Optimal contracts with adverse selection and moral hazard:
Are incentives high- or low-powered?

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February 20, 2023

Abstract

A risk-neutral principal contracts with a risk-neutral agent protected by limited liability to undertake production. The value of the production can be enhanced in parallel by a fixed additional benefit the occurrence of which depends stochastically on the agent's effort. The agent receives a bonus in the event of its occurrence. The agent has private information about his marginal cost and his effort, which do not interact with each other. However, we find that the optimal contract is not dichotomous. Compared with adverse selection alone, it exhibits higher-powered incentives for production and lower-powered incentives for bonuses. Compared with moral hazard alone, the optimal contract exhibits lower-powered incentives for production and higher or lower-powered incentives for bonuses. Moreover, the contract is almost separating: when pooling is optimal, this occurs only for high marginal costs. Extending the analysis to a variable additional benefit shows that countervailing incentives arise, creating higher-powered

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incentives for both production and bonuses for some types.
Keywords: Adverse selection, moral hazard, limited liability, power of incentives.

1 Introduction

Most adverse-selection models are structured so that the contract between the principal and the agent must determine the volume of trade while taking account of the asymmetry of information on the agent's exact cost structure. However, the principal may wish to enrich production contracts using tools that incentivize the agent to undertake actions that influence the gains from trade.[1] Such economic contexts abound. A regulatory agency contracting for production with a public utility cannot disentangle the level of production, depending in particular on its marginal cost, from the quality of the service consumers benefit from. A landlord can no longer ignore the gains it can obtain due to actions by its tenant to preserve the environment (reducing the use of pesticides, maintaining natural habitats, etc.) even if the tenant's ability remains of prime importance in determining the amount of production.[2] These examples show that production building on adverse-selection issues is followed by beneficial actions relying on moral-hazard questions. Thus, a generalized agency (or a mixed) problem occurs between the principal and the agent.

Theoretical background and results. The main idea of this paper is to determine the optimal generalized agency contract by mixing two textbook models in incentive theory: one with adverse selection, the other with moral hazard.[3] We consider a risk-neutral principal who contracts with a risk-neutral agent to undertake a certain level of production. However, the value of the production induced can be enhanced in parallel by a fixed additional benefit the occurrence of which depends stochastically on the agent's effort. If the additional benefit arises, the agent receives a contingent payment, i.e., a bonus. The agent incurs both a production cost depending on his private knowledge type and a disutility due to his private knowledge effort. We consider that these two elements are separable, that is, they do not interact with each other. Moreover, the agent is protected by limited liability: he cannot end the relationship with a negative payoff in the absence of the additional benefit.

We first study contracts with only one asymmetry of information that will constitute benchmarks. Because of the separability of the production cost and the disutility of effort,

[1] Alternatively, one can think of actions that avoid damage. From an incentive point of view, the modeling of one or the other is equivalent.

[2] There are many other specific economic contexts in which the general framework we have just described applies. For instance, contractual agreements for quality improvements, for reclaiming a polluted site (see Lappi (2020)), for public-private partnerships (see Iossa and Martimort (2015)). Optional scope contracts in software development outsourcing, or performance-based contracting in the utilities sector can also be related to our general analysis.

[3] Laffont and Martimort (2002), chapters 2 and 5.

these contracts are dichotomous regarding the distortions with respect to the efficient levels of production and bonuses:

- The adverse-selection contract implies distortions on production only, not on bonuses,
- The moral-hazard contract involves the reverse.

These distortions serve the principal's goal that consists in extracting the agency costs. When there is only adverse selection (resp. moral hazard), lower quantities (resp. bonuses) mean a reduction in the information (resp. limited-liability) rent. This dichotomy outcome derives directly from the separability assumption.

In a generalized agency context where there are both adverse selection and moral hazard, our main result is that the dichotomy vanishes. Indeed, we show that if an agent gets a limited-liability rent only, a novel trade-off arises between efficiency and limited-liability rent extraction because it is constrained by the truthful revelation of the type. The choice of the quantity produced cannot be separated from the bonus proposed. They are no longer independent of each other.

Let us describe such a trade-off. The interaction between production and bonuses influences the marginal cost of the rent. Indeed, consider that a rent of $1 is left to an agent who only obtains a limited-liability rent. On the one hand, all agents who are more efficient will also benefit from this dollar. This is the usual agency cost for the principal in the adverse-selection case. On the other hand, this relaxes the limited-liability constraint for some more efficient agents also obtaining the limited-liability rent. This involves a marginal benefit for the principal. It is measured by the cumulative of the shadow prices of the limited-liability constraint only for types concerned by this standard agency cost in the moral-hazard case. All in all, the marginal cost of the rent is reduced with respect to adverse selection alone.

It follows that the optimal quantity exhibits *higher-powered incentives* with respect to adverse selection alone. However, the optimal quantity always exhibits *lower-powered incentives* with respect to moral hazard alone since this latter case achieves the first-best production. Indeed, all in all, the marginal cost of the rent remains positive, except for the costliest type for which it is zero. Indeed, leaving $1 of rent to this type is not costly. If it were, limited-liability rent would be set at zero. But this would imply no bonus, and so no effort by the agent. This would imply a first-order loss for the principal whereas the rent left involves a second-order loss only. Thus, the least efficient type also produces the first-best quantity.

As a result, the moral-hazard bonus level is the central bonus level around which the optimal bonus is distributed. More precisely, the bonus has to be decreased to satisfy truthful revelation. So, while always being lower than its efficient level, it first exhibits higher-powered incentives, then lower-powered incentives with respect to the moral-hazard case. We can therefore see that, compared to adverse selection and moral-hazard benchmarks, quantities and bonuses are distorted in opposite directions when the agent is protected by his limited-liability. This shows the principal uses both instruments in the optimal contract as substitutes.

To pursue the analysis of this novel trade-off, it is important to recall that it applies only to agents that get a limited-liability rent only. Yet, it is possible that, under certain conditions (roughly speaking, an additional benefit that is not excessive) sufficiently efficient agents get an information rent on top of a limited-liability rent. In this case, for these types, the trade-off between efficiency and information rent extraction mentioned above applies and the dichotomy phenomenon reappears.

Finally, until now, the description of our novel trade-off implies separating contracts. We also show that pooling can occur. However, it can only arise on the right of the distribution of the types. This is due to the fact that the costliest type must produce the first-best quantity. This may conflict with the desire to distort downward the efficient quantity for more efficient types. In this case, the quantity could be increasing in type on the right of the distribution. This would contradict the implementability condition (i.e., a non-increasing quantity) required to ensure truthful revelation.

Thereafter, we extend the analysis in three directions. First, the agent has a positive asset. The main consequence is that it relaxes the limited-liability constraint and thus the participation constraint can have an effect on seeking the optimal contract. All in all, we show that there are eleven possible cases that lead to five contracts with salient properties. We find that the novel trade-off just mentioned occurs if the agent's asset is sufficiently low. If the asset is sufficiently high, the optimal contract is the adverse selection one. For intermediate values of the asset, we find intermediate contracts. In a second extension, we show that our results straightforwardly apply to an agency relationship where the agent may exert an effort to reduce the occurrence of a fixed cost overrun, added to the agent's production cost, instead of an additional benefit. The main feature of this setting is that the bonus becomes a malus. In a third extension, we consider that the additional benefit is quantity-dependent. It follows that the dichotomy property vanishes and the limited-liability becomes type-dependent. We find that the optimal contracts combine our main results and those with type-dependent participation constraints, for which mainly countervailing incentives arise.

Related literature. The development of the literature on generalized agency models or mixed models with true moral hazard is relatively recent. Our article makes a contribution to three strands of this literature.

First, a set of articles focuses on a pure moral-hazard model that is made more complicated by the presence of adverse selection. When the contract concerns a risk-neutral agent under liability constraints, one can cite theoretical works by Lewis and Sappington (2001), Ollier (2007), Ollier and Thomas (2013) with two states of nature only (i.e., success or failure), and by Gottlieb and Moreira (2022) for more than two states. These theoretical settings have been extended and applied to financial contracting issues by At and Thomas (2019), to sharecropping in At and Thomas (2017), and to hospital pricing contracts in Maréchal and Thomas (2021). The common result of these papers is that *pooling is optimal* because of liability constraints even under weak regularity assumptions. However, there are many sources of pooling. When distortions are linked to reductions in both informational and limited-liability rents, they create an incompatibility with the implementability condition. This is the case in Ollier and Thomas (2013) for instance. When informational and limited-liability rents have exactly the same rate of change, then the constraint that represents the implementability condition is binding, as in At and Thomas (2019). Finally, pooling may arise when the moral-hazard outcome alone exhibits non-responsiveness (Maréchal and Thomas (2021)).
In our present paper, the optimal policy is globally separating even when the agent obtains a limited-liability rent only. This is a novelty in the existing literature on mixed models. Moreover, when pooling arises, it is only on the right-hand side of the distribution. This source of pooling is new in this trend of literature and can be related to the one depicted in Rochet and Stole (2002) for a nonlinear pricing setting, where pooling can only occur at the bottom even if the type distribution is log-concave.

In this first strand of literature, some papers consider that the contracting party is a risk-averse agent. Faynzilberg and Kumar (2000) study the conditions under which truthful revelation and obedience are compatible with each other. Castro-Pires and Moreira (2021) include liability protection and find that pooling is optimal. In the absence of such protection, Maréchal and Thomas (2018) show that the agent's prudence, on top of its efficiency, can imply higher-powered incentives on effort despite risk aversion. In our paper, we also find the idea of higher-powered incentives for effort and for output but only when the limited-liability constraint of the agent is binding.

A second strand of literature on mixed agency models is based on the "false" moral-hazard approach provided by Laffont and Tirole (1986) for monopoly regulation, and in

which “true” moral-hazard issues are included. These are the cases of Laffont (1995) and Hiriart and Thomas (2017) in which a safety care effort to avoid environmental damage is analyzed. In these papers, the effort is binary and they focus on the case where the greatest effort is always optimal. The first paper highlights the potential conflict between safety care and cost minimization. The second shows that this conflict can be resolved but not uniformly. It depends on the nature of the rents foregone, which themselves are a function of the supermodularity (or submodularity) of the average disutility of efforts.
In a model *à la* Baron and Myerson (1982), we assume the separability of production costs and disutility of effort. This allows us to avoid some unnecessary complications due to entangling two efforts (cost reduction and safety care) in the same disutility function. This makes it possible to approach the problem with a continuous effort instead of a binary one and to characterize all possible contracts, which are five in number.

Third, the closest paper to ours is Hiriart and Martimort (2006). In this paper, based on an adverse-selection model *à la* Baron and Myerson (1982), a care effort is introduced to avoid environmental damage. Our model generalizes this article in two ways: firstly by considering a continuum of types instead of a two-types case, and secondly by situations where the limited liability constraint is not binding regardless of the type. Thus, depending on which constraint is binding (between limited liability, incentive compatibility, and participation), we obtain eleven configurations that lead to one of the five optimal contracts we establish.

