie to the left of 4. This indicates that standard HAC-based inference on linear autocorrelations, $\rho'_{R,R}(1)$, is *invalid* for several of the data series.

Columns 4 and 7 in Table 2 contain full-sample estimates $\hat{\rho}'_{R,|R|^s\text{sign}(R)}$ of the correlations $\rho'_{R,|R|^s\text{sign}(R)}(1)$ (as in (i')) for different values of s, and column 10 contains estimates of the multi-period (auto)correlation MAC(5) (a weighted sum of the correlation coefficients $\rho'_{R,|R|^s\text{sign}(R)}(h)$ of order $h = 1, \ldots, 5$) used in Baltussen et al. (2019). Column 5 provides, similar to Baltussen et al. (2019), HAC-based t-statistics for the nullity of the linear autocorrelation $\rho'_{R,R}(1)$ (as in (i)), whereas column 6 contains the t-statistics in (20) based on $q = 8$ group estimates of $\rho'_{R,R}(1)$. In column 7, for each return time series, the power s is chosen based on the left end-points of the confidence intervals for the tail index of the time series (column 3 of Table 2). Specifically, following Examples 2.2 and 2.3, if the left end-point exceeds 3, we set $s = 0.5$; if the end-point lies between 2.5 and 3.0, we set $s = 0.25$; and if the end-point lies between 2.2 and 2.5, we set $s = 0.1$. Under the above values of powers s (and assuming that the true tail index ζ belongs to the reported confidence intervals), the moment conditions for asymptotic normality of $\hat{\rho}'_{R,|R|^s\text{sign}(R)}(1)$ and asymptotic independence and normality of the group estimators (under the hypothesis $H_0 : \rho'_{R,|R|^s \text{sign}(R)}(1) = 0$ in consideration) in Lemmas 2.2 and 2.3 are satisfied. Consequently, by Theorem 2.1, the robust $t-$statistic approaches for testing $H_0 : \rho'_{R,|R|^s \text{sign}(R)}(1) = 0$ are asymptotically valid.

The reported HAC t-statistics and $t-$statistics in group estimates as in (20) in columns 5 and 6 suggest that the hypothesis of zero linear autocorrelation, $H_0 : \rho'_{R,R}(1) = 0$, cannot be rejected for most of the series at conventional significance levels. However, the HAC and robust $t-$statistic approaches are not theoretically justified in the case $s = 1$.

Further, based on the theoretically justified $t-$statistics in group estimates (as in (20)) in column 9, the hypothesis $H_0 : \rho'_{R,|R|^s \text{sign}(R)}(1) = 0$ is rejected only for six of the series. Hence, based on the theoretically justified robust $t-$statistic approaches, we find evidence of zero correlations in most of the series in contrast to the conclusions in Baltussen et al. (2019). On the other hand, similar to Baltussen et al. (2019), we find somewhat stronger evidence for non-zero weighted autocorrelations (MAC(5)), based on the robust t-statistics in group estimates reported in column 12, where the null of MAC(5) = 0 is rejected for 11 out of 20 series.

[Table 2 about here.]

Table 3 contains the results on testing for nonlinear dependence and volatility clustering in the return time series considered using the 95% confidence intervals constructed on the base of robust $t-$statistic approaches applied to inference on the autocorrelations $\rho_{|R|^p}(5)$ with $p = 0.1, 0.5, 1, 2$. The table also presents the results on tail index estimation and HAC-based confidence intervals for the dependence measures considered. The t-statistic approaches are theoretically justified for the powers $p = 0.1, 0.5$ (but not for $p = 1, 2$) provided that the true tail indices belong to the confidence intervals reported in the table. One should further note that, for the above tail indices, the HAC approaches are theoretically justified only under $p = 0.1$. Overall, the results in the table confirm the presence of nonlinear dependence and volatility clustering in the returns on the most of the financial indices. Exceptions are the ASX 200 and Russell 2000 indices, where $t-$statistic approaches with $q = 8$ applied to $\rho_{|R|^{0.1}}(5)$ indicate, somewhat surprisingly, absence of volatility clustering.

[Table 3 about here.]

5 Conclusion and suggestions for further research

The paper proposes new approaches to inference on measures of market (non-)efficiency, volatility clustering and nonlinear dependence in the case of general heavy-tailed dependent time series, including GARCH-type processes. We provide the results that motivate the use of measures of market (non-)efficiency and volatility clustering based on (small) powers of absolute returns and their signed versions.

The inference approaches dealt with in the paper are based on robust $t-$statistic tests and several new results on their applicability in the settings considered. Theoretical and numerical results and empirical applications in the paper confirm validity, appealing finite sample properties, and wide applicability of the proposed inference approaches.

The future research may focus on the development of the two-sample analogues of the approaches to robust inference on market (non-)efficiency and volatility clustering dealt with in the paper using the results in Ibragimov and Müller (2016). The results in this direction may be used in testing for structural breaks in the dynamics of key economic and financial time series, including financial returns and foreign exchange rates, and comparisons of the properties of the dynamics of different economic and financial markets. The research on the above inference problems is currently under way by the authors, and will be presented elsewhere.

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Table 1: Size of tests for absence of autocorrelations, $h = 1$

	HAC				$t-$statistic approach															
	${\rho'}_{R,R}(h)$	${\rho'}_{R,\|R\|^{0.5}sign(R)}(h)$	${\rho'}_{R,\|R\|^{0.25}sign(R)}(h)$	${\rho'}_{R,\|R\|^{0.1}sign(R)}(h)$	${\rho'}_{R,R}(h)$				${\rho'}_{R,\|R\|^{0.5}sign(R)}(h)$				${\rho'}_{R,\|R\|^{0.25}sign(R)}(h)$				${\rho'}_{R,\|R\|^{0.1}sign(R)}(h)$			
q					4	8	12	16	4	8	12	16	4	8	12	16	4	8	12	16
ARCH(1), $N(0,1)$	7.9	6.1	6.2	6.1	4.6	4.5	4.8	4.8	4.9	4.9	5.1	5.2	5.3	5.1	5.2	5.5	5.4	5.0	5.2	5.4
ARCH(1), $t(50,0.5)$	8.6	5.9	5.8	5.8	4.4	5.1	5.6	6.2	4.6	5.1	5.3	6.1	4.9	5.4	5.5	5.8	5.0	5.3	5.5	5.7
ARCH(1), $t(3,0.5)$	17.3	9.7	8.2	7.6	6.6	9.5	11.5	13.6	5.8	7.3	8.2	9.3	5.5	6.5	7.1	8.0	5.2	5.9	6.3	7.3

