A comprehensive revealed preference approach to approximate utility maximisation*

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Abstract

We develop a comprehensive revealed preference method for studying preferences of individuals whose choices are inconsistent with utility maximisation. We assume that only the directly revealed *strict* preference relation reflects true preferences of the agent, while the weak one may be subject to imprecision, vagueness of judgement, or incommensurability. As a result, a natural consistency condition, *acyclicity*, is imposed on the former alone. We show that this restriction is necessary and sufficient for the data to be rationalisable with *approximate utility maximisation*, where an alternative is selected from a menu only if its utility is not significantly lower than that of any other available option. More importantly, although the individual may fail to maximise their utility exactly, it is possible to recover their true preferences from observations, make out-of-sample predictions and welfare comparisons. Our results require minimal assumptions on the empirical framework and are applicable, amongst others, to the study of choices over consumption bundles, state-contingent consumption, and lotteries.

Keywords: revealed preference, strict preference, acyclicity, approximate utility maximisation, non-transitive indifferences, recoverability
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1 Introduction

We introduce a method for studying preferences of agents whose choices are inconsistent with deterministic utility maximisation. This most pervasive model of rational consumer

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choice is a cornerstone to a large and fruitful body of the economic analysis. Apart from the descriptive appeal, it has a clear normative underpinning, as it characterises choices of individuals with transitive preferences. In fact, whenever the researcher observes how the individual ranks *any* two alternatives, transitivity (essentially) exhausts all testable restrictions of utility maximisation.

In virtually all empirical applications it is impractical (or, rather, impossible) to observe all conceivable comparisons. In such an incomplete setting, transitivity of preferences (revealed through the subject's choices) is no longer sufficient for the data to be consistent with utility maximisation. Instead, one has to consider a stronger restriction: the *generalised axiom of revealed preference* (GARP).[1] Formally, let $\mathcal{O}$ be a dataset consisting of a finite number of pairs (A, x), where x represents an alternative selected from the menu A; and $\rhd$ be a strict transitive ranking that imposes an objective structure on the consumption space.[2] An option x is *directly revealed preferred* to y, denoted by xR^*y, if the former was chosen whenever the latter was available, i.e., $(A, x) \in \mathcal{O}$ and $y \in A$. The relation is *strict* and denoted by xP^*y, if x was chosen over something objectively better than y, i.e., we have $(A, x) \in \mathcal{O}$ and $z \rhd y$, for some $z \in A$. GARP requires that there are no strict cycles induced by the revealed preference relation R^* and P^*. This is equivalent to existence of a utility u that strictly increases with respect to $\rhd$ and rationalises the dataset $\mathcal{O}$, that is, we have

$$(A, x) \in \mathcal{O} \text{ and } y \in A \text{ implies } u(x) \geq u(y).$$[3]

More importantly, whenever the data satisfies GARP, the revealed preference toolkit allows to elicit preferences of the individual, make our-of-sample predictions, and conduct welfare analysis, while fully embracing heterogeneity of the agent.

An overwhelming empirical evidence suggests that choices of individuals are not consistent enough to satisfy GARP.[4] Naturally, this poses the fundamental question whether utility maximisation is an appropriate description of human behaviour. From a more

[1] See Afriat (1967), Diewert (1973), Varian (1982), as well as Forges and Minelli (2009) and Nishimura et al. (2017). For a handbook treatment, see Chapter 3 in Chambers and Echenique (2016).

[2] For example, $\rhd$ may correspond to the coordinate-wise ordering $>$ (or $\gg$) over $\mathbb{R}^\ell_+$, capturing the idea that "more is better" within the domain of consumption bundles. When studying choices over lotteries, one may identify $\rhd$ with the first order stochastic dominance.

[3] The utility u *strictly increases* with respect to $\rhd$ if $x \rhd y$ implies $u(x) > u(y)$.

[4] See Chapter 5 in Chambers and Echenique (2016) and more recently Halevy et al. (2018), Echenique et al. (2019), Feldman and Rehbeck (2022), Zrill (2020), Dembo et al. (2021), and Cappelen et al. (2021).

practical perspective, violation of this condition renders the revealed preference toolkit inapplicable, since it critically depends on the data satisfying GARP.

Given the deterministic nature of GARP, which is either satisfied by the data or not (in a purely binary way), one may be reluctant to reject utility maximisation based solely on such evidence. After all, just like any other scientific theory, this model serves merely as an approximation of individual behaviour, rather than its exact description. One possible step towards relaxing the test would be to include a stochastic element into the analysis by, e.g., allowing for measurement errors or mistakes made by the subject. Although such an approach is most sensible, the existing methods either require that the econometrician knows precisely the error distribution (see Varian, 1985),[5] or are designed to study choices of entire populations of agents, rather than particular subjects (see Aguiar and Kashaev, 2020). Thus, the inherent scarcity of individual choice data makes it difficult for the researcher to fully explore consumer heterogeneity, which (arguably) is the main advantage of the revealed preference analysis.

In this paper we take a different approach. We maintain the deterministic nature of the revealed preference test, but investigate a weaker condition: *acyclicity* of the directly revealed strict preference relation P^*. Our restriction implicitly assumes that only the revealed strict comparisons convey a reliable information about the true preferences of the individual, while the weak ones may be subject to imprecision, vagueness of judgement, or incommensurability. After all, we have xP^*y when x is chosen over something *objectively* better than y. Therefore, only the strict relation is required to exhibit some form of consistency. In our main result, we show that acyclicity of P^* is equivalent to the dataset being rationalisable with *approximate utility maximisation*. That is, there is a utility u and a positive threshold function δ such that

$$(A, x) \in \mathcal{O} \text{ and } y \in A \quad \text{implies} \quad u(x) + \delta(y) \geq u(y).$$

The alternative x is selected from the menu A only if its utility is at most $\delta(y)$ utils lower than that of any other available option y. This representation appeals to the idea of imperfect discrimination, suggesting that the individual discerns between two alternatives only if they yield a sufficiently different utility.[6] Unless the difference between values

[5] This assumes that the error is drawn from a normal distribution with known mean and variance.

[6] The reader may recognise that our model is analogous to the *interval order* representation of preferences proposed in Fishburn (1970). We address this in Section 5. Since this paper focuses on the utility u, we find the term *approximate utility maximisation* more appropriate.

attributed to x and y exceeds the particular threshold, the agent behaves as if they were indifferent, even if the options are strictly ordered with u. In addition, we show that the strict relation P^* is consistent with the utility u, that we interpret as the "true" preferences of the agent. However, this no longer applies to the weak relation R^*, which captures the idea that only P^* reveals the actual tastes of the consumer.

Most importantly, the model above provides us with a versatile framework for studying preferences of consumers whose choices are inconsistent with utility maximisation. The critical feature of this approach is that it separates the choice of the individual (governed by the approximate utility maximisation) from their preferences (that we identify with the utility u). Somewhat surprisingly, even though the observable choices may violate GARP, we show that one can still elicit the utility u. Similarly, as in the classic revealed preference theory, the model allows for out-of-sample predictions and a meaningful, data-driven welfare analysis. Finally, through an appropriate choice of the objective relation $\triangleright$, our approach admits an intuitive way of measuring the extent to which the data departs from utility maximisation, by appealing to the notion of imperfect discrimination. Our results abstract away from specific economic environments and require only a few assumptions on the consumption space and the data available to the researcher, which makes them applicable to a variety of empirical settings.

The idea of imperfect discrimination is well-known to Economics since Georgescu-Roegen (1936), Armstrong (1939, 1950), and Luce (1956). Inspired by the research in psychophysics, these papers acknowledge the inability of human beings to discern between close quantities of goods and claim that any descriptive theory of choice should allow for "imperfect powers of discrimination of the human mind whereby inequalities become recognizable only when of sufficient magnitude" (Armstrong, 1950, page 122). Specifically, they highlight the importance of non-transitive indifferences implied by such a behaviour. Although x may not be noticeably different from y, and y from z, the alternative x may be sufficiently distinct from z for the agent to tell them apart. See Aleskerov et al. (2007) for detailed discussion on these topics.

Inconsistencies of the weak revealed preference relations could also be related to vagueness of judgement or preference imprecision, studied, e.g., in Butler and Loomes (2007) and Cubitt et al. (2015).[7] Alternatively, we show that (in certain settings) such a phe-

[7] These authors observe that subjects tend to evaluate lotteries using *intervals* of certainty equivalents, rather than specific values. Given strict preference over money, this violates transitive indifferences.

nomenon is consistent with the satisficing behaviour as in Simon (1947), where the subject fails to maximise their utility due to an unobserved mental or physical cost.

We do not claim that approximate utility maximisation is the ultimate explanation for *any* deviation from the classic notion of rationality. Inevitably, some departures require a qualitatively different approach to modelling consumer choice. Rather, the point of this paper is to develop a versatile framework for studying preferences of individuals when the empirical data exhibit minor inconsistencies with the classic theory.

Organisation of the paper In Section 2 we introduce our setup and basic notation. Section 3 contains our main results: In Theorem 1 we show that the directly revealed strict preference relation P^* is acyclic if, and only if, the dataset can be rationalised with approximate utility maximisation; in Theorem 2 we discuss how to elicit the utility u from choices that are consistent with approximate utility maximisation; finally, Theorem 3 proposes a robust approach to welfare analysis.

We consider our method to be well-suited for the study of imperfect discrimination within the revealed preference framework. In Section 4, we discuss how this phenomenon can be analysed using the toolkit developed in this paper. In Section 5 we compare approximate utility maximisation to related models in the literature, including interval order maximisation, satisficing, and maximisation of an acyclic relation. Section 6 is devoted to some direct applications of our method. First, we extend the results in Polisson et al. (2020) to study approximate utility maximisation within a broad class of models of choice under risk. Then, we revisit the result in Dziewulski (2020) and show the tight relation between our model and one of the most widespread measures of departures from rationality — the critical cost-efficiency index of Afriat (1973).

Proofs omitted in the main body are postponed until the Appendix. Related results and extensions are discussed in the Online supplement, including an alternative, constructive take on Theorem 1 that pertains to linear programming.

2 Preliminaries

We begin our discussion by introducing the notation and terminology.

2.1 Basic definitions

Let X be the *consumption space*, i.e., the grand set of mutually exclusive choice alternatives. A *menu* is a non-empty subset A of X, and $\mathcal{A} = 2^X \setminus \{\emptyset\}$ denotes the set of all menus. A *dataset* (or a set of observations) $\mathcal{O}$ is a *finite* collection of pairs (A, x), where $x \in A$ is a choice from the menu $A \in \mathcal{A}$. Unless stated otherwise, we impose no additional assumptions on the space X or the dataset $\mathcal{O}$.

Our setup includes the consumer demand framework as in Afriat (1967), Diewert (1973), or Varian (1982), where in each observation $(A, x) \in \mathcal{O}$ an individual chooses an ℓ-dimensional bundle $x \in X = \mathbb{R}^\ell_+$ from a linear budget $A = \{y \in X : p \cdot y \leq m\}$, for some prices $p \in \mathbb{R}^\ell_{++}$ and income $m \geq 0$.[8] Similarly, it encompasses the setup of Forges and Minelli (2009), that allows for non-linear budget sets. Numerous empirical studies, including the famous Allais experiment, investigate choices of subjects over lotteries. In such a case, the space of alternatives X is the probability simplex and a menu can be given by an arbitrary (possibly finite) subset of X. One could also consider "budget sets" of lotteries as in Sopher and Narramore (2000) or Feldman and Rehbeck (2022).

Rationalisability In this paper we characterise datasets that can be rationalised with different models of choice. For this reason, we find it convenient to introduce a general notation that can be applied to each case. A *choice correspondence* (a model) is a set-valued mapping $c : \mathcal{A} \rightrightarrows X$ that assigns a menu $A \in \mathcal{A}$ to a set $c(A) \subseteq A$. The set $c(A)$ denotes all possible choices that the agent would consider from the menu A.[9] The correspondence c *rationalises* the set of observations $\mathcal{O}$ if

$$(A, x) \in \mathcal{O} \quad \text{implies} \quad x \in c(A), \tag{1}$$

i.e., the data are consistent with the model c.

This definition highlights two important aspects of our analysis. First, as the choice correspondence c is set-valued, we allow for the consumers to exhibit indifferences (or incomparabilities). This is inevitable in general consumption spaces, as shown in Nishimura and Ok (2014). Moreover, we assume that the dataset available to the researcher is in-

[8] It is usual in this literature to assume that the purchase x exhausts the available budget, i.e., $m = p \cdot x$, for all observations (A, x). However, this plays no role in our analysis.

[9] In principle, the set $c(A)$ may be empty, for some menu $A \in \mathcal{A}$. However, within the class of choice models discussed in this paper, one may assume that $c(A)$ is non-empty for any finite set A, without loss of generality. In Section 3 we discuss conditions under which $c(A)$ is non-empty for any compact A.

complete, i.e., they monitor only some elements of $c(A)$ (usually, one), for only a finite number of menus $A \in \mathcal{A}$. These are most natural features of any empirical study.

Choice monotonicity Our notion of rationalisability is too weak to induce any testable implications for the models considered in this paper. To make our research question non-vacuous, we strengthen the definition by requiring for the model c to be monotone.[10] Given a strict partial order $\rhd$ over X,[11] a choice model $c : \mathcal{A} \rightrightarrows X$ is $\rhd$-*monotone* if for any (possibly unobserved) alternative $x \in X$ and any menu $A \in \mathcal{A}$, we have

$$y \in A \text{ and } y \rhd x \quad \text{implies} \quad x \notin c(A). \tag{2}$$

The relation $\rhd$ is an objective ranking over elements in X that summarises additional restrictions imposed on the choice model, independently from the data.[12] An alternative x is never chosen if something objectively better was available.

Examples of $\rhd$ include coordinate-wise ordering $>$ (or $\gg$) over $X = \mathbb{R}^{\ell}_{+}$, capturing the idea that "more is better" within the domain of consumption bundles.[13] When studying choices over lotteries, one may identify $\rhd$ with the first order stochastic dominance; thus, imposing affinity for gambles in which greater rewards are more likely.[14]

Our method is particularly useful when analysing the problem of imperfect discrimination in consumer choice. It is well-established in the psychophysics literature that individuals distinguish between intensities of a physical stimulus (e.g., light, touch, sound) only if they are significantly different. The well-established Weber-Fechner law stipulates that people perceive the change whenever the ratio of the intensities exceeds a particular constant, the so-called *just-noticeable difference*. Dehaene (2008) shows that the same law applies to human perception of numbers, quantities, and numerosities, which is particularly relevant in economic decisions. This can be incorporated into our analysis through the relation $\rhd$. We discuss this thoroughly in Section 4 but, to fix ideas, we provide a few examples of $\rhd$ that capture the notion of imperfect discrimination.