Finally, when considering a variable additional benefit, the limited-liability constraint is akin to a type-dependent participation constraint. We appeal to Jullien (2000) to conduct the analysis. The main difference is that the type-dependent limited-liability rent is endogenous. So the challenge is to identify the levels of this rent that matter along the interval of types.

Section 2 presents the model and some basic results, while section 3 delivers some benchmarks. In Section 4, we solve and comment on the main problem where type and effort are not observable by the principal. The last section considers the extensions.

Details of the proofs are covered in an Appendix.

2 The model

2.1 Background

We ground our analysis on textbook models with adverse selection and moral hazard (see Laffont and Martimort (2002), chapters 2 and 5).

Consider a risk-neutral principal who contracts with a risk-neutral agent to undertake the production of q units of a good. These units have gross value $S(q)$ for the principal, with $S' > 0$, $S'' < 0$, and $S(0) = 0$. However the agent's activity may increase this value by an additional benefit $B > 0$.

The cost of producing the q units is $C(\theta, q) = \theta q$ where the parameter $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$ is the agent's private knowledge efficiency (or his type). It is the realization of a random variable with a common knowledge and continuous density $f(\theta) > 0$. The cumulative is $F(\theta)$. We assume the monotone hazard rate property $(\dot{\frac{F(\theta)}{f(\theta)}}) > 0$.

By exerting a nonobservable effort $e \geq 0$, the agent may induce the occurrence of B. This effort increases the probability that B is achievable. To be parsimonious, we assume that $p(e) = e \in [0, 1]$. This assumption is sufficiently general to depict the idea that the effort has a stochastic effect on the surplus for the principal. Exerting an effort e entails the agent incurring the disutility $\psi(e)$. The disutility is assumed to be a strictly increasing and convex function of e, with $\psi(0) = 0$, $\psi''(e) > 0$ and $\lim_{e\to 0}\psi'(e) = 0$ and $\lim_{e\to 1}\psi'(e) = +\infty$ in order to ensure interior solutions. We also consider that $\psi'''(e) \geq 0$. We also denote $r(e) = e\psi''(e) > 0$, so $r'(e) > 0$.

We denote the agent's asset by l. However, to better encompass the economic interpretations in play, we will consider until section 5.1 that the agent has a null asset, that is $l = 0$.

The principal proposes a contract that defines a production q, a non-contingent payment t, and a contingent payment (a bonus) $w \geq 0$ if benefit occurs. Therefore for any realization of θ, the utility of the principal is

$$W = S(q) + eB - t - ew,$$

and the utility of the agent is

$$U = t + ew - \theta q - \psi(e).$$

The timing. The game has the following timing:

- the agent observes his efficiency,
- the principal proposes the contract,
- the agent accepts or refuses,
- the agent exerts an effort,
- the quantity and maybe the additional surplus are observed,
- the payments are made.

The contract. The offer is made whereas the agent knows his efficiency type at the time the contract is signed. Following Myerson (1982), there is no loss of generality in focusing on direct revelation mechanisms. The principal both designs the scheme $\langle q(\hat{\theta}), t(\hat{\theta}), w(\hat{\theta})\rangle$ specifying the production level and the payments for any agent's report $\hat{\theta}$, and recommends to the agent to exert an effort $e(\hat{\theta})$.

The payoffs. Faced with the scheme $\langle q(\hat{\theta}), t(\hat{\theta}), w(\hat{\theta})\rangle$, the utility of the agent θ, reporting $\hat{\theta}$ and exerting effort e is

$$u(\theta, \hat{\theta}, e) = t(\hat{\theta}) + ew(\hat{\theta}) - \theta q(\hat{\theta}) - \psi(e). \tag{1}$$

The constraints. The principal faces three kinds of constraints: incentives, participation, and limited liability.

Incentives. The principal must ensure truthful revelation and obedience. Let $U(\theta)$ be the rent given up to the agent. Using (1), the incentive constraints are, $\forall \hat{\theta}, \theta \in \Theta$, and $e, e(\theta) \in [0,1]$

$$U(\theta) = u(\theta, \theta, e(\theta)) \geq u(\theta, \hat{\theta}, e). \tag{2}$$

This constraint implies that the rent obtained by the agent θ while reporting the truth and following the recommended effort $e(\theta)$ must be greater than the profit if the report is $\hat{\theta} \neq \theta$ and the exerted effort is e.

Participation constraint. The constraints are $\forall \theta \in \Theta$

$$U(\theta) \geq 0, \tag{3}$$

where the agent's outside option is normalized to 0. To accept the contract, the agent cannot obtain less than his outside option.

Limited-liability constraint. The contract must call for limited-liability clauses. That is, the payoff in the event that the additional benefit is not achieved cannot be set beyond his asset, i.e., $\forall\theta \in \Theta$

$$t(\theta) - \theta q(\theta) \geq 0. \tag{4}$$

The objective function. The objective pursued by the principal is

$$EW = \int_{\underline{\theta}}^{\overline{\theta}} \{S(q(\theta)) + e(\theta)B - t(\theta) - e(\theta)w(\theta))\} f(\theta)d\theta. \tag{5}$$

The principal must design the contract in order to maximize EW subject to the incentives, the participation, and the limited-liability constraints.

2.2 Reformulation

We present the set of incentive-feasible allocations.

Lemma 1. *The set of incentive-feasible allocations $\mathcal{S}^*$ is given by $\forall\theta \in [\underline{\theta}, \overline{\theta}]$*

$$\begin{aligned}
& e(\theta) = \epsilon(w(\theta)) \text{ such that} \\
& \qquad w(\theta) - \psi'(\epsilon(w(\theta))) = 0, && (6) \\
& \dot{U}(\theta) = -q(\theta) < 0, && (7) \\
& \dot{q}(\theta) \leq 0, && (8) \\
& U(\overline{\theta}) \geq 0, && (9) \\
& U(\theta) \geq R(w(\theta)), \text{ with} && (10) \\
& \qquad R(w(\theta)) = \epsilon(w(\theta))w(\theta) - \psi(\epsilon(w(\theta))).
\end{aligned}$$

Proof. The proof is given in appendix 7.1. □

The economic reading of Lemma 1 is the following. Equation (6) is the incentive constraint for the effort. This corresponds to the best effort for the agent. It trades off the marginal benefit of enhancing the surplus, corresponding to the bonus, w, and the marginal disutility of exerting such an effort e, ψ'. Thus, the bonus represents the power of incentives. Notice that $\epsilon(w)$ is increasing and concave in w as

$$\epsilon'(w) = \frac{1}{\psi''(\epsilon(w))} > 0 \text{ and } \epsilon''(w) = \frac{-\psi'''(\epsilon(w))\epsilon'(w)}{\psi''(\epsilon(w))^2} \leq 0. \tag{11}$$

To ensure the agent's incentive-compatibility to reveal his efficiency, equations (7) and (8) must shape the contract. The first one indicates the negative rate that the rent U must follow to compensate for the incentive to lie, and the second is the implementability condition that constrains the quantity q to be non-increasing.

The participation constraint boils down to (9) which tells us that the rent of the least efficient agent must be higher than the outside option.[4]

The agent's rent is thus

$$U(\theta) = t(\theta) + \epsilon(w(\theta))w(\theta) - \theta q(\theta) - \psi(\epsilon(w(\theta))). \tag{12}$$

Combining (4) and (12), the limited-liability constraints can be rewritten as (10). This means that the rent of the agent U is bounded by the limited-liability rent, $R(w)$. This is the rent due to limited liability. Hence the informational part of the rent is the excess of the rent $U(\theta)$ over $R(w(\theta))$. Notice that $R(w)$ increases with the bonus since, by the envelope theorem, $R'(w) = \epsilon(w) > 0$.

With the help of (1) at $\hat{\theta} = \theta$, let us replace in (5) the expected payment $t + ew$ by the rent plus the cost and the disutility $\theta q + \psi(e) + U$. The objective function can be rewritten using (6)

$$EW = \int_{\underline{\theta}}^{\overline{\theta}} \left\{ S(q(\theta)) + \epsilon(w(\theta))B - \theta q(\theta) - \psi(\epsilon(w(\theta))) - U(\theta) \right\} f(\theta) d\theta. \tag{13}$$

The principal's problem is to maximize (13) w.r.t. $q(\theta)$, $U(\theta)$, and $w(\theta)$ among $\mathcal{S}^*$.

Notice that the integrand is the sum of the social surplus less the agent's rent. Thus, the rent is costly for the principal. It follows that she seeks to find an allocation trading-off between efficiency and rent extraction.

[4]Indeed, since $\dot{U}(\theta) < 0$, it is sufficient to satisfy (9) to ensure that all other types participate.

3 Benchmarks

To be compared with the future analysis, we present three benchmarks. In the two first, limited-liability issues are ignored.

3.1 Symmetric information

When there are no informational issues (θ and e are observable and verifiable), the efficient or first-best contract is given by

$$\begin{cases} U^{fb}(\theta) = 0, \\ S'(q^{fb}(\theta)) - \theta = 0, \\ B - \psi'(e^{fb}) = 0. \end{cases} \tag{14}$$

As is well known, the first-best contract sets the agent's utility at 0 because it is costly, and quantity and effort are efficient because their marginal benefit minus their marginal cost or marginal disutility is equal to 0.

Combining (6) and (14) allows us to define a relevant bonus level w^{fb} by

$$B - w^{fb} = 0,$$

the superscript fb meaning that this bonus implements the efficient effort. Note that such a contract runs up against the agent's limited liability as $R(\epsilon(w^{fb})) > U^{fb}(\theta) = 0$ so (10) is violated.

3.2 Asymmetric information

Adverse selection. When there are no moral-hazard issues, the principal only needs truthful revelation. It follows that the set of incentive-feasible allocations $\mathcal{S}^{as}$ is given by constraints (7)-(9). The principal's problem consists in maximizing (13) among $\mathcal{S}^{as}$ w.r.t.

$q(\theta), U(\theta)$, and $\epsilon(w) = e$ since effort is observable. The optimal contract is given by, $\forall \theta \in \Theta$

$$\begin{cases} U^{as}(\theta) = U^{as}(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} q^{as}(\tau) d\tau \text{ with } U^{as}(\overline{\theta}) = 0, \\ S'(q^{as}(\theta)) - \theta = \frac{F(\theta}{f(\theta)}, \\ e^{as} = e^{fb}. \end{cases} \tag{15}$$

To obtain truthful revelation, the principal must leave a rent. But, as mentioned above, the rent, termed informational rent here, U is costly for the principal. Thus the level of the rent left to the worst type $\overline{\theta}$ is null, as in the fb. Better types get a positive rent that positively depends on the output. Thus, to reduce it, the quantity q is distorted downward below its fb level, except for $\underline{\theta}$. Accordingly, F relates *the marginal cost of the information rent*. Indeed, if \$1 rent is left to a given type, the principal must give this dollar to all better types, which are in proportion F. This is the usual trade-off between efficiency and information rent extraction. One can check that under the monotone hazard rate property, the contract is separating. By contrast, since effort has no influence on the rent, the efficient effort is attained. So, if a bonus were paid to the agent, it would be such that $w^{as} = w^{fb} = B$.

Moral-hazard. When the type is observable but effort is not, the principal only needs obedience. The set of the incentive-feasible allocations $\mathcal{S}^{mh}$ is reduced to the constraints (6) and (10).[5] The principal has to maximize the function in the parentheses of the integrand in (13) w.r.t. $q(\theta)$, $U(\theta)$, and $w(\theta)$ among $\mathcal{S}^{mh}$.

This leads to the following contract, $\forall \theta \in \Theta$

$$\begin{cases} U^{mh}(\theta) = R(w^{mh}), \\ q^{mh}(\theta) = q^{fb}(\theta) \\ B - \psi'(\epsilon(w^{mh})) = \frac{\epsilon(w^{mh})}{\epsilon'(w^{mh})}. \end{cases} \tag{16}$$

The agent obtains a positive rent, named here limited liability. Since it is costly and depends positively on the bonus, it is optimal to distort the bonus downward below its efficient level w^{fb}. Here, $\epsilon(w)$ measures the marginal cost of the limited-liability rent $R(w)$, whereas $(B - \psi'(\epsilon(w)))\epsilon'(w)$ measures the marginal benefit. That is a standard trade-off between efficiency and limited-liability rent extraction.[6] However, the quantity is fixed at

[5]Notice that the participation constraints vanish because $R(w(\theta)) > 0$.

[6]For readers more familiar when e is used instead of w to solve the moral-hazard problem, notice that the

its first-best level because it does not influence the rent.

Moreover, when looking more closely at the limited-liability constraint, it contains both U and w. Thus, the shadow price of this constraint must be equal:

- to 1, because \$1 of rent left to the agent reduces the principal's utility by exactly 1,
- and also to $\frac{(B-\psi'(\epsilon(w))\epsilon'(w)}{\epsilon(w)}$, which is equal to $\frac{B-w}{r(\epsilon(w))}$ by (6) and (11).

For the sake of presentation, let us denote

$$\rho(w) = \frac{B-w}{r(\epsilon(w))}$$

Thus when only moral hazard matters, the *shadow price of the limited liability constraint* must be such that

$$\rho(w) = 1, \forall \theta \in \Theta. \tag{17}$$

Highlights. Before closing this section, it is important to highlight two general characteristics of contracts under asymmetric information.

First, we observe that, when there is only one source of asymmetric information, the *optimal contracts are dichotomous* because either the output q or the bonus w is distorted with respect to its first best level. So, distortions never occur simultaneously on the quantity/bonus couple.

Second, we will reuse the ratio $\rho(w) = \frac{B-w}{r(\epsilon(w))}$. So, appealing to (11), notice that $r(\epsilon(w))$ is an increasing function of w, then the derivative of $\rho'(w)$ is negative provided that $B - w \geq 0$.

Now we are equipped to analyze the properties of the solution to the principal's problem.

4 Generalized agency contracts

Now we consider that the principal is plagued by asymmetric information about the agent's type and effort. According to the preceding section, she potentially must leave an information rent to the agent. This occurs when $U(\theta) > R(w(\theta))$. Otherwise, the agent receives a limited-liability rent only.

limited-liability rent is $\mathcal{R}(e) = e\psi'(e) - \psi(e)$ so that $\mathcal{R}'(e) = r(e)$. The optimal effort is thus $B - \psi'(e^{mh}) = r(e^{mh})$ which is equivalent to (16).

To begin, we focus on separating equilibria.

4.1 Separating contracts

We write contracts in the plural because one originality of this economic environment is that there are two optimal contracts. The first is presented in the following proposition.

Proposition 1. *Let $\theta^* \in (\underline{\theta}, \overline{\theta})$ and θ^{mh} be the type such that its bonus is equal to w^{mh}. The contract entails*

$$\bullet\ U_1^*(\theta) = \begin{cases} R(w^{fb}) + \int_{\theta}^{\theta^*} q_1^*(\tau)d\tau & \text{if } \theta \leq \theta^*, \\ R(w_1^*(\theta)) = R(w_1^*(\overline{\theta})) + \int_{\theta}^{\overline{\theta}} q_1^*(\tau)d\tau, \ \text{with } R(w_1^*(\overline{\theta})) > 0 & \text{otherwise}; \end{cases} \quad (18)$$

$$\bullet\ S'(q_1^*(\theta)) - \theta = \begin{cases} \frac{F(\theta)}{f(\theta)} & \text{if } \theta \leq \theta^*, \\ \frac{F(\theta) - \int_{\theta^*}^{\theta} \rho\left(w_1^*(\tau)\right) f(\tau)d\tau}{f(\theta)} \geq 0 \ \text{with equality holding at } \overline{\theta} & \text{otherwise}; \end{cases} \quad (19)$$

$$\bullet\ w_1^*(\theta) = \begin{cases} w^{fb} & \text{if } \theta \leq \theta^*, \\ R^{-1}(U_1^*(\theta)) \ \text{with } w^{fb} > w_1^*(\theta) \gtreqless w^{mh} \ \text{if } \theta \lesseqgtr \theta^{mh} & \text{otherwise}; \end{cases} \quad (20)$$

if $U_1^*(\underline{\theta}) > R(w^{fb})$.

Proof. The proof is given in appendix 7.2. □

Let us give the economic interpretations of this proposition. First, the optimal contract is characterized by a threshold type θ^*. Below this cost level, the agent is just facing a second-best contract as if the relationship between the principal and the agent were only plagued by an adverse-selection issue. Equation (18) indicates that most efficient types, that are below this threshold, obtain the informational rent, i.e., a greater rent than the highest limited-liability rent the agent could obtain, $R(w^{fb})$. Less efficient agents get limited-liability rents only and in particular even the worst type obtains a positive rent.

Therefore, the principal proposes a quantity/bonus couple that combines efficiency and rent reduction and that is adapted to the exact nature of the rent left. For efficient agents (i.e., those $\theta \leq \theta^*$) that obtain an information rent, that is $U > R$, (19) and (20) show that the offered pair is that of adverse selection alone. Indeed, only the reduction of information rents matters for the principal. The best possible trade-off is therefore between efficiency and information rent extraction. The dichotomy outcome we had remains relevant.

By contrast, for high-cost agents (i.e., those with $\theta > \theta^*$) that obtain a limited-liability rent (i.e., $U = R$), the dichotomy property disappears. On this interval, we thus have a novel trade-off between efficiency and rent extraction from limited liability constrained by the truthful revelation of the type.

The choice of the quantity produced cannot be separated from the bonus proposed, and they are no longer independent of each other. We discuss this entanglement through the four following steps. First, by means of (19), we note that the marginal cost of the rent is now $F(\theta) - \int_{\theta^*}^{\theta} \rho\left(w_1^*(\tau)\right) f(\tau)d\tau$. It is weaker than F, that of adverse selection alone. Indeed, let us consider that a rent of \$1 is left to a given agent. On the one hand, all of the more efficient agents, that are in proportion F, will also benefit from this dollar. On the other hand, this relaxes the limited-liability constraint for some of the more efficient agents, the ones who only get the limited-liability rent. This marginal benefit is measured by the cumulative of the shadow prices of the limited-liability constraint that are above θ^*. For a given type, this shadow price is equal to $\rho\left(w\right) f$, so regarding more efficient types, the cumulative is $\int_{\theta^*}^{\theta} \rho\left(w_1^*(\tau)\right) f(\tau)d\tau$.

Second, it follows that the optimal quantity $q_1^*\left(\theta\right)$ is higher than or equal to $q^{as}\left(\theta\right)$, the quantity with adverse selection only: there are higher-powered incentives on the quantity outcome. However, it remains lower than the optimal quantity with moral hazard only as the first-best is achieved. This is because the marginal cost of the rent remains positive, except for the costliest type $(\bar{\theta})$ for which it is zero. Let us explain why going from \$0 of rent to \$1 for this type is not costly in the end. If it were, limited liability rent would be set at its lowest possible level, that is zero. But this would be equivalent to paying no bonus to this agent who accordingly would make no effort. This would imply a first-order loss for the principal whereas the rent left to the worst type involves a second-order loss only.

Third, regarding the bonus, as indicated by (20), it is distorted away from the first-best level to limit the limited-liability rent, but it is not set at w^{mh} that would prevail with moral hazard alone. In particular, it is no longer constant. Indeed, the bonus is necessarily decreasing as the production cost goes up. Satisfying the incentive compatibility condition (7), then $\dot{U} = \dot{R}$, and it follows $\dot{w} = -q/\epsilon < 0$. However, the bonus keeps properties that are close to the moral-hazard outcome. Indeed, w^{mh} is the central bonus level around which bonus distortions are distributed. So there are higher-powered incentives for $\theta \in (\theta^*, \theta^{mh})$ but lower-powered incentives otherwise. This is because the marginal cost of rent is zero for the $\bar{\theta}$-type agent, and then $\int_{\theta^*}^{\bar{\theta}} \rho\left(w_1^*(\theta)\right) f(\theta)d\tau = 1$. Compared to the moral-hazard setting, the shadow price for each type is no longer equal to 1 but its cumulative is.[7]

Fourth, looking at (19) shows that the principal uses both instruments in the contract as substitutes when the agent is protected by his limited liability, whereas they are independent otherwise. Indeed, one can easily see that any increase in a bonus granted to an agent $\theta \geq \theta^*$, is linked to a decrease in the desired production. According to the trade-off highlighted above, giving a larger bonus to an agent decreases the expected shadow price of the limited-liability constraint and therefore calls for a reduction of the production expected from this agent, as the marginal surplus goes up. Indeed, were the bonus set to its first-best level, the production would be heavily distorted downward toward the adverse selection outcome, at a prohibitive cost for the principal in terms of limited-liability rents left to all agents. On the contrary, were the bonus to be largely reduced, for instance to the moral-hazard level, the production would be efficient but at a cost for the principal in terms of informational rents (as agents would not report their true cost).

Figures 1 and 2 illustrate the shapes of these contracts.[8]

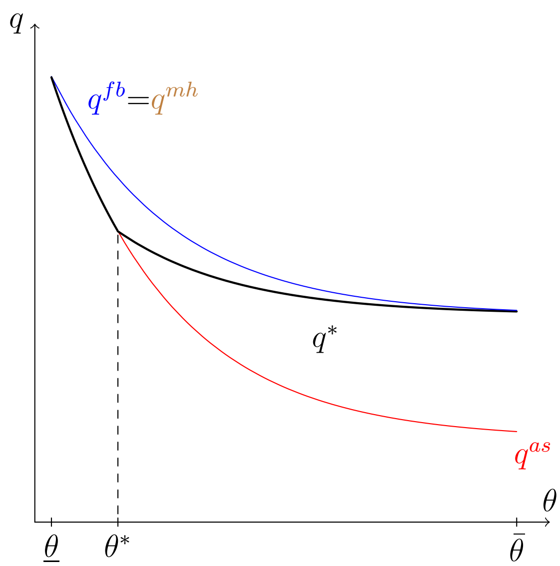

Figure 1: Quantities in Proposition 1

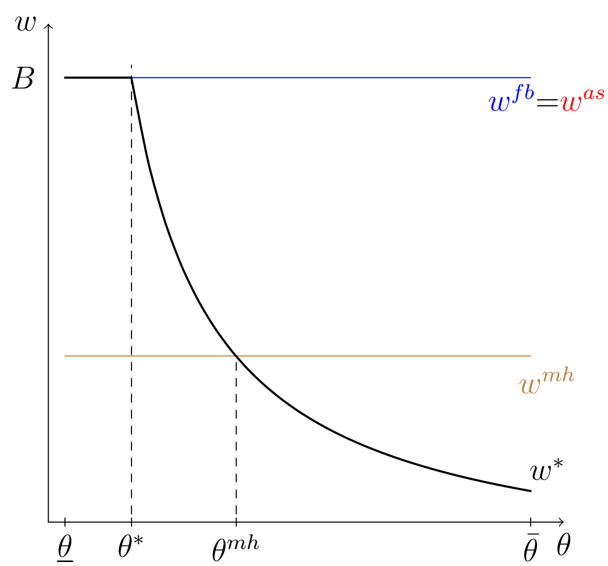

Figure 2: Bonuses in Proposition 1

As the condition at the end of Proposition 1 makes clear, this optimal contract is only

[7]Following Jullien (2000), we note that the function $P(\theta) = \int_{\underline{\theta}}^{\theta} \rho(w(\tau)) f(\tau) d\tau$ has the properties of a cumulative distribution function over the non-degenerate interval where the limited-liability constraint is binding. Thus $p(\theta) = \rho(w(\theta)) f(\theta)$ has the properties of a density probability function.

[8]Note that the results do not necessarily show convex profiles as we have adopted in the figures.

valid if $U_1^*(\underline{\theta}) > R(w^{fb})$. This ensures that all agents more efficient than θ^* get an information rent. If this is no longer the case, all agents get a limited-liability rent, and the optimal contract must be modified. We present this case in the proposition that follows.

Proposition 2. *The contract entails* $\theta^* = \underline{\theta}$ *and*

- $$U_2^*(\theta) = R(w_2^*(\theta)) = R(w_2^*(\overline{\theta})) + \int_\theta^{\overline{\theta}} q_2^*(\tau)d\tau, \text{ with } R(w_2^*(\overline{\theta})) > 0; \tag{21}$$
- $$S'(q_2^*(\theta)) - \theta = \frac{F(\theta) - \int_{\underline{\theta}}^{\theta} \rho\left(w_2^*(\tau)\right) f(\tau)d\tau}{f\left(\theta\right)} \geq 0 \text{ with equality holding at } \overline{\theta}; \tag{22}$$
- $w_2^*(\theta) = R^{-1}(U_2^*(\theta))$ *with*
$$w^{fb} \geq w_2^*(\theta) \gtreqless w^{mh} \text{ if } \theta \lesseqgtr \theta^{mh} \text{ and equality holding if } U_1^*(\underline{\theta}) = R(w^{fb}); \tag{23}$$

if $U_1^*(\underline{\theta}) \leq R(w^{fb})$.

Proof. The proof is given in appendix 7.2. □

This proposition is qualitatively the same as the former, except that, even the most efficient agents obtain a limited-liability rent while satisfying the truthful revelation constraint (i.e., $\dot{U} = -q$): this is equation (21) and the novel trade-off we identified above is again at work. All the comments above may be repeated about (22) and (23).
However, there is a major difference because of (23). Now this makes it possible that, for the most efficient agent, the bonus is set below the efficient level, i.e., $w_2^*(\underline{\theta}) < w^{fb}$. Indeed, if $U_1^*(\underline{\theta}) < R(w^{fb})$, then the rent level is too low to satisfy the limited-liability constraint. For this constraint to be verified, the principal needs to lower the level of limited-liability rent for this type, which requires lowering its bonus accordingly, below the first-best level w^{fb}. However, the bonus must be kept above w^{mh} in order to preserve the "distribution" of bonuses around this value as described above.

To complete the analysis, it remains for us to explain under which conditions this latter proposition is more likely to occur than the former. Recall that $w^{fb} = B$ and that $R'(w) > 0$, so that the limited-liability rent, R, increases with the additional benefit, B. Thus, *ceteris paribus*, the necessary condition to get Proposition 2, c.f. $U_1^*(\underline{\theta}) \leq R(w^{fb})$, has a greater chance of occurring when B is high.

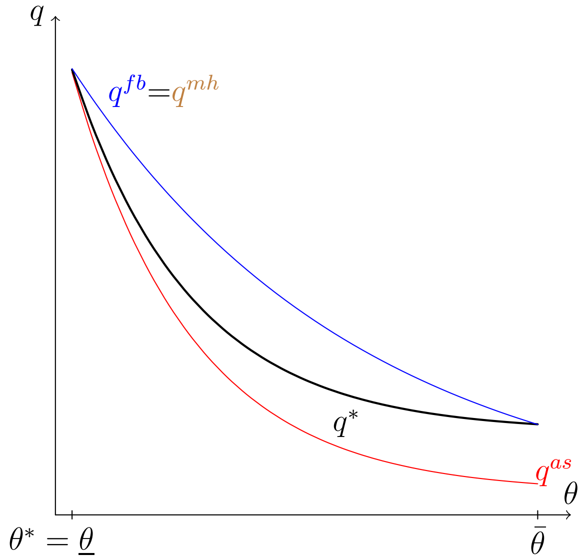

Figure 3: Quantities in Proposition 2

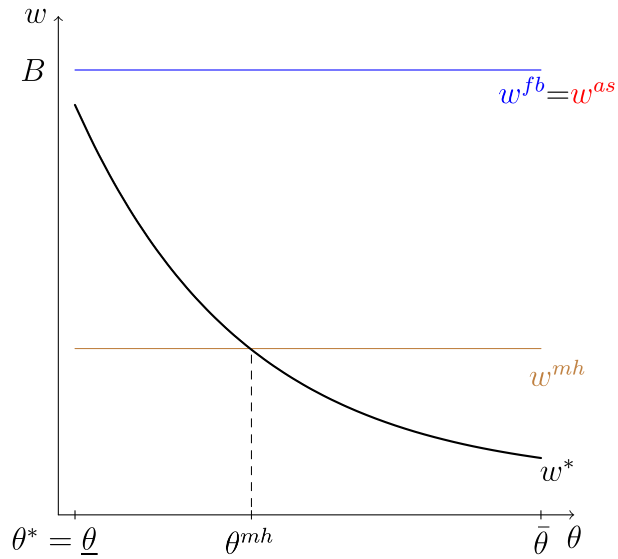

Figure 4: Bonuses in Proposition 2

4.2 Pooling

Propositions 1 and 2 assumed separating contracts i.e., $\dot{q_i}^*(\theta) < 0, i = 1, 2$. In this paragraph, we study the relevance of this assumption.

Generally speaking, as shown in Appendix 7.3, a separating contract emerges if the adjusted marginal cost[9] increases with the type. It follows that the question of pooling arises only for $\theta > \theta^*$, since below this threshold, the quantity is that of adverse selection. Thus, the separation of types is ensured because the monotone hazard rate property is a sufficient condition to imply that the rate of the adjusted marginal cost, $2 - \frac{\dot{f}(\theta)}{f(\theta)}\frac{F(\theta)}{f(\theta)}$, is positive.
The comparable condition for $\theta > \theta^*$ is

$$2 - \frac{\dot{f}(\theta)}{f(\theta)}\frac{F(\theta)}{f(\theta)} > \rho\left(w(\theta)\right) - \frac{\dot{f}(\theta)}{f(\theta)}\frac{\int_{\theta^*}^{\theta}\rho\left(w(\tau)\right)f(\tau)d\tau}{f(\theta)}. \tag{24}$$

Of course, this condition is subject in practice to the specifications given to the primitives of the model. However, we can state the following result.

Proposition 3. *If the quantity defined in Propositions 1 and 2 has to be increasing, this can*

[9]According to Baron and Myerson (1982), this is the sum of the technological marginal cost and the informational marginal cost

only occur for $\theta \geq \theta_b$, where θ_b is a critical value such that $\theta_b > \theta^{mh}$.

Proof. The proof is given in appendix 7.3. □

In other words, if pooling is needed, it only occurs on the right of the distribution. This result can be easily highlighted if we consider that f is uniform, i.e., $\dot{f} = 0$. In this case, condition (24) boils down to

$$2 > \rho\left(w(\overline{\theta})\right)$$

because this last ratio increases with θ.[10] By contrast, if (24) fails, one can check that $\theta_b > \theta^{mh}$ since, by definition, $\rho\left(w(\theta^{mh})\right) = 1 < 2$.

If pooling occurs, standard techniques lead to an optimal solution such that the quantity given in (19) and (22) is satisfied on average over the interval $[\theta_s, \overline{\theta}]$ where $\theta_s < \theta_b$. A consequence is that the worst type no longer produces an efficient quantity. This case is depicted in Figures 5 and 6.[11]

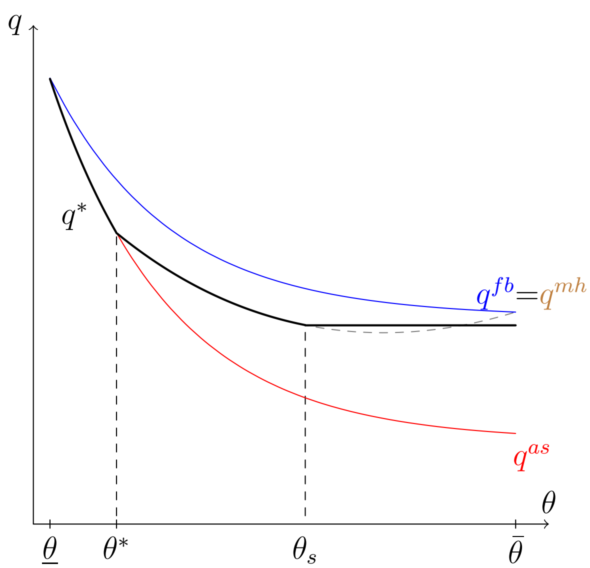

Figure 5: Quantities in Proposition 3

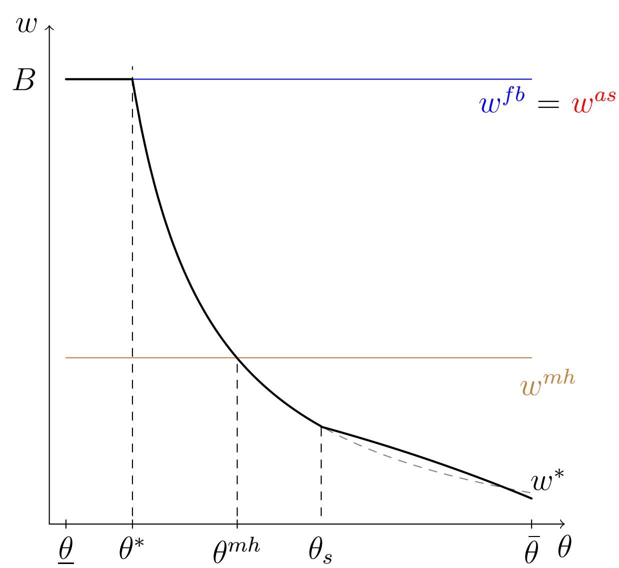

Figure 6: Bonuses in Proposition 3

Let us give some economic intuitions for pooling. On the one hand, it is straightforward that pooling is inevitable if $\theta^* \geq \theta_{as}$ where θ_{as} is such that $q^{as}(\theta_{as}) = q^*(\overline{\theta})$. On the other hand, the lower θ^*, the earlier the path $q^*(\theta)$ splits from $q^{as}(\theta)$, and thus the more easily

[10]Recall that the ratio decreases with w and w decreases with θ.

[11]Notice that beyond θ_{π^*} the bonus has necessarily the concave profile shown in the figure.

it reaches $q^{fb}(\overline{\theta})$ while satisfying $\dot{q}(\theta) \leq 0$. It follows that pooling has a greater chance of occurring as θ^* increases *ceteris paribus*. In the light of the last comment in the preceding paragraph, this will tend to be the case if the additional benefit B is low.

4.3 Explicit examples

In order to illustrate Propositions 1 to 3, we turn now to specific examples of our model. Let us assume that $S(q) = 1 + \frac{1}{4}\ln q$ and f is uniform on $\Theta = [0,1]$. Then it is straightforward to find the benchmark levels for quantities that is $q^{fb}(\theta) = \frac{1}{\theta}$ and $q^{as}(\theta) = \frac{1}{2\theta}$. Concerning efforts, the disutility is set to $\psi(e) = 4e^2$ so that $r(e) = 8e$ and $\epsilon(w) = \frac{w}{8}$. One can easily determine the benchmark levels for bonuses $w^{fb} = B$ and $w^{mh} = \frac{1}{2}B$.

Now taking different levels of the additional benefit B leads to each configuration in Propositions 1 to 3. For values of B such that $B \geq 7.04$, Proposition 2 applies and no pooling appears. For instance, if $B = 8$, we find[12] $w^{fb} = 8 > w_2^*(\underline{\theta}) = 7.39 > w^{mh} = 4 > w_2^*(\overline{\theta}) = 3.29$, with $\theta^{mh} \simeq 0.43$.

For lower values of the benefit, i.e., if $3.91 \leq B < 7.04$ then Proposition 1 applies and no pooling appears. For instance, if $B = 4$, we find $\theta^* = 0.0024$ and $w^{mh} = 2 > w_1^*(\overline{\theta}) = 1.36$, with $\theta^{mh} \simeq 0.52$.

Finally, for the lowest values of B such that $B < 3.91$, Proposition 3 applies as the quantity outcome defined in (19) turns out to be increasing with θ for some values of θ. For instance if $B = 2$, we get $\theta^* = 0.167$ and the quantity q_1^* is increasing for $\theta \geq \theta_b \simeq 0.84 > \theta^{mh} \simeq 0.63$. So pooling is necessary as shown in Table 1 since $q^*(\theta^{mh}) = 0.213 < q^*(\overline{\theta}) = 0.25$.

For each value of B given here, we provide quantity outcomes in Table 1.

B	8		4			2		
θ	θ^{mh}	$\overline{\theta}$	θ^*	θ^{mh}	$\overline{\theta}$	θ^*	θ^{mh}	$\overline{\theta}$
q^{fb}	0.57	0.25	103.5	0.48	0.25	1.49	0.39	0.25
q^*	0.49	0.25	51.8	0.37	0.25	0.74	0.213	0.25
q^{as}	0.28	0.125	51.8	0.24	0.125	0.74	0.19	0.125

Table 1: Simulated quantity outcomes

[12] The following values are derived from numerical simulations carried out using Maple 6 software. Discretization of the interval Θ (over 400 points) has been necessary to solve the integral equations in (19) and (22).

5 Extensions

We now provide three extensions of our main analysis. In a first, we complete our framework using the initially omitted assumption of a positive asset for the agent. In the second extension, we show that our main results readily apply to a different configuration in which the agent's effort may result in a fixed cost overrun instead of an additional benefit. The third studies the implications of a quantity-dependent additional benefit. To simplify, we restrict our attention to separating contracts.

5.1 Positive assets

Let us further generalize our analysis, looking for optimal contracts when the agent has a positive asset, i.e., $l > 0$.

For the agent, this implies that the limited-liability constraints are modified. The constraints (4) become $\forall\theta \in \Theta$, $t(\theta) - \theta q(\theta) \geq -l$, whereas their reformulation (10) is, $\forall\theta \in \Theta$

$$U(\theta) \geq R(w(\theta)) - l. \tag{25}$$

When the agent holds a positive asset, the optimal quality and bonus identified above remain qualitatively the same. They are such that:

$$\begin{cases} S'(q^*(\theta)) - \theta = \frac{F(\theta) - \left(\mathbb{1}|_{\theta \geq \theta^*}\right) \int_{\theta^*}^{\theta} \rho(w^*(\tau)) f(\tau) d\tau}{f(\theta)}, \\ w^*(\theta) \leq w^{fb} \text{ with equality holding if } \theta \leq \theta^*; \end{cases}$$

where $\mathbb{1}$ is an indicator function and θ^* is still the type on which, when necessary, the limited-liability constraint is binding. In this regard, it is obvious that the positive asset relaxes the limited-liability constraints $U(\theta) \geq R(w(\theta)) - l$. The main consequences play on the threshold level θ^* and on the rent left to the worst type $U(\overline{\theta})$.

These are stated in the following proposition. Let us define two asset levels $l^{fb} = R(w^{fb})$ and $l^{mh} = R(w^{mh})$.

Proposition 4. *There exist $l^{fb} > l^{mh} > \overline{l} > 0$ and $\underline{l} < l^{fb}$ such that:*

- *if $\underline{l} > l$, then $\theta^* = \underline{\theta}$,*
- *if $l^{fb} > l > \underline{l}$, then $\overline{\theta} > \theta^* > \underline{\theta}$,*

- *if* $l > l^{fb}$*, then* $\theta^* = \overline{\theta}$*;*

and

- *if* $\overline{l} > l$*, then* $U^*(\overline{\theta}) = R(w^*(\overline{\theta})) - l > 0$*,*
- *if* $l > \overline{l}$*, then* $U^*(\overline{\theta}) = 0$*.*

Proof. The proof completes those of Propositions 1 and 2. □

All in all, there are eleven possible cases that lead to five contracts with salient properties. Let us present them:

$$\mathbf{A} = \begin{cases} \theta^* = \overline{\theta} \\ U(\underline{\theta}) > R(w^{fb}) - l \\ U(\overline{\theta}) = 0 > R(w^{fb}) - l, \end{cases} \quad \mathbf{C} = \begin{cases} \theta^* = \underline{\theta} \\ U(\underline{\theta}) = R(\underline{w}) - l > 0 \\ U(\overline{\theta}) = R(\overline{w}) - l = 0, \end{cases} \quad \mathbf{E} = \begin{cases} \theta^* = \underline{\theta} \\ U(\underline{\theta}) = R(\underline{w}) - l \\ U(\overline{\theta}) = R(\overline{w}) - l > 0. \end{cases}$$

$$\mathbf{B} = \begin{cases} \theta^* \in (\underline{\theta}, \overline{\theta}) \\ U(\underline{\theta}) > R(w^{fb}) - l \\ U(\overline{\theta}) = R(w(\overline{\theta})) - l = 0, \end{cases} \quad \mathbf{D} = \begin{cases} \theta^* \in (\underline{\theta}, \overline{\theta}) \\ U(\underline{\theta}) > R(w^{fb}) - l \\ U(\overline{\theta}) = R(\overline{w}) - l > 0, \end{cases}$$

The double-entry table, Table 2, allows us to state the occurrence of these five contracts:[13]

if $\underline{l} > 0$	and $\underline{l} > \overline{l}$	l	0 $\overline{l}$ $\underline{l}$ l^{fb}
		Contracts	**E** **C** **B** **A**
	and $\underline{l} < \overline{l}$	l	0 $\underline{l}$ $\overline{l}$ l^{fb}
		Contracts	**E** **D** **B** **A**
if $\underline{l} < 0$	-	l	0 $\overline{l}$ l^{fb}
		Contracts	**D** **B** **A**

Table 2: Occurrence of contracts with respect to l

Let us make some comments about those contracts. When l is sufficiently high, i.e., higher than l^{fb}, the agent may absorb huge losses and the moral-hazard problem is not significant

[13]Notice that the values of $\overline{l}$ are not necessarily equal depending on the row in the table. For each configuration concerning $\underline{l}$ and $\overline{l}$, the rows on the extreme right of the table report the values of l at the top and the contracts at the bottom.

for the principal. So the optimal contract is **A**, it is dichotomous and corresponds to the adverse-selection contract. When the asset is lower than l^{fb}, the moral hazard problem is relevant, just as it was in our main analysis. Optimal contracts described in Propositions 1, 2, and 3 can occur even if l is positive. They are equivalent to contracts **D**, respectively **E**. This is the case when l is at least lower than the critical value $\bar{l}$. Whereas when l is at least higher than $\bar{l}$ but lower than l^{fb}, contracts **B** and **C** are intermediate contracts. Contract **B** (resp. **C**) looks like those in Proposition 1 (resp. 2) except that the most inefficient type obtains no rent because his asset is high enough to absorb losses.

From these comments, we can conclude that the consequences of a positive asset on higher or lower-powered incentives are qualitatively the same as those where the agent has no asset.

5.2 Private cost overrun

In this extension, we show that we can straightforwardly extend our results to an agency relationship where the agent may exert an effort to reduce the occurrence of a fixed cost overrun, denoted $K > 0$, added to his production cost.

Compared to the main model, we assume that $B = 0$. The utility of the agent is $U = t + ew - \theta q - \psi(e) - (1 - e)\,K$. Note that the agent has *private incentives* to exert effort: without a contingent payment (i.e., if $w = 0$), the agent exerts the effort e_K defined by $\psi'(e_K) = K$. On top of that, the limited-liability constraint is now $\forall\theta \in \Theta$, $t(\theta) - \theta q(\theta) - K \geq 0$, so our Lemma 1 still applies except that (6) and (10) become, respectively:

$$
\begin{aligned}
&w(\theta) + K - \psi'(\epsilon(w(\theta))) = 0,\\
&U(\theta) \geq R(w(\theta)) = \epsilon(w(\theta))\,(w(\theta) + K) - \psi(\epsilon(w(\theta))).
\end{aligned}
$$

The dichotomy property continues to apply in the benchmarks, even if effort and contingent payment levels are modified in this new setting. More precisely, the first-best effort is $e^{fb} = e_K$, which defines a relevant contingent payment $w^{fb} = 0$. Providing more efforts than the private-based ones is inefficient, and no contingent payment is granted at the first-best.

Regarding the moral-hazard contingent payment,[14] it changes as the shadow price of the

[14]The adverse-selection benchmark is unchanged.

limited-liability constraint is now such that

$$\rho\left(w^{mh}\right) = \frac{-w^{mh}}{r(\epsilon(w^{mh}))} = 1, \forall\theta \in \Theta.$$

As a result, we get $w^{mh} < 0$. That is, the contingent payment becomes a penalty or a malus. Due to liability protection, the cost overrun, if any, is systematically reimbursed, so the principal does not want the agent to put in more effort that he would do on his own initiative.

It is now easy to see that according to our Propositions, the contingent payment is also a malus when both adverse selection and moral hazard plague the relation, with similar properties to those in our main analysis. Quantities and contingent payments identified above remain qualitatively the same and are such that:

$$\begin{cases} S'(q^*(\theta)) - \theta = \frac{F(\theta) - \left(\mathbb{1}|_{\theta \geq \theta^*}\right)\int_{\theta^*}^{\theta} \rho(w^*(\tau))f(\tau)d\tau}{f(\theta)}, \\ w^*(\theta) \leq 0 \text{ with equality holding if } \theta \leq \theta^*. \end{cases}$$

In terms of power of incentives, they are still higher for production, but considerably lower for effort.

5.3 Variable additional benefit

Now, we consider that the additional benefit depends on the quantity. Let $B = \beta q, \beta > 0$.[15] The benchmark quantities and bonuses become under asymmetric information, $\forall\theta \in \Theta$

$$\begin{cases} S'(q^{as}(\theta)) + \beta\epsilon\left(w^{as}\left(\theta\right)\right) - \theta = \frac{F(\theta)}{f(\theta)}, \\ \beta q^{as}\left(\theta\right) = w^{as}\left(\theta\right); \end{cases} \qquad \begin{cases} S'(q^{mh}(\theta)) + \beta\epsilon\left(w^{mh}\left(\theta\right)\right) - \theta = 0, \\ \frac{\beta q^{mh}(\theta) - w^{mh}(\theta)}{r(\epsilon(w^{mh}(\theta)))} = 1. \end{cases}$$

We first observe that the moral-hazard shadow price of the limited-liability constraint, $\rho(q(\theta), w(\theta)) = \frac{\beta q(\theta) - w(\theta)}{r(\epsilon(w(\theta)))}$ is also dependent on quantity, as expected. Second, quantities and bonuses are no longer independent. Third, this intrication has as its consequence that the limited-liability rent is type-dependent. Finally, assuming that $\epsilon'\left(w\right)\beta^2 < -S''\left(q\right)$,[16] both outcomes are still decreasing with respect to the cost θ, and they are still separating.

[15] This assumption fits with the setting in Laffont and Martimort (2002), chapter 7.

[16] Indeed, the expected additional surplus, $\beta\epsilon(w)q$, is concave in w but linear in q. So the concavity of the principal's objective ensured by the initial gross surplus $S\left(q\right)$ is challenged by this additional benefit. This is not the case if β is not too high.

Of course, this influences the generalized agency framework since it becomes close to those of type-dependent participation constraint problem (Jullien (2000)). The novelty is that the bound provided by the limited-liability rent is endogenous. So we must identify the levels of R that matter along the interval $[\underline{\theta}, \overline{\theta}]$.

To facilitate this challenge, let $(q_\pi(\theta), w_\pi(\theta))$ be defined by the following system:

$$\begin{cases} S'(q_\pi(\theta)) + \beta\epsilon\left(w_\pi\left(\theta\right)\right) - \theta = \frac{F(\theta)-\pi}{f(\theta)}, \\ \beta q_\pi\left(\theta\right) = w_\pi\left(\theta\right); \end{cases} \tag{26}$$

where π has some value in $[0, 1]$. By comparison with the adverse-selection case above (i.e., where $\pi = 0$), we expect that (26) gives the optimal quantity and bonus at θ when the limited-liability constraint is free for any constant π. This constant enables us to determine the optimal level of the limited-liability rent, so we denote its corresponding value $R(w_\pi(\theta))$. It is decreasing in θ.[17] Let us assume that it is also convex, i.e., $\ddot{R}(w_\pi) = \epsilon'(w_\pi)(\dot{w}_\pi)^2 + \epsilon(w_\pi)\ddot{w}_\pi > 0$.[18]

Thus, since $U(\theta)$ and $R(w_\pi(\theta))$ are decreasing and convex, the former can cross the latter at most twice. So we consider two thresholds, θ^* and θ_*, such that θ^* (resp. θ_*) gets a limited-liability rent only whereas the type $\theta^* - d\theta$ (resp. $\theta^* + d\theta$) obtains an information rent and $\theta^* + d\theta$ (resp. $\theta^* - d\theta$) a limited-liability rent.

Finally, according to footnote 7, recall that the function $P = \int \rho f d\theta$ has the properties of a cumulative distribution function over the non-degenerate interval where the limited-liability constraint is binding.

We restrict our attention to the cases where there are two interior thresholds. Let us analyze in turn case A where $\theta^* < \theta_*$ and case B where $\theta_* < \theta^*$.

Case A. By construction (i.e., the paths of $U(\theta)$ and $R(w_\pi(\theta))$ and the definitions of θ^* and θ_*), types belonging to $[\underline{\theta}, \theta^*)$ get a decreasing information rent, those belonging to $[\theta^*, \theta_*]$ a limited-liability rent, and those belonging to $(\theta_*, \overline{\theta}]$ an increasing information rent.

By the properties of P, we deduce that $P(\theta^*) = 0$ and $P(\theta_*) = 1$. Thus, the limited-

[17] Indeed as $\dot{q}^{as}, \dot{w}^{as} < 0$ under the concavity condition $\epsilon'(w)\beta^2 < -S''(q)$, this is also the case for $\dot{q}_\pi, \dot{w}_\pi$ and using the envelope theorem, $\dot{R}(w_\pi) = \epsilon(w_\pi)\dot{w}_\pi < 0$

[18] This convexity is readily ensured if $\ddot{w}_P \geq 0$, but this is not necessarily true from the assumptions made so far. To ensure such a convex shape, further restrictions would be needed on the third derivatives of $S(q)$ and $\psi(e)$, omitted here. For instance, with a quadratic function for $S(q)$ and $\psi(e)$, convexity of $R(w_\pi(\theta))$ is fulfilled as $\ddot{w}_\pi = 0$.

liability rent $R(w_\pi(\theta))$ that constrains the rent $U(\theta)$ when the limited-liability constraint is slack must be that parameterized by $\pi = 0$ on $[\underline{\theta}, \theta^*]$ and by $\pi = 1$ on $[\theta_*, \overline{\theta}]$.

Proposition 5. *If $\theta^* < \theta_*$, the optimal contract entails $P^*(\theta^*) = 0$ and $P^*(\theta_*) = 1$ and*

- $\begin{cases} (q^*(\theta), w^*(\theta)) = (q_\pi(\theta), w_\pi(\theta))|_{\pi=0}, \\ U^*(\theta) = R(w^*(\theta^*)) + \int_\theta^{\theta^*} q^*(\tau)d\tau, \end{cases}$ *if* $\theta \leq \theta^*$;

- $\begin{cases} S'(q^*(\theta)) + \beta\epsilon\,(w^*(\theta)) - \theta = \frac{F(\theta) - \int_{\theta^*}^{\theta} p^*(\tau)d\tau}{f(\theta)}, \\ U^*(\theta) = R(w^*(\theta)), \end{cases}$ *if* $\theta^* < \theta < \theta_*$;

- $\begin{cases} (q^*(\theta), w^*(\theta)) = (q_\pi(\theta), w_\pi(\theta))|_{\pi=1}, \\ U^*(\theta) = R(w^*(\theta_*)) - \int_{\theta_*}^{\theta} q^*(\tau)d\tau, \end{cases}$ *if* $\theta_* \leq \theta$;

with $p^*(\theta) = \rho(q^*(\theta), w^*(\theta))f(\theta)$.

On $[\underline{\theta}, \theta^*]$, the contract exhibits the usual downward distortions on quantities because the agent is tempted to underestimate its type (i.e., usual incentives). Thus bonuses are also reduced. Power of incentives are thus lowered. The reverse occurs on $[\theta_*, \overline{\theta}]$ since countervailing incentives are at work (i.e., overestimation of the type). Over $[\theta^*, \theta_*]$, the rent starts at $R(w_\pi(\theta^*))|_{\pi=0}$ and ends at $R(w_\pi(\theta_*))|_{\pi=1}$. Powers of incentives increase over this interval.

Since the most and the least efficient agents obtain the first-best allocation, the contract occurs if $U^*(\underline{\theta}) > R(w_\pi(\underline{\theta}))|_{\pi=0} = R(w^{fb}(\underline{\theta}))$ and $U^*(\overline{\theta}) > R(w_\pi(\overline{\theta}))|_{\pi=1} = R(w^{fb}(\overline{\theta}))$.

If either of these conditions fails, the contract boils down to the contract with usual incentives only (which looks like those in Proposition 1 except that ρ now depends on q) or its symmetric counterpart with countervailing incentives only. If both conditions fail, the contract is akin to those in Proposition 2.

Case B. By construction again, we must have $U(\theta) < R(w_\pi(\theta))$ on $[\underline{\theta}, \theta_*) \cup (\theta^*, \overline{\theta}]$ and $U(\theta) > R(w_\pi(\theta))$ on (θ_*, θ^*). By the properties of P, we have $0 < P(\theta_*) = P(\theta^*) < 1$. Moreover, there must be some $\theta_{\pi^*} \in (\theta_*, \theta^*)$ such that the information rent increases with θ on $[\theta_*, \theta_{\pi^*}]$ (countervailing incentives) and decreases on $[\theta_{\pi^*}, \theta^*]$ (usual incentives).

Combining these two elements, the limited-liability rent that constrains the rent when the limited-liability constraint is slack must be parameterized by some $\pi = F(\theta_{\pi^*}) \in (F(\theta_*), F(\theta^*))$.

Proposition 6. *If $\theta_* < \theta^*$, the optimal contract entails $P^*(\underline{\theta}) = 0$, $P^*(\theta_*) = P^*(\theta^*) = F(\theta_{\pi^*})$, $P^*(\overline{\theta}) = 1$, and*

- $$\begin{cases} S'(q^*(\theta)) + \beta\epsilon\left(w^*\left(\theta\right)\right) - \theta = \frac{F(\theta) - \int_{\underline{\theta}}^{\theta} p^*(\tau)d\tau}{f(\theta)}, \\ U^*(\theta) = R(w^*(\theta)), \end{cases} \quad \text{if } \theta < \theta_*;$$

- $$\begin{cases} (q^*(\theta), w^*(\theta)) = (q_\pi(\theta), w_\pi(\theta))|_{\pi = F(\theta_{\pi^*})}, \\ U^*(\theta) = R(w^*(\theta^*)) + \int_{\theta}^{\theta^*} q^*(\tau)d\tau, \end{cases} \quad \text{if } \theta_* \geq \theta \geq \theta^*;$$

- $$\begin{cases} S'(q^*(\theta)) + \beta\epsilon\left(w^*\left(\theta\right)\right) - \theta = \frac{F(\theta) - \left(F(\theta_{\pi^*}) + \int_{\theta^*}^{\theta} p^*(\tau)d\tau\right)}{f(\theta)}, \\ U^*(\theta) = R(w^*(\theta)), \end{cases} \quad \text{if } \theta^* > \theta;$$

with $p^*(\theta) = \rho(q^*(\theta), w^*(\theta))f(\theta)$.

Comments are qualitatively the same as for the preceding proposition, they only need to be adapted to the relevant interval. This case occurs if both $U^*(\underline{\theta}) < R(w^{fb}(\underline{\theta}))$ and $U^*(\overline{\theta}) < R(w^{fb}(\overline{\theta}))$.

6 Conclusion

This paper combines two textbook models of the theory of incentives, one with adverse selection, the other with moral hazard, to form a generalized agency model. We study a contractual relationship between a principal and an agent who may undertake a certain level of production and effort. The production yields a basic level of benefit for the principal but the effort can stochastically enhance this benefit. Although we adopted the separability of the cost function and the disutility of effort, our main result showed that the optimal contract is not dichotomous because both types of information asymmetry interact when limited liability applies for the agent. This gives rise to a new type of trade-off: reducing the bonus to reduce the limited-liability rent *versus* lowering production to reduce the informational rent. In the end, the optimal contract yields substitutability between production and bonuses from the point of view of the principal, which helps reduce the marginal cost of the rent left to the agent. Consequently, the optimal contract shows high-powered incentives for the quantity outcome but low-powered ones for the bonus, but only for the least efficient agents. Therefore the moral-hazard bonus level is the central level around which bonus distortions are distributed. Another interesting feature of the optimal contract is that it is generally separating. Pooling may occur (for inefficient agents only) but it is not necessary.

Finally, we provide three extensions for our results. First, when the agent has a positive asset, we identify five possible contracts, covering the polar configurations where either all types get a limited-liability rent or all types obtain an informational rent. Second, we show that our results readily apply to a setting for which the agent exerts the effort to reduce the occurrence of a fixed-cost overrun, instead of a windfall benefit. The main change is that the contingent payment becomes a malus. As a result, the malus level may be very significantly high compared to the moral-hazard malus level for the least inefficient agents. Third, we consider that the additional benefit is quantity-dependent. It follows that the dichotomy property vanishes and the limited-liability becomes type-dependent. We find that the optimal contracts mix our results with those of type-dependent participation constraints. This gives rise to countervailing incentives.

We now present a possible avenue of research based on our framework (which also forms a companion paper). In the present article, we assume a separability condition: the resources mobilized to exert effort have no impact on the production cost. It seems realistic enough to consider that this separability hypothesis is not always true. Consider the contrary. Suppose for example that the cost function becomes $c(\theta, e)q$. It follows immediately that the incentive constraints grow in complexity because not only does production affect moral-hazard incentives but effort also affects adverse-selection incentives.

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7 Appendix

7.1 Proof of Lemma 1

Faced with the contract $\langle q(\hat{\theta}), t(\hat{\theta}), w(\hat{\theta})\rangle$, the agent undertakes an effort such that, according to (2)

$$w(\hat{\theta}) - \psi'(\epsilon(w(\hat{\theta}))) = 0. \tag{A.1}$$

Let $v(\hat{\theta}, \theta)$ be the indirect rent of the agent θ reporting $\hat{\theta}$. We get

$$v(\hat{\theta}, \theta) = t(\hat{\theta}) + \epsilon(w(\hat{\theta}))w(\hat{\theta}) - \theta q(\hat{\theta}) - \psi(\epsilon(w(\hat{\theta}))).$$

The agent may select a contract such that $\hat{\theta} = \theta$ as $\theta = \arg\max_{\hat{\theta}} v(\hat{\theta}, \theta)$, i.e., $v_{\hat{\theta}}(\hat{\theta}, \theta) = 0$ and $v_{\hat{\theta}\hat{\theta}}(\hat{\theta}, \theta) \leq 0$. These conditions imply $\dot{U}(\theta) = v_\theta(\theta, \theta)$ (since $U(\theta) = v(\theta, \theta)$) and $v_{\hat{\theta},\theta}(\theta, \theta) \geq 0$. Using standard developments lead to (7) and (8). Moreover, (6) follows from (A.1) at $\hat{\theta} = \theta$ and usual calculus show that (8) is also globally sufficient. Relation (9) follows directly from the combination of (3) and (7). Finally, using (4) with (1) at $\hat{\theta} = \theta$ when $e = \epsilon\left(w\left(\theta\right)\right)$, leads to (10).

7.2 Proof of Propositions 1, 2, and 4

Let us ignore argument θ for simplicity and consider (25) as limited-liability constraints. We face a dynamic optimization program with

- two states, U and q,
- a control κ such that $\dot{q} = \kappa \leq 0$,
- a decision variable w,
- a mixed constraint $U - (R(w) - l) \geq 0$,
- and a terminal state condition $U(\overline{\theta}) \geq 0$.

The Hamiltonian with respective costates λ and δ writes

$$H = \{S(q) + \epsilon(w)B - \theta q - \psi(\epsilon(w)) - U\} f - \lambda q + \delta\kappa$$

and the Lagrangian with $\mu \geq 0$ is

$$L = H + \mu(U - (R(w) - l))$$

and $R(w) = \epsilon(w)w - \psi(\epsilon(w))$.

Necessary conditions are, using (6)

$$\kappa\frac{\partial L}{\partial \kappa} = \delta\kappa = 0 \text{ and } \frac{\partial L}{\partial \kappa} = \delta \geq 0, \tag{A.2}$$

$$\frac{\partial L}{\partial w} = 0 = (B - w)\epsilon'(w)f - \mu\epsilon(w), \tag{A.3}$$

and

$$-\frac{\partial L}{\partial U} = \dot{\lambda} = f - \mu, \tag{A.4}$$

$$-\frac{\partial L}{\partial q} = \dot{\delta} = -(S'(q) - \theta)f + \lambda. \tag{A.5}$$

The complementarity slackness condition is

$$\mu \geq 0,\ \mu(U - (R(w) - l)) = 0, \tag{A.6}$$

and transversality conditions are

$$\lambda(\underline{\theta}) = \delta(\underline{\theta}) = 0, \tag{A.7}$$

$$\lambda(\overline{\theta})U(\overline{\theta}) = 0 \text{ and } \lambda(\overline{\theta}) \geq 0, \text{ and } \delta(\overline{\theta}) = 0. \tag{A.8}$$

The mixed constraint is always qualified as the jacobian of $U - R(w) + l$ with respect to the controls κ and w is not zero, so no jumps in costates are needed. Concavity of L in U, q, w, κ is ensured as the hessian of L writes $\nabla^2 L = \operatorname{diag}(0, S''(q)\, f, h(w)f - \mu\epsilon''(w), 0)$, where $h(w) = \epsilon''(w)(B - w) - \epsilon'(w) \leq 0$ if $B \geq w$. Consequently using (11) and $\mu \geq 0$, we have $h(w)f - \mu\epsilon'(w) \leq 0$, and L is semi-definite negative. The above conditions are sufficient.

From now on, as it is usual in incentives literature, we begin the analysis by looking for a separating solution, i.e., $\kappa > 0$.

Let us present intermediate results in the following steps.

Step 0: If $U = R(w) - l$, this occurs on the right of Θ. Indeed, since U decreases with θ whereas $R(w^{fb}) - l$ and $R(w^{mh}) - l$ are constant, the limited-liability constraints can be slack only on $[\theta^*, \overline{\theta}]$.

Step 1: value of μ. Notice that $U = R(w) - l$ implies $\dot{U} = R'(w)\dot{w}$, or, provided that $q > 0$, $\dot{w} = -\frac{q}{\epsilon(w)} < 0$. Moreover, it is easy to check that $\frac{\partial(\frac{B-w}{r(\epsilon(w))})}{\partial w} < 0$. Using these two inequalities and combining (A.3) and (A.6), we get

$$\mu = \frac{B-w}{r(\epsilon(w))} f \begin{cases} = 0 \Rightarrow w = w^{fb} \text{ if } \theta < \theta^*, \\ \geq 0 \Rightarrow w \leq w^{fb} \text{ and } \dot{w} < 0 \text{ if } \theta^* \leq \theta. \end{cases} \tag{A.9}$$

Moreover, remark that

$$\mu = f \Rightarrow w = w^{mh}. \tag{A.10}$$

Step 2: value of q. From (A.2) and $\kappa > 0$, we must have $\delta = 0 \Rightarrow \dot{\delta} = 0$ and by (A.5):

$$S'(q) - \theta = \frac{\lambda}{f}. \tag{A.11}$$

Combining (A.4), (A.7), and (A.9) we obtain

$$\lambda = F - (\mathbb{1}|_{\theta \geq \theta^*}) \int_{\theta^*}^{\theta} \frac{B-w}{r(\epsilon(w))} f d\tau, \tag{A.12}$$

and inserting this in (A.11), we obtain

$$S'(q) - \theta = \frac{F - (\mathbb{1}|_{\theta \geq \theta^*}) \int_{\theta^*}^{\theta} \frac{B-w}{r(\epsilon(w))} f d\tau}{f}. \tag{A.13}$$

Step 3: value of U. Using the bound condition $U(\overline{\theta}) \geq 0$ to integrate $\dot{U}$ leads to

$$U = U(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} q d\tau. \tag{A.14}$$

Step 4: condition ensuring $\theta^* > \underline{\theta}$. By continuity of μ at θ^*, (A.9) leads to $\mu(\theta^*) = 0$ and thus $w(\theta^*) = w^{fb}$. Therefore since $\theta^* \in (\underline{\theta}, \overline{\theta})$, we must have $U(\theta^*) = R(w^{fb}) - l$. Hence, $\dot{U} < 0$ implies that $\theta^* > \underline{\theta}$ if $U(\underline{\theta}) > R(w^{fb}) - l$. We can define $\underline{l}$ by

$$\underline{l} = R(w^{fb}) - U(\underline{\theta}), \tag{A.15}$$

and such that $\theta^* > \underline{\theta}$ needs

$$l > \underline{l}. \tag{A.16}$$

Step 5: condition ensuring $\overline{\theta} > \theta^*$. Once again, by definition of θ^* and by the continuity of μ at θ^*, this arises if $R(w^{fb}) - l > 0$, i.e., if

$$R(w^{fb}) = l^{fb} > l. \tag{A.17}$$

Step 6: condition ensuring $U(\overline{\theta}) > 0$. Since $U(\overline{\theta}) > 0$, then from (A.8) $\lambda(\overline{\theta}) = 0$ and by (A.12)

$$\int_{\theta^*}^{\overline{\theta}} \frac{B - w}{r(\epsilon(w))} f d\tau = 1. \tag{A.18}$$

Integrating (A.18) by parts, one can derive since $w(\theta^*) = w^{fb} = B$ if $\theta^* > \underline{\theta}$ and/or $F(\underline{\theta}) = 0$ if $\theta^* = \underline{\theta}$

$$\frac{B - w(\overline{\theta})}{r(\epsilon(w(\overline{\theta})))} = 1 + \int_{\theta^*}^{\overline{\theta}} \frac{\partial(\frac{B-w}{r(\epsilon(w))})}{\partial w} \dot{w} f d\tau > 1$$

so, using (A.10)

$$w(\overline{\theta}) < w^{mh}. \tag{A.19}$$

Because w is necessarily positive by (A.3) since $\epsilon'(w) > 0$ and $\epsilon(0) = 0$, we can set $\overline{l} = R(w(\overline{\theta}))$ such that $U(\overline{\theta}) = R(w(\overline{\theta})) - l > 0$ requires

$$\overline{l} > l. \tag{A.20}$$

Proof of Propositions 1 and 2. Let $l = 0$.

- *Proof of the values U, q, and w.*
  From (A.17) (resp. (A.20)), we get $\theta^* < \overline{\theta}$ (resp. $U(\overline{\theta}) = R(w(\overline{\theta})) > 0$). Moreover, two cases arise according to the sign of $\underline{l}$. If it is negative (resp. positive), the condition for Proposition 1 (resp. 2) is satisfied using (A.15). By (A.16), we obtain $\theta^* > \underline{\theta}$ (resp $\theta^* = \underline{\theta}$).

  Combining these results with (A.9), (A.13), and (A.14), we obtain the values of U, q, and w in (18)-(23), keeping in mind from step 0 that U is also equal to R if $\theta > \theta^*$.

- *Proof of properties of q and w.*

  Let us begin by q. Using (A.5), (A.9) and (A.10), $\dot{\lambda}$ can cross 0 at most once, and

$$\dot{\lambda} \gtreqless 0 \Leftrightarrow f \gtreqless \mu \Leftrightarrow w \gtreqless w^{mh}.$$

  Thus, using the transversality conditions (A.7) and (A.8), λ is non negative. Moreover, we have $\lambda = 0$ at bounds since $U(\overline{\theta}) > 0$, which proves, using (A.11), the sign property in (19) and (22).

  Now, consider w. The similar properties in (20) and (23) directly follow from the second part of (A.9) and (A.19).

  Then, consider w in Proposition 2 only. Let $U(\underline{\theta}) \leq R(w^{fb})$, we get $\theta^* = \underline{\theta}$ and $\mu(\underline{\theta}) \geq 0$ by (A.9), or

$$w(\underline{\theta}) \leq w^{fb}. \tag{A.21}$$

  with equality holding at $U(\underline{\theta}) = R(w^{fb})$ by continuity of μ.

  Invoking the mean theorem for (A.18), it necessarily exists $\tilde{\theta} \in (\underline{\theta}, \overline{\theta})$:

$$\int_{\underline{\theta}}^{\overline{\theta}} \left\{ \frac{B - w}{r(\epsilon(w))} f \right\} d\tau = \frac{B - w(\tilde{\theta})}{r(\epsilon(w(\tilde{\theta})))} \int_{\underline{\theta}}^{\overline{\theta}} f d\tau = 1$$

  so $w(\tilde{\theta}) = w^{mh}$. Since $\dot{w} < 0$, this implies

$$w^{mh} < w(\underline{\theta}). \tag{A.22}$$

  (A.21) and (A.22) prove that $w^{mh} < w_2^*(\underline{\theta}) \leq w^{fb}$ with equality holding if $U_1(\underline{\theta}) = R(w^{fb})$.

Proof of Proposition 4. The proof comes from (A.16), (A.17), (A.19), and (A.20).

7.3 Pooling

Now let us check the conditions ensuring that the solution q is separating, i.e., $\dot{q} = \kappa < 0$. Differentiating (A.11) with respect to θ leads to

$$S''(q)\kappa = 1 + \left(\frac{\dot{\lambda}}{f}\right).$$

Since $S''(q) < 0$, using (A.4) implies, after simplifications

$$\kappa > 0 \Leftrightarrow 2 > \frac{\mu}{f} + \frac{\dot{f}\lambda}{f^2}.$$

Using (A.9) and (A.12) leads to (24) in the text. Note that one can write (24) as

$$1 + (\frac{\dot{F}}{f}) > (\mathbb{1}|_{\theta \geq \theta^*}) \left[\frac{B-w}{r(\epsilon(w))} - \frac{\dot{f}}{f^2}\int_{\theta^*}^{\theta} \frac{B-w}{r(\epsilon(w))} f d\tau\right]. \tag{A.23}$$

Using the fact that $(\frac{\dot{F}}{f}) = 1 - \frac{\dot{f}F}{f^2} \Leftrightarrow -\frac{\dot{f}}{f^2} = \frac{(\frac{\dot{F}}{f})-1}{F}$, (A.23) can be also written as

$$1 + (\frac{\dot{F}}{f}) + \frac{1}{F}\int_{\theta^*}^{\theta} \frac{B-w}{r(\epsilon(w))} f d\tau > \frac{B-w}{r(\epsilon(w))} + \frac{(\frac{\dot{F}}{f})}{F}\int_{\theta^*}^{\theta} \frac{B-w}{r(\epsilon(w))} f d\tau.$$

Using mean theorem it then exists $\hat{\tau} \in (\theta^*, \theta)$ this leads to

$$\begin{aligned} & 1 + (\frac{\dot{F(\theta)}}{f(\theta)}) \left(1 - \frac{B - w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1 - \frac{F(\theta^*)}{F(\theta)}\right)\right) \\ > & \frac{B - w(\theta)}{r(\epsilon(w(\theta)))} - \frac{B - w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1 - \frac{F(\theta^*)}{F(\theta)}\right) \geq 0. \end{aligned} \tag{A.24}$$

where $\hat{\tau} \in (\theta^*, \theta)$ which increases with θ. Indeed, differentiating $\frac{1}{F(\theta)}\int_{\theta^*}^{\theta} \frac{B-w(\tau)}{r(\epsilon(w(\tau)))} f(\tau)\, d\tau$ with respect to θ leads to

$$\begin{aligned} \left(\frac{B - w(\theta)}{r(\epsilon(w(\theta)))} - \frac{B - w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\right) f(\theta) &= \frac{d\hat{\tau}}{d\theta} w'(\hat{\tau}) \frac{\partial}{\partial w}\left(\frac{B - w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\right)(F(\theta) - F(\theta^*)) \\ \Rightarrow \frac{d\hat{\tau}}{d\theta} &= \frac{\left(\frac{B-w(\theta)}{r(\epsilon(w(\theta)))} - \frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\right) f(\theta)}{w'(\hat{\tau}) \frac{\partial}{\partial w}\left(\frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\right)(F(\theta) - F(\theta^*))} > 0 \end{aligned}$$

So verifying the necessary condition (A.24) implies that the solution is separating.

First, note that $\frac{B-w(\theta)}{r(\epsilon(w(\theta)))}$ and $\frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1-\frac{F(\theta_*)}{F(\theta)}\right)$ increases with θ. From Proposition 1 and 1 when $\theta^* \geq \underline{\theta}$, it always exists $\theta^{mh} > \theta^* : w\left(\theta^{mh}\right) = w^{mh}$. So for all

- $\theta \leq \theta^{mh}$ then $\hat{\tau} < \theta \leq \theta^{mh}$, such that $w(\hat{\tau}) > w(\theta) > w^{mh}$ and

  $$1 \geq \frac{B-w(\theta)}{r(\epsilon(w(\theta)))} > \frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))} > \frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1-\frac{F(\theta^*)}{F(\theta)}\right) > 0$$

  so

  $$1 > \frac{B-w(\theta)}{r(\epsilon(w(\theta)))} - \frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1-\frac{F(\theta^*)}{F(\theta)}\right) > 0$$

  as a result (A.24) is verified.

- $\theta > \theta^{mh}$,

  - if $\theta > \theta^{mh} \geq \hat{\tau} > \theta^*$ then $w^{fb} > w(\hat{\tau}) \geq w^{mh} > w(\theta)$ and

    $$\frac{B-w(\theta)}{r(\epsilon(w(\theta)))} > 1 \geq \frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))} > \frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1-\frac{F(\theta^*)}{F(\theta)}\right) > 0$$

    and

    $$\begin{aligned}(\frac{\dot{F}(\theta)}{f(\theta)})\left(1-\frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1-\frac{F(\theta^*)}{F(\theta)}\right)\right) &> 0 \\ \frac{B-w(\theta)}{r(\epsilon(w(\theta)))} - \frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1-\frac{F(\theta^*)}{F(\theta)}\right) &> 0\end{aligned}$$

    So it may exist a $\theta_b : \theta_b > \theta^{mh} \geq \hat{\tau}$ such that

    $$1 + \frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1-\frac{F(\theta^*)}{F(\theta_b)}\right) = \frac{B-w(\theta_b)}{r(\epsilon(w(\theta_b)))}$$

    and (A.24) is verified, for $\theta \leq \theta_b$.

  - if $\theta > \hat{\tau} > \theta^{mh}$ then $w^{mh} > w(\hat{\tau}) > w(\theta)$ then

    $$1 - \frac{B-w(\theta)}{r(\epsilon(w(\theta)))} < 1 - \frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))} < 0$$

which implies it may exist a $\theta_b' : \theta_b' > \theta_b$ such that

$$1-\left[\frac{B-w(\theta_b')}{r(\epsilon(w(\theta_b')))}-\frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1-\frac{F\left(\theta^*\right)}{F\left(\theta_b'\right)}\right)\right]+(\frac{\dot{F}(\theta_b')}{f(\theta_b')})\left(1-\frac{B-w(\hat{\tau})}{r(\epsilon(w(\hat{\tau})))}\left(1-\frac{F\left(\theta^*\right)}{F\left(\theta_b'\right)}\right)\right)=0$$

Then (A.24) is never verified, for $\theta \geq \theta_b'$ so the solution is no more separating.

- Then for $\theta \in (\theta_b, \theta_b')$ there is no more sufficient condition that holds, the the necessary condition (A.24) has to be checked.