Table 2: Empirical results, testing for efficient market hypothesis, $h = 1$

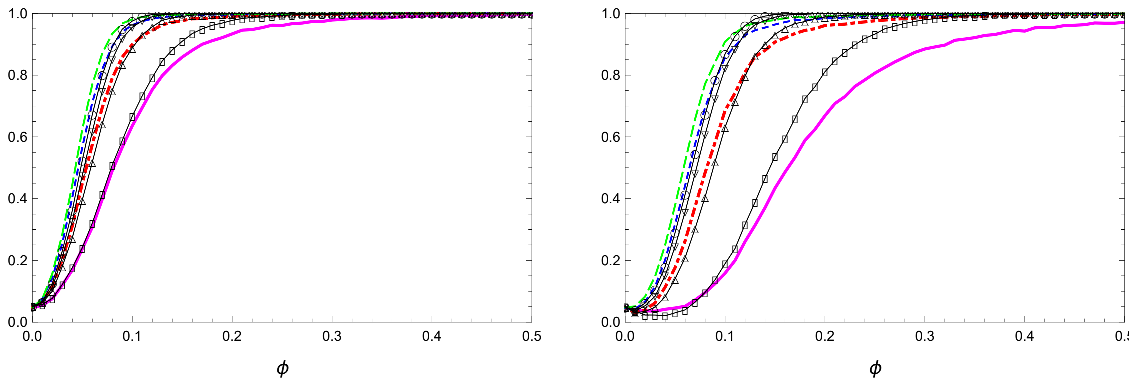
Series	Estimate of tail index	CI for tail index	$\hat{\rho}'_{R,R}(h)$	t-statistic for $\rho'_{R,R}(h) = 0$ (HAC)	t_β for $\rho_{R,R}(h) = 0$ $(q = 8)$	$\hat{\rho}'_{R,\|R\|^s sign(R)}(h)$	t-statistic for $\rho'_{R,\|R\|^s sign(R)}(h) = 0$ (HAC)	t_β for $\rho'_{R,\|R\|^s sign(R)}(h) = 0$ $(q = 8)$	Estimate of MAC(5)	t-statistic for MAC(5) = 0 (HAC)	t_β for MAC(5) = 0 $(q = 8)$
S&P 500	3.26	[2.66,3.86]	-0.074	-3.44***	-3.12***	-0.056	-3.74***	-3.32***	-0.085	-4.41***	-5.02***
FTSE 100	3.48	[2.84,4.12]	-0.040	-2.24***	-1.75	-0.029	-1.96**	-1.69	-0.075	-4.31***	-3.12***
DJESI 50	3.73	[3.05,4.42]	-0.029	-1.66**	-2.1**	-0.027	-1.71**	-2.03**	-0.057	-3.59***	-3.31***
TOPIX	3.31	[2.69,3.93]	0.014	0.66	1.63	0.036	2.26***	2.05**	-0.014	-0.87	1.42
ASX 200	3.41	[2.78,4.04]	-0.034	-1.77**	-1.23	-0.018	-1.16	-0.71	-0.038	-1.86**	-0.96
TSE 60	3.13	[2.55,3.70]	-0.039	-1.7**	-0.15	-0.003	-0.2	0.06	-0.058	-2.93***	-1.55
CAC 40	3.64	[2.97,4.30]	-0.027	-1.51	-1.77	-0.025	-1.65	-1.72	-0.061	-3.57***	-2.76***
DAX	3.62	[2.96,4.29]	-0.013	-0.8	-0.64	-0.008	-0.53	-0.6	-0.026	-1.54	-1.48
IBEX 35	3.77	[3.07,4.46]	0.004	0.23	-0.09	0.005	0.29	-0.04	-0.036	-2.12***	-1.91**
MIB	3.75	[3.06,4.44]	-0.024	-1.48	-1.91**	-0.027	-1.78**	-1.91**	-0.031	-1.79**	-2.93***
AEX Index	3.32	[2.71,3.93]	-0.007	-0.37	-0.19	0.002	0.12	0.14	-0.026	-1.58	-1.14
OMX Stockholm	4.18	[3.41,4.95]	0.001	0.06	-0.86	0.013	0.83	-0.03	-0.030	-2.07***	-1.59
SMI	3.19	[2.60,3.78]	0.029	1.49	0.89	0.017	1.15	0.69	-0.016	-0.78	-1.24
Nikkei 225	3.32	[2.70,3.95]	-0.042	-1.95**	-2.05**	-0.024	-1.48	-1.54	-0.046	-2.96***	-2.26**
HSI	3.41	[2.78,4.05]	-0.004	-0.2	0.51	0.026	1.69**	1.88	-0.014	-0.9	0.76
Nasdaq 100	3.64	[2.97,4.32]	-0.059	-3.17***	-2.18**	-0.036	-2.6***	-2.59***	-0.087	-4.72***	-3.36***
NYSE	3.05	[2.49,3.62]	-0.057	-2.57***	-1.56	-0.025	-1.77**	-1.23	-0.067	-3.54***	-2.42***
Russell 2000	3.51	[2.86,4.16]	-0.058	-2.62***	-1.41	-0.026	-1.58	-1.24	-0.060	-3.48***	-2.37***
S&P 400	3.38	[2.76,4.01]	-0.029	-1.34	-0.4	0.005	0.3	0.59	-0.053	-2.93***	-2.4***
KOSPI 200	3.88	[3.16,4.61]	0.023	1.23	2.05**	0.031	1.88**	2.71***	-0.014	-0.96	0.17

Note: *, ** and *** denote the significance at 10, 5 and 1% levels respectively.