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[10] In contrast, Balakrishnan et al. (2021) propose a method of estimating the entire set $c(A)$ and determining which comparisons represent indifferences (or incomparabilities) and which correspond to strict preferences. However, their approach requires rich datasets and applies only to finite menus.

[11] A *strict partial order* $\rhd \subseteq X \times X$ is a binary relation that is *irreflexive*, i.e., $x \not\rhd x$, for all $x \in X$; and *transitive*, i.e., $x \rhd y$ and $y \rhd z$ implies $x \rhd z$, for all $x, y, z \in X$.

[12] An empty $\rhd$ is equivalent to imposing no additional restrictions on c.

[13] We denote $x \geq y$ if $x_i \geq y_i$, for all $i = 1, \ldots, \ell$. The relation is *strict*, and denoted by $x > y$, if $x \geq y$ and $x \neq y$. Finally, we have $x \gg y$ if $x_i > y_i$, for all $i = 1, \ldots, \ell$.

[14] Lottery y *first order stochastically dominates* x if for any increasing function $f : S \to \mathbb{R}$, we have $\sum_{s \in S} f(s)y(s) \geq \sum_{s \in S} f(s)x(s)$. The relation is strict if the inequality is strict for some f.

Example 1. Let $X \subseteq \mathbb{R}^\ell_+$. Following the idea of Weber and Fechner, let

$$x \rhd y \quad \text{if} \quad x \geq y \text{ and } x_i \geq \lambda_i y_i, \text{ for some } i = 1, \ldots, \ell,$$

where the number $\lambda_i > 1$ specifies the just-noticeable difference for good $i = 1, \ldots, \ell$, i.e., the relative change in the amount of the good that allows the agent to differentiate between bundles. See also Figure 1 (left) on page 14.

Dziewulski (2020) considers a simplified version of Example 1 where, given some $\lambda > 1$, we have $x \rhd y$ if $x = \lambda' y$, for some $\lambda' \geq \lambda$. Thus, the relative amount of *all* goods must be sufficiently greater. We discuss the importance of this formulation in Section 6.2.

Example 2. Let X be a space of probability measures over $S \subseteq \mathbb{R}$. In such a case, one can model imperfect discrimination by specifying

$$x \rhd y \quad \text{if} \quad x \text{ first order stochastically dominates } y \text{ and } d(x, y) \geq \lambda,$$

for some $\lambda > 0$, where d is a metric on X. Hence, a lottery x is objectively superior to any probability distribution that is stochastically dominated and sufficiently distant from it. Alternatively, one could explore the idea of Rubinstein (1988) and impose conditions on ratios of probabilities and prizes that are sufficient to distinguish between lotteries.[15]

Note that $\rhd$-monotonicity is imposed on choices of the individual and *not* their preferences. This distinction will be critical in the remainder of the paper.[16]

2.2 Classic revealed preference analysis

To provide a better context for our main analysis, we present the basic revealed preference result that establishes the necessary and sufficient condition under which a dataset is rationalisable with (the exact) *utility maximisation*. Formally, we determine when there is a function $u : X \to \mathbb{R}$ such that the correspondence

$$c(A) \;:=\; \Big\{ x \in A : u(x) \geq u(y), \text{ for all } y \in A \Big\} \tag{3}$$

[15] Rubinstein (1988) discusses a model of choice over simple lotteries that assign a probability p to a monetary prize and $(1 - p)$ to receiving nothing. Roughly speaking, the lotteries are considered to be distinguishable if the ratios of either prizes or probabilities exceed a particular constant.

[16] It is critical to point out that Weber-Fechner law is a statistical property attributed to distributions of choices. In contrast, Examples 1 and 2 specify imperfect discrimination in deterministic terms. Since our setup assumes limited choice data, it would be impossible to falsify any form of stochastic choice and, thus, the statistical definition of just-noticeable difference would be vacuous. Instead, we interpret the law literally. As a result, our notions of just-noticeable difference capture an upper bound for the insensitivity of the subject, rather than its precise value.

rationalises the dataset $\mathcal{O}$ as in (1). Since any set of observations can be trivially rationalised with a constant function u, this question is vacuous. To remedy this, it is common strengthen the notion of rationalisability by imposing a form of strict monotonicity on u. In the remainder of this subsection, we require that u is strictly increasing with respect to the strict partial order $\rhd$; i.e., if $x \rhd y$ then $u(x) > u(y)$, for any $x, y \in X$. One can easily verify that, within utility maximisation models, this condition is equivalent to $\rhd$-monotonicity of c, thus, motivating our definition.[17]

Under what conditions on the dataset $\mathcal{O}$ one could rationalise it with a $\rhd$-monotone utility maximisation? It is convenient to address this question by referring to the revealed preference relations. An alternative x is *directly revealed preferred* to y, if there is an observation in which both x and y were available and x was chosen. Formally,

$$xR^*y \quad \text{if} \quad (A, x) \in \mathcal{O} \text{ and } y \in A.$$

We think of this relation in terms of weak preference.

To construct the strict counterpart of R^*, we employ the relation $\rhd$. An alternative x is *directly revealed strictly preferred* to y, if there is an observation in which x was chosen over something objectively better than y, i.e.,

$$xP^*y \quad \text{if} \quad (A, x) \in \mathcal{O} \text{ and } z \rhd y, \text{ for some } z \in A.$^{[18]}$$

One can quickly notice that the relations R^*, P^* are consistent with *any* utility u that rationalises the dataset $\mathcal{O}$ as in (3); i.e., xR^*y implies $u(x) \geq u(y)$, and xP^*y implies $u(x) > u(y)$.[19] Hence, an immediate testable restriction for utility maximisation is the following (GARP): For any sequence $z^1, \ldots, z^n$ of alternatives in X such that either $z^i R^* z^{i+1}$ or $z^i P^* z^{i+1}$, for all $i = 1, \ldots, (n-1)$, it may never be that $z^n P^* z^1$. Hence, there is no revealed preference cycle in which any two alternatives are strictly ordered. Otherwise, we would have $u(z^1) \geq \ldots \geq u(z^n)$ and $u(z^n) > u(z^1)$, yielding a contradiction. Theorem 2 in Nishimura et al. (2017) assures that, under some regularity conditions, this is also a sufficient condition for a dataset to be rationalisable in this sense.[20]

[17] Indeed, if $x \rhd y$ implies $u(x) > u(y)$ then the model in (3) must be $\rhd$-monotone. Conversely, suppose that $x \rhd y$. Whenever c is $\rhd$-monotone, then $y \notin c(\{x, y\})$, which requires $u(x) > u(y)$.

[18] Within the consumer demand framework à la Afriat, the relations R^* and P^* are equivalent to the revealed preferences as in Varian (1982). Let $X = \mathbb{R}^\ell_+$, $\rhd = >$, and $A = \{y \in \mathbb{R}^\ell_+ : p \cdot y \leq p \cdot x\}$, for some prices $p \in \mathbb{R}^\ell_{++}$, for any observation $(A, x) \in \mathcal{O}$. In such a case, xR^*y is equivalent to $p \cdot y \leq p \cdot x$, and xP^*y to $p \cdot y < p \cdot x$, for an observed choice x and some alternative y.

[19] Consistency of R^* with u is immediate. Whenever xP^*y then $(A, x) \in \mathcal{O}$ and $z \rhd y$, for some $z \in A$, which implies that $u(x) \geq u(z) > u(y)$, since xR^*y and u is $\rhd$-monotone.

[20] Although our notation differs, this condition is equivalent to *cyclical* $\unrhd$*-consistency* in Nishimura

3 The main results

We proceed with our main results. First, we focus on the relationship between acyclicity of P^* and approximate utility maximisation. Then, we discuss how to elicit preferences of the individual and perform welfare analysis in this setting.

3.1 Approximate utility maximisation

Suppose that only the directly revealed strict relation P^* conveys a reliable information about preferences of the individual, while the weak one R^* may be subject to imprecision, vagueness of judgement, or incommensurability. In such a case, it is possible to observe revealed preference cycles along which some alternatives are ordered with P^*. However, as we maintain transitivity of the strict preference, the relation P^* must be *acyclic*. That is, there is no sequence $z^1, z^2, \dots, z^n$ in X such that

$$z^1 P^* z^2,\ z^2 P^* z^3,\ \dots\ ,\ z^{n-1} P^* z^n,\ \text{and}\ z^n P^* z^1.$$

This excludes any revealed preference cycles that are induced by the revealed strict relation P^* alone. Although acyclicity of P^* remains necessary for the dataset to be rationalisable with utility maximisation, it is no longer sufficient, as it allows for cycles that are generated by the weak R^* and the strict P^* relations jointly.

Before exploring implications of an acyclic P^*, we impose a weak separability assumption on the relation $\rhd$, which is purely technical.

Assumption 1. There is a countable set $D \subseteq X$ such that, for any $x, y \in X$ satisfying $x \rhd y$, there is $z \in D$ for which either (i) $z \rhd y$, and $z' \rhd x$ implies $z' \rhd z$, for all $z' \in X$; or (ii) $x \rhd z$, and $z' \rhd z$ implies $z' \rhd y$, for all $z' \in X$.

This condition holds trivially whenever X is countable, since one can always choose $D = X$ and set $z = x$ or $z = y$, for any $x, y \in X$ satisfying $x \rhd y$. However, this assumption is indispensable when considering general spaces.

Theorem 1. *For any dataset $\mathcal{O}$ and a strict partial order $\rhd$ satisfying Assumption 1, the following statements are equivalent.*

(i) The directly revealed strict preference relation P^* is acyclic.

et al. (2017) for the preorder $\trianglerighteq := \rhd \cup \{(x, x) : x \in X\}$. Within the classic demand framework of Afriat, this coincides with the *generalised axiom of revealed preference* (or GARP). See Varian (1982).

(ii) There is a utility $u : X \to \mathbb{R}$ and a positive threshold function $\delta : X \to \mathbb{R}_+$ such that the choice correspondence $c : \mathcal{A} \rightrightarrows X$, given by

$$c(A) \;:=\; \Big\{x \in A : u(x) + \delta(y) \geq u(y),\ \textit{for all } y \in A\Big\}, \tag{4}$$

is $\rhd$-monotone and rationalises the set of observations $\mathcal{O}$. [Proof]

Acyclicity of the strict revealed preference relation P^* fully characterises the model of approximate utility maximisation in (4), where an alternative x is selected from a menu A if its utility is at most $\delta(y)$ utils lower than that of any other available option y. This appeals to the idea of imperfect discrimination. The individual discerns between two alternatives only if they yield a sufficiently different utility. Unless the difference between values attributed to x and y exceed the particular threshold, the agent behaves as if they were indifferent, even when the options are strictly ordered with u.

Perhaps somewhat surprisingly, although the agent may not maximise their utility, one can still learn about their true preferences. Theorem 1 follows from the fact that the revealed relation P^* is consistent with *any* utility u that rationalises the data as in (4); i.e., if xP^*y then $u(x) > u(y)$.[21] Indeed, recall that xP^*y only if $(A, x) \in \mathcal{O}$ and $z \rhd y$, for some $z \in A$. Since the data is rationalisable as in (4), it must be that $u(x) \geq u(z) - \delta(z)$. Moreover, $\rhd$-monotonicity requires that $y \notin c\big(\{y, z\}\big)$, which is true only if $u(z) - \delta(z) > u(y)$. Combining the two inequalities yields $u(x) > u(y)$.

It immediately follows that (ii) implies (i). If the relation P^* were not acyclic, there would be a sequence of alternatives $z^1, \ldots, z^n$ such that $u(z^1) > \ldots > u(z^n) > u(z^1)$, leading to a contradiction. Showing the converse is more demanding and postponed until the Appendix. Our argument consists of two steps. First, we show that whenever the strict relation P^* is acyclic, there is a utility function u such that either $x \rhd y$ or xP^*y imply $u(x) > u(y)$. This (and only this) part of the proof requires Assumption 1. The second step is summarised in the following proposition.

Proposition 1. *For any set of observations $\mathcal{O}$, any strict partial order $\rhd$, and any utility $u : X \to \mathbb{R}$ the following statements are equivalent.*

(i) If $x \rhd y$ or xP^*y then $u(x) > u(y)$, for any $x, y \in X$.

[21] However, this is no longer true for the weak relation R^*, capturing the idea that only the strict relation P^* reflects the true preferences of the individual.

(ii) There is a positive threshold function $\delta : X \to \mathbb{R}_+$ such that the correspondence $c : \mathcal{A} \rightrightarrows X$ in (4) is $\rhd$-monotone and rationalises the dataset $\mathcal{O}$. [Proof]

This result is of interest in itself. Theorem 1 specifies the necessary and sufficient condition under which there *exists* a utility u that approximately rationalises the data as in (4). However, in many applications the researcher is interested whether choices of the individual are consistent with a particular function u. For example, whenever X is the space of ℓ-dimensional consumption bundles, it may be desirable to determine if there is a concave function u that rationalises the data. Alternatively, if X is the space of lotteries, one may be interested if u admits the expected utility specification. Proposition 1 stipulates that any such test is equivalent to verifying if the particular utility u is consistent with $\rhd$ and P^*. In Section 6.1 we employ this result to study preferences over state-contingent consumption under risk.

3.2 Continuous approximate utility maximisation

Given the generality of our setup, Theorem 1 does not specify any particular properties of the function u that rationalises the data as in (4). Specifically, the utility need not be continuous, even in well-behaved choice environments.

Example 3. Let $X = \mathbb{R}^2_+$ and $\mathcal{O} = \big\{(A^1, x^1), (A^2, x^2)\big\}$, with $A^t := \big\{y \in X : p^t \cdot y \leq 1\big\}$, for $t = 1, 2$ and $p^1 = (1, 2)$, $p^2 = (2, 1)$, $x^1 = (0, 1/2)$ $x^2 = (1/3, 1/3)$. Moreover, let $\rhd = \gg$, which satisfies Assumption 1.[22] It is easy to check that this dataset is rationalisable as in (4). Since $p^2 \cdot x^1 < 1$, there is $z \in A^2$ such that $z \gg x^1$. Thus, we have $x^2 P^* x^1$. Given that $p^1 \cdot x^2 = 1$, it is *not* true that $x^1 P^* x^2$, which suffices for the relation P^* to be acyclic.[23] Therefore, there is a utility u and a threshold δ that rationalise the dataset as in (4). However, any such function u must be discontinuous at x^2.

Indeed, $x^2 P^* x^1$ and $x^1 R^* x^2$ imply $u(x^2) > u(x^1)$ and $u(x^1) \geq u(x^2) - \delta(x^2)$, respectively, which requires that $\delta(x^2) > 0$. Take any sequence $\{z^n\}$ converging to x^2 such that $x^2 \rhd z^n$, for all n. By $\rhd$-monotonicity, it must be that $u(x^2) - u(z^n) > \delta(x^2)$, for all n, which holds for a *continuous* utility u only if $\delta(x^2) = 0$.

Given the importance of continuity for establishing non-emptiness of the set $c(A)$

[22] See Lemma 4.1 in Peleg (1970).

[23] Since $x^1 R^* x^2 P^* x^1$, this set is *not* rationalisable with an *exact* utility maximisation.

or eliciting preferences from limited data (see Chambers et al., 2021), it is desirable to determine conditions under which there is a continuous rationalisation.

Assumption 2 (Continuity)**.** Suppose that X is a locally compact and separable metric space, and the strict partial order $\rhd$ satisfies the following conditions:

(i) The set (preorder) $\rhd \cup \{(x, x) : x \in X\}$ is closed.

(ii) For any compact $Z \subseteq X$, the set $\{x \in X : z \rhd x, \text{ for some } z \in Z\}$ is compact.

These continuity restrictions on X and $\rhd$ are sufficient to prove that acyclicity of P^* is equivalent to a *continuous* approximate rationalisation.

Proposition 2. *Let X and $\rhd$ satisfy Assumption 2, and the menu A be compact, for all $(A, x) \in \mathcal{O}$. The relation P^* is acyclic if, and only if, there is a continuous utility u that rationalises $\mathcal{O}$ as in (4), for some positive threshold function δ.* [Proof]

The necessity part follows immediately from Theorem 1, since it is independent of any ancillary assumptions. To prove the converse, we apply Levin's Theorem (see Levin, 1983 or the appendix in Nishimura et al., 2017) to show that acyclicity of P^* is sufficient for existence of a *continuous* utility u such that either $x \rhd y$ or xP^*y implies $u(x) > u(y)$, for any $x, y \in X$. The rest follows from Proposition 1.

The dataset in Example 3 can not be rationalised with approximate utility maximisation for a continuous function u precisely because the relation $\rhd$ violates Assumption 2 — specifically part (ii).[24] In contrast, the assumption is satisfied by the relation in Example 1. Thus, one would be able to rationalise the data with a continuous utility u if they considered a weaker form of choice monotonicity. We discuss this further in Section 4.

3.3 Recovering preferences from almost optimal choices

In Theorem 1 we established that acyclicity of the strict revealed preference relation P^* fully characterises approximate utility maximisation. In this subsection we turn to an alternative question: Assuming that the observed choices are generated by such a model, how can we estimate the true preferences u of the individual?[25]

[24] Indeed, Assumption 2(ii) is critical. Suppose that $\rhd = >$. Although it obeys Assumption 2(i), it is not sufficient to rationalise the dataset in Example 3 with a *continuous* $\rhd$-monotone approximate utility maximisation. Nevertheless, in the Online supplement we show that, given this specification of $\rhd$, the dataset is approximately rationalisable with an upper semi-continuous utility.

[25] Our question is analogous to the one discussed in Varian (1982), Halevy et al. (2017), or Nishimura et al. (2017) regarding the exact utility maximisation.

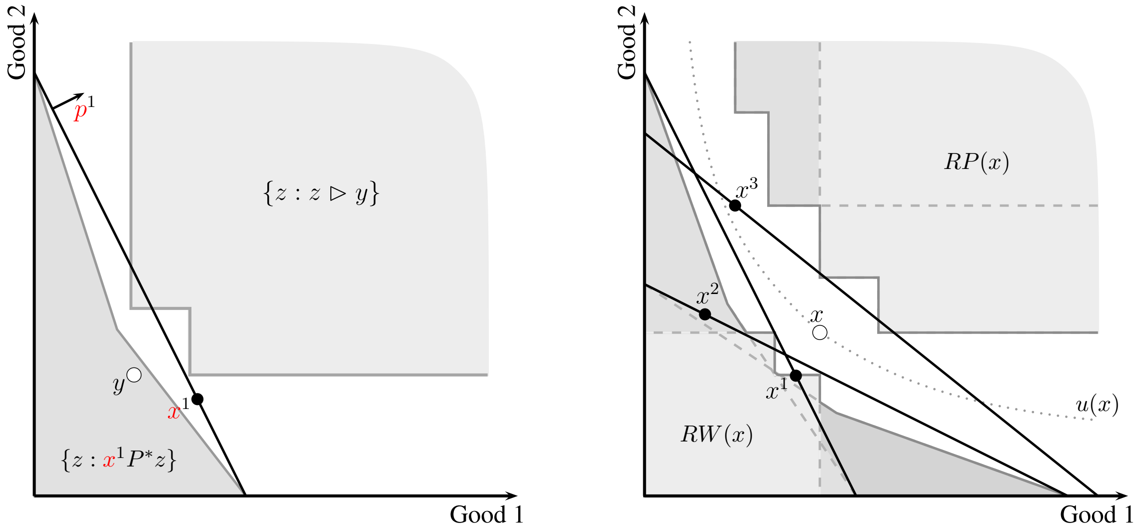

Figure 1: Revealed worst and revealed preferred set for x.

Throughout this section we take the dataset $\mathcal{O}$ and the relation $\rhd$ as the premise. Moreover, we assume that $\mathcal{O}$ is rationalisable with a $\rhd$-monotone approximate utility maximisation as in (4), for some unobserved utility u and threshold δ. By P^* we denote the directly revealed strict preference relation, defined in Section 2.

It is convenient to refer to the notion of the *revealed strict preference relation* P, i.e., the transitive closure of P^*. Formally, we have xPy whenever there is a sequence of alternatives $z^1, z^2, \ldots, z^n$ in X such that $z^1 = x$, $z^n = y$, and

$$z^1 P^* z^2, \; z^2 P^* z^3, \; \ldots \;, \; z^{n-2} P^* z^{n-1}, \; \text{and} \; z^{n-1} P^* z^n.$$

Obviously, the directly revealed relation P^* is acyclic if, and only if, its transitive closure P is irreflexive, i.e., we have *not* xPx, for all $x \in X$. By Theorem 1, this is equivalent to the data being rationalisable as in (4).

We proceed with our discussion on recoverability of preferences. Take an arbitrary alternative $x \in X$, not necessarily observed in the dataset. First, we are interested in evaluating the set of all alternatives that are strictly inferior to x with respect to the latent utility u. Define the *revealed worst set* by

$$RW(x) \;:=\; \Big\{ y \in X : xPy; \; \text{or } x \rhd y; \; \text{or } x \rhd z \text{ and } zPy, \; \text{for some } z \in X \Big\}.$$

Consider the dataset depicted in Figure 1 (right), where the consumption space is $X = \mathbb{R}^2_+$ and each observed menu is given by $A^t = \{ y \in X : p^t \cdot y \leq 1 \}$, for some prices $p^t \in \mathbb{R}^2_{++}$ and $t = 1, 2, 3$. Moreover, let $\rhd$ be given as in Example 1, i.e., $x \rhd y$ if $x \geq y$

and $x_i \geq \lambda_i y_i$, for some $i = 1, 2$ and $\lambda_1, \lambda_2 > 1$. The lower gray area represents the set $RW(x)$ for the bundle x. Indeed, the set contains all elements y such that $x \rhd y$. In particular, this includes x^1. Therefore, any alternative that is revealed strictly inferior to x^1 also belongs to $RW(x)$, i.e., any $y \in X$ such that $x^1 P^* y$ or $x^1 P^* x^2 P^* y$.

Analogously, one can define the *revealed preferred set* as

$$RP(x) \;:=\; \Big\{ y \in X : yPx; \text{ or } y \rhd x; \text{ or } y \rhd z \text{ and } zPx, \text{ for some } z \in X \Big\}.$$

Revisit Figure 1, where the revealed preferred set is represented by the top shaded area. Clearly, the set includes every element y that satisfies $y \rhd x$. Moreover, since there is some $z \in A^3$ such that $z \rhd x$, we have $x^3 P^* x$. Hence, both x^3 and any $y \rhd x$ are also in the set. The next result follows immediately from the definitions of the two sets.

Corollary 1. *For any $x, y \in X$, we have $y \in RW(x)$ if, and only if, $x \in RP(y)$.*

The two sets are essential in estimating the unobserved preferences from the data. In particular, $RP(x)$ provides a tight bound for the set of all alternatives that are strictly preferable to x with respect to the unobserved utility u. We formalise this below. Before stating the result, define $V_u(x) := \{y \in X : u(y) > u(x)\}$, for any utility function $u : X \to \mathbb{R}$. Moreover, let $NRW(x)$ be the complement of $RW(x)$.

Theorem 2. *For any alternative $x \in X$ and any utility u that rationalises the dataset $\mathcal{O}$ as in (4) for some threshold δ, we have $RP(x) \subseteq V_u(x) \subseteq NRW(x)$. Moreover, under Assumption 1, we have $y \notin RP(x)$ and $y \neq x$ only if $\mathcal{O}$ is rationalisable as in (4) for a utility u satisfying $u(x) > u(y)$ and some threshold δ.* [Proof]

Following Proposition 1, it is clear that $y \in RP(x)$ implies $u(y) > u(x)$, for any utility that rationalises $\mathcal{O}$ as in (4). Analogously, since $y \in RW(x)$ implies $u(y) < u(x)$, the set $V_u(x)$ must be nested in $NRW(x)$. Thus, the sets $RW(x)$ and $RP(x)$ are indeed bounds for the true preferences, since any indifference curve intersecting x must belong to the complement of $RW(x) \cup RP(x)$. In addition, the bounds are *tight* in the following sense: If y is not in $RP(x)$, one can always rationalise the data with an approximate maximisation of some utility u that ranks x strictly above y. Thus, it is impossible to improve the estimate without excluding some preferences that could explain the data. By Corollary 1, an analogous result holds for the set $RW(x)$.

Remark 1. Theorem 2 can be extended to the class of *continuous* utility functions u. Suppose that the menu A is compact, for all observations $(A, x) \in \mathcal{O}$. Through a combination of the arguments supporting Proposition 2 and Theorem 2, one can show that under Assumption 2, we have $y \notin RP(x)$ and $y \neq x$ only if the dataset is rationalisable as in (4) with a *continuous* utility u satisfying $u(x) > u(y)$.

3.4 Robust welfare analysis

Theorem 2 shows how to estimate the unobserved utility u that rationalises the data as in (4). This allows us to partially rank alternatives in X with respect to the true yet unobserved preferences of the individual. However, when performing welfare analysis it is much more natural to compare sets of alternatives rather than particular options. For example, when evaluating different tax structures, one is interested in ranking budget sets the consumer would face under each regime. Here, we introduce and characterise an intuitive ordering over menus that allows us to make meaningful, data-driven welfare statements under approximate utility maximisation.

The main difficulty in evaluating welfare within our framework follows from the separation of choice (guided by the choice model c in (4)) from the agent's well-being (summarised by the utility u). Suppose that choices of the consumer are determined with the correspondence $c(A) := \big\{x \in A : u(x) + \delta(y) \geq u(y),\ \text{for all } y \in A\big\}$, for some utility u and threshold δ. For the time being, we assume that the two functions are known. For any two menus $A, A' \subseteq X$, the set A' is *preferred* to A if any choice from A' is strictly preferable to any choice from A with respect to the utility u. Formally, for any $x \in c(A')$ and $y \in c(A)$, we have $u(x) > u(y)$.[26] Although we accept that the agent may choose options that are not maximising their utility u exactly (due to imperfect discrimination, imprecision, or satisficing) we identify welfare with their true preferences.

Since the dataset $\mathcal{O}$ is finite and incomplete, it can be supported by multiple functions u, δ. We address this issue by focusing on a robust comparison over menus. As in the previous section, we take the dataset $\mathcal{O}$ and the relation $\rhd$ as the premise. Moreover, we assume that the set $\mathcal{O}$ is rationalisable as in (4), for some unobserved utility u and threshold δ. For any two menus $A, A' \in \mathcal{A}$ (not necessarily observed in the data), we say

[26] Unlike for the exact utility maximisation, under approximate utility maximisation the elements of the choice set $c(A)$ may be assigned different values of the utility u.

that A' is *robustly preferred* to A, if for *any* functions u, δ that rationalise $\mathcal{O}$ as in (4), the set A' is preferred to A in the sense defined previously. Therefore, for any model of approximate utility maximisation that is consistent with the data, any choice from A' has to be superior to any choice from A with respect to the utility u.

In this subsection we characterise this robust ordering over menus by employing the revealed preference relations P^* and P. First, for an arbitrary menu A (not necessarily observed in the dataset), we identify the set of all possible choices from A that would be consistent with the set of observations $\mathcal{O}$. For any menu $A \in \mathcal{A}$, define

$$S(A) \;:=\; \Big\{y \in A : z \not P^* y, \text{ for all } z \in A; \text{ and } xPy \text{ implies } z \not P^* x, \text{ for all } z \in A\Big\},$$

where P denotes the revealed strict preference relation induced by $\mathcal{O}$, i.e., it is the transitive closure of the directly revealed strict preference relation P^*.

Proposition 3. *Under Assumption 1, for any $A \in \mathcal{A}$ and $y \in A$, the hypothetical dataset $\mathcal{O} \cup \{(A, y)\}$ is rationalisable as in (4) if, and only if, $y \in S(A)$.* [Proof]

The above result is of interest in itself. It states that the set $S(A)$ contains all (both within and out-of-sample) choices from the menu A that are consistent with the dataset $\mathcal{O}$. Hence, it contains all predictions consistent with the data. This is particularly useful when performing a counterfactual analysis.

Given the generality of our setup, we can not guarantee that the set $S(A)$ is non-empty, for all $A \in \mathcal{A}$. However, it is easy to show that this is always true when A is finite. Similarly, whenever the dataset $\mathcal{O}$ is rationalisable as in (4) for a continuous function u, then $S(A)$ is non-empty for any compact A. See Remark 2 below.

Theorem 3. *Under Assumption 1, for any menus $A, A' \in \mathcal{A}$, the set A' is robustly preferable to A if, and only if, for any $y \in A$, either* (i) *$z \not P^* y$, for all $z \in A'$; or* (ii) *zPy implies $z' \not P^* z$, for all $z' \in A'$; or* (iii) *xPy, for all $x \in S(A')$.* [Proof]

The robust comparison over menus is partial and, in general, does not rank any two sets of alternatives. In fact, unlike for the exact utility maximisation, it is possible that two menus A, A' are unordered, even when A is a subset of A'. Since choices are not necessarily maximising the utility u, there may be alternatives in A that are strictly preferable to some options selected from the set A'. However, once A' dominates A in the robust sense, any alternative that would be selected from A is inferior to any alternative chosen from A', even when the individual fails to maximise their utility.

Remark 2. Suppose that Assumption 2 is satisfied and the menu A is compact, for each observation $(A, x) \in \mathcal{O}$. Since $\mathcal{O}$ is rationalisable with a $\rhd$-monotone approximate utility maximisation, Proposition 2 guarantees that the corresponding utility u is continuous, without loss. In particular, this suffices for the set $S(A)$ to be non-empty, for any compact menu A. By combining the arguments supporting Proposition 2 and Theorem 3, one can prove the following: *For any compact menus $A, A' \in \mathcal{A}$, the set A' is robustly preferable to A if, and only if, for any* continuous *utility u and some threshold δ that rationalise $\mathcal{O}$ as in (4), the set A' is preferred to A in the sense defined previously.*

4 Imperfect discrimination

It is most common in the revealed preference analysis to assume that choices of individuals satisfy a strong form of monotonicity. Whether these are strict orderings $>, \gg$ over the space of bundles, or first order stochastic dominance over the space of lotteries, the researcher requires that the objectively better alternative is always chosen over the inferior one, even if the difference between them is infinitesimal. Undoubtedly, such forms of monotonicity have a great normative appeal. However, there is growing empirical evidence suggesting that such a requirement may be too demanding.

Sippel (1997) presents an experimental study of consumer choice within the standard Afriat-like framework, in which subjects were making purchases of multiple consumption goods subject to various budget constraints. Even though the individuals were incentivised to exhaust their budgets, a significant number of them failed to do so, thus, directly violating that "more is better".[27] More recently, Nielsen and Rehbeck (2022) reported direct violations of first order stochastic dominance in choices over lotteries. In their experimental study, 90% of subjects expressed the desire to obey first order stochastic dominance, yet 85% of those violated the condition at least once in the subsequent choice experiment. This is in line with Dembo et al. (2021), who find that violations of the expected utility theory are caused predominantly by inconsistencies of choice with

[27] Unlike Sippel (1997), other experimental studies that employ an Afriat-like setup restrict choices to the budget line *only*. Therefore, the design makes it impossible to observe direct violations of strict monotonicity. See, e.g., Harbaugh et al. (2001), Andreoni and Miller (2002), Choi et al. (2007), Fisman et al. (2007), Andreoni and Sprenger (2012a,b), Ahn et al. (2014), Choi et al. (2014), Halevy et al. (2018), Echenique et al. (2019), Zrill (2020), Cappelen et al. (2021), and Dembo et al. (2021). Similarly, one could not violate stochastic dominance directly in Feldman and Rehbeck (2022).

first order stochastic dominance, rather than independence.

We do not postulate that, based on this evidence, one should abandon the idea of monotonicity entirely. However, when studying choices that involve small stakes, like in experimental settings or day-to-day consumption decisions, it may be sensible to consider weaker forms of monotonicity that admit some level of insensitivity to small differences among alternatives, and describe the observed behaviour more accurately.

As pointed out in Section 2, imperfect discrimination is well documented in the psychophysics literature.[28] Individuals perceive differences between intensities of a physical stimulus (e.g., light, touch, sound) only if they are significantly different; according to the well-established Weber-Fechner law, this occurs whenever the ratio of their intensities exceeds a particular constant, the so-called *just-noticeable difference*. Interestingly, following Dehaene (2008), the same logarithmic law applies to human perception of quantities and numerosities, which is particularly relevant in economic decision making. Since such a behaviour occurs even in incentivised experiments, it seems to be a physiological phenomenon, independent of the true preferences of individuals.

We consider our approach to be well-suited for the study of imperfect discrimination within the revealed preference framework. First of all, the model separates choices of the subject, captured by the correspondence c in (4), from their actual preferences summarised by the corresponding utility u. In addition, the notion of $\rhd$-monotonicity is imposed on the choice model c, rather than the corresponding utility u. Although the choices of the decision-maker may be subject to some degree of insensitivity to differences among alternatives (e.g., represented by the relations $\rhd$ in Examples 1 and 2) this does *not* preclude the function u from being increasing in a stronger sense, capturing the normative affinity for even the most infinitesimal increases in consumption or improvement of odds in a gamble. This contrasts with the exact utility maximisation, where monotonicity of choice and preferences always coincide (recall the discussion in Section 2.2). Most importantly, despite the separation, one can still elicit the utility u from the data and make welfare statements, as shown in Section 3.[29]

The separation of choice and preferences has a footing in empirical evidence. The aforementioned experiment in Nielsen and Rehbeck (2022) shows a systematic inconsistency between the decision-theoretic rules that individuals consider to be desirable

[28] See Gescheider (1997) for a handbook treatment of this topic.

[29] See Section B.1.2 in the Online supplement for additional discussion.

(including first order stochastic dominance) and their actual choices. The authors conclude that "even though individuals may want to follow [stochastic dominance], this may not translate to them making choices consistent with it even when given an explanation of how the axiom applies to a decision problem."[30] The descriptive and normative aspects of consumer choice are disjoint, which can be captured in our approach.

We do not deny that, with sufficient care and attention, individuals are capable of identifying which alternative is strictly greater in the particular sense, even when the difference between them is infinitesimal. Rather, we hypothesise that when it comes to every-day consumption or choices with small stakes in experimental settings, decisions may follow intuitive judgements based on approximate quantities involved.[31] Violations of strict monotonicity could also result from unobserved mental or physical costs of switching from an inferior to a dominant alternative, as subjects may not find the change worthwhile, unless it yields sufficiently more utility. This is in line with the idea of satisficing by Simon (1947). In the next section we discuss the relationship between the latter and approximate utility maximisation more closely.

5 Interval orders, satisficing, and acyclicity

In this section we compare the model of approximate utility maximisation to related models of consumer choice that admit non-transitive indifferences.

5.1 Interval orders

Approximate utility maximisation is tightly related to the notion of interval orders introduced in Wiener (1914) and Fishburn (1970). An *interval order* is a binary relation $\succ$ over the consumption space X that is irreflexive, i.e., $x \not\succ x$, for all $x \in X$, and satisfies the interval order condition (or Ferrer's property), i.e., if $x \succ y$ and $x' \succ y'$ then either $x \succ y'$ or $x' \succ y$, for any $x, x', y, y' \in X$. Fishburn (1970) shows that any interval order defined over a *countable* space X can be represented by a utility u and a positive threshold δ as follows: $x \succ y$ if, and only if, $u(x) + \delta(y) > u(y)$.[32]

[30] Nielsen and Rehbeck (2022, p. 2252). Similar distinction between choice and preferences of a decision-maker is discussed in Mandler (2005), Nishimura (2018), and Nishimura and Ok (2020).

[31] See Dehaene (2008).

[32] Fishburn (1973) and Bridges (1985, 1986) specify conditions under which interval orders admit such a representation over a general space X.

It is straightforward to show that whenever a set of observations is rationalisable with a $\rhd$-monotone approximate utility maximisation for some functions u, δ, there is an interval order $\succ$ such that the correspondence $c : \mathcal{A} \rightrightarrows X$, given by

$$c(A) \;:=\; \Big\{x \in A : y \not\succ x, \text{ for all } y \in A\Big\}, \tag{5}$$

is $\rhd$-monotone and rationalises the data. Therefore, under Assumption 1 and/or 2, acyclicity of P^* is sufficient for the data to be rationalisable with an interval order maximisation. Below we claim that it is also necessary.

Proposition 4. *For any dataset $\mathcal{O}$ and any strict partial order $\rhd$, there is an interval order $\succ$ such that the correspondence c in (5) is $\rhd$-monotone and rationalises $\mathcal{O}$ only if the directly revealed strict preference relation P^* is acyclic.* [Proof]

This result complements Fishburn (1975) that characterises choice correspondences generated by an interval order maximisation under the assumption that the researcher observes the *entire* set $c(A)$ for *all* possible menus $A \in \mathcal{A}$. In contrast, we assume that the data are incomplete. Our result requires no assumptions on the space X, dataset $\mathcal{O}$, or the relation $\rhd$. In particular, since we allow for the consumption space X to be uncountable, the interval order $\succ$ may not have a representation as in Fishburn (1970). Moreover, there is no direct relation between the revealed preference P^* and the interval order $\succ$ supporting the data. Specifically, xP^*y does *not* imply $x \succ y$.

5.2 Satisficing

In some settings, approximate utility maximisation can be interpreted in terms of satisficing à la Simon (1947), where the individual selects alternatives that are "good enough" with respect to some criterion.[33] Formally, a choice correspondence $c : \mathcal{A} \rightrightarrows X$ represents the *satisficing* behaviour if there is a utility $u : X \to \mathbb{R}$ such that $x \in c(A)$ and $u(y) \geq u(x)$ implies $y \in c(A)$, for any $y \in A$ and $A \in \mathcal{A}$

One can easily verify that approximate utility maximisation is a special case of satisficing. Indeed, suppose that $c(A) = \big\{x \in A : u(x) + \delta(y) \geq u(y), \text{ for all } y \in A\big\}$, for some utility u and threshold function δ. Since $x \in c(A)$ implies $u(x) \geq u(z) - \delta(z)$, for all $z \in A$, then $u(y) \geq u(x)$ only if $y \in c(A)$, for any $y \in A$. By Theorem 1, it immediately

[33] See also Tyson (2008).

follows that, for any dataset $\mathcal{O}$ and a strict partial order $\rhd$ satisfying Assumption 1, acyclicity of the revealed preference relation P^* is sufficient for the observations to be rationalisable with a $\rhd$-monotone satisficing behaviour.

The converse is *not* true. Suppose that $X = \{a, b, c, d\}$ and $b \rhd a$, $d \rhd c$, with other pairs being incomparable. Hence, since X is finite, the relation $\rhd$ is a strict partial order satisfying Assumption 1. Consider the dataset $\mathcal{O}$ consisting of observations $\big(\{a, d\}, a\big)$ and $\big(\{b, c\}, c\big)$. Since aP^*c and cP^*a, the set $\mathcal{O}$ is not rationalisable as in (4). Nevertheless, it is consistent with a $\rhd$-monotone satisficing behaviour.[34]

Although, in general, the testable implications of the two models differ, there is an important class of choice environments in which they are indistinguishable.

Proposition 5. *Take a dataset $\mathcal{O}$ and a partial order $\rhd$ obeying Assumption 1, and suppose that $z \rhd y$ and $z \in A$ implies $y \in A$, for all $(A, x) \in \mathcal{O}$ and $y \in X$. Then, the set $\mathcal{O}$ is rationalisable with a $\rhd$-monotone approximate utility maximisation as in (4) if, and only if, it is rationalisable with a $\rhd$-monotone satisficing behaviour.* [Proof]

The additional assumption in the proposition is satisfied in various choice environments. Suppose that $X = \mathbb{R}^\ell_+$ and the menu A is downward comprehensive, for each observation $(A, x) \in \mathcal{O}$.[35] Specifically, this holds within the classical consumer demand setting à la Afriat and in the general framework of Forges and Minelli (2009). In addition, if $x \rhd y$ implies $x \geq y$, for all $x, y \in X$, then the assumption in Proposition 5 is satisfied. This includes the relation in Example 1, as well as the specification in Dziewulski (2020). In either case, the testable implications of approximate utility maximisation and satisficing are equivalent. We explore this further in Section 6.2.

5.3 On acyclicity

Although tempting, one should *not* equate acyclicity of the directly revealed preference relation P^* with the dataset being rationalisable with an *acyclic* relation. Formally, the dataset $\mathcal{O}$ is rationalisable in this sense if there is an acyclic relation $\succ$ such that the correspondence in (5) is $\rhd$-monotone and rationalises the set $\mathcal{O}$.

[34] For example, take any utility u such that $u(b) > u(a) > u(d) > u(c)$, and a $\rhd$-monotone correspondence c satisfying $c\big(\{a, d\}\big) = \{a\}$ and $c\big(\{b, c\}\big) = \{b, c\}$.

[35] A set $A \subseteq X \subseteq \mathbb{R}^\ell$ is *downward comprehensive* if $x \in A$ and $y \leq x$ implies $y \in A$, for all $y \in X$, where $\geq$ denotes the coordinate-wise ordering.

Since any interval order is a strict partial order, Proposition 4 guarantees that acyclicity of P^* is a sufficient condition for such a form of rationalisation. However, acyclicity of P^* is not a necessary condition, as we highlight in the result below.[36]

Proposition 6. *For any dataset $\mathcal{O}$ and a strict partial order $\rhd$, these are equivalent.*

(i) The directly revealed strict preference relation P^* is irreflexive.

(ii) There is an acyclic relation $\succ$ such that the correspondence c in (5) is $\rhd$-monotone and rationalises the set of observations $\mathcal{O}$.

(iii) There is a strict partial order $\succ$ such that the correspondence c in (5) is $\rhd$-monotone and rationalises the set of observations $\mathcal{O}$.

(iv) There is a complete and quasitransitive relation $\succeq$ such that the correspondence $\hat{c} : \mathcal{A} \rightrightarrows X$, given by $\hat{c}(A) := \big\{x \in A : x \succeq y, \text{ for all } y \in A\big\}$, is $\rhd$-monotone and rationalises the set of observations $\mathcal{O}$.[37]

(v) The set $\mathcal{O}$ is rationalisable with a $\rhd$-monotone choice correspondence c. [Proof]

A dataset is rationalisable with an acyclic relation if, and only if, the relation P^* is irreflexive. Clearly, this is a weaker condition than acyclicity of P^* required in Theorem 1. Moreover, by statements (ii)–(iv), acyclic rationalisation is observationally equivalent to maximisation of a strict partial order and a complete, quasitransitive relation. However, as pointed out in (v), the empirical content of either model is limited to consistency of the observed choices with the pre-imposed objective ordering $\rhd$. This highlights the triviality of such representations. Unlike approximate utility maximisation, non-transitive indifferences are too weak to be useful in any analysis.

6 Further applications

We conclude this paper with two more applications of our main results.

6.1 State-contingent consumption under risk

Proposition 1 implies that any utility u that is consistent with the relation $\rhd$ and the directly revealed strict preference relation P^* can rationalise the set of observations as

[36] This is an extended version of Theorem 2.21 in Chambers and Echenique (2016).

[37] The relation $\succeq$ is *complete* if either $x \succeq y$ or $y \succeq x$, for all $x, y \in X$. The relation is *quasitransitive* whenever its asymmetric (strict) part $\succ$ is transitive.

in (4), for some threshold function δ. Since, in general, the relations $\rhd$ and P^* induce an infinite number of binary comparisons, verifying whether a utility u is consistent with either of them may be difficult. In this subsection we apply Proposition 1 to an important class of preferences over state-contingent consumption under risk. We extend the method of *generalised restriction of infinite domains* (GRID) by Polisson et al. (2020) to show that, within a broad class of models, checking for consistency with $\rhd$ and P^* can be reduced to a finite number of comparisons (or even a linear program).

Suppose there is a finite set of states $S = \{1, 2, \ldots, \ell\}$ and the probability π_s of each state $s \in S$ is known to the consumer and the observer. The contingent consumption space is $X = \mathbb{R}^{\ell}_{+}$, where the s'th entry x_s of the vector $x \in X$ denotes the consumption level in the state $s \in S$. A set of observations is given by $\mathcal{O} = \big\{(A^t, x^t) : t \in T\big\}$, where $x^t \in A^t$ denotes the state-contingent consumption bundle selected from the menu A^t. Here we require that A^t is bounded, for all $t \in T$.

Choices over contingent consumption were studied in, e.g., Choi et al. (2007, 2014) Ahn et al. (2014), Halevy et al. (2018), Zrill (2020), Cappelen et al. (2021), and Dembo et al. (2021).[38] In these particular experiments, the subjects were making multiple choices from budget lines $A^t = \big\{y \in \mathbb{R}^{\ell}_{+} : p^t \cdot y = 1\big\}$, given state-contingent prices $p^t \in \mathbb{R}^{\ell}_{++}$, for all $t \in T$, making it similar to the classic Afriat-like setup. Nevertheless, the following approach is applicable to arbitrary bounded menus.

In this subsection we employ Theorem 1 and Proposition 1 to provide an easy-to-apply test for approximate utility maximisation as in (4), where the corresponding function u is given by a particular formulation of risk preference. Many such utilities can be represented as $u(y) := F\big(v(y_1), v(y_2), \ldots, v(y_\ell)\big)$, where $v : \mathbb{R}_+ \to \mathbb{R}_+$ is a Bernoulli function and $F : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ is an aggregator. For example, given the state probabilities π_s, for all $s \in S$, the expected utility formulation is

$$u(y) \;=\; F\big(v(y_1), v(y_2), \ldots, v(y_\ell)\big) \;=\; \sum_{s=1}^{\ell} \pi_s v(y_s), \tag{6}$$

where the aggregator F takes the form $F(z) = \sum_{s=1}^{\ell} \pi_s z_s$, for $z \in \mathbb{R}^{\ell}_{+}$. Similarly, the model of rank dependent expected utility in Quiggin (1982) and disappointment aversion preferences in Gul (1991) admit such a representations for a particular aggregator F. See Section I.D in Polisson et al. (2020) for details.

[38] See also Gneezy and Potters (1997) and Hey and Pace (2014).

For simplicity, we focus on the case where the aggregator F is the same across all observations $t \in T$. Clearly, this is not without loss of generality. For example, when studying the expected utility as in (6), this would require that state probabilities π_s remain constant across all observations. Nevertheless, our result can be easily generalised to accommodate a variable aggregator F, as shown in the Online supplement. Below we extend Theorem 1 in Polisson et al. (2020) to approximate utility maximisation over state-contingent consumption. Let $\mathcal{X} := \{0\} \cup \big\{x_i^t : \text{ for some } i = 1, \ldots, \ell \text{ and } t \in T\big\}$ be the finite set of all consumption levels observed in the dataset and 0.

Proposition 7. *For any dataset $\mathcal{O} = \big\{(A^t, x^t) : t \in T\big\}$ with bounded menus A^t, for all $t \in T$, a continuous and strictly increasing aggregator F,*[39] *and a relation $\rhd$ such that $y \rhd x$ implies $y > x$, for any $x, y \in X$, the following statements are equivalent.*

(i) There is a strictly increasing Bernoulli function $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mathcal{O}$ is rationalisable as in (4) for the utility $u(y) := F\big(v(y_1), v(y_2), \ldots, v(y_\ell)\big)$ and some threshold function δ. Moreover, v is upper-semicontinuous without loss.[40]

(ii) There is a strictly increasing function $\bar{v} : \mathcal{X} \to \mathbb{R}_+$ satisfying

$$F\big(\bar{v}(x_1), \bar{v}(x_2), \ldots, \bar{v}(x_\ell)\big) \;>\; F\big(\bar{v}(y_1), \bar{v}(y_2), \ldots, \bar{v}(y_\ell)\big),$$

for any $x, y \in \mathcal{X}^\ell$ such that xP^*z and $z \geq y$, for some $z \in X$.

We postpone the proof until the Online supplement. In order to verify if the data is rationalisable as in (4) for a utility $u(y) := F\big(v(y_1), v(y_2), \ldots, v(y_\ell)\big)$, for some Bernoulli function v, it suffices to check if it is rationalisable over the finite grid $\mathcal{X}^\ell$. This simplifies the test significantly and, in the case of expected utility, rank dependent expected utility, and disappointment aversion, reduces it to a linear program.[41]

Proposition 7 crucially depends on the assumption that $y \rhd x$ implies $y > x$, for all $x, y \in X$. Clearly, this is satisfied by the mappings in Example 1 and the correspondence discussed in Dziewulski (2020). Otherwise, we impose no restrictions on $\rhd$. In particular, neither of the assumptions presented in Section 3 are required for this result to hold. Whenever the condition is violated, Proposition 7 is not applicable, and consistency of the function u with $\rhd$ and P^* has to be verified differently.

[39] A function $F : X \to \mathbb{R}$ defined over $X \subseteq \mathbb{R}^\ell$ is *strictly increasing* if $x > y$ implies $F(x) > F(y)$.
[40] The function v is *upper semi-continuous* if the set $\big\{y \in \mathbb{R}_+ : v(y) \geq a\big\}$ is closed, for any number a.
[41] This can be shown by re-purposing the approach in Sections I.B and I.D in Polisson et al. (2020).

As it was pointed out in Section 3, it is not always possible to approximately rationalise a set of observations with a continuous function u. Similarly, Proposition 7 does not guarantee that the Bernoulli function v and, thus, $y \to F\big(v(y_1), v(y_2), \ldots, v(y_\ell)\big)$ are continuous. In the Online supplement, we show that whenever the menu A^t is compact, for each observation $t \in T$, and the correspondence $\rhd$ satisfies Assumption 2, one can assume that the function v is continuous, without loss of generality.

6.2 A general measure of departures from rationality

It is a common observation in numerous empirical studies that choices of individuals are not consistent enough to be congruent with the *exact* utility maximisation. As a result, a significant part of the revealed preference literature is devoted to measures that evaluate how severely the data departs from the classic notion of rationality. Arguably, the most common of them all is the *critical cost-efficiency index* (CCEI, also known as *Afriat's efficiency index*), introduced in Afriat (1973) to evaluate violations of utility maximisation within the standard consumer demand framework.[42]

Throughout this subsection, let $X = \mathbb{R}^\ell_+$ and, for any observation $t \in T$, the corresponding menu be given by $A^t = \big\{y \in \mathbb{R}^\ell_+ : p^t \cdot y \leq p^t \cdot x^t\big\}$, for some prices $p^t \in \mathbb{R}^\ell_{++}$. The dataset $\mathcal{O} = \big\{(A^t, x^t) : t \in T\big\}$ is rationalisable for an *efficiency parameter* $e \in [0, 1]$ (a number) if there is a strictly increasing utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that

$$e(p^t \cdot x^t) \geq p^t \cdot y \;\text{ implies }\; u(x^t) \geq u(y),$$

for all $t \in T$. That is, the observed bundle x^t is preferable to all alternatives that are cheaper than the fraction e of x^t, given prices p^t, for all $t \in T$. Clearly, for $e = 1$, this coincides with the exact utility maximisation. CCEI is equal to the supremum over all efficiency parameters e for which the above condition holds.

Dziewulski (2020) provides a behavioural foundation for this measure. Namely, CCEI is the reciprocal of the infimum over all numbers $\lambda > 1$ for which the dataset is rationalisable as in (4), for a strictly increasing utility u and threshold δ, where the relation $\rhd$ is given by: $x \rhd y$ if $x = \lambda' y$, for some $\lambda' \geq \lambda$. Therefore, CCEI attributes violations of the exact utility maximisation to the particular form of imperfect discrimination. This

[42] Among others, CCEI was employed in Sippel (1997), Harbaugh et al. (2001), Andreoni and Miller (2002), Choi et al. (2007), Fisman et al. (2007), Ahn et al. (2014), Choi et al. (2014), Cherchye et al. (2017), Echenique et al. (2019), Cherchye et al. (2020), Dembo et al. (2021), and Cappelen et al. (2021).

equivalence result is established for the general specification of the utility function u. However, in numerous applications CCEI is used to measure departures from a specific formulation of the utility u. For example, Cherchye et al. (2017, 2020) apply an analogous measure to a multiperson household model; Polisson et al. (2020) evaluate CCEI for departures from expected utility, rank dependent utility, and disappointment aversion; Cappelen et al. (2021) and Dembo et al. (2021) employ it to estimate deviations from the model of probabilistic sophistication and expected utility maximisation. We apply Proposition 1 to extend the equivalence result to an arbitrary sub-class of utilities.

Proposition 8. *For any dataset $\mathcal{O}$, any strictly increasing utility $u : \mathbb{R}^\ell_+ \to \mathbb{R}$,*[43] *and any number $e^* \in (0, 1]$, the following statements are equivalent.*

(i) For any $e < e^*$, if $e(p^t \cdot x^t) \geq p^t \cdot y$ then $u(x^t) \geq u(y)$, for any $y \in \mathbb{R}^\ell_+$ and $t \in T$.

(ii) For any $\lambda > 1/e^*$, the dataset $\mathcal{O}$ is rationalisable as in (4) for the relation $\rhd$, the utility u, and some threshold δ, where $x \rhd y$ if $x = \lambda' y$, for some $\lambda' \geq \lambda$. [Proof]

It immediately follows that for *any* strictly increasing utility u, the CCEI under which the function supports the data is equal to the reciprocal of the infimum over all λs for which approximate maximisation of the same utility rationalises the data as in (4), for the relation $\rhd$. In addition, Proposition 5 implies that the model of *satisficing*, discussed in Section 5.2, provides an alternative foundation for CCEI.

Corollary 2. *For any dataset $\mathcal{O}$ the corresponding CCEI is equal to the infimum over all numbers $\lambda > 1$ for which the observations are rationalisable with a $\rhd$-monotone model of satisficing, where $x \rhd y$ if $x = \lambda' y$, for some $\lambda' \geq \lambda$.*

Most measures in the existing literature focus on departures from rationality within the classic consumer demand framework à la Afriat. This includes Afriat (1973), Varian (1990), Echenique et al. (2011), Dean and Martin (2016), Echenique et al. (2018, 2020), Allen and Rehbeck (2020, 2021), and de Clippel and Rozen (2021). In addition, the measure developed in Echenique et al. (2018, 2020) is designed for a particular class of additively separable models of time preference and choice under risk and uncertainty. Allen and Rehbeck (2021) focus solely on departures from quasilinear utility maximisation. In contrast, Apesteguia and Ballester (2015) develop an index that is suitable for

[43] That is, if $x > y$ then $u(x) > u(y)$.

environments beyond Afriat's, but their approach is applicable only to choices over finite domains. Finally, a versatile take on this issue was proposed in Houtman and Maks (1985), yet it lacks an appealing economic interpretation.

Our observations allow for a natural extension of CCEI not only to arbitrary utilities u, but also to empirical settings beyond the classic demand framework à la Afriat. Given any dataset $\mathcal{O}$ with arbitrary menus A, one can establish the severity of departures from rationality with the least $\lambda > 1$ for which the data can be rationalised as in (4) for the relation $\rhd$, where $x \rhd y$ if $x = \lambda' y$, for some $\lambda' \geq \lambda$. Unlike the original interpretation, our take on CCEI does not depend on the linear specification of the budget sets and allows for a meaningful comparison across different choice environments. Moreover, given the results in Section 3.3 and 3.4, it permits not only to measure departures from rationality, but also to elicit the true preferences of the individual, make out-of sample predictions, and evaluate welfare when the data are not consistent with utility maximisation.

A Appendix

Here we present proofs that were omitted in the main body of the paper. Before stating the argument supporting Theorem 1, it is convenient to prove Proposition 1.

A.1 Proof of Proposition 1

To prove implication (ii) $\Rightarrow$ (i), suppose that the function u rationalises the observations as in (4), for some threshold δ. If $x \rhd y$ then $u(x) > u(y) + \delta(x) \geq u(y)$, where the first inequality follows from $\rhd$-monotonicity of c, and the second is implied by $\delta(x) \geq 0$. Whenever xP^*y, there is a menu A such that $(A, x) \in \mathcal{O}$ and $z \rhd y$, for some $z \in A$. In particular, we have $u(x) \geq u(z) - \delta(z) > u(y)$, for any such z.

To prove the converse, take any utility u specified as in the proposition and define the function δ as follows: If $y \in A$, for some $(A, x) \in \mathcal{O}$, then

$$\delta(y) \; := \; \max\Big\{ \max\big\{u(y) - u(x), 0\big\} : (A, x) \in \mathcal{O} \text{ and } y \in A\Big\}.$$

Otherwise, let $\delta(y) = 0$. Hence, the function is well-defined and positive.

First, we claim that the resulting choice correspondence c is $\rhd$-monotone. Take any menu A and $x \in c(A)$. Towards contradiction, suppose there is some $y \in A$ such that

$y \rhd x$. By assumption, this implies that $u(y) > u(x)$. If $\delta(y) = 0$, then $x \in c(A)$ implies $u(x) \geq u(y) - \delta(y) > u(x)$, yielding a contradiction. Alternatively, suppose that $\delta(y) > 0$. By construction, this holds only if $y \in A'$ for some $(A', x') \in \mathcal{O}$. Since $y \rhd x$, this implies $x'P^*x$, and so $u(x') > u(x)$, for any such $(A', x') \in \mathcal{O}$. In particular, for some (A', x'),

$$u(x) + \delta(y) \;=\; u(x) + u(y) - u(x') \;<\; u(y),$$

which contradicts that $x \in c(A)$. Thus, the correspondence c is $\rhd$-monotone.

To prove that c rationalises $\mathcal{O}$, take any observation (A, x) and $y \in A$. By construction of the threshold δ, we have $\delta(y) \geq \max\{u(y) - u(x), 0\} \geq u(y) - u(x)$. This suffices for x to be an element of $c(A)$, which concludes the proof.

A.2 Proof of Theorem 1

We prove that statement (i) implies (ii). Given Proposition 1, it suffices to show that there is a utility $u : X \to \mathbb{R}$ such that if $x \rhd y$ or xP^*y then $u(x) > u(y)$. Before we proceed with our argument, we introduce an auxiliary result.

Lemma A.1. *Let $\succ$ be an irreflexive, transitive binary relation, and $D \subseteq X$ be a countable set such that $x \succ y$ implies either $z \not\succ x$ and $z \succ y$, or $x \succ z$ and $y \not\succ z$, for some $z \in D$. There is a function $u : X \to \mathbb{R}$ such that $x \succ y$ implies $u(x) > u(y)$.*

Proof. Take any countable set D specified as in the proposition and enumerate its elements so that $D = \{z^k\}_{k=1}^{\infty}$. For any $x \in X$ define the set $M(x) := \{k : x \succ z^k\}$ and $N(x) := \{k : z^k \succ x\}$. One can easily show that $x \succ y$ implies $M(y) \subseteq M(x)$ and $N(y) \supseteq N(x)$, for any $x, y \in X$. Moreover, at least one of the set inclusions must be strict. Indeed, if $x \succ z$ and $y \not\succ z$, for some $z \in D$, then $M(y) \subset M(x)$, while $z \not\succ x$ and $z \succ y$ implies $N(y) \supset N(x)$. Define the function $u : X \to \mathbb{R}$ by

$$u(x) \;:=\; \sum_{k \in M(x)} 2^{-k} - \sum_{k \in N(x)} 2^{-k},$$

which is well-defined and, by our previous observation, consistent with $\succ$. □

We continue with the main proof. We assume throughout that the relation P^* is acyclic, thus, its transitive closure P is irreflexive.

Lemma A.2. *If $z' \rhd y$ implies $z' \rhd z$, for all $z' \in X$, then xPy implies xPz.*

Proof. Suppose that xPy. By definition, there is (A', x') such that $x'P^*y$ or, equivalently, $z' \rhd y$, for some $z' \in A'$. Since $z' \rhd y$ implies $z' \rhd z$, we have $x'P^*z$. If $x = x'$, we are done. Otherwise, we have xPx' and $x'P^*z$, which implies xPz. □

The next lemma is an immediate corollary to the previous result.

Lemma A.3. *If $x\,P\,y \rhd z$ then xPz.*[44]

Indeed, by transitivity of $\rhd$, $z' \rhd y$ and $y \rhd z$ implies $z' \rhd z$, for all $z' \in X$. Therefore, by Lemma A.2, xPy implies xPz. In the reminder of this section, let $\succ$ denote the transitive closure of $P \cup \rhd$. The next lemma is critical to our argument.

Lemma A.4. *The relation $\succ$ is equal to $P \cup \rhd \cup (\rhd \circ P)$.*[45]

Proof. Clearly, $P \cup \rhd \cup (\rhd \circ P)$ is a subset of $\succ$. To prove the converse, suppose that $x \succ y$. Since P and $\rhd$ are transitive, this holds in four instances: Either (i) xPy or (ii) $x \rhd y$. Alternatively, (iii) there are elements $z^1, z^2, \ldots, z^n$ in X such that

$$x \;=\; z^1 \; P \; z^2 \;\rhd\; z^3 \; P \; z^4 \;\rhd\; \ldots \;\rhd\; z^{n-2} \; P \; z^{n-1} \;\rhd\; z^n \;=\; y.$$

By Lemma A.3 and transitivity of P, this implies xPy. Finally, (iv) we have

$$x \;=\; z^1 \;\rhd\; z^2 \; P \; z^3 \;\rhd\; z^4 \; P \; \ldots \; P \; z^{n-2} \;\rhd\; z^{n-1} \; P \; z^n \;=\; y,$$

for some alternatives $z^1, z^2, \ldots, z^n$ in X. Similarly, by Lemma A.3 and transitivity of P this implies that $x \rhd z^2 \; P \, y$. It is straightforward to show that any other case can be reduced to one of the four above. This concludes our proof. □

Lemma A.5. *The transitive closure $\succ$ of $P \cup \rhd$ is irreflexive.*

Proof. Given Lemma A.4 and the fact that P and $\rhd$ are irreflexive, it suffices to show that $\rhd \circ P$ is irreflexive. Suppose that $x \rhd z\; P\; x$, for some $x, z \in X$. Since this is equivalent to $z\; P\; x \rhd z$, and so zPz (by Lemma A.3), it contradicts that P is irreflexive. □

Below we present a useful extension of Lemma A.2.

Lemma A.6. *If $z' \rhd y$ implies $z' \rhd z$, for all $z' \in X$, then $x \succ y$ implies $x \succ z$.*

[44] Throughout, we denote $x\;P\;y \rhd z$ in place of xPy and $y \rhd z$, for any $x, y, z \in X$.

[45] We denote $(\rhd \circ P) := \{(x, y) : x \rhd z \; P \; y, \text{ for some } z \in X\}$.

Proof. Suppose that $x \succ y$. By Lemma A.4, this holds is three instances. If xPy then xPz, by Lemma A.2. Following the same argument, if $x \rhd z'' \, P \, y$, for some $z'' \in X$, then $x \rhd z'' \, P \, z$. Finally, we have $x \rhd y$ only if $x \rhd z$. Either way, we obtain $x \succ z$. □

Consider the final auxiliary result.

Lemma A.7. *Under Assumption 1, there is a countable set $D \subseteq X$ such that $x \succ y$ implies either $z \not\succ x$ and $z \succ y$, or $x \succ z$ and $y \not\succ z$, for some $z \in D$.*

Proof. Take any set $D \subseteq X$ as in Assumption 1 and define $D' := D \cup \{x : (A, x) \in \mathcal{O}\}$, which is countable (since $\mathcal{O}$ is finite). Suppose that $x \succ y$. By Lemma A.4, it suffices to consider three instances. If xPy then $z \not\succ x$ and $z \succ y$, for $z = x \in D'$. Whenever $x \rhd z \, P \, y$, for some $z \in X$, then $z \not\succ x$ and $z \succ y$, where $z \in D'$.

Finally, suppose that $x \rhd y$. By Assumption 1, there is $z \in D$ such that either (i) $z \rhd y$, and $z' \rhd x$ implies $z' \rhd z$, for all $z' \in X$; or (ii) $x \rhd z$, and $z' \rhd z$ implies $z' \rhd y$, for all $z' \in X$. If (i) is true, then $z \succ y$. We show that $z \not\succ x$ by contradiction. By Lemma A.6, if $z \succ x$ then $z \succ z$, which contradicts that $\succ$ is irreflexive. Analogously, we show that condition (ii) implies $x \succ z$ and $y \not\succ z$. □

By Lemmas A.5 and A.7, the relation $\succ$ is irreflexive, transitive, and satisfies the separability condition. By Lemma A.1, there is a utility $u : X \to \mathbb{R}$ such that $x \succ y$ implies $u(x) > u(y)$. In particular, if $x \rhd y$ or xP^*y then $u(x) > u(y)$. By Proposition 1, there is a threshold δ for which the dataset $\mathcal{O}$ is rationalisable as in (4).

A.3 Proof of Proposition 2

Implication ($\Leftarrow$) follows from Theorem 1, since it is true independently of ancillary assumptions. To show the converse, suppose that X is a locally compact and separable metric space. Moreover, for any $(A, x) \in \mathcal{O}$, let the menu A be compact. Finally, the directly revealed strict preference relation P^* is acyclic, thus, its transitive closure P is irreflexive. By $\succ$ we denote the transitive closure of $P \cup \rhd$.

To prove the result, we show that $\succeq := \succ \cup \{(x, x) : x \in X\}$ is a closed-continuous preorder, i.e., a closed, reflexive, and transitive binary relation. We then apply Levin's Theorem to prove that there is a continuous function $u : X \to \mathbb{R}$ that extends $\succ$, i.e., $x \succ y$ implies $u(x) > u(y)$. See the original result in Levin (1983), or the appendix in Nishimura et al. (2017). The rest follows from Proposition 1.

We proceed with the proof. It is straightforward to show that $\succeq$ is a preorder. We show that it is closed-continuous via two lemmas.

Lemma A.8. *Under Assumption 2, the revealed strict preference relation P is compact.*

Proof. We begin the proof by showing that the *directly* revealed strict preference relation P^* is compact. Indeed, we have $P^* = \bigcup_{(A,x)\in\mathcal{O}} \{(x,y) : z \rhd y, \text{ for some } z \in A\}$. Since menu A is compact, Assumption 2 implies that so is $\{(x,y) : z \rhd y, \text{ for some } z \in A\}$. Given that $\mathcal{O}$ is finite, the relation P^* is compact as well.

We show that P is compact by induction. Let $E^0 = P^*$ and

$$E^n := \bigcup_{(A,x)\in\mathcal{O}} \Big\{(x,y) : xE^{n-1}x' \text{ and } x'P^*y, \text{ for some } (A',x') \in \mathcal{O}\Big\},$$

for any $n \geq 1$. Since E^0 and P^* are compact, the set E^n is a finite union of compact sets, thus, itself compact, for any $n \geq 1$. Hence, the set $P = \bigcup_{n=0}^{|\mathcal{O}|} E^n$ is compact. □

The above result implies the following observation.

Lemma A.9. *Under Assumption 2, the relation $\succeq$ is closed.*

Proof. By Lemma A.3, it suffices to show that $P \cup \rhd \cup (\rhd \circ P) \cup \{(x,x) : x \in X\}$ is closed. By Assumption 2, the union $\rhd^* := \rhd \cup \{(x,x) : x \in X\}$ is closed. Moreover, Lemma A.8 implies that P is compact. Following Lemma C in Nishimura et al. (2017), the relation $\rhd^* \circ P = (\rhd \circ P) \cup P$ is closed, thus, so is $(\rhd \circ P) \cup P \cup \rhd^* = P \cup \rhd \cup (\rhd \circ P) \cup \{(x,x) : x \in X\}$. This completes the proof. □

Since $\succeq$ is a closed-continuous preorder, Levin's Theorem guarantees that there is a continuous function $u : X \to \mathbb{R}$ such that $x \succ y$ implies $u(x) > u(y)$. In particular, both $x \rhd y$ and xP^*y imply $u(x) > u(y)$. The rest follows from Proposition 1.

A.4 Proof of Theorem 2

We prove only the second part. Let $\succ$ be the transitive closure of $P \cup \rhd$. By Lemmas A.4 and A.5, the relation $\succ$ is irreflexive and equal to $P \cup \rhd \cup (\rhd \circ P)$. Throughout this section we denote the transitive closure of $\succ \cup \{(x,y)\}$ by $\hat{\succ}$.

Lemma A.10. *The binary relation $\hat{\succ}$ is irreflexive.*

Proof. Since $y \notin RW(x)$ and $\succ = P \cup \rhd \cup (\rhd \circ P)$, we have $y \not\succ x$, by definition of $RW(x)$. We consider two cases. If $x \succ y$ then $\hat{\succ} = \succ$, which is irreflexive. Otherwise, the relation $\hat{\succ}$ fails to be irreflexive only if $z \succ x$ and $y \succ z$, for some $z \in X$. However, this implies $y \succ x$, which contradicts our initial claim. □

The following lemma shows that $\hat{\succ}$ satisfies the separability condition.

Lemma A.11. *Under Assumptions 1, there is a countable set $D \subseteq X$ such that $z' \hat{\succ} z$ implies either $z' \hat{\succ} z''$ and $z \hat{\not\succ} z''$, or $z'' \hat{\not\succ} z'$ and $z'' \hat{\succ} z$, for some $z'' \in D$*

Proof. Take any set D specified in Assumption 1 and define

$$D' \ := \ D \cup \big\{x' : (A', x') \in \mathcal{O}\big\} \cup \{x, y\},$$

which is countable. Suppose that $z' \hat{\succ} z$. If $z' \not\succ z$, then either $z' = x$, $z = y$, or $z' \succ x$ and $y \succ z$. Clearly, the required condition is satisfied for $z'' = x$ or $z'' = y$.

Alternatively, suppose that $z' \succ z$. By Lemma A.4, this holds in three instances. If $z' P z$, let $z'' = z' \in D'$. Since $\hat{\succ}$ is irreflexive, it must be that $z'' \hat{\not\succ} z'$ and $z'' \hat{\succ} z'$. Similarly, if $z' \rhd z'' \, P \, z$, for some $z'' \in X$, then $z'' \hat{\not\succ} z'$ and $z'' \hat{\succ} z'$, where $z'' \in D'$.

Suppose that $z' \rhd z$. By Assumption 1, there is $z'' \in D$ such that either (i) $z'' \rhd z$, and $z''' \rhd z'$ implies $z''' \rhd z''$, for all $z''' \in X$; or (ii) $z' \rhd z''$, and $z''' \rhd z''$ implies $z''' \rhd z$, for all $z''' \in X$. If (i) is true, then $z'' \rhd z$, and so $z'' \hat{\succ} z$. Towards contradiction, let $z'' \hat{\succ} z'$. If $z'' \succ z'$, then $z'' \succ z''$ (by Lemma A.6), yielding a contradiction. Similarly, $z'' \succ x$ and $y \succ z'$ implies $y \succ z'' \succ x$, contradicting that $y \not\succ x$. Thus, we have $z'' \hat{\not\succ} z'$ and $z'' \hat{\succ} z$. Analogously, (ii) implies $z' \hat{\succ} z''$ and $z \hat{\not\succ} z''$, for some $z'' \in D$. □

By combination of Lemmas A.10, A.11, and A.1, there is utility $u : X \to \mathbb{R}$ such that $z' \tilde{\succ} z$ implies $u(z') > u(z)$. Therefore, both $z' \rhd z$ and $z' P^* z$ imply $u(z') > u(z)$, as well as $u(x) > u(y)$. The rest follows from Proposition 1.

A.5 Proof of Proposition 3

Denote $\tilde{\mathcal{O}} = \mathcal{O} \cup \big\{(A, y)\big\}$ and let $\tilde{P}^*, \tilde{P}$ be the revealed relations induced by $\tilde{\mathcal{O}}$. In particular, we have $P \subseteq \tilde{P}$. Clearly, the set $\tilde{\mathcal{O}}$ is rationalisable only if $y \in S(A)$. Otherwise, $z \rhd y$ for some $z \in A$ would imply $y \tilde{P}^* y$, while xPy and $z \rhd x$ for some $z \in A$ would imply $y \tilde{P} y$. Either way, this would contradict that the relation $\tilde{P}$ is irreflexive.

We prove the converse by contradiction. Suppose that $y \in S(A)$, but the set $\tilde{\mathcal{O}}$ is not rationalisable. Given that $\mathcal{O}$ is rationalisable by assumption and, thus, the relation P is irreflexive, this holds only if $y\tilde{P}y$, which can take place in two instances: If (i) $y\tilde{P}^*y$, then $z \rhd y$ for some $z \in A$; if (ii) $y\tilde{P}^*x$ and xPy, then $z \rhd x$ and xPy, for some $z \in A$ and $x \in X$. Either way, this contradicts that $y \in S(A)$ and completes our proof.

A.6 Proof of Theorem 3

Implication ($\Leftarrow$) is straightforward. Indeed, for any u, δ that rationalise $\mathcal{O}$ as in (4), and any $x \in c(A')$, each of the conditions (i)–(iii) would imply $u(x) > u(y)$, for all $y \in A$.

We prove the converse by contradiction. Suppose that A' is robustly preferred to A, but there is some $y \in A$ that violates each of the conditions (i)–(iii). In particular, there is some $x \in S(A')$ such that *not* xPy. Take any such x and denote $\tilde{\mathcal{O}} := \mathcal{O} \cup \{(A', x)\}$. By Proposition 3, the set $\tilde{\mathcal{O}}$ is rationalisable as in (4). Let $\tilde{P}$ denote the revealed strict preference relation induced by $\tilde{\mathcal{O}}$, and $\tilde{RW}(x)$ be the corresponding revealed worst set for x. We claim that $y \notin \tilde{RW}(x)$. Indeed, it can not be that $x\,\tilde{P}y$, since this would imply one of the conditions (i)–(iii). Similarly, we can exclude $x \rhd y$. Suppose that $x \rhd z$ and $z\tilde{P}y$, for some $z \in X$. In such a case, we have $x\tilde{P}^*z$. If $z\tilde{P}y$, then either zPy, or zPx and $x\tilde{P}y$. Thus, either y obeys condition (ii), or $x\tilde{P}x$, contradicting that $x \in S(A')$.

Since $y \notin \tilde{RW}(x)$, Theorem 2 guarantees that there are functions u, δ that rationalise $\tilde{\mathcal{O}}$ as in (4) and $u(y) > u(x)$. This contradicts that A' is robustly preferred to A.

A.7 Proof of Proposition 4

Suppose that the correspondence $c(A) := \{x \in A : y \not\succ x, \text{ for all } y \in A\}$ rationalises the set of observations, for some interval order $\succ$. We show that the directly revealed strict preference relation P^* is acyclic. First, define a binary relation Q as: xQy if $z \succ y$ and $z \not\succ x$, for some $z \in X$. Following Lemma 3.1 in Aleskerov et al. (2007), Q is asymmetric and negatively transitive.[46] Given that the correspondence c is $\rhd$-monotone, it must be that $x \rhd y$ implies $x \succ y$. Otherwise, we would have $y \in c(\{x, y\})$, contradicting that c is $\rhd$-monotone. We claim that xP^*y implies xQy. Take any observation $(A, x) \in \mathcal{O}$ and $z \in A$ such that $z \rhd y$. Clearly, it must be that $z \succ y$ and $z \not\succ x$, which implies xQy.

[46] A relation R on X is *asymmetric* if xRy implies *not* yRx. The relation *negatively transitive* if *not* xRy and *not* yRz implies *not* xRz.

To show that P^* is acyclic, take any sequence $z^1, z^2, \ldots, z^n$ in X such that $z^i P^* z^{i+1}$, for all $i = 1, \ldots, (n-1)$. Thus, given the observation above, we obtain $z^i Q z^{i+1}$, or *not* $z^{i+1} Q z^i$, for all $i = 1, \ldots, (n-1)$ (by asymmetry of Q). By negative transitivity of Q, it must be that *not* $z^n Q z^1$, and so *not* $z^n P^* z^1$.

A.8 Proof of Proposition 5

We only prove the "if" part. Suppose that the set $\mathcal{O}$ is rationalisable with a $\rhd$-monotone model c of satisficing behaviour. There is a function $u : X \to \mathbb{R}$ such that $x \in c(A)$ and $u(y) \geq u(x)$ implies $y \in c(A)$, for any $A \in \mathcal{A}$ and $y \in A$.

We claim that xP^*y implies $u(x) > u(y)$. By definition, we have $z \rhd y$, for some $(A, x) \in \mathcal{O}$ and $z \in A$. By assumption, this implies $y \in A$. Since c is $\rhd$-monotone and rationalises the data, it must be that $u(x) > u(y)$. If not, then $x \in c(A)$ and $u(y) \geq u(x)$ would imply $y \in c(A)$, contradicting that c is $\rhd$-monotone.

By the above observation, the directly revealed strict preference relation P^* must be acyclic. Therefore, Theorem 1 guarantees that the dataset $\mathcal{O}$ is rationalisable with a $\rhd$-monotone approximate utility maximisation as in (4).

A.9 Proof of Proposition 6

First, we show that (i) implies (iv). Define $\succeq$ by: $x \succeq y$ if *not* $y \rhd x$, which is complete and quasitransitive, since the asymmetric part of $\succeq$ is equal to $\rhd$ (which is transitive). To prove that $\hat{c}$ is $\rhd$-monotone, take any $x, y \in A \in \mathcal{A}$. If $y \rhd x$ then $y \succ x$, which implies that $x \notin \hat{c}(A)$. Finally, take any $(A, x) \in \mathcal{O}$. We have $x \notin \hat{c}(A)$ only if $y \succ x$, for some $y \in A$. Since this implies $y \rhd x$, we have xP^*x, contradicting (i).

To show that (iv) implies (iii), let $\succeq$ be a complete and quasitransitive relation in (iv), with its strict part $\succ$. By definition, $\succ$ is irreflexive. Moreover, by quasitransitivity of $\succeq$, it is transitive. Thus, $\succ$ is a strict partial order. Since, in this case, $\hat{c} = c$, the correspondence in (5) is $\rhd$-monotone and rationalises the data.

Implication (iii) $\Rightarrow$ (ii) $\Rightarrow$ (v) is immediate. We prove that (v) implies (i) by contradiction. If xP^*x, there is some $(A, x) \in \mathcal{O}$ and $y \in A$ such that $y \rhd x$. Moreover, since c rationalises the data, it must be that $x \in c(A)$, which violates $\rhd$-monotonicity.

A.10 Proof of Proposition 8

To show that (i) implies (ii), take any $\lambda > 1/e^*$. Following (i), there is some e such that $\lambda > 1/e > 1/e^*$, and $e(p^t \cdot x^t) \geq p^t \cdot y$ implies $u(x^t) \geq u(y)$. By monotonicity of u, this guarantees that $e(p^t \cdot x^t) > p^t \cdot y$ only if $u(x^t) > u(y)$. Note that, there is some $z \in A$ such that $z \rhd y$, and so $x^t P^* y$ if, and only if, $p^t \cdot x^t \geq p^t \cdot (\lambda y)$. Since $1/\lambda < e$, this suffices for $x^t P^* y$ to imply $u(x^t) > u(y)$. Moreover, monotonicity of u implies $u(\lambda y) > u(y)$, for any $\lambda > 1$. By Proposition 1, the data is rationalisable as in (4) for the utility u.

To show the converse, take any $e < e^*$. By (ii), there is some number λ such that $e < 1/\lambda < e^*$. By the argument above and Proposition 1, we know that $p^t \cdot x^t \geq p^t \cdot (\lambda y)$ implies $u(x^t) > u(y)$. Since $e < 1/\lambda$, this suffices for (i) to hold.

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Online supplement to: "A comprehensive revealed preference approach to approximate utility maximisation"

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Abstract

This supplement contains additional results related to Dziewulski (2022). These notes should be read in conjunction with the main paper.

Here we include results that complement the findings presented in the main paper. In Section B.1 we discuss an alternative, constructive take on Theorem 1 based on linear programming methods. In particular, we determine properties of the utility function u that are *not* testable in certain choice environments. In Section B.2 we state proofs of the results presented in Section 6.1 of the main paper, regarding approximate utility maximisation over state-contingent consumption under risk.

Throughout this supplement we employ the notation introduced in the main paper. In order to keep our exposition compact, we say that a dataset $\mathcal{O}$ is *approximately rationalisable*, if there is a utility u and a threshold function δ that rationalise the observations in the sense specified in Theorem 1, given a strict partial order $\rhd$.

B.1 The constructive approach

Theorem 1 and Proposition 3 in the main paper establish equivalence between acyclic direct revealed strict preference P^* and approximate utility maximisation in a general

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setting. However, the lack of a tractable constructive argument makes it difficult to establish any properties of the functions u and δ that rationalise the data. Here, we impose additional structure on our framework to present an alternative take on our results.

We assume throughout that the Euclidean consumption space $X = \mathbb{R}^\ell_+$ is endowed with the natural product order $\geq$.[1] A dataset is denoted by $\mathcal{O} = \{(A^t, x^t) : t \in T\}$, where T denotes a finite set of labels. We focus on choices from *generalised budget sets*, as in Forges and Minelli (2009). That is, for any observation $(A^t, x^t) \in \mathcal{O}$, there is a well-defined and strictly increasing function $f^t : X \to \mathbb{R}$ such that

$$A^t = \Big\{y \in X : f^t(y) \leq 0\Big\}.$$[2]

As pointed out in Section 2 of the main paper, this includes the classic consumer choice setup discussed in Afriat (1967), Diewert (1973), and Varian (1982).

Given the strict partial order $\rhd$, we find it convenient to define the *upper contour* correspondence $\Gamma_\rhd : X \rightrightarrows X$ by $\Gamma_\rhd(x) := \{y \in X : y \rhd x\}$. Throughout the supplement, we shall impose conditions directly on $\Gamma_\rhd$, rather than $\rhd$.

Assumption B.1. *For all $x \in X$, the set $\Gamma_\rhd(x)$ is non-empty. Moreover, if $y \in \Gamma_\rhd(x)$ and z is in the closure of $\Gamma_\rhd(y)$ then $z' < z$, for some $z' \in \Gamma_\rhd(x)$.*

It is critical for our constructive argument that the correspondence $\Gamma_\rhd$ is well-defined. The second part of the assumption imposes a specific form of monotonicity on the correspondence. In particular, it implies that $x \notin \Gamma_\rhd(x)$, for all $x \in X$.[3]

Remark B.1. It will become clear from our exposition that all the results presented in this section can be generalised to any space X that is endowed with some preorder $\geq_X$, and where X is either finite or bounded from below with respect to the ordering $\geq_X$, i.e., there is some $y \in X$ such that $x \in X$ implies $x \geq_X y$. This includes the space of probability distributions over $S = \mathbb{R}_+$, endowed with the first order stochastic dominance.

[1] We denote $x \geq y$ if $x_i \geq y_i$, for all $i = 1, \dots, \ell$, then $x \geq y$. The relation is *strict*, and denoted by $x > y$, if $x \geq y$ and $x \neq y$. Finally, we have $x \gg y$ if $x_i > y_i$, for all $i = 1, \dots, \ell$.

[2] If A^t can be represented as $A^t = \{y \in X : f^t_i(y) \leq 0, \text{ for all } i = 1, \dots, n\}$ for multiple well-defined and strictly increasing functions $f^t_i : X \to \mathbb{R}$, for all $i = 1, \dots, n$, then $A^t = \{y \in X : f^t(y) \leq 0\}$, where the function $f^t(y) := \max\{f^t_i(y) : i = 1, \dots, \ell\}$ is well-defined and strictly increasing.

[3] Clearly, if $x \in \Gamma_\rhd(x)$ then, for any z' in the closure of $\Gamma_\rhd(x)$, there would have to be some $z \in \Gamma_\rhd(x)$ such that $z' > z$, which yields a contradiction.

B.1.1 Constructive rationalisation

Given our discussion in Section 3 of the main paper, it is clear that whenever the set of observations $\mathcal{O}$ is rationalisable with approximate utility maximisation then the corresponding directly revealed strict preference relation P^* is acyclic. This observation follows directly from the definition of the relation, and is independent of ancillary assumptions. In this subsection we provide a constructive argument supporting the converse. We propose a utility u and a threshold δ that rationalise the data in this sense.

We begin our construction by defining the function $g^t : X \to \mathbb{R}$ as

$$g^t(x) \;:=\; \begin{cases} f^t(x) & \text{if } f^t(x) \leq 0; \\ f^t(x) + \epsilon & \text{otherwise;} \end{cases} \tag{B.1}$$

for some $\epsilon > 0$, where f^t is the well-defined and strictly increasing function that represents the menu A^t, for all $t \in T$. Thus, the function g^t is also well-defined and strictly increasing. Moreover, we have $g^t(y) \leq 0$ if, and only if, $y \in A^t$, for all $t \in T$. Define function $h^t : X \to \mathbb{R}$ as $h^t(x) := \inf\big\{g^t(y) : y \in \Gamma_{\triangleright}(x)\big\}$, for all $t \in T$.

Lemma B.1. *For all $t \in T$, we have $h^t(x) \leq 0$ if, and only if, $\Gamma_{\triangleright}(x) \cap A^t \neq \emptyset$.*

Proof. If $y \in \Gamma_{\triangleright}(x) \cap A^t \neq \emptyset$ then $0 \geq f^t(y) = g^t(y) \geq h^t(x)$. To show the converse, suppose that $h^t(x) \leq 0$ and $\Gamma_{\triangleright}(x) \cap A^t = \emptyset$. In particular, for any $y \in \Gamma_{\triangleright}(x)$, we have $g^t(y) = f^t(y) + \epsilon > \epsilon$. This implies $h^t(x) \geq \epsilon > 0$, yielding a contradiction. □

It is easy to show that the revealed relation P^* is acyclic if, and only if, for any cycle $\mathcal{C} = \big\{(a, b), (b, c), \ldots, (z, a)\big\}$ in $T \times T$, we have $\Gamma_{\triangleright}(x^s) \cap A^t = \emptyset$, for some $(t, s) \in \mathcal{C}$. By the lemma above, this is equivalent to $h^t(x^s) > 0$, for some $(t, s) \in \mathcal{C}$. In the following result, we show that this suffices to solve a particular linear system.

Lemma B.2. *The relation P^* is acyclic only if there are numbers $(\phi^t)_{t\in T}$ and strictly positive numbers $(\mu^t)_{t\in T}$ such that $\phi^s < \phi^t + \mu^t h^t(x^s)$, for all $t, s \in T$.*

The system of inequalities presented in this lemma is very similar to the so-called *Afriat inequalities*. However, it requires for all the inequalities to be strict. The result itself is analogous to Lemma A.2 in Dziewulski (2020) and can be proven using an argument presented in Section 2 of Fostel et al. (2004). We introduce one final lemma.

Lemma B.3. *Under Assumption B.1, if $y \in \Gamma_{\triangleright}(x)$ then $h^t(y) > h^t(x)$, for any $t \in T$.*

Proof. By monotonicity of g^t and definition of h^t, there is some z in the closure of $\Gamma_{\triangleright}(y)$ such that $h^t(y) \geq g^t(z)$. Following Assumption B.1, there is $z' \in \Gamma_{\triangleright}(x)$ satisfying $z' < z$. Since g^t is strictly increasing, we obtain $h^t(y) \geq g^t(z) > g^t(z') \geq h^t(x)$. □

The main theorem of this section presents a particular utility u and a threshold function δ that approximately rationalise the set of observations $\mathcal{O}$.

Theorem B.1. *Under Assumption B.1, the dataset $\mathcal{O}$ is approximately rationalisable with the utility $u : X \to \mathbb{R}$, given by*

$$u(y) \;:=\; \min\left\{\phi^t + \mu^t h^t(y) : t \in T\right\},$$

and the threshold $\delta : X \to \mathbb{R}_+$, given by

$$\delta(y) \;:=\; \max\left\{0;\, \max\left\{u(y) - \mu^t g^t(y) - u(x^t) : t \in T\right\}\right\},$$

for any numbers $(\phi^t)_{t\in T}$ and strictly positive numbers $(\mu^t)_{t\in T}$ as in Lemma B.2.

Proof. Clearly, both u and δ are well-defined. Let the function $v : X \to \mathbb{R}$ be given by $v(y) := \min\left\{u(y); \min\{u(x^t) + \mu^t g^t(y) : t \in T\}\right\}$. Thus, $u(y) \geq v(y)$, for all $y \in X$.

We claim that $y \in \Gamma_{\triangleright}(x)$ implies $v(y) > u(x)$. Indeed, by Lemma B.3, we have

$$u(x) \;=\; \min\left\{\phi^t + \mu^t h^t(x) : t \in T\right\} \;<\; \min\left\{\phi^t + \mu^t h^t(y) : t \in T\right\} \;=\; u(y),$$

since μ^t is strictly positive, for all $t \in T$. On the other hand, by construction of the numbers $(\phi^t)_{t\in T}$, $(\mu^t)_{t\in T}$, we have $\phi^t < u(x^t)$, for all $t \in T$. This implies

$$u(x) \;=\; \min\left\{\phi^t + \mu^t h^t(x) : t \in T\right\} \;<\; \min\left\{u(x^t) + \mu^t g^t(y) : t \in T\right\},$$

since $y \in \Gamma_{\triangleright}(x)$ implies $h^t(x) \leq g^t(y)$. The two observations guarantee $u(x) < v(y)$.

Since $g^t(y) = f^t(y) \leq 0$ implies $v(y) \leq u(x^t) + \mu^t f^t(y) \leq u(x^t)$, we have $u(x^t) \geq v(y)$, for all $y \in A^t$ and $t \in T$. Given that $v(y) = u(y) - \delta(y)$, the proof is complete. □

The next corollary follows immediately from the above construction.

Corollary B.1. *Suppose that the function f^t representing the menu A^t is continuous, for each observation $t \in T$, and $\triangleright = >$.*[4] *Then, the dataset $\mathcal{O}$ is approximately rationalisable for an upper semi-continuous utility u and some threshold δ, without loss of generality.*[5]

[4] Clearly, the same result holds for $\triangleright = \gg$.

[5] The function u is *upper semi-continuous* if the set $\{x \in X : u(x) \geq a\}$ is closed, for any number a.

Proof. Since $\Gamma_{\triangleright}$ satisfies Assumption B.1, Theorem B.1 guarantees that the dataset $\mathcal{O}$ is rationalisable with the utility function $u(y) := \min\left\{\phi^t + \mu^t h^t(y) : t \in T\right\}$ and a threshold δ. By strict monotonicity of f^t, the function h^t is equal to $h^t(y) = f^t(y)$, if $f^t(y) < 0$, and $h^t(y) = f^t(y) + \epsilon$ otherwise, for some $\epsilon > 0$. Clearly, it is upper semi-continuous. In particular, the function $y \to \left[\phi^t + \mu^t h^t(y)\right]$ is upper semi-continuous, for any number ϕ^t and strictly positive number μ^t, for all $t \in T$. Since the *min* operator preserves upper semi-continuity, the function u is upper semi-continuous. □

Recall Example 3 in the main paper. In that example the directly revealed strict preference relation induced by the observed choices was acyclic. Given that budget sets A^t were represented with a strictly increasing and continuous function f^t, the above corollary guarantees that this particular dataset could be approximately rationalised with an upper semi-continuous utility function u, without loss of generality.

B.1.2 Limits to testability

Proposition 2 of the main paper specifies conditions, under which the utility u that approximately rationalises the data is continuous, without loss of generality. Hence, in such environments, continuity is *not* testable. The construction of the utility u and the threshold δ in Theorem B.1 allows us to further investigate properties of these functions and identify choice environments $(\mathcal{O}, \triangleright)$ for which they are not falsifiable.

Throughout this subsection, we take the dataset $\mathcal{O}$ and correspondence $\Gamma_{\triangleright}$ (or the relation $\triangleright$) as the premise. In addition, we assume that $\mathcal{O}$ is approximately rationalisable.

Continuity We begin our discussion by presenting sufficient conditions under which the data can be explained with continuous functions u and δ.

Assumption B.2. *The* lower bound *correspondence* $\partial\Gamma^{\downarrow}_{\triangleright} : X \rightrightarrows X$, *given by*

$$\partial\Gamma^{\downarrow}_{\triangleright}(x) \;:=\; \Big\{y \in \Gamma_{\triangleright}(x) : z < y \text{ implies } z \notin \Gamma_{\triangleright}(x), \text{ for all } z \in X\Big\},$$

is well-defined, compact-valued, and continuous.[6]

Recall the correspondence $\Gamma_{\triangleright}(x) := \left\{y \in X : y > x\right\}$. In this case, the lower bound $\partial\Gamma_{\downarrow}(x)$ is empty, for all $x \in X$, which violates the above assumption.[7]

[6] See Definition 17.4 in Aliprantis and Border (2006) for a definition of a *continuous* correspondence.
[7] The same applies to the correspondence $\Gamma_{\triangleright}(x) := \left\{y \in X : y \gg x\right\}$.

Proposition B.1. *Under Assumptions B.1 and B.2, if the function f^t representing the menu A^t is continuous, for all $t \in T$, then the dataset $\mathcal{O}$ is approximately rationalisable, for a continuous utility u and a continuous threshold δ, without loss of generality.*

Proof. Define function $g^t : X \to \mathbb{R}$ as $g^t(x) := f^t(x)$, which is well-defined, strictly increasing, and continuous, for all $t \in T$. Moreover, let the function $h^t : X \to \mathbb{R}$ be given as in Section B.1.1, for all $t \in T$. By continuity and strict monotonicity of g^t, and compactness of $\partial\Gamma^{\downarrow}_{\triangleright}(x)$, we have $h^t(x) = \min\{g^t(y) : y \in \partial\Gamma^{\downarrow}_{\triangleright}(x)\}$. Since the function g^t and the correspondence $\partial\Gamma^{\downarrow}_{\triangleright}$ are continuous, Berge's Maximum Theorem guarantees that h^t is continuous (see, e.g., Theorem 17.31 in Aliprantis and Border, 2006).

We claim that $h^t(x) \leq 0$ if, and only if, $\Gamma_{\triangleright}(x) \cap A^t \neq \emptyset$. Clearly, if $y \in \Gamma_{\triangleright}(x) \cap A^t$ then $0 \geq f^t(y) = g^t(y) \geq h^t(x)$. Conversely, if $h^t(x) \leq 0$ then $g^t(y) = f^t(y) \leq 0$, for some $y \in \partial\Gamma^{\downarrow}_{\triangleright}(x) \subseteq \Gamma_{\triangleright}(x)$, which can be satisfied only if $\Gamma_{\triangleright}(x) \cap A^t \neq \emptyset$.

This observation guarantees that the dataset $\mathcal{O}$ is rationalisable if, and only if, for any cycle $\mathcal{C} = \{(a,b),(b,c),\ldots,(z,a)\}$ in $T \times T$, we have $h^t(x^s) > 0$, for some $(t,s) \in \mathcal{C}$. Following the argument in Section B.1.1, this suffices for the set of observations to be rationalisable with the functions u and δ specified in Theorem B.1. Since g^t and h^t are continuous, for all $t \in T$, so is the utility u and the threshold δ. □

In relation to Proposition 2 of the main paper, this result introduces alternative assumptions under which a dataset is rationalisable with a continuous utility. Moreover, the same conditions guarantee a continuous threshold δ.

Monotonicity Next, we address the question of monotonicity of the utility u. As stated in Proposition 1, any function u that rationalises the data with approximate utility maximisation, must be consistent with the correspondence $\Gamma_{\triangleright}$, i.e., if $y \in \Gamma_{\triangleright}(x)$ then $u(y) > u(x)$.[8] However, unlike for the exact utility maximisation, the utility u can satisfy a stronger notion of monotonicity and still rationalise the observed choices.

A correspondence $\Gamma_{\triangleright}$ is *increasing* if, for any $x' \geq x$ and $y' \in \Gamma_{\triangleright}(x')$, there is some $y \in \Gamma_{\triangleright}(x)$ such that $y' \geq y$. The correspondence is *strictly increasing* if, for any $x' > x'$ and y' in the closure of $\Gamma_{\triangleright}(x')$, there is some $y \in \Gamma_{\triangleright}(x)$ such that $y' > y$. Similarly, we will say that the utility u is (strictly) *increasing* if $x \geq (>)\, y$ implies $u(x) \geq (>)\, u(y)$. That is, the function is monotone with respect to the standard order $\geq (>)$ over $\mathbb{R}^{\ell}$.

[8]Recall that $y \in \Gamma_{\triangleright}(x)$ is equivalent to $y \triangleright x$.

Proposition B.2. *Under Assumption B.1, if $\Gamma_\triangleright$ is (strictly) increasing, then the dataset $\mathcal{O}$ is approximately rationalisable with a (strictly) increasing utility u.*[9]

Proof. Define functions g^t and h^t as in Section B.1.1, for all $t \in T$. First, we show the result outside the brackets. Whenever $\Gamma_\triangleright$ is increasing, for any $x' \geq x$ and $y' \in \Gamma_\triangleright(x')$, there is some $y \in \Gamma_\triangleright(x)$ such that $y' \geq y$. Since g^t increases, this implies $g^t(y') \geq g^t(y) \geq h^t(x)$, and so $h^t(x') \geq h^t(x)$. Hence, the function h^t is increasing, for all $t \in T$. This suffices to show that the utility u in Theorem B.1 is also increasing.

To prove the result within the brackets, take any $x' > x$ and y' in the closure of $\Gamma_\triangleright(x')$ satisfying $h^t(x') \geq g^t(y')$. By assumption, there is some $y \in \Gamma_\triangleright(x)$ such that $y' > y$, and so strict monotonicity of g^t implies $h^t(x') \geq g^t(y') > g^t(y) \geq h^t(x)$. Therefore, the function h^t is strictly increasing, which suffices for the utility u to be strictly increasing. □

This result highlights the distinction between monotonicity of choice and the utility u under approximate utility maximisation, discussed in Section 4. Preferences u of the individual can be strictly monotone, yet this need not translate to the choice. For example, since the relation $\triangleright$ in Example 1 induces a strictly increasing correspondence $\Gamma_\triangleright$, any dataset that is approximately rationalisable can be supported with a strictly increasing utility u, without loss, even though the choice itself admits a degree of insensitivity to differences among alternatives. Although the agent may agree that more is better from the normative standpoint, they may fail to follow this rule due to imperfect discrimination or imprecision, similarly to the observation in Nielsen and Rehbeck (2020).

Convexity We conclude this section by addressing convexity of preferences. It is well-known since Afriat (1967), Diewert (1973), and Varian (1982) that, within the classic consumer choice framework, any dataset $\mathcal{O}$ that is rationalisable with exact maximisation of a strictly increasing utility, can be supported in this sense with a concave utility u, without loss. We extend this result to approximate utility maximisation.

We say the correspondence $\Gamma_\triangleright$ is *quasiconcave* whenever, for any $x, x' \in X$, $\alpha \in [0,1]$, and $y \in \Gamma_\triangleright\big(\alpha x + (1-\alpha)x'\big)$ there is $z \in \Gamma_\triangleright(x)$, $z' \in \Gamma_\triangleright(x')$, and $\beta \in [0,1]$ such that $y \geq \beta z + (1-\beta)z'$. The correspondence is *concave* if this condition holds for $\beta = \alpha$. Note that, this definition does not require for the values of $\Gamma_\triangleright$ to be convex.

[9] The function u is (strictly) *increasing* if $x\,(>) \geq y$ implies $u(x)\,(>) \geq u(y)$.

Proposition B.3. *Under Assumption B.1, if the function f^t representing the menu A^t is quasiconcave, for all $t \in T$, and the correspondence $\Gamma_{\triangleright}$ is quasiconcave, then the dataset $\mathcal{O}$ is approximately rationalisable for a quasiconcave utility u.*[10]

Proof. Define functions g^t and h^t as in Section B.1.1, for all $t \in T$. Since the function f^t is strictly increasing and quasiconcave, so is g^t, for all $t \in T$. Take any $x, x' \in X$, $\alpha \in [0,1]$, and $z \in \Gamma_{\triangleright}\big(\alpha x + (1-\alpha)x'\big)$. By assumption, there is some $z \in \Gamma_{\triangleright}(x)$, $z' \in \Gamma_{\triangleright}(x')$, and $\beta \in [0,1]$ such that $y \geq \beta z + (1-\beta)z'$. This implies that

$$g^t(y) \;\geq\; g^t\big(\beta z + (1-\beta)z'\big) \;\geq\; \min\big\{g^t(z), g^t(z')\big\} \;\geq\; \min\big\{h^t(x), h^t(x')\big\},$$

where the inequalities follow from monotonicity of g^t, quasiconcavity of g^t, and the definition of h^t, respectively. By taking the infimum over the left hand-side, we conclude that $h^t\big(\alpha x + (1-\alpha)x'\big) \geq \min\big\{h^t(x), h^t(x')\big\}$. Hence, the function h^t is quasiconcave, for all $t \in T$. Given that quasiconcavity is preserved by the *min* operator, this suffices for the utility u specified in Theorem B.1 to be quasiconcave. □

Under some additional assumptions, we can guarantee that the utility u that approximately rationalises the observations is concave, without loss of generality.

Proposition B.4. *Under Assumptions B.1 and B.2, if the function f^t representing the menu A^t is continuous and concave, for all $t \in T$, and the correspondence $\Gamma_{\triangleright}$ is concave, then the dataset $\mathcal{O}$ is approