Negative Bubbles*

Kosuke Aoki[†] Kalin Nikolov[‡]

Abstract

We develop a macroeconomic model with credit frictions in which firms' ability to borrow depends on the value of their equity. Under irreversibility of capital investment, this framework admits negative asset price bubbles in addition to the positive ones already emphasized in the literature. We characterize the macroeconomic effects of negative bubbles and show that, depending on the cost of seizing the firm in bankruptcy, they may be contractionary or expansionary. The surprising result that negative bubbles may be expansionary arises due to two offsetting effects. Negative bubbles reduce overall collateral which is contractionary when credit constraints bind. However, the contraction in aggregate collateral encourages the production of tangible collateral (capital) which is expansionary. When capital is highly pleadgeable to debt-holders and therefore good collateral, the second effect dominates and negative bubbles expand real economic activity.

JEL classification:
Keywords: Financial Frictions, Negative Bubbles

*We thank XYZ for helpful comments. The views expressed in this paper are those of the authors and do not necessarily represent the views of the European Central Bank

[†]University of Tokyo, Department of Economics.

[‡]European Central Bank, Directorate General Research - Financial Research.

1 Introduction

Asset markets go through periods of boom and bust which economists and financial analysts struggle to connect with fundamentals. Several rational expectations theories have therefore been developed to explain the volatility of asset prices in terms of ups and downs in investor sentiment which lead to deviations from fundamentals called 'bubbles'. In these models (Tirole (1985), Martin and Ventura (2012) and Farhi and Tirole (2012) among many others) investors buy overvalued assets because they expect to be able to sell them to others in the future. Credit constraints that lead to low risk-free interest rates compared to returns on capital create the conditions for bubbles to exist and to be expansionary.[1] Models in which the market value of the firm is used as collateral also admit to bubbly solutions (Miao and Wang (2015, 2018), Azariadis et al. (2016)).

One limitation all these bubble models share is that they can only generate positive bubbles. In the rational bubbles literature, bubbly assets can never have negative value as long as they can be costlessly disposed of (Martin and Ventura (2018)). And other papers (e.g. Miao and Wang (2015, 2018)) only stress the possibility of positive bubbles. In contrast, there is evidence (Shiller (2000)) that asset prices can undershoot fundamentals and that sentiment can also be negative as well as positive.[2]

In this paper, we show that models in which firms' borrowing limits depends on their market value (first developed by Gertler and Karadi (2011) and Miao and Wang (2015, 2018)) admit both positive and negative bubbles depending on parameter values. We build a simplified version of Miao and Wang (2018): an otherwise standard neoclassical model in which only a fraction of firms have investment opportunties in each period and when they do, they must collateralize all borrowing with the value of the firm.[3] The tightness of credit constraints is determined by how much of the firm's tangible assets (capital) can be recovered by creditors in the even of bankruptcy. Like in Miao and Wang (2015, 2018), bubbles on the firm's value can exist because a more valuable firm generates more credit and more profits in the event of an investment opportunity. It is this rise in profits that can then justify the higher firm value thus making the bubbly solution self fulfilling even when all investment in the economy is dynamically efficient.

[1] Kollmann (2022) generates bubbles without credit constraints by appealing to self-fulfilling beliefs about the future investment rate.

[2] Shiller (2000) presents evidence for the existence of both positive and negative bubbles. Based on a questionnaire of institutional investors, he identifies 'negative bubbles' as periods in which investors expect stocks to decline sharply causing them to sell assets before the decline occurs.

[3] To keep the exposition simple, we do not use the model of Gertler and Karadi (2011) in this paper. However, we have developed a version of it with negative bubbles.

In the original model of Miao and Wang (2015, 2018), negative bubbles cannot exist because capital is costlessly transferrable between firms. A firm whose equity is trading below the market value of capital could always sell the capital and close the firm. What our paper shows is that, if we assume that capital is firm-specific once installed, this argument does not hold and negative bubbles can exist due to the same intuition as positive ones. Pessimism about firm equity valuations leads to the firm's equity trading below the value of capital. This pessimism is self-fulfilling when it leads to reduced credit access and hence lower profits. However, the capital cannot be sold to another firm. The option to close the firm is also not exercised as long as the total value of the firm is positive.

In our model, bubbly equilibria always exist. They are positive, expansionary and welfare enhancing only if credit constraints are sufficiently tight. With looser credit constraints, bubbles turn negative and become contractionary. Moreover, since they push output further below the first best, they reduce welfare. Once credit is very ample because capital is highly pledgeable, credit constraints do not bind and the economy is at the first best in the bubbleless equilibrium. Negative bubbles still exist but become expansionary beyond a certain level of the pledgability of capital, causing output to rise above the first best.

The intuition for this surprising result is that when the bubble (i.e. intangible collateral) is negative, it shrinks overall collateral supply and raises the market valuation of installed physical capital (i.e. tangible collateral). This has two offsetting effects on the quantity of capital demanded by firms. A higher capital price means that firms require a higher capital rental rate in order to maintain the 'dividend yield' of capital unchanged. This effect is contractionary. However, the rise in the price of new capital goods makes their production more profitable and increases the collateral premium attached to capital goods which can be pledged to obtain more debt funding. Hence firms are willing to accumulate capital even with a lower conventional 'dividend yield' because they can obtain liquidity against it when an investment opportunity arises. When the pledgeability of capital is sufficiently high, the second collateral premium effect dominates and the overall level of capital (and hence output) can rise above the first best. However, even though such negative bubbles are expansionary, they are still welfare-reducing because they lead to overinvestment and dynamic inefficiency.

We also examine the policy implications of our model. We show that a subsidy/tax on debt can restore the economy to the first best when there is under/overinvestment regardless of whether the economy is in a bubbly or bubbleless equilibrium. The optimal subsidy decreases following a switch to an equilibrium with positive bubbles. The subsidy increases if there is a negative contractionary bubble. Finally, a tax is imposed if there is a negative expansionary bubble which leads to overinvestment.

Negative bubbles can also exist in models with non-rational expectations. For instance, in the framework of Adam and Marcet (2011), private investors are assumed to postulate an autoregressive law of motion for the price-dividend ratio of an asset and to estimate the parameters of the law of motion from the (model-generated) data. The framework in Adam and Marcet (2011) generates momentum in asset prices leading to overvaluations when expectations are optimistic and undervaluations when they are pessimistic.

The bubbles in our framework are a type of intangible (reputational) collateral. There is a literature on this: Azariadis et al. (2016) and Cui and Kaas (2021) have a similar setting in which unsecured credit increases the value of the firm thus improving incentives to repay. These papers do not consider negative bubbles because they assume that capital can be freely sold by each firm.

Looking at the literature more broadly, there are other models where changes in collateral values or collateral pledgability can lead to changes in firm investment and asset demand. In Kiyotaki and Moore (2019), limited pledgability and resaleability of physical capital creates demand for liquid assets which can be sold when the profitable possibility to invest in new capital goods arises. In our framework, capital cannot be resold at all and it has limited pledgeability but we abstract from monetary assets. For us liquidity is equivalent to pledgeability which is why capital rises in value during negative bubble episodes when overall credit supply is restricted. When capital is very pledgeable, it is also accumulated to some extent due to its liquidity properties (despite having a low conventional rate of return). In the Appendix we also develop an extension of our modelling framework with liquid short term government debt (money) and show that this addition does not remove the existence of negative bubbles.[4]

The paper is organized as follows. Section 2 outlines the baseline model, section 3 solves for the steady state with and without bubbles and discusses the conditions under which a bubbly equilibrium exists and is expansionary or contractionary for real economic activity. Section 4 shows the transitional dynamics to steady states with positive or negative bubbles. Section 5 discusses the policy interventions that can bring the economy to the first best. Section 6

[4] Miao and Wang (2014) consider (positive) sectoral asset price bubbles and show that they cause a reallocation of capital towards the sector with the bubble. In Matsuyama (2013), changes in firm net worth cause a shift in the composition of investment along the productivity/pledgeability spectrum.

concludes.

2 The Model

We have a closed economy populated by a continuum of measure one of risk-neutral households who supply a unit of labour inelastically. Their preferences are given by the following value function:

$$W_t = C_t + \beta W_{t+1}, \quad 0 < \beta < 1, \tag{1}$$

where W_t is lifetime welfare, C_t is period consumption and β is the time discount factor.

Production uses capital and labour

$$Y_t = K_t^{\alpha} L^{1-\alpha}, \tag{2}$$

where $L = 1$ and we drop it from subsequent notation.

Capital depreciates at the rate of δ per period. There is a continuum of firms which own the capital stock. They start each period with their capital from the previous period and produce. Then, within the period, with a probability π, they have the possibility to produce new capital goods. To finance this capital goods production, the firm can access only the rental income from its old capital as well as intra-period loans from households.

$$i_t = r_t k_t + d_t. \tag{3}$$

Those loans carry an interest of unity but are limited to the value of the firm in bankruptcy:

$$d_t \leqq V(\lambda k_t). \tag{4}$$

In bankruptcy, lenders can recover only a fraction λ of the firm's capital but, crucially, all of its bubbles if they exist.[5]

Capital is irreversible at the firm level. This implies that capital goods are fully generic when newly produced and any firm can purchase them and install them. However, once capital is installed in one firm, it cannot be subsequently re-sold to another because it becomes fully

[5]For simplicity we abstract from liquid assets such as government debt which can be accumulated by firms in anticipation of investment opportunities (Kiyotaki and Moore (2019)). In the Appendix we develop a version of the model with government bonds and show that this does not affect the results we will derive in this section.

firm-specific. This implies that the individual firm faces a capital accumulation equation

$$k_{t+1} = (1-\delta)\, k_t + m_t,$$

where

$$m_t \geqslant 0$$

is new capital purchases.

The value function of the firm is therefore given below:

$$V(k_t) = \max_{m_t, i_t} r_t k_t - q_t m_t + \pi\,(q_t - 1)\, i_t + \beta V(k_{t+1}).$$

The firm decides how much of new capital goods i_t to produce if it gets the productive opportunity and how much new capital goods m_t to install. The maximization is subject to the flow of funds constraint

$$i_t = r_t k_t + d_t,$$

the borrowing constraint

$$d_t \leqslant V\,(\lambda k_t)\,,$$

the capital accumulation constraint

$$k_{t+1} = (1-\delta)\, k_t + m_t,$$

the irreversibility constraint

$$m_t \geqslant 0,$$

and the limited liability constraint

$$V(k_t) \geqslant 0.$$

Now we can proceed to solve the firm's problem for the cases when the irreversibility constraint binds or does not bind.

2.1 Non-binding irreversibility and borrowing constraint

When $q_t = 1$, the firm is indifferent between producing new capital goods and not. When $m_t > 0$, the irreversibility constraint does not bind and we can substitute the capital evolution equation into the value function as follows:

$$V\left(k_t\right) = \max_{k_{t+1}} \left\{ (r_t + 1 - \delta)k_t - k_{t+1} + \beta V\left(k_{t+1}\right) \right\}. \tag{5}$$

Guessing that the value function is linear in capital

$$V\left(k_t\right) = \phi_t k_t, \tag{6}$$

we can easily derive the standard capital first order condition

$$1 = \beta\left(r_{t+1} + 1 - \delta\right),$$

and the value of installed capital is given by

$$\phi_t = r_t + 1 - \delta. \tag{7}$$

2.2 Non-binding irreversibility constraint and binding borrowing constraint

When $q_t > 1$, the firm would like to choose an infinite level of capital production and the collateral constraint binds. Let us guess that the value function has the following form (including a bubble term):

$$V\left(k_t\right) = \phi_t k_t + b_t. \tag{8}$$

Substituting this into the collateral constraint (4), we obtain

$$d_t = \lambda \phi_t k_t + b_t.$$

Subsituting this into the value function we get the following functional equation in ϕ_t:

$$\phi_t k_t + b_t = \max_{k_{t+1}} \left\{ \left(r_t + q_t(1-\delta)\right) k_t - q_t k_{t+1} + \pi\left(q_t - 1\right)\left(r_t k_t + \lambda \phi_t k_t + b_t\right) + \beta\left(\phi_{t+1} k_{t+1} + b_{t+1}\right) \right\}. \tag{9}$$

The first order condition with respect to k_{t+1} is given by

$$-q_t + \beta \phi_{t+1} = 0. \tag{10}$$

The envelope condition is given by

$$\phi_t = r_t + q_t(1-\delta) + \pi\left(q_t - 1\right)\left(r_t + \lambda\phi_t\right),$$

which can be written as

$$\phi_t = \frac{\left(1 + \pi\left(q_t - 1\right)\right) r_t + q_t(1-\delta)}{1 - \lambda\pi\left(q_t - 1\right)}. \tag{11}$$

To get a bit more intuition about the meaning of the above expression, we can rewrite it as follows:

$$\phi_t = \frac{1 + \pi\left(q_t - 1\right)}{1 - \lambda\pi\left(q_t - 1\right)} r_t + \frac{\beta}{1 - \lambda\pi\left(q_t - 1\right)}(1-\delta)\phi_{t+1}. \tag{12}$$

When credit constraints do not bind and $q_t = 1$, equation (11) reduces to

$$\phi_t = r_t + \beta(1-\delta)\phi_{t+1}. \tag{13}$$

We can therefore see that when the investment flow is restricted and $q_t > 1$, this boosts the dividend term in r_t by a factor of $\frac{1+\pi(q_t-1)}{1-\lambda\pi(q_t-1)}$ through the opportunity to use rental income from capital to finance further investment (the numerator) and from the possibility to use leverage when doing so (the denominator).

In addition, even though the firm cannot sell already installed capital to fund new investment, its market value can be used as collateral for loans from the household. This has the effect of reducing the effective discount rate from β to $\frac{\beta}{1-\lambda\pi(q_t-1)}$ in equation (12). The firm is willing to hold capital at a lower required rate of return than β^{-1} because capital can be pledged during an investment opportunity giving rise to potential profits from the production of new capital goods. It therefore enjoys a collateral (or liquidity) premium driven by the 'spread' $q_t - 1$.

Finally, the process for the bubble (if it exists) is the following:

$$b_t = \pi\left(q_t - 1\right) b_t + \beta b_{t+1}. \tag{14}$$

Just as in Miao and Wang (2012), the bubble delivers a 'dividend' to the firm because it increases collateral and allows the firm to borrow and invest more in the event that it has an investment opportunity. This is profitable as long as $q_t > 1$. In this framework the bubble does not arise due to dynamic inefficiency and the real interest rate is equal to β^{-1} which is above the economy's growth rate (which is assumed to be zero). The bubble therefore cannot deliver investors' required rate of return while remaining stable as a share of national income through capital gains alone. A 'dividend' is needed and the bubble delivers it by relaxing credit constraints.

Crucially, our modified framework with irreversibility of capital at the firm level allows for negative bubbles as long as they are not so large so as to violate the limited liability constraint:

$$V(k_t) = \phi_t k_t + b_t \geqslant 0.$$

If capital were fully tradeable across firms, the firm could always sell its capital to another firm and then shut down thus killing the negative bubble. But with irreversible capital, such a possibility no longer exists and negative bubbles can be present. We will investigate the consequences of such negative bubbles for the macroeconomy in subsequent analysis.

2.3 Binding irreversibility constraint

When

$$q_t > \beta \left(r_{t+1} + (1-\delta)\, q_{t+1} \right),$$

individual firms do not wish to buy new capital and the irreversibility constraint binds. Since firms are identical, when the constraint binds for one, it binds for all firms and capital purchases fall to zero. By market clearing, investment is zero too and the price of capital falls below its replacement cost of unity.

The firm's value function is therefore characterized by the following functional equations while the irreversibility constraint binds:

$$\phi_t = r_t + \beta\,(1-\delta)\,\phi_{t+1},$$

and the evolution of b_t satisfies

$$b_t = \beta b_{t+1}.$$

For the parameter values we consider in the subsequent numerical solution of our model economy, the irreversibility constraint does not bind even when a large positive bubble collapses. In Figure A1 in the Appendix, we consider the collapse of a bubble when $\lambda = 0.1$. Such a bubble is positive and expansionary and its collapse reduces collateral and investment. Nevertheless, as Figure A1 shows, the fall in investment is not large enough to hit the irreversibility constraint. Hence we do not consider this constraint in the numerical exercises we perform subsequently.

2.4 Aggregate equilibrium conditions

In the aggregate economy, we have three more aggregate equilibrium conditions in addition to equation (11) and the aggregate version of equation (14). The capital rental rate is:

$$r_t = \alpha K_t^{\alpha-1}, \tag{15}$$

and the aggregate capital stock evolves as follows:

$$K_{t+1} = (1-\delta) K_t + I_t \tag{16}$$

where investment is zero when the irreversibility constraint binds and is given by the aggregate collateral constraint when this is binding and $q > 1$:

$$I_t = \pi \left(\left(r_t + \lambda \beta^{-1} q_t \right) K_t + B_t \right). \tag{17}$$

2.5 Definition of equilibrium

Rational expectations equilibrium is a sequence of endogenous variables $r_t, q_t, \phi_t, B_t, K_t, I_t, Y_t$ which satisfy (2), (10), (11), (14), (15) - (17)

3 Steady state

We start by characterizing analytically the steady state of the model. Our aim in this section is to derive the conditions under which bubbles exist, are expansionary and are negative or positive.

3.1 Bubbleless steady state

3.1.1 Steady state with binding borrowing constraints

Firstly we characterize the steady state in which the borrowing constraint is binding. Combining equation (10) and (11) and imposing steady state, we get the following equation for the value of capital as a function of itself as well as the rental rate. We denote variables in the bubbleless steady state with superscript 'N'. Then we obtain

$$q^N = \beta \frac{\left(1 + \pi \left(q^N - 1\right)\right) r^N + q^N (1-\delta)}{1 - \lambda \pi \left(q^N - 1\right)}. \tag{18}$$

This is a quadratic equation in (q,r) space due to the effect of leverage (the denominator in equation (18)). Equation (10) evaluated at the steady state is given by

$$q^N = \beta\phi^N. \tag{19}$$

The capital stock evolution (16) evaluated at the steady state is given by

$$\delta = \pi\left(r^N + \lambda\phi^N\right). \tag{20}$$

From equation (18), (19) and (20), we obtain the close-form solutions of the steady state values of ϕ and r as

$$\phi^N = \frac{\delta(1-\pi)}{\pi(1-\beta+\lambda)}, \tag{21}$$

$$q^N = \beta\frac{\delta(1-\pi)}{\pi(1-\beta+\lambda)}, \tag{22}$$

$$r^N = \frac{\delta(1-\beta+\lambda\pi)}{\pi(1-\beta+\lambda)}. \tag{23}$$

Equation (22) and (23) imply that q^N and r^N are both monotonically decreasing in λ. The capital stock can be computed from the expression for the rental rate for capital.

$$r^N = \alpha(K^N)^{\alpha-1}. \tag{24}$$

Condition for binding borrowing constraints Finally we derive the condition under which the borrowing constraint is binding. The borrowing constraint binds in the steady state only if $q^N > 1$. From equation (22), the necessary and sufficient condition for $q^N > 1$ is given by

$$\lambda \leq \beta(1-\pi)\frac{\delta}{\pi} - (1-\beta) \equiv \bar{\lambda}. \tag{25}$$

Condition for $0 < \bar{\lambda} < 1$. In what follows we consider the case in which $0 < \bar{\lambda} < 1$. From equation (25), we have $\bar{\lambda} > 0$ if and only if

$$\pi < \frac{\beta\delta}{1-\beta(1-\delta)} \equiv \bar{\pi}. \tag{26}$$

Intuitively, in order for the credit constraint to be binding, the probability of investment opportunity must be small enough. Otherwise, agents would accumulate enough net worth to become self-financing. Similarly, we have $\bar{\lambda} < 1$ if and only if

$$\pi > \frac{\beta\delta}{2 - \beta(1-\delta)} \equiv \underline{\pi}. \tag{27}$$

Intuitively, if the probability of investment opportunity is large enough, agents can accumulate enough net worth so that their borrowing constraint becomes slack if it is loose enough ($\lambda > \bar{\lambda}$). Combining these two conditions, we obtain $0 < \bar{\lambda} < 1$ if and only if

$$\underline{\pi} < \pi < \bar{\pi}. \tag{28}$$

In what follows, we consider the parameter space in which equation (28) holds.

3.1.2 Steady state with non-binding borrowing constraints.

Secondly we characterize the unconstrained steady state. If $\lambda > \bar{\lambda}$ then the steady state equilibrium is unconstrained. In the unconstrained steady state equilibrium,

$$q^N = 1, \tag{29}$$

$$r^N = \beta^{-1} - (1-\delta). \tag{30}$$

3.2 Bubbly steady state

Next we characterize the bubbly equilibrium. We denote variables in the bubbly steady state with superscript 'B'. The bubble arbitrage condition (14) pins down the price of capital:

$$q^B = 1 + \frac{1-\beta}{\pi} > 1. \tag{31}$$

The price of capital delivers exactly the right amount of 'dividends' per unit of the bubble so that the bubble holder (who is also the shareholder in the representative firm) receives their required rate of return β^{-1} while the bubble remains constant as a share of national income.

Using equation (18) and substituting for q from equation (31) delivers the following expression for the capital rental rate:

$$r^B = \left(1 + \frac{1-\beta}{\pi}\right)\left(\frac{1 - \lambda(1-\beta) - \beta(1-\delta)}{\beta\,(2-\beta)}\right) \tag{32}$$

with the capital stock given by the capital production input first order condition:

$$r^B = \alpha(K^B)^{\alpha-1}. \tag{33}$$

Finally, the capital stock evolution equation pins down the bubble as a share of the capital stock:

$$B = \frac{\left(\delta - \pi\left(r^B + \frac{\lambda}{\beta}q^B\right)\right)}{\pi}K^B. \tag{34}$$

We can see that equation (34) allows the possibility of negative bubbles especially when λ is relatively large.

Threshold value of λ for $q^N = q^B$. When characterizing the region where bubbles are negative, the point where the capital price in the bubbly equilibrium exceeds that in the bubbleless equilibrium will turn out to be important. Therefore it will be useful in subsequent analysis to derive the value of λ that satisfies $q^N = q^B$. From equation (22) and (31), this is given by

$$\lambda^* \equiv \beta(1-\pi)\frac{\delta}{\pi + 1 - \beta} - (1-\beta). \tag{35}$$

Comparing equation (25) and (35), the following inequality holds:

$$\bar{\lambda} > \lambda^*. \tag{36}$$

This means that the parameter values for which the bubbly and bubbleless equilibria coincide occurs when borrowing constraints bind. We consider a general case in which λ^* satisfies

$$0 < \lambda^* < 1. \tag{37}$$

Since $\lambda^* < \bar{\lambda}$ holds, we have $\lambda^* < 1$ under condition (27). From equation (35), the condition for $\lambda^* > 0$ is given by

$$\pi < \pi^* \equiv \frac{\delta\beta - (1-\beta)^2}{1 - \beta(1-\delta)}. \tag{38}$$

We note from equation (26) and (38) that $\pi^* < \bar{\pi}$.

We further need the condition under which $\underline{\pi} < \pi^*$ in order to ensure that the parameter space for $0 < \lambda^* < \bar{\lambda}$ is non-empty. From equation (27) and (38), we have $\underline{\pi} < \pi^*$ if and only if

$$\delta > \frac{(1-\beta)^2(2-\beta)}{\beta-(1-\beta)^2}. \tag{39}$$

This condition easily holds when β is close to unity.

[Figure 1 here]

Therefore, in what follows, we consider the parameter space in which $\underline{\pi} < \pi < \pi^*$ so that $0 < \lambda^* < \bar{\lambda}$ holds. (See Figure 1.)

3.2.1 Negative bubbles

Equation (34) implies that the bubble can be negative when:

$$\delta - \pi\left(r^B + \frac{\lambda}{\beta}q^B\right) < 0, \tag{40}$$

where q^B and r^B are respectively given by equation (31) and (32).

Proposition 1 *Consider the bubbly steady state and the no-bubble steady state. Assume that $\underline{\pi} < \pi < \pi^*$ so that $0 < \lambda^* < \bar{\lambda}$ holds. Then,*

1. *If $\lambda \leq \lambda^*$, then,*
$$B \geq 0, \quad q^N \geq q^B. \tag{41}$$

2. *If $\lambda > \lambda^*$, then,*
$$B < 0, \quad q^N < q^B. \tag{42}$$

Proof. Equation (22) implies that q in the bubbleless steady state is monotonically decreasing in $\lambda \in [0, \bar{\lambda}]$, and equation (29) implies that q is constant and equals to unity for $\lambda \in [\bar{\lambda}, 1]$. Equation (31) implies that q in the bubbly steady state is constant at $1 + (1-\beta)/\pi$ regardless of the value of $\lambda \in [0, 1]$. Therefore, we establish that, under the condition $\underline{\pi} < \pi < \bar{\pi}$, $q^N > q^B$, and vice versa.

Next we turn to the sign of the bubble in the bubbly steady state. $B < 0$ if and only if inequality (40) holds. By substituting equation (31) and (32) into inequality (40), we establish that equation (40) holds if and only if

$$\lambda > \lambda^*. \tag{43}$$

■

Note that condition (38) is equivalent to the condition for the positive bubble to exist. Otherwise, λ^* (defined by equation (35)) becomes negative and, as a result, the bubble is negative for all $\lambda \in [0, 1]$.

[Figure 2 here]

Bubbles are positive for parameter values where the price of capital is higher in the bubbleless steady state than in the bubbly one and negative when the opposite is true (see Figure 2). Intuitively, when the price of capital is very high in the bubbleless steady state ($\lambda \leq \lambda^*$), this shows that collateral is extremely scarce. This provides a very big 'dividend' to the bubble and, for equilibrium to be restored in the collateral market, the bubble grows and adds to the aggregate stock of collateral, driving the price of capital down to $q^B = 1 + \frac{1-\beta}{\pi}$.

The opposite happens when the price of capital is relatively low in the bubbleless steady state ($\lambda > \lambda^*$). This is an economy in which collateral is relatively abundant in the bubbleless equilibrium and the price of capital is not sufficient to generate a high enough dividend so that the bubbly part of the firm earns a rate of return of β^{-1}. A negative bubble therefore appears which decreases aggregate collateral supply thus lifting the price of capital to q^B.

3.2.2 Expansionary/contractionary effects of the bubble

We turn next to the question of whether the bubble is expansionary or contractionary. Despite having a model with credit frictions, the answer to this question turns out to be subtly different from the question of whether the bubble is positive or negative. While positive bubbles turn out to be always expansionary in our framework, negative ones may be contractionary or, more surprisingly, expansionary, depending on parameter values.

In what follows, we focus on effects of the bubble on the capital rental rate r which is equal to the marginal product of capital. Since the production function is concave and firms face identical productivity, production expands whenever the emergence of the bubble decreases the rental rate of capital.

Capital rental rate in the bubbleless equilibrium. The capital rental rate in the bubbleless equilibrium is given by equation (23) for $0 \leq \lambda \leq \bar{\lambda}$ (constrained equilibrium, i.e., the borrowing constraint (4) is binding). It is monotonically decreasing in λ. For $\bar{\lambda} < \lambda \leq 1$ (unconstrained equilibrium), the rental rate is given by equation (30), which is independent of

λ.

Capital rental rate in the bubbly equilibrium. The interest rate in the bubbleless equilibrium is given by equation (32) for $\lambda \in [0, 1]$. Note that the economy is constrained in the bubbly equilibrium for all $\lambda \in [0, 1]$. Equation (32) implies that the capital rental rate is also monotonically decreasing in λ.

The following proposition is useful for the comparison of the interest rate in the bubbleless steady state and the bubbly steady state.

Proposition 2 *Suppose that* $\underline{\pi} < \pi < \pi^*$. *Then,*

$$r^N < r^B \quad \textit{for } \lambda = 0, \tag{44}$$

and

$$r^N > r^B \quad \textit{for } \lambda = 1. \tag{45}$$

Proof. Firstly, we derive the condition for $r^N < r^B$ when $\lambda = 0$. By substituting $\lambda = 0$ into equation (23) and (32), we obtain

$$r^N_{|\lambda=0} = \frac{\delta}{\pi}, \tag{46}$$

$$r^B_{|\lambda=0} = (1 + \pi - \beta)\frac{1 - (1 - \delta)\beta}{\pi\beta(2 - \beta)}, \tag{47}$$

where $r^N_{|\lambda=0}$ and $r^B_{|\lambda=0}$ respectively denote the rental rate in the bubbleless and bubbly steady state when $\lambda = 0$. From equation (46) and (47), the necessary and sufficient condition for $r^N_{|\lambda=0} < r^B_{|\lambda=0}$ reduces to

$$\pi < \frac{\delta\beta - (1 - \beta)^2}{1 - \beta(1 - \delta)} = \pi^*. \tag{48}$$

Secondly, we derive the condition for $r^N > r^B$ when $\lambda = 1$. When $\lambda = 1$ the bubbleless steady state is unconstrained and the rental rate is given by equation (30). We can use equation equation (32) to compute the rental rate in the bubbly steady state when $\lambda = 1$. Therefore we obtain

$$r^N_{|\lambda=1} = \beta^{-1} - (1 - \delta), \tag{49}$$

$$r^B_{|\lambda=1} = \frac{1 + \pi - \beta}{2 - \beta}\frac{\delta}{\pi}, \tag{50}$$

where $r^N_{|\lambda=1}$ and $r^B_{|\lambda=1}$ respectively denote the rental rate in the bubbleless and bubbly steady state when $\lambda = 1$. From equation (49) and ((50)), we have $r^N_{|\lambda=1} > r^B_{|\lambda=1}$ if and only if

$$\pi > \frac{\beta\delta}{2 - \beta(1-\delta)} = \underline{\pi}. \tag{51}$$

■

Note that $r^N = r^B$ for $\lambda = \lambda^*$. In addition to this, define λ^{**} such that

$$r^B_{\lambda=\lambda^{**}} = \beta^{-1} + (1-\delta). \tag{52}$$

Then, the comparison of the capital rental rate can be summarized by the following propositon.

Proposition 3 *Assume that $\underline{\pi} < \pi < \pi^*$. Then,*

1. *If $0 < \lambda < \lambda^*$, then $B > 0$ and $r^B < r^N$. Therefore the bubble is expansionary.*
2. *If $\lambda^* < \lambda < \lambda^{**}$, then $B < 0$ and $r^B > r^N$. Therefore the bubble is contractionary.*
3. *If $\lambda^{**} < \lambda < 1$, then $B < 0$ and $r^B < r^N$. Therefore the bubble is expansionary.*

[Figure 3 here]

Figure 3 summarizes graphically the result of Proposition 3. When the credit constraint is very tight ($\lambda < \lambda^*$), the bubble is positive and expansionary (as in Miao-Wang model). The effect of the negative bubble on the economy depends on the severity of credit constraint. When $\lambda^* \leq \lambda \leq \lambda^{**}$, the negative bubble is contractionary, and it is expansionary when $\lambda^{**} \leq \lambda \leq 1$.

This is a somewhat surprising result. Despite the bubble being negative and reducing the quantity of collateral overall, when $\lambda^{**} \leq \lambda \leq 1$, it becomes expansionary causing capital and output to rise above the first best. We know that when $\lambda > \lambda^*$ the bubble is negative and it reduces overall collateral supply leading to a higher price of capital. But what effect does a higher price have on the rental rate of capital and hence on the level of capital and output? Above λ^{**} the negative bubble and increased capital price actually crowd capital in rather than out, while, below λ^{**}, the opposite is true.

To understand the intuition for this, we can rewrite equation (18) so that it expresses the rental rate of capital as a function of the price of capital:

$$r_t = q_t \left(\frac{\beta^{-1}\left(1 - \pi\lambda\left(q_t - 1\right)\right) - (1-\delta)}{1 + \pi\lambda\left(q_t - 1\right)} \right). \tag{53}$$

We can see that there are two ways in which the rental rate r depends on the price q. First, there is a linear term (holding $\frac{\beta^{-1}(1-\pi\lambda(q_t-1))-(1-\delta)}{1+\pi\lambda(q_t-1)}$ fixed). This implies that higher q should lead to a higher rental rate in order to maintain the dividend yield constant. This effect is contractionary — a higher capital price requires a lower capital quantity in order to boost the rental rate.

However, there is an important second collateral effect which works in the opposite direction. The term in the brackets in equation (53) above captures the effects of the collateral premium attached to capital and this is decreasing in the capital price. Capital generates income that can be used in an investment opportunity and its market value can also be pledged to obtain loans. This makes it more valuable. The size of the collateral premium is controlled by $\lambda(q_t - 1)$: the bigger the profits the firm can make in an investment opportunity and the more pledgeable capital is, the more valuable it is to hold for a given conventional dividend yield r_t/q_t. In equilibrium, this implies that the firm is willing to hold capital at a low dividend yield and as the profits from capital production $(q_t - 1)$ increase, the equilibrium dividend yield declines to a greater extent the more pledgeable capital is (i.e. under a higher λ).

Below λ^{**}, the first (conventional) effect dominates and the negative bubble leads to a higher rental rate and a lower capital stock. Above λ^{**}, the second (collateral premium) effect dominates and the negative bubble leads to a lower rental rate and a higher capital stock.

4 Transition to the bubbly equilibrium

Having analyzed the steady state of the model, we now show how the model economy transits from a bubbleless to a bubbly equilibrium. We cannot do this using analytical solutions so we instead examine perfect foresight solutions following one-time switches from the bubbleless to the bubbly equilibrium.

We use a numerical solution with an illustrative parameterization with parameter values popular in the wider literature. The aim is not to generate a quantitative realistic model simulation but more to illustrate the workings of the model better by also appealing to some numerical solutions. We set the discount factor β to 0.99 on a quarterly basis. The quarterly depreciation rate δ is 0.03 implying a 12% annual depreciation rate. The quarterly probability of a firm having an investment opportunity is 0.1. The share of capital in production α is 0.3.

We conduct the simulations under a number of values for the pledgeability of capital λ which have been motivated by the analysis in the previous section. We examine transitions in the three main bubble regions we discussed in the steady state analysis of our model - positive expansionary bubbles ($0 < \lambda < \lambda^*$), negative contractionary bubbles ($\lambda^* \leq \lambda \leq \lambda^{**}$) and negative expansionary bubbles ($\lambda^{**} < \lambda < 1$).

4.1 Transition to a positive expansionary bubble ($\lambda = 0.1$)

We start from a value of $\lambda = 0.1$ which, in the bubbleless steady state, leads to capital and output levels significantly below the first best steady state (shown in the dashed line in Figure 4). The price of capital is also significantly higher than that in the bubbly equilibrium which we already showed to be the precondition for the existence of a positive bubble. The solid line shows the transition dynamics starting from the bubbleless steady state (the first point of the solid line) to the bubbly steady state.

[Figure 4 here]

Once sentiment becomes optimistic, the bubble jumps and gradually settles on its long-term positive value. This increases the debt limit of firms who increase their borrowing to take advantage of profitable investment in new capital. The greater aggregate supply of collateral (due to the bubble) decreases the price of capital sharply on impact — this causes a switch in the composition of collateral from tangible (the value of capital) to intangible sources (the value of the bubble). More collateral under a binding credit constraint also increases investment, the capital stock and output. Production gets closer to the first best but stays below it in the long run as credit constraints continue to bind, limiting investment.

Consumption declines on impact to make room for investment but increases in the long run. Welfare jumps as the economy moves closer to the first best. Thus the long run gains in consumption outweigh any temporary declines over the transition path to the bubbly equilibrium.

4.2 Transition to a negative contractionary bubble ($\lambda = 0.25$)

We now investigate the macroeconomic impact of negative bubbles. We start from a bubbleless steady state where credit constraints either bind but not by much (this case is shown in Figure 5) or are non-binding but close to binding.

[Figure 5 here]

The bubble jumps to a negative value which is larger for higher values of the pledgeability of capital (λ). The firm's debt limit declines and starts to bind more tightly in the bubbly equilibrium leading to a fall of the amount of debt funding the firm is able to obtain. The value of capital jumps as tangible collateral becomes more valuable now that intangible collateral (the bubble) has turned negative. However, due to the relatively low pledgeability of capital (low value of λ), the rise in the equilibrium value of tangible collateral requires also an increase in the rental rate. Hence, overall capital accumulation declines and output and investment fall.

On impact, the decline in investment leads to a short-lived increase in consumption but eventually it declines in line with the fall in capital and output. Welfare falls on impact and continues to decline as production falls further below the first best.

4.3 Transition to a negative expansionary bubble ($\lambda = 0.4$)

Our last example is of the case where the borrowing constraint is very loose in the bubbleless steady state — $\lambda = 0.4$. In this example, the borrowing needs of the firm are more than covered by the collateral value of capital and output and capital are at the first best levels.

Yet again, the spread between the price of capital and its replacement cost must increase in the bubbly equilibrium so the equilibrium bubble is negative. However, it now becomes expansionary even though it reduces firms' overall borrowing limits (shown in the second panel of Figure 6).

[Figure 6 here]

The reason for this surprising result is that the collateral value of capital becomes highly sensitive to spreads when pledgeability is high. When capital is highly pledgeable (i.e. when λ is high), the holders of capital require a very low rate of return in order to hold it. This is because it provides good access to credit and allows the holder to take advantage of highly profitable capital production opportunities. Thus, the rise in the profitability of capital in a bubbly steady state is only compatible with equilibrium once the capital rental rate has fallen below its value in the bubbleless (first best) steady state. This happens as firms accumulate additional capital goods despite negative bubbles. The collapse in intangible collateral crowds in tangible collateral when the latter is sufficiently pledgeable. This happens even to the point of over-investment which reduces welfare.

5 Policy Analysis

In the baseline version of the model, the intra-period loans between firms and households carry an interest rate of unity. We now consider a macroprudential tax or subsidy to debt. This makes the interest rate faced by firms equal to $1+\tau$ where τ is the tax/subsidy rate. When $\tau > 0$ there is a tax, otherwise debt is subsidised. The policy is financed with lump-sum taxes on households in the event of a macroprudential subsidy. When debt is taxed, households receive lump sum transfers.

Under the subsidy/tax, the expression for the value of installed capital becomes

$$\phi_t = r_t + q_t(1-\delta) + \pi\left(q_t - 1 - \tau_t\right)\left(r_t + \lambda\phi_t\right).$$

We now characterize the level of the subsidy/tax required to implement the first best in the bubbleless and bubbly steady state.

5.1 Bubbleless equilibrium

In the bubbleless steady state with a subsidy, we have the following steady state solution as a function of the subsidy/tax τ:

$$\phi^N(\tau) = \frac{\delta(1-\pi(1+\tau))}{\pi(1-\beta+\lambda)}, \tag{54}$$

$$q^N(\tau) = \beta\frac{\delta(1-\pi(1+\tau))}{\pi(1-\beta+\lambda)}, \tag{55}$$

$$r^N(\tau) = \frac{\delta(1-\beta+\lambda\pi(1+\tau))}{\pi(1-\beta+\lambda)}. \tag{56}$$

To reach the first best, the social planner must therefore implement a debt subsidy which is described in the following proposition:

Proposition 4 *Assume that $\underline{\pi} < \pi < \pi^*$. If $\lambda < \bar{\lambda}$, then the collateral constraint binds and output is below the first best.*

Then a debt subsidy equal to $\tau = \frac{\left(\beta^{-1}-1+\delta\right)\pi(1-\beta+\lambda)-(1-\beta)\delta}{\delta\lambda\pi} - 1$ implements the first best allocation.

Proof. Follows by setting equation (56) equal to the first best value of the capital rental rate $\beta^{-1} - 1 + \delta$ and solving for τ. ■

Proposition 4 shows that a suitably chosen subsidy can implement the first best in the bubbleless equilibrium with binding credit constraints. Since output is below the first best, a subsidy is always required to restore efficiency financed by a lump sum tax on households.

5.2 Bubbly equilibrium

Under the tax, the rental rate of capital in the bubbly equilibrium as a function of the subsidy/tax is given by the expression below.

$$r^{B}(\tau) = \left(1 + \frac{1-\beta}{\pi}\right)\left(\frac{1-\lambda(1-\beta-\pi\tau_t)-\beta(1-\delta)}{\beta\left(2-\beta-\pi\tau_t\right)}\right). \tag{57}$$

It is clear that the subsidy (setting $\tau < 0$) would reduce the capital rental rate and would be expansionary. Then the following Proposition follows

Proposition 5 *Assume that $\underline{\pi} < \pi < \pi^*$.*

Then, a debt subsidy/tax equal to $\tau_t = \frac{\left(\beta^{-1}-1+\delta\right)\beta(2-\beta)-(1-\beta(1-\delta))\left(1+\frac{1-\beta}{\pi}\right)}{\pi\left(\lambda\left(1+\frac{1-\beta}{\pi}\right)+\left(\beta^{-1}-1+\delta\right)\beta\right)}$ implements the first best allocation.

1. *If $0 < \lambda < \lambda^*$, then $B > 0$ and $r^B < r^N$. The bubble is expansionary and the optimal debt subsidy is lower in the bubbly equilibrium.*

2. *If $\lambda^* < \lambda < \lambda^{**}$, then $B < 0$ and $r^B > r^N$. The bubble is contractionary and the optimal debt subsidy is higher in the bubbly equilibrium.*

3. *If $\lambda^{**} < \lambda < 1$, then $B < 0$ and $r^B < r^N$. The bubble is expansionary and a debt tax is imposed in the bubbly equilibrium.*

Proof. Follows by setting equation (57) equal to the first best value of the capital rental rate $\beta^{-1} - 1 + \delta$ and solving for τ. ■

When output is below the first best in the bubbly equilibrium ($\lambda < \lambda^{**}$), a subsidy is needed. When output is above the first best ($\lambda > \lambda^{**}$), a tax is needed. The proposition also shows that when the bubble is positive and expansionary ($0 < \lambda < \lambda^*$), the optimal subsidy is lower in the bubbly equilibrium compared to the bubbleless one. Also, when there is a negative expansionary bubble ($\lambda^{**} < \lambda < 1$), a debt tax is imposed in the bubbly equilibrium in order to remove the over-investment. In the intermediate region ($\lambda^* < \lambda < \lambda^{**}$), there is a negative contractionary bubble and the optimal subsidy increases in the bubbly equilibrium in order to correct the increased underinvestment in this part of the parameter space. Therefore the optimal subsidy/tax policy has a countercyclical character similarly to real life regulatory tools such as the Countercyclical Capital Buffer in Basel III. Proposition 5 shows that a suitably chosen subsidy or tax can implement the first best in the bubbly equilibrium

We now provide a numerical example of the way the economy adjusts to the imposition of a debt tax. We consider the case of $\lambda = 0.4$ as well as the parameter values used in the previous section. In this case, we have a negative expansionary bubble which generates over-investment and requires a tax on debt in order to bring the economy back to the first best. Figure 7 shows the transition from a bubbly equilibrium without a tax/subsidy to a bubbly equilibrium where a tax implements the first best. The figure shows that the tax reduces firms' limits and hence their investment. Capital gradually declines to the first best (shown as the solid red line in the figure). The reduction in investment initially boosts consumption although in the long run, lower output leads to lower consumption. Welfare increases on impact because the tax reduces over-investment and the short term increase in consumption dominates the long term (and hence discounted) decline in consumption.

[Figure 7 here]

6 Conclusions

We build a model economy in which firms' credit limits depend on stock market valuations which themselves depend on access to credit. This gives rise to bubbly equilibria where stock market values depart from strict fundamentals due to a self-fulfilling optimism or pessimism about firms' credit access. We show that, when capital investment is irreversible at the firm level, these bubbles are not just positive as discussed in the wider literature but also negative. During a negative bubbly episode, equity investors become pessimistic about the firm's access to credit and hence its future profitability. This reduces the firm's share price thus confirming investors' initial pessimism. The irreversibility of capital at the firm level ensures that the value of the firm can fall below the price of new capital goods.

Positive bubbles in the model appear when credit constraints are tight (i.e. when much of the firm's capital is destroyed in the event of bankruptcy). They are also always expansionary as in the Miao and Wang (2015, 2018) framework. Negative bubbles appear when credit constraints are moderate or loose (i.e. when most the firm's capital survives in bankruptcy). They are contractionary under moderate credit constraints. The value of the firm falls, firms invest less and the economy contracts. More surprisingly, however, negative bubbles also turn out to be expansionary when credit constraints are very loose and capital is good collateral. During an expansionary negative bubble, output actually increases above the first best and the economy enters an over-investment regime. The reason for this finding is that the negative bubble leads to tighter borrowing limits and to the increase in the spread firms' earn from producing capital goods. When capital is good collateral, this boosts its collateral premium so much that it actually leads to the overproduction of capital goods as the economy tries to compensate for the way the negative bubble undervalues firms.

Finally, we also consider what policy interventions can restore efficiency in our economy model. A debt subsidy/tax can restore output and consumption to the first best in steady states where those are below/above the first best. When a bubble appears, the subsidy/tax needs to be adjusted in a counter-cyclical manner. For a positive expansionary bubble, the debt subsidy is optimally reduced relative to the bubbleless equilibrium. In the event of a negative contractionary bubble, the debt subsidy needs to be increased while for negative expansionary bubbles, a debt tax is needed to correct the over-investment.

References

Adam, K. and Marcet, A. (2011). Internal rationality, imperfect market knowledge and asset prices. *Journal of Economic Theory*, 146(3):1224–1252.

Azariadis, C., Kaas, L., and Wen, Y. (2016). Self-fulfilling credit cycles. *The Review of Economic Studies*, 83(4):1364–1405.

Cui, W. and Kaas, L. (2021). Default cycles. *Journal of Monetary Economics*, 117:377–394.

Farhi, E. and Tirole, J. (2012). Bubbly liquidity. *The Review of economic studies*, 79(2):678–706.

Gertler, M. and Karadi, P. (2011). A model of unconventional monetary policy. *Journal of monetary Economics*, 58(1):17–34.

Kiyotaki, N. and Moore, J. (2019). Liquidity, business cycles, and monetary policy. *Journal of Political Economy*, 127(6):2926–2966.

Kollmann, R. (2022). Speculative bubbles and aggregate boom bust cycles: Closed and open economies. *mimeo*.

Martin, A. and Ventura, J. (2012). Economic growth with bubbles. *American Economic Review*, 102(6):3033–58.

Martin, A. and Ventura, J. (2018). The macroeconomics of rational bubbles: A user's guide. *Annual Review of Economics*, 10(1):505–539.

Matsuyama, K. (2013). The good, the bad, and the ugly: An inquiry into the causes and nature of credit cycles. *Theoretical Economics*, 8(3):623–651.

Miao, J. and Wang, P. (2012). Bubbles and total factor productivity. *American Economic Review*, 102(3):82–87.

Miao, J. and Wang, P. (2014). Sectoral bubbles, misallocation, and endogenous growth. *Journal of Mathematical Economics*, 53:153–163.

Miao, J. and Wang, P. (2015). Banking bubbles and financial crises. *Journal of Economic Theory*, 157:763–792.

Miao, J. and Wang, P. (2018). Asset bubbles and credit constraints. *American Economic Review*, 108(9):2590–2628.

Shiller, R. J. (2000). Measuring bubble expectations and investor confidence. *The Journal of Psychology and Financial Markets*, 1(1):49–60.

Tirole, J. (1985). Asset bubbles and overlapping generations. *Econometrica: Journal of the Econometric Society*, pages 1499–1528.

A Extension: adding government bonds to the model

In this section we investigate the consequences of adding pure discount government bonds s_t to the model. We show that adding such bonds does not affect the existence and properties of negative bubbles.

The bonds pay a unit of consumption next period and cost p_t today. Then the value of the firm is

$$V_t = \phi_t k_t + \psi_t s_t + b_t,$$

where b_t is a potential bubble. We again assume that only λ fraction of capital can be recovered. In contrast, we assume that government bonds cannot be diverted and are fully recoverable by creditors. The collateral constraint is therefore:

$$d_t \leqq \lambda \phi_t k_t + \psi_t s_t + b_t.$$

Substituting into the value function, we get the following:

$$
\begin{aligned}
&\phi_t k_t + \psi_t s_t + b_t \\
&= \max_{s_{t+1},k_{t+1}} \Bigg\{ (r_t + q_t(1-\delta))\, k_t + s_t - q_t k_{t+1} - p_t s_{t+1} \\
&+ \pi\,(q_t - 1)\,(r_t k_t + \lambda \phi_t k_t + \psi_t s_t + b_t) + \beta\,(\phi_{t+1} k_{t+1} + \psi_{t+1} s_{t+1} + b_{t+1}) \Bigg\}.
\end{aligned}
$$

The envelope condition for capital gives us

$$\phi_t = (r_t + q_t(1-\delta)) + \pi\,(q_t - 1)\,(r_t + \lambda\phi_t)\,,$$

which can be written as

$$\phi_t = \frac{(1 + \pi\,(q_t - 1))\, r_t + q_t(1-\delta)}{1 - \lambda\pi\,(q_t - 1)}.$$

The envelope condition for government bonds is given by

$$\psi_t = 1 + \pi\,(q_t - 1)\,\psi_t,$$

which is written as

$$\psi_t = \frac{1}{1 - \pi\,(q_t - 1)}.$$

The first order condition for capital is given by:

$$q_t = \beta\phi_{t+1},$$

while the first order condition for government bonds is:

$$
\begin{aligned}
p_t &= \beta\psi_{t+1} \\
&= \frac{\beta}{1 - \pi\,(q_{t+1} - 1)}.
\end{aligned}
$$

Finally, we have the bubble valuation equation:

$$b_t = \pi\,(q_t - 1)\, b_t + \beta b_{t+1}.$$

A.1 Bubbleless equilibrium

We can see that, because $q > 1$, the price of bonds in the steady state is below the first best price of β. Because they are good collateral, the bonds earn a liquidity premium. And since the bonds earn less than the $1/\beta$ required rate of return for households, they are all held by the firms. Assuming M units of debt in the economy, the capital evolution equation is given below:

$$\delta = \pi \left(r + \frac{\lambda}{\beta} q + \frac{\beta}{1 - \pi (q - 1)} M \right). \tag{58}$$

Increasing M will be expansionary and drive down q. **[Proof to be completed.]**

A.2 Bubbly equilibrium

In the bubbly steady state equilibrium, the bubble valuation equation implies:

$$\pi (q - 1) = 1 - \beta.$$

This means that the price of debt is equal to unity.

$$p = \frac{\beta}{1 - \pi (q - 1)} = 1.$$

Then the capital evolution equation is as follows:

$$\delta = \pi \left(r + \frac{\lambda}{\beta} \left(1 + \frac{1 - \beta}{\pi} \right) + \frac{1}{\beta} M + B \right). \tag{59}$$

We know that this equation pins down the value of the bubble in the absence of government debt. With government debt, it pins down $\frac{1}{\beta} M + B$. This means that a higher quantity of government debt will reduce the value of the bubble (i.e. make it more negative). Bubbles still always exist but are more likely to be negative. However, the λ thresholds for bubbles to be expansionary or contractionary remain the same regardless of the quantity of government debt. **[Proof to be completed.]**

Figure 1: Parameter space

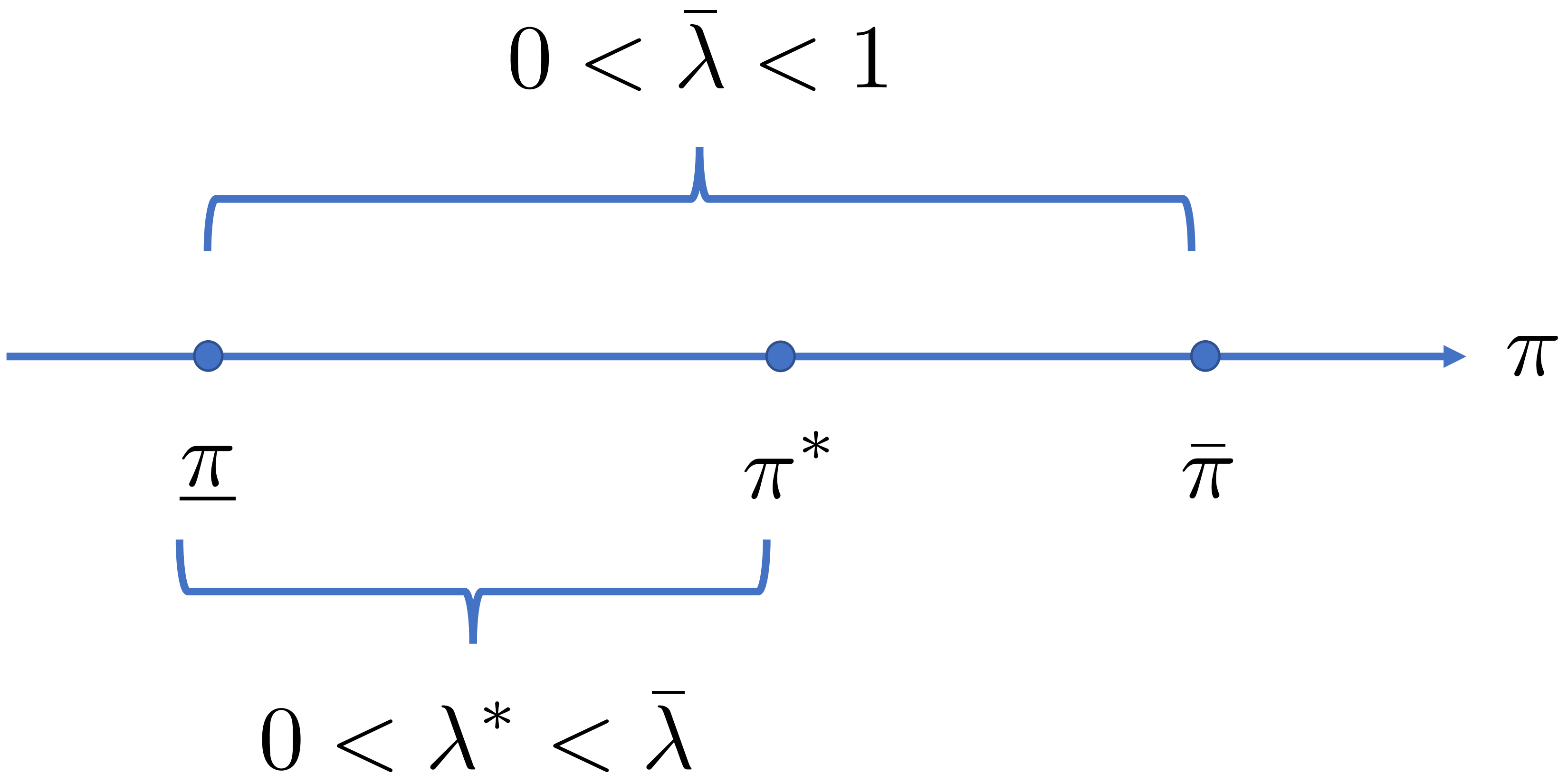

Figure 2: Equilibrium and the degree of credit friction

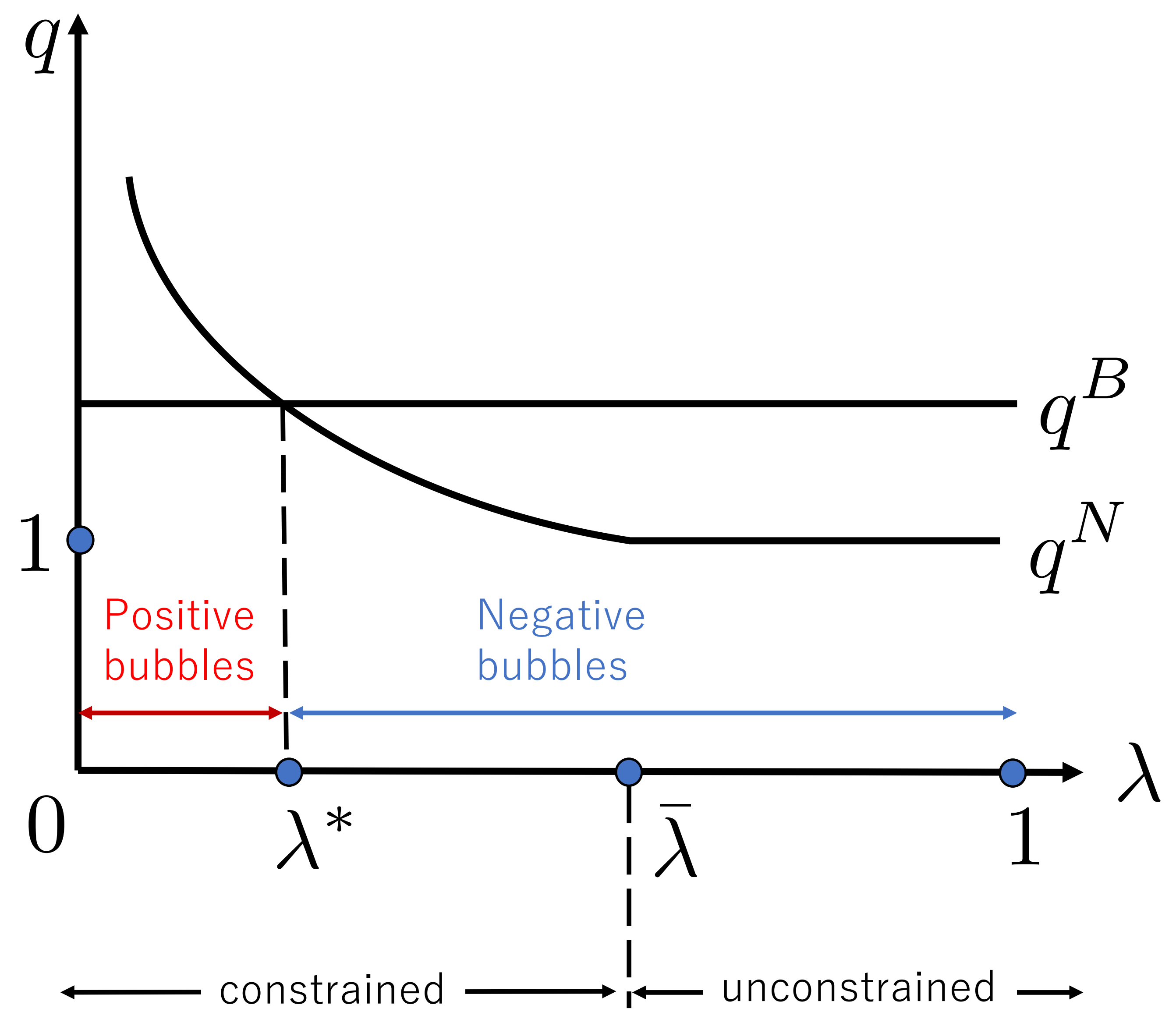

Figure 3: Expansionary/contractionary effects of the bubble

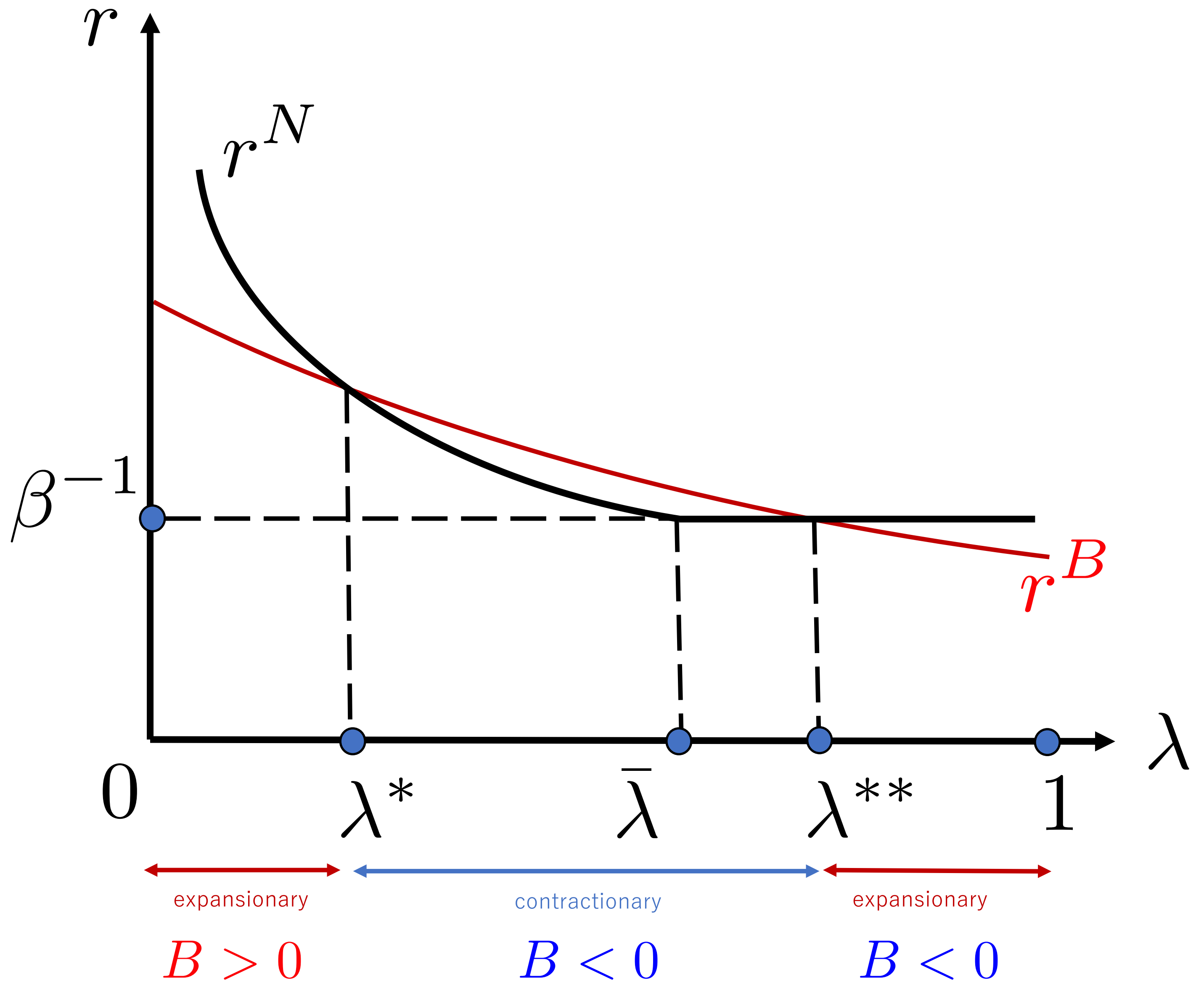

Figure 4: Transition from the bubbleless to the bubbly steady state under tight borrowing constraints ($\lambda = 0.1$)

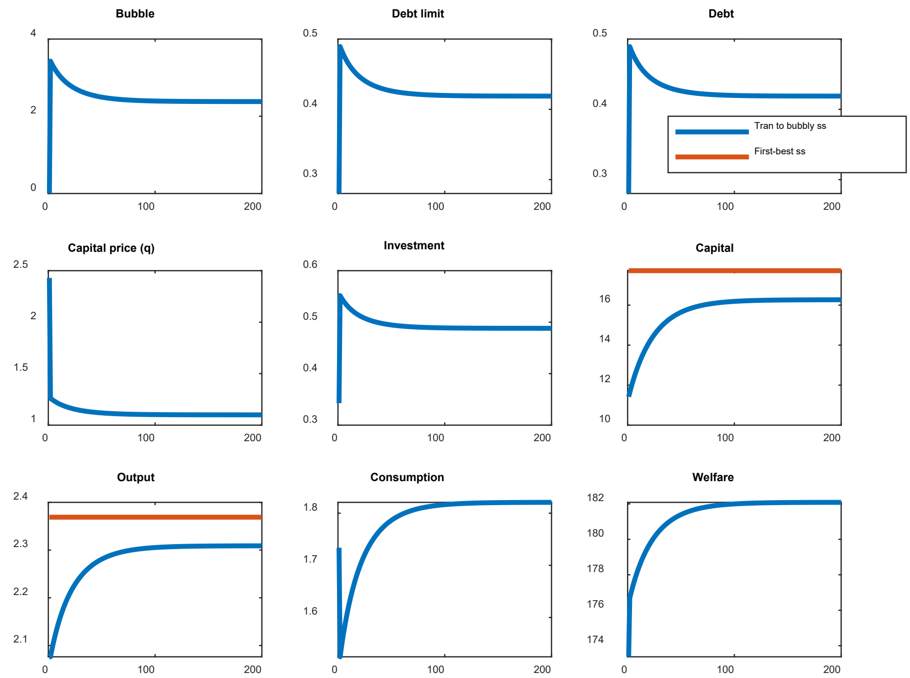

Figure 5: Transition from the bubbleless to the bubbly steady state under moderate borrowing constraints ($\lambda = 0.25$)

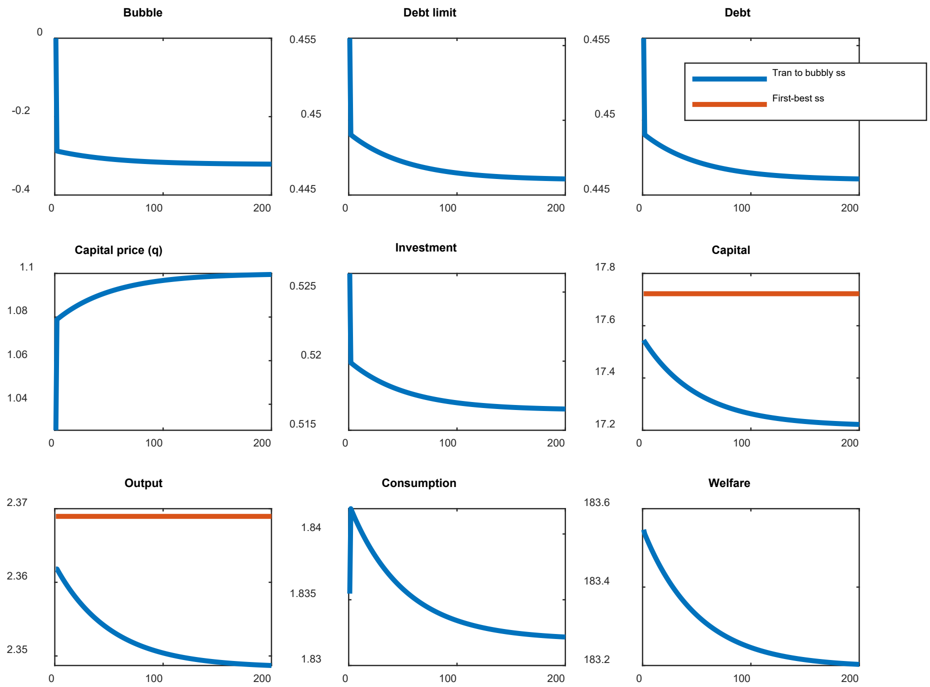

Figure 6: Transition from the bubbleless to the bubbly steady state under loose borrowing constraints (λ = 0.4)

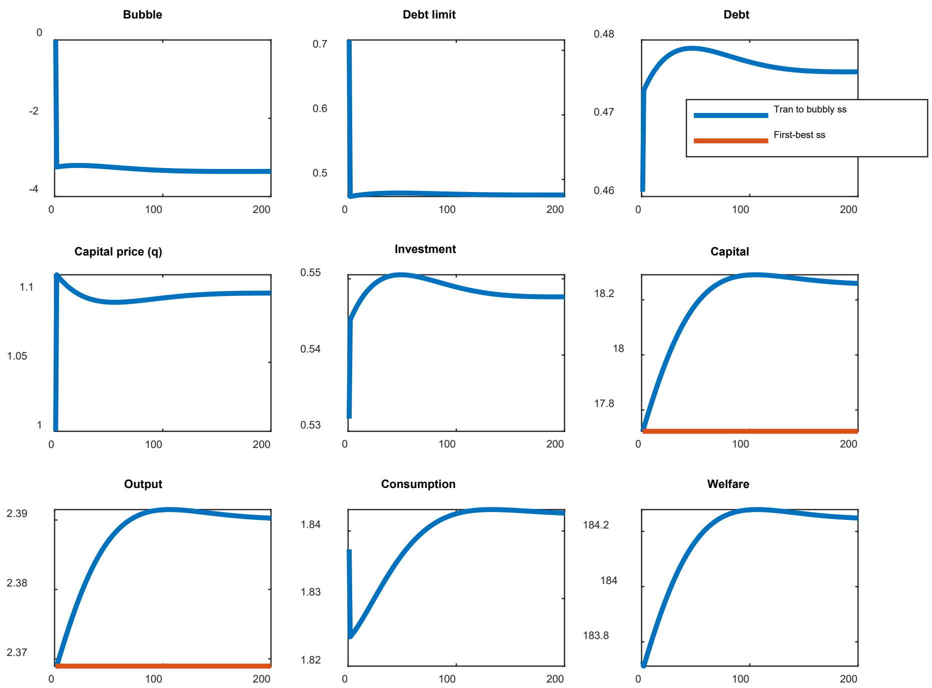

Figure 7: Transition following the imposition of a 2.25% debt tax in the bubbly equilibrium when λ = 0.4

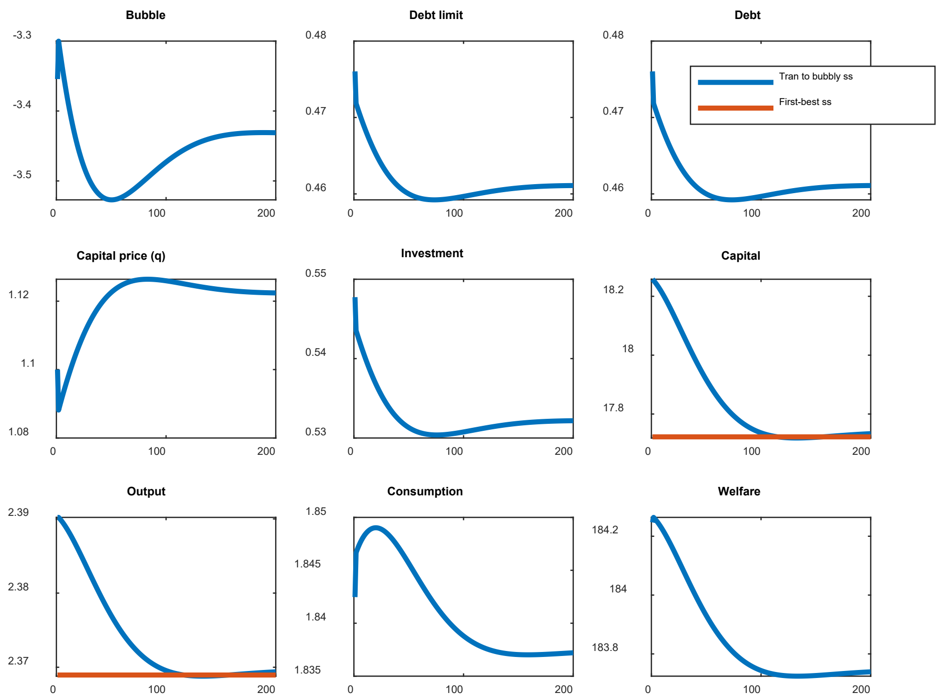

Figure A1: Transition from the bubbly to the bubbleless steady state for λ = 0.1 [Chart which shows that the irreversibility constraint does not bind following the collapse of the bubble]