Table 3: Empirical results, testing for non-linearity, $h = 5$

Series	$\hat{\rho}_{R^2}(h)$	CI for $\rho_{R^2}(h)$ (HAC)	CI for $\rho_{R^2}(h)$ $(q=8)$	$\hat{\rho}_{\|R\|}(h)$	CI for $\rho_{\|R\|}(h)$ (HAC)	CI for $\rho_{\|R\|}(h)$ $(q=8)$	$\hat{\rho}_{\|R\|^{0.5}}(h)$	CI for $\rho_{\|R\|^{0.5}}(h)$ (HAC)	CI for $\rho_{\|R\|^{0.5}}(h)$ $(q=8)$	$\hat{\rho}_{\|R\|^{0.1}}(h)$	CI for $\rho_{\|R\|^{0.1}}(h)$ (HAC)	CI for $\rho_{\|R\|^{0.1}}(h)$ $(q=8)$
S&P 500	0.316	[0.17,0.47]*	[0.05,0.207]*	0.331	[0.26,0.4]*	[0.05,0.262]*	0.261	[0.21,0.31]*	[0.04,0.24]*	0.177	[0.15,0.21]*	[0.02,0.19]*
FTSE 100	0.330	[0.17,0.49]*	[0.09,0.243]*	0.302	[0.23,0.37]*	[0.11,0.256]*	0.240	[0.2,0.28]*	[0.09,0.21]*	0.143	[0.11,0.17]*	[0.03,0.12]*
DJESI 50	0.259	[0.16,0.35]*	[0.02,0.211]*	0.255	[0.2,0.31]*	[0.05,0.215]*	0.203	[0.16,0.24]*	[0.05,0.17]*	0.126	[0.09,0.16]*	[0.02,0.1]*
TOPIX	0.203	[0.11,0.3]*	[0.05,0.19]*	0.219	[0.14,0.29]*	[0.06,0.235]*	0.170	[0.12,0.22]*	[0.05,0.2]*	0.109	[0.08,0.14]*	[0.03,0.14]*
ASX 200	0.228	[0.12,0.34]*	[0.01,0.147]*	0.243	[0.18,0.3]*	[0.03,0.17]*	0.188	[0.14,0.23]*	[0.02,0.14]*	0.107	[0.08,0.14]*	[-0.01,0.09]
TSE 60	0.350	[0.22,0.48]*	[0.04,0.25]*	0.354	[0.26,0.44]*	[0.04,0.27]*	0.272	[0.22,0.33]*	[0.03,0.23]*	0.165	[0.13,0.2]*	[0.01,0.16]*
CAC 40	0.262	[0.15,0.37]*	[0.01,0.20]*	0.248	[0.19,0.3]*	[0.05,0.21]*	0.201	[0.16,0.24]*	[0.05,0.18]*	0.142	[0.11,0.17]*	[0.04,0.13]*
DAX	0.246	[0.16,0.34]	[0.05,0.203]*	0.274	[0.23,0.32]*	[0.08,0.226]*	0.234	[0.2,0.27]*	[0.08,0.19]*	0.169	[0.14,0.2]*	[0.05,0.14]*
IBEX 35	0.159	[0.05,0.26]*	[0.02,0.13]*	0.217	[0.16,0.27]*	[0.04,0.17]*	0.191	[0.15,0.23]*	[0.04,0.15]*	0.141	[0.11,0.17]*	[0.03,0.11]*
MIB	0.212	[0.12,0.3]*	[0.04,0.19]*	0.251	[0.2,0.3]*	[0.05,0.21]*	0.220	[0.18,0.26]*	[0.04,0.18]*	0.159	[0.13,0.19]*	[0.03,0.13]*
AEX Index	0.377	[0.25,0.51]*	[0.03,0.28]*	0.325	[0.26,0.39]*	[0.06,0.27]*	0.248	[0.2,0.29]*	[0.06,0.22]*	0.152	[0.12,0.18]*	[0.02,0.15]*
OMX Stockholm	0.232	[0.14,0.32]*	[0.06,0.188]*	0.240	[0.19,0.29]*	[0.09,0.194]*	0.208	[0.17,0.24]*	[0.07,0.16]*	0.150	[0.12,0.18]*	[0.03,0.13]*
SMI	0.283	[0.19,0.38]*	[0.07,0.246]*	0.302	[0.23,0.37]*	[0.1,0.28]*	0.244	[0.2,0.29]*	[0.09,0.23]*	0.163	[0.13,0.19]*	[0.05,0.16]*
Nikkei 225	0.214	[0.11,0.32]*	[0.03,0.18]*	0.219	[0.14,0.3]*	[0.05,0.22]*	0.174	[0.13,0.22]*	[0.05,0.2]*	0.126	[0.09,0.16]*	[0.05,0.15]*
HSI	0.177	[0.1,0.25]*	[0.02,0.22]*	0.235	[0.18,0.29]*	[0.03,0.21]*	0.195	[0.15,0.24]*	[0.01,0.17]*	0.143	[0.11,0.17]*	[0.003,0.13]*
Nasdaq 100	0.224	[0.11,0.34]*	[0.04,0.17]*	0.346	[0.3,0.39]*	[0.06,0.22]*	0.310	[0.27,0.35]*	[0.06,0.19]*	0.217	[0.19,0.25]*	[0.05,0.11]*
NYSE	0.366	[0.19,0.54]*	[0.04,0.23]*	0.347	[0.27,0.43]*	[0.05,0.27]*	0.261	[0.21,0.31]*	[0.05,0.24]*	0.178	[0.15,0.21]*	[0.03,0.19]*
Russell 2000	0.315	[0.22,0.41]*	[0.04,0.20]*	0.288	[0.22,0.36]*	[0.03,0.24]*	0.206	[0.15,0.26]*	[0.01,0.21]*	0.113	[0.06,0.16]*	[-0.001,0.16]
S&P 400	0.338	[0.22,0.46]*	[0.06,0.23]*	0.326	[0.26,0.4]*	[0.06,0.25]*	0.245	[0.19,0.29]*	[0.05,0.22]*	0.161	[0.13,0.19]*	[0.04,0.16]*
KOSPI 200	0.218	[0.13,0.3]*	[0.07,0.256]*	0.263	[0.22,0.31]*	[0.07,0.23]*	0.238	[0.2,0.27]*	[0.05,0.19]*	0.144	[0.11,0.17]*	[0.002,0.12]*

Note: * denote the significance at 5% level.



(a) $N(0,1)$ noise (b) $t(3,0.5)$ noise

Figure 1: Size-adjusted power
$\rho'_{R,R}(h)$, HAC : ━━ , $\rho'_{R,|R|^{0.5}sign(R)}(h)$, HAC : ━ · ━ , $\rho'_{R,|R|^{0.25}sign(R)}(h)$, HAC : - - ,
$\rho'_{R,|R|^{0.1}sign(R)}(h)$, HAC : — — , $\rho'_{R,R}(h)$, $q=8$: —□— , $\rho'_{R,|R|^{0.5}sign(R)}(h)$, $q=8$: —△— ,
$\rho'_{R,|R|^{0.25}sign(R)}(h)$, $q=8$: —▽— , $\rho'_{R,|R|^{0.1}sign(R)}(h)$, $q=8$: —○—

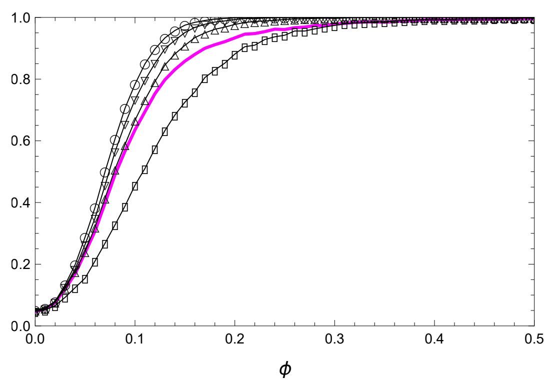
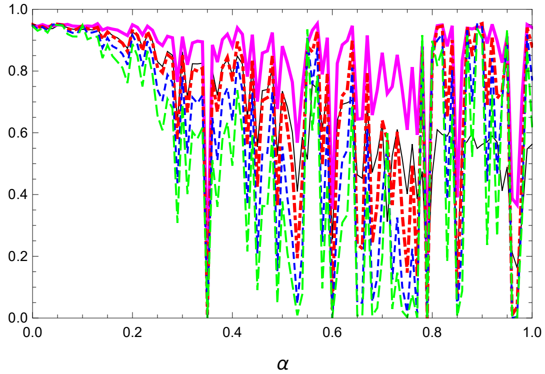
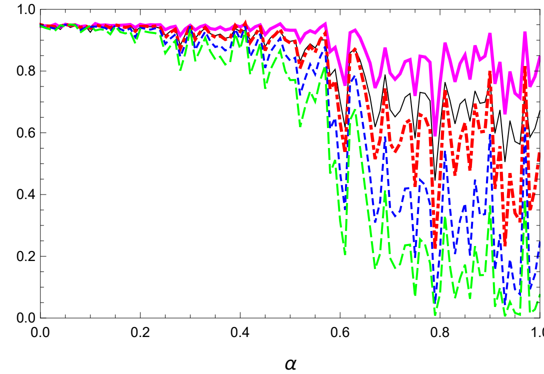


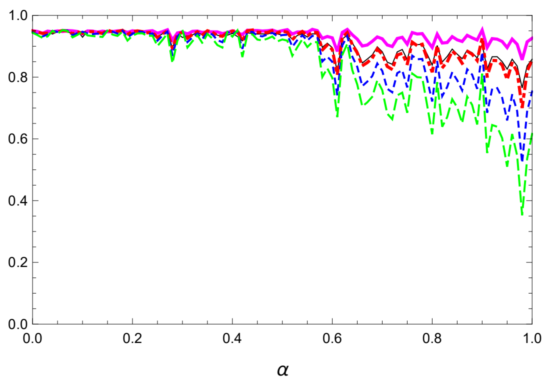
Figure 2: Size-adjusted power for ARCH(1) with $N(0,1)$ noise, $\rho'_{R,R}(h)$.
HAC: ━━ , $q=4$: —□— , $q=8$: —△— , $q=12$: —▽— , $q=16$: —○—



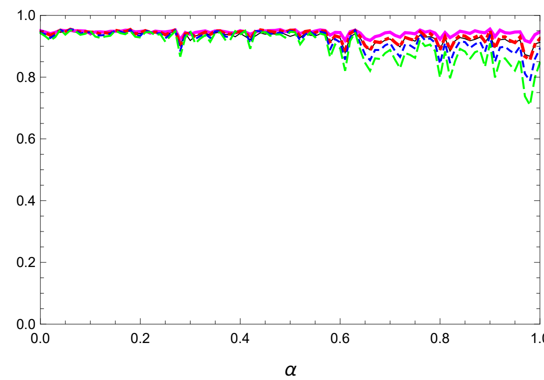
(a) $\rho_{R^2}(h)$



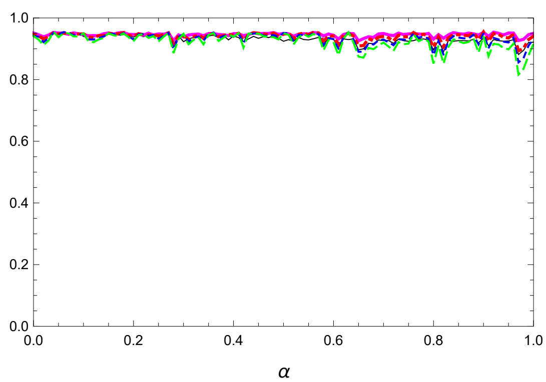
(b) $\rho_{|R|}(h)$



(c) $\rho_{|R|^{0.5}}(h)$



(d) $\rho_{|R|^{0.25}}(h)$



(e) $\rho_{|R|^{0.1}}(h)$

Figure 3: Coverage level for ARCH(1) with $N(0,1)$ noise
HAC: , $q = 4$: , $q = 8$: $q = 12$: , $q = 16$:

Online Appendix

to

New Approaches to Robust Inference on Market (Non-)Efficiency, Volatility Clustering and Nonlinear Dependence

S.1 Asymptotic normality of general sample covariances and correlations

The following Lemma S.1 provides the results on asymptotic normality of sample covariances and correlations for arbitrary functions of stationary processes (see also Francq and Zakoïan, 2006, and Lindner, 2009, in relation to GARCH processes). The result relies on applying a central limit theorem (CLT) for stationary α-mixing (i.e. strongly mixing) processes. We refer to Rio (2017) for a detailed treatment of mixing processes. We note that Lemmas 2.1 and 2.2 follow immediately from general Lemma S.1.

Lemma S.1 *Let $(R_t)_{t\in\mathbb{Z}}$ be an $\mathbb{R}$-valued stationary and strongly mixing process with mxing coefficients $\alpha(n), n \in \mathbb{Z}$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be measurable functions. Consider the sample covariance of $f(R_t)$ and $g(R_{t-h)}$ for some fixed $h \geq 0$,*

$$\hat{\gamma}_{T,f(R),g(R)}(h) = \frac{1}{T}\sum_{t=1}^{T} f(R_t)g(R_{t-h}) - (\frac{1}{T}\sum_{t=1}^{T} f(R_t))(\frac{1}{T}\sum_{t=1}^{T} g(R_{t-h})), \tag{S.1}$$

and its population equivalent,

$$\gamma_{f(R),g(R)}(h) = \mathrm{Cov}(f(R_t), g(R_{t-h})) = E[f(R_t)g(R_{t-h})] - E[f(R_t)]E[g(R_{t-h})]. \tag{S.2}$$

Likewise, consider the sample correlation,

$$\hat{\rho}_{T,f(R),g(R)}(h) = \frac{\hat{\gamma}_{T,f(R),g(R)}(h)}{\sqrt{\hat{\gamma}_{T,f(R),f(R)}(0)\hat{\gamma}_{T,g(R),g(R)}(0)}}, \quad h \geq 1, \tag{S.3}$$

and its population equivalent

$$\rho_{f(R),g(R)}(h) = \frac{\gamma_{f(R),g(R)}(h)}{\sqrt{\gamma_{f(R),f(R)}(0)\gamma_{g(R),g(R)}(0)}}. \tag{S.4}$$

Suppose that there exists a $\delta > 0$ such that

$$\max\{E[|f(R_t))|^{2+\delta}], E[|g(R_t))|^{2+\delta}]\} < \infty \text{ and } \max_{h=0,\ldots,m}\{E[|f(R_t)g(R_{t-h})|^{2+\delta}]\} < \infty, \tag{S.5}$$

and such that the mixing coefficients of $(R_t)_{t\in\mathbb{Z}}$ *satisfy*

$$\sum_{n=1}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty. \tag{S.6}$$

Then

$$\sqrt{T}(\hat{\gamma}_{T,f(R),g(R)}(h) - \gamma_{f(R),g(R)}(h))_{h=0,\ldots,m} \xrightarrow{d} (G_{h,f(R),g(R)})_{h=0,\ldots,m}, \tag{S.7}$$

where $(G_{h,f(R),g(R)})_{h=0,\ldots,m}$ *is an* $(m+1)$*-dimensional Gaussian vector with zero mean and the covariance matrix given by*

$$\Gamma \;=\; \mathrm{Var}(Y_0) + 2\sum_{k=1}^{\infty} \mathrm{Cov}(Y_0, Y_k), \tag{S.8}$$

where $Y_t = (Y_{t,h})_{h=0,\ldots,m}$, $Y_{t,h} = (f(R_t) - E[f(R_t)])(g(R_{t-h}) - E[g(R_{t-h})]) - \gamma_{f(R),g(R)}(h)$.

If $\gamma_{f(R),g(R)}(h))_{h=1,\ldots,m} = (0,\ldots,0)$, *then*

$$\sqrt{T}(\hat{\rho}_{T,f(R),g(R)}(h))_{h=1,\ldots,m} \xrightarrow{d} \left((\gamma_{f(R),f(R)}(0)\gamma_{g(R),g(R)}(0))^{-1/2} G_{h,f(R),g(R)}\right)_{h=1,\ldots,m}. \tag{S.9}$$

If there exists a $\delta > 0$ *such that*

$$\max\{E[|f(R_t)|^{4+\delta}], E[|g(R_t)|^{4+\delta}]\} < \infty \tag{S.10}$$

and such that (S.6) *holds, then*

$$\sqrt{T}(\hat{\rho}_{T,f(R),g(R)}(h) - \rho_{f(R),g(R)}(h))_{h=1,\ldots,m} \xrightarrow{d} (\tilde{G}_{h,f(R),g(R)})_{h=1,\ldots,m}, \tag{S.11}$$

where $(\tilde{G}_{h,f(R),g(R)})_{h=0,\ldots,m}$ *is an* $(m+1)-$*dimensional Gaussian vector with zero mean and the covariance matrix given by* $A\Gamma^{\dagger}A'$, *where* A *is constant matrix defined in* (S.14) *and*

$$\Gamma^{\dagger} \;=\; \mathrm{Var}(Y_0^{\dagger}) + 2\sum_{k=1}^{\infty} \mathrm{Cov}(Y_0^{\dagger}, Y_k^{\dagger}), \tag{S.12}$$

with $Y_t^{\dagger} = (Y_t', V_{t,1}, V_{t,2})'$, $V_{t,1} = (f(R_t) - E[f(R_t)])^2 - \gamma_{f(R),f(R)}(0)$, $V_{t,2} = (g(R_t) - E[g(R_t)])^2 - \gamma_{g(R),g(R)}(0)$.

Proof: Firstly, note that strong mixing is a property about the σ-field generated by $(R_t)_{t\in\mathbb{Z}}$. Since f and g are measurable, with $V_t = (f(R_t), g(R_t), g(R_{t-1}), \ldots, g(R_{t-m}))$, it follows that the process $(V_t)_{t\in\mathbb{Z}}$ is stationary and strongly mixing with mixing coefficients satisfying (S.6) (see also Francq and Zakoïan, 2006). Likewise, the same property applies to the process $(\tilde{V}_{t,h})_{t\in\mathbb{Z}}$ with $\tilde{V}_{t,h} = f(R_t)g(R_{t-h})$, $h = 0,\ldots,m$. This allows us to apply Theorem 18.5.3 of Ibragimov and Linnik (1971) [the CLT, henceforth] to the processes $(V_t)_{t\in\mathbb{Z}}$ and $(\tilde{V}_{t,h})_{t\in\mathbb{Z}}$, under the moments conditions in the lemma. Next, note that

$$\begin{aligned}\hat{\gamma}_{T,f(R),g(R)}(h) - \gamma_{f(R),g(R)}(h) &= \frac{1}{T}\sum_{t=1}^{T}(f(R_t) - E[f(R_t)])(g(R_{t-h}) - E[g(R_{t-h})]) - \gamma_{f(R),g(R)}(h)\\ &\quad -(\frac{1}{T}\sum_{t=1}^{T}(f(R_t) - E[f(R_t)]))(\frac{1}{T}\sum_{t=1}^{T}(g(R_{t-h}) - E[g(R_{t-h})).\end{aligned}$$

Suppose that (S.5) holds. By the CLT, $\frac{1}{T}\sum_{t=1}^{T} f(R_t) - E[f(R_t)]$ and $\frac{1}{T}\sum_{t=1}^{T} g(R_{t-h}) - E[g(R_{t-h})]$ are $O_p(T^{-1/2})$, so that

$$\begin{aligned}\sqrt{T}(\hat{\gamma}_{T,f(R),g(R)}(h) - \gamma_{f(R),g(R)}(h)) &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}((f(R_t) - E[f(R_t)])(g(R_{t-h}) - E[g(R_{t-h})])\\ &\quad -\gamma_{f(R),g(R)}(h)) + o_p(1).\end{aligned}$$

Let $Y_{t,h} = (f(R_t)-E[f(R_t)])(g(R_{t-h})-E[g(R_{t-h})])-\gamma_{f(R),g(R)}(h)$. Using (S.5), there exists a $\delta > 0$ such that, by Hölder's inequality, $E[|Y_{t,h}|^{2+\delta}] < \infty$. Since $E[Y_{t,h}] = 0$ and $(Y_{t,h} : t \in \mathbb{Z})$ is strongly mixing satisfying (S.6), $\sqrt{T}(\hat{\gamma}_{T,f(R),g(R)}(h) - \gamma_{f(R),g(R)}(h)) \xrightarrow{d} G_{h,f(R),g(R)}$ by the CLT. By similar arguments applied to linear combinations of
$\sqrt{T}(\hat{\gamma}_{T,f(R),g(R)}(h) - \gamma_{f(R),g(R)}(h))_{h=0,\ldots,m}$, and the Cramér-Wold device, (S.7) holds.

Turning to the sample correlations, the limiting distribution for the case $\gamma_{f(R),g(R)}(h))_{h=1,\ldots,m} = 0$ is immediate, by noting that $(\hat{\gamma}_{T,f(R),f(R)}(0)-\gamma_{f(R),f(R)}(0))$ and $(\hat{\gamma}_{T,g(R),g(R)}(0)-\gamma_{g(R),g(R)}(0))$ are $o_p(1)$ and using Slutsky's theorem. Next, using (S.10) and arguments as above,

$$\sqrt{T}\begin{bmatrix}(\hat{\gamma}_{T,f(R),g(R)}(h) - \gamma_{f(R),g(R)}(h))_{h=0,\ldots,m}\\ \hat{\gamma}_{T,f(R),f(R)}(0) - \gamma_{f(R),f(R)}(0)\\ \hat{\gamma}_{T,g(R),g(R)}(0) - \gamma_{g(R),g(R)}(0)\end{bmatrix} \xrightarrow{d} G^{\dagger}, \tag{S.13}$$

where $G^\dagger$ is an $(m+3)$-dimensional Gaussian vector with zero mean and the covariance matrix given by $\Gamma^\dagger$. Let $x = (x_1, \dots, x_{m+3})' \in \mathbb{R}^{m+3}$ and define the function $\tilde{g} : \mathbb{R}^{m+3} \to \mathbb{R}^{m+1}$ as $\tilde{g}(x) = (\frac{x_1}{\sqrt{x_{m+2}x_{m+3}}}, \dots, \frac{x_{m+1}}{\sqrt{x_{m+2}x_{m+3}}})'$. Define the matrix

$$A = \left.\frac{\partial \tilde{g}(x)}{\partial x'}\right|_{x=\gamma^\dagger}, \quad \gamma^\dagger = ((\gamma_{f(R),g(R)}(h))'_{h=0,\dots,m}, \gamma_{f(R),f(R)}(0), \gamma_{g(R),g(R)}(0))'. \tag{S.14}$$

The convergence in (S.11) is then obtained by an application of the delta method. □

S.2 Proof of Lemma 2.3

Suppose that $(R_t)_{t\in\mathbb{Z}}$ is stationary and β-mixing. For the sake of clarity we focus on the asymptotic independence of group-based estimators for covariances. The sample correlations are dealt with in a similar fashion. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be measurable functions, and define $y_t = (y_{t,1}, y_{t,2})' = (f(R_t), g(R_{t-1}))'$. Since f and g are measurable, $(y_t)_{t\in\mathbb{Z}}$ is stationary and β-mixing. With $i, j \in \mathbb{Z}$, $0 \le i < j$, let

$$\hat{\phi}_{i,j} := (j-i+1)^{-1} \sum_{t=i}^{j} y_{t,1}y_{t,2} - \left((j-i+1)^{-1} \sum_{t=i}^{j} y_{t,1}\right)\left((j-i+1)^{-1} \sum_{t=i}^{j} y_{t,2}\right) - \phi_0,$$

where $\phi_0 = E[y_{t,1}y_{t,2}] - E[y_{t,1}]E[y_{t,2}]$. Suppose that for some deterministic sequence (a_T), satisfying $a_T \to \infty$,

$$a_T\hat{\phi}_{1,T} \to_d Z,, \tag{S.15}$$

for some r.v. Z (potentially non-Gaussian). Consider the case of two equi-sized groups, such that we have the group-based estimators $\hat{\phi}_{1,\lfloor T/2\rfloor}$ and $\hat{\phi}_{\lfloor T/2\rfloor+1,2\lfloor T/2\rfloor}$. (In the case of more groups, one has to repeat the coupling argument.) We seek to show that $a_{\lfloor T/2\rfloor}\hat{\phi}_{1,\lfloor T/2\rfloor}$ and $a_{\lfloor T/2\rfloor}\hat{\phi}_{\lfloor T/2\rfloor+1,2\lfloor T/2\rfloor}$ are asymptotically independent. By the Cramér-Wold device, the asymptotic independence holds, if we show that for any constants $(k_1, k_2) \in \mathbb{R}^2$, $k_1a_{\lfloor T/2\rfloor}\hat{\phi}_{1,\lfloor T/2\rfloor} + k_2a_{\lfloor T/2\rfloor}\hat{\phi}_{\lfloor T/2\rfloor+1,2\lfloor T/2\rfloor} \to_d k_1Z^{(1)} + k_2Z^{(2)}$ where $Z^{(1)}$ and $Z^{(2)}$ are independent copies of Z. Let $\tilde{T} := \tilde{T}(T) \to \infty$ be a sequence of positive integers satisfying $\tilde{T} = o(T)$ as $T \to \infty$. It holds that

$$a_{\lfloor T/2\rfloor}\hat{\phi}_{\lfloor T/2\rfloor+1,2\lfloor T/2\rfloor} = a_{\lfloor T/2\rfloor}\hat{\phi}_{\lfloor T/2\rfloor+1,\lfloor T/2\rfloor+1+\tilde{T}} + a_{\lfloor T/2\rfloor}\hat{\phi}_{\lfloor T/2\rfloor+2+\tilde{T},2\lfloor T/2\rfloor}$$
$$=: S_T^{(1)} + S_T^{(2)}, \tag{S.16}$$

where it holds, due to (S.15), that

$$S_T^{(1)} = o_p(1). \tag{S.17}$$

Let $(y_t^\star : t \in \mathbb{Z})$ be a sequence of r.v.'s with the same distribution as that of $(y_t : t \in \mathbb{Z})$ and independent of $\mathcal{F}_{\lfloor T/2\rfloor} = \sigma(y_t : t \leq \lfloor T/2\rfloor)$. By Theorem 5.1 of Rio (2017), and using that $(y_t : t \in \mathbb{Z})$ is stationary,

$$P(y_t^\star \neq y_t \text{ for some } t \geq \lfloor T/2\rfloor + k) = \beta(k), \tag{S.18}$$

where $(\beta(k) : t \in \mathbb{Z})$ denotes the sequence of β-mixing coefficients. Let

$$\hat{\phi}_{i,j}^\star := (j-i+1)^{-1}\sum_{t=i}^{j} y_{t,1}^\star y_{t,2}^\star - \left((j-i+1)^{-1}\sum_{t=i}^{j} y_{t,1}^\star\right)\left((j-i+1)^{-1}\sum_{t=i}^{j} y_{t,2}^\star\right) - \phi_0.$$

Note that $S_T^{(2)} = a_{\lfloor T/2\rfloor}\hat{\phi}^\star_{\lfloor T/2\rfloor+2+\tilde{T},2\lfloor T/2\rfloor} + (a_{\lfloor T/2\rfloor}\hat{\phi}_{\lfloor T/2\rfloor+2+\tilde{T},2\lfloor T/2\rfloor} - a_{\lfloor T/2\rfloor}\hat{\phi}^\star_{\lfloor T/2\rfloor+2+\tilde{T},2\lfloor T/2\rfloor})$.

For any $\varepsilon > 0$, using (S.18) and that $(y_t : t \in \mathbb{Z})$ is stationary and β-mixing,

$$P\left[\left|a_{\lfloor T/2\rfloor}\hat{\phi}_{\lfloor T/2\rfloor+2+\tilde{T},2\lfloor T/2\rfloor} - a_{\lfloor T/2\rfloor}\hat{\phi}^\star_{\lfloor T/2\rfloor+2+\tilde{T},2\lfloor T/2\rfloor}\right| > \varepsilon\right]$$
$$\leq P\left[y_t^\star \neq y_t \text{ for some } t \geq \lfloor T/2\rfloor + 2 + \tilde{T}\right] = \beta(2+\tilde{T}) = o(1),$$

so that

$$S_n^{(2)} = a_{\lfloor T/2\rfloor}\hat{\phi}^\star_{\lfloor T/2\rfloor+2+\tilde{T},2\lfloor T/2\rfloor} + o_p(1). \tag{S.19}$$

Hence, combining (S.16), (S.17), and (S.19),

$$a_{\lfloor T/2\rfloor}\hat{\phi}_{\lfloor T/2\rfloor+1,2\lfloor T/2\rfloor} = a_{\lfloor T/2\rfloor}\hat{\phi}^\star_{\lfloor T/2\rfloor+2+\tilde{T},2\lfloor T/2\rfloor} + o_p(1).$$

Using (S.15), we then obtain that for any $(k_1, k_2) \in \mathbb{R}^2$,

$$\begin{aligned} k_1 a_{\lfloor T/2 \rfloor} \hat{\phi}_{1,\lfloor T/2 \rfloor} + k_2 a_{\lfloor T/2 \rfloor} \hat{\phi}_{\lfloor T/2 \rfloor+1,2\lfloor T/2 \rfloor} &= k_1 a_{\lfloor T/2 \rfloor} \hat{\phi}_{1,\lfloor T/2 \rfloor} + k_2 a_{\lfloor T/2 \rfloor} \hat{\phi}^{\star}_{\lfloor T/2 \rfloor+2+\tilde{T},2\lfloor T/2 \rfloor} + o_p(1) \\ &\overset{w}{\to} k_1 Z^{(1)} + k_2 Z^{(2)}, \end{aligned}$$

where $Z^{(1)}$ and $Z^{(2)}$ are copies of Z, and $Z^{(1)}$ and $Z^{(2)}$ are independent since $a_{\lfloor T/2 \rfloor} \hat{\phi}^{\star}_{\lfloor T/2 \rfloor+2+\tilde{T},2\lfloor T/2 \rfloor}$ is independent of $\mathcal{F}_{\lfloor T/2 \rfloor}$.

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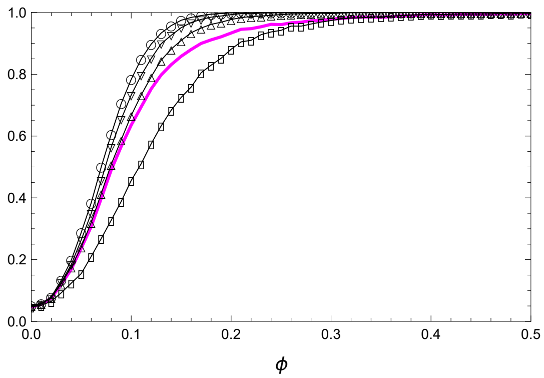
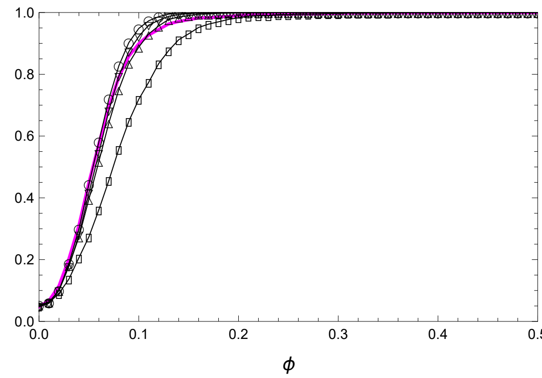
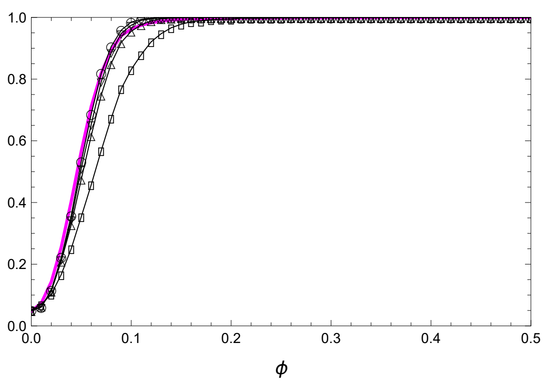
S.3 Additional figures



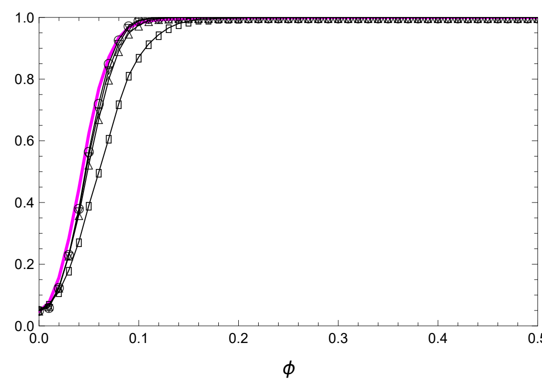
(a) $\rho'_{R,R}(h)$



(b) $\rho'_{R,|R|^{0.5} sign(R)}(h)$

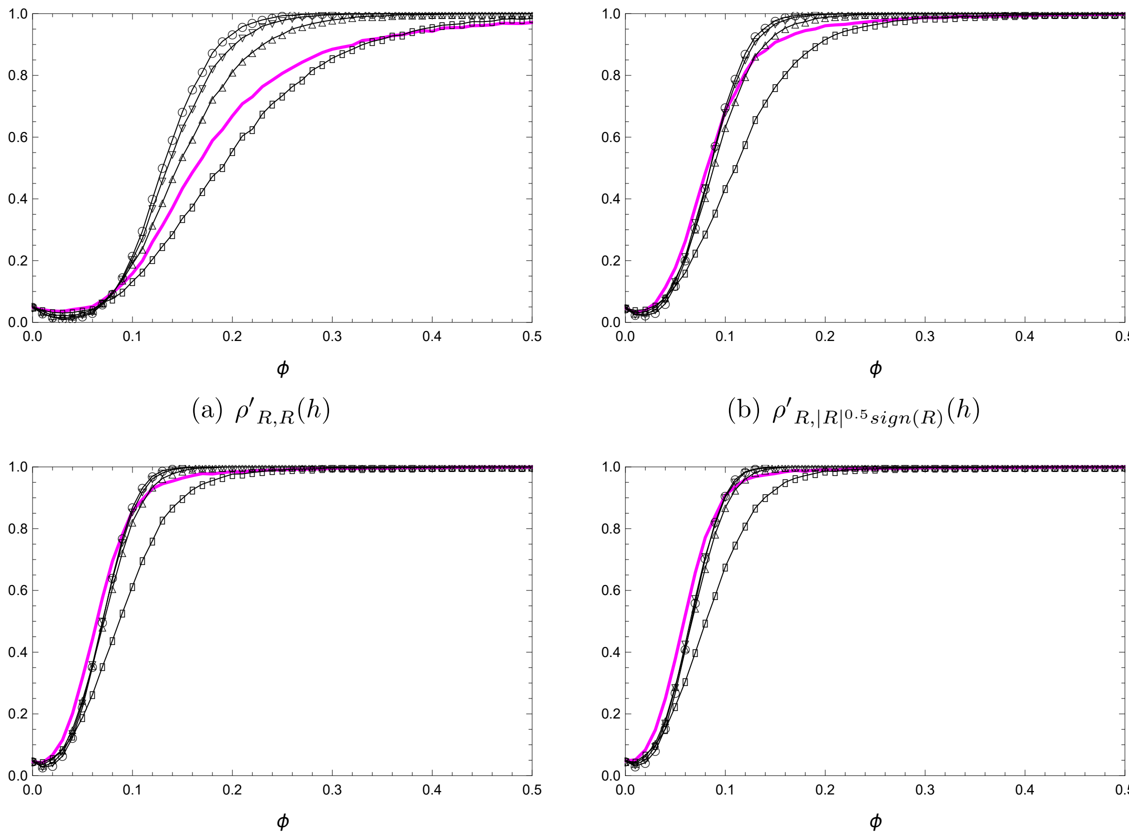


(c) $\rho'_{R,|R|^{0.25} sign(R)}(h)$



(d) $\rho'_{R,|R|^{0.1} sign(R)}(h)$

Figure S.1: Size-adjusted power for ARCH(1) with $N(0,1)$ noise
HAC: , $q = 4$: , $q = 8$: , $q = 12$: , $q = 16$:



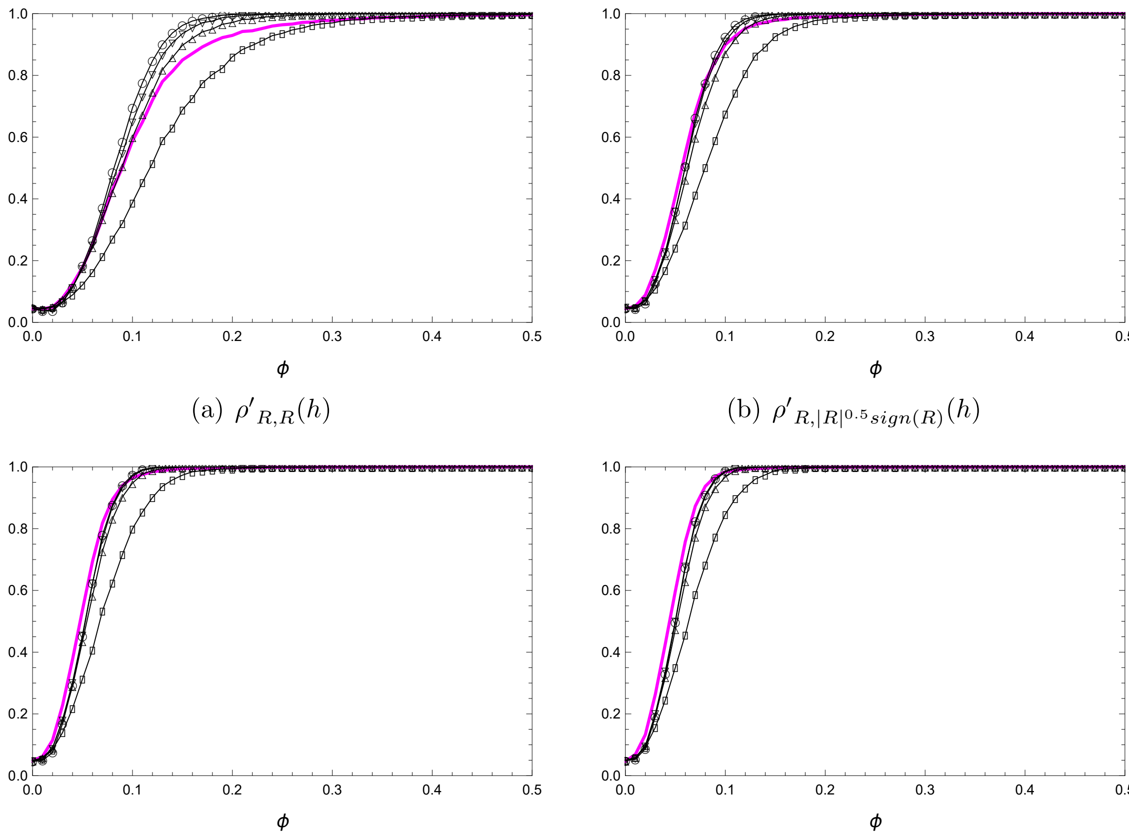
(a) $\rho'_{R,R}(h)$

(b) $\rho'_{R,|R|^{0.5}sign(R)}(h)$

(c) $\rho'_{R,|R|^{0.25}sign(R)}(h)$

(d) $\rho'_{R,|R|^{0.1}sign(R)}(h)$

Figure S.2: Size-adjusted power for ARCH(1) with $t(3, 0.5)$ noise noise
HAC: , $q = 4$: , $q = 8$: , $q = 12$: , $q = 16$:



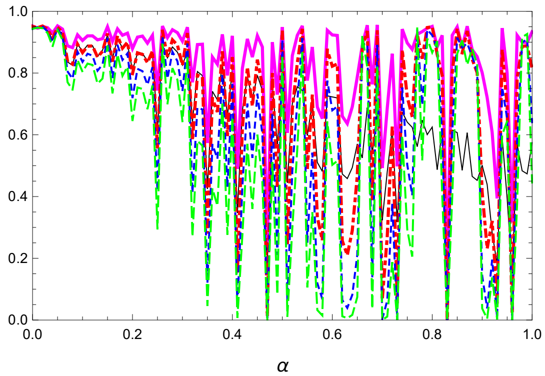
(a) $\rho'_{R,R}(h)$

(b) $\rho'_{R,|R|^{0.5}sign(R)}(h)$

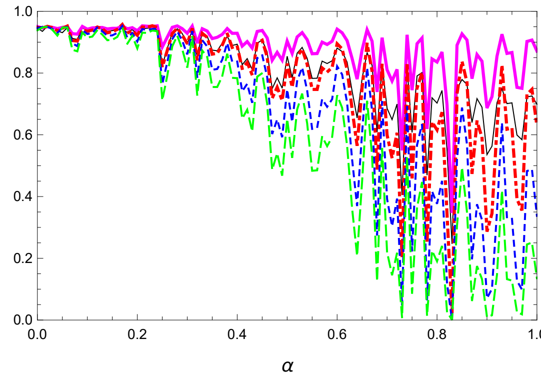
(c) $\rho'_{R,|R|^{0.25}sign(R)}(h)$

(d) $\rho'_{R,|R|^{0.1}sign(R)}(h)$

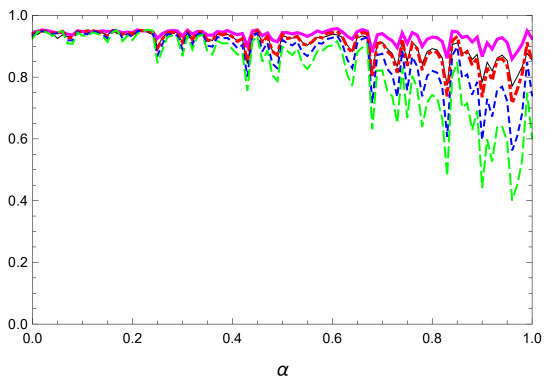
Figure S.3: Size-adjusted power for ARCH(1) with $t(50, 0.5)$ noise
HAC: —— , $q = 4$: —□— , $q = 8$: —△— , $q = 12$: —▽— , $q = 16$: —○—



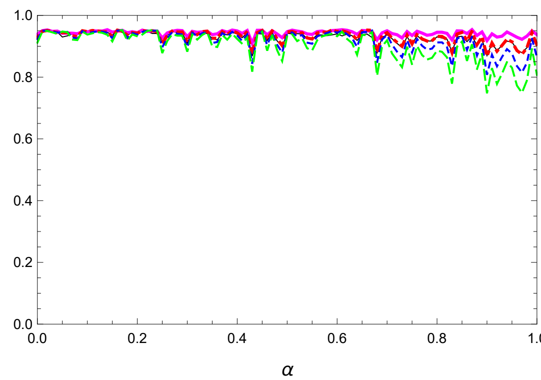
(a) $\rho_{R^2}(h)$



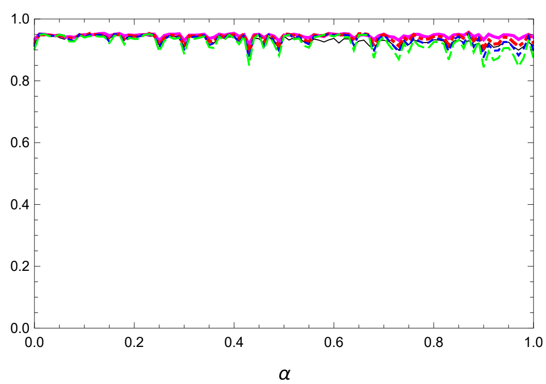
(b) $\rho_{|R|}(h)$



(c) $\rho_{|R|^{0.25}}(h)$

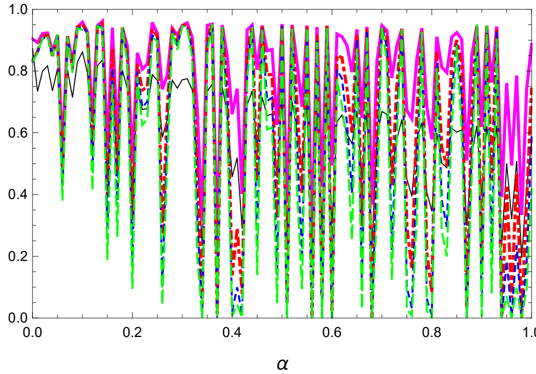


(d) $\rho_{|R|^{0.1}}(h)$

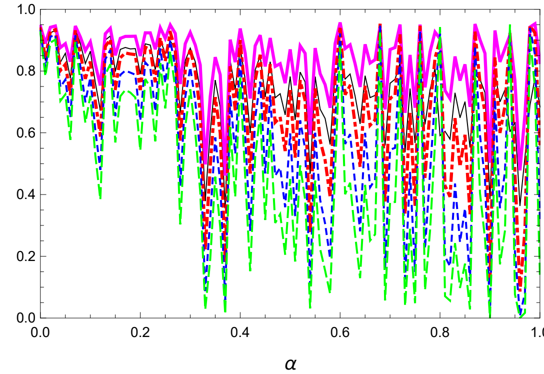


(e) $\rho_{|R|^{0.1}}(h)$

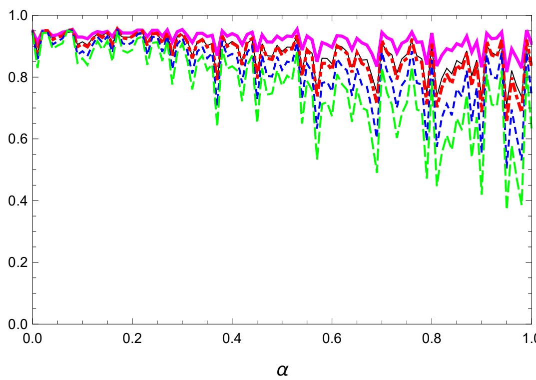
Figure S.4: Coverage level for ARCH(1) with $t(50, 0.5)$ noise
HAC: , $q = 4$: , $q = 8$: $q = 12$: , $q = 16$:



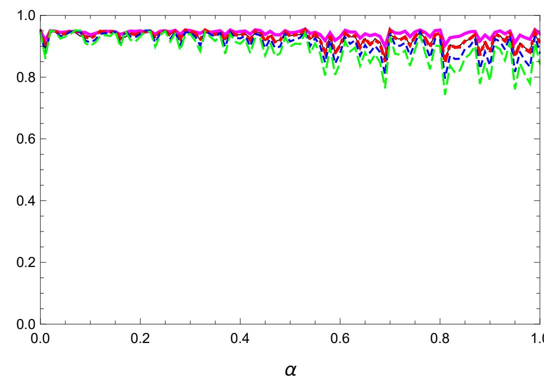
(a) $\rho_{R^2}(h)$



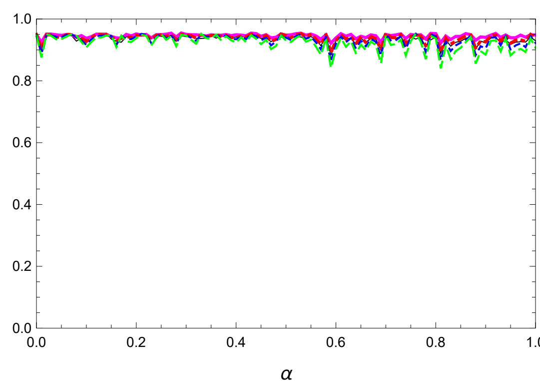
(b) $\rho_{|R|}(h)$



(c) $\rho_{|R|^{0.5}}(h)$



(d) $\rho_{|R|^{0.25}}(h)$



(e) $\rho_{|R|^{0.1}}(h)$

Figure S.5: Coverage level for ARCH(1) with $t(3, 0.5)$ noise
HAC:, $q = 4$:, $q = 8$: $q = 12$:, $q = 16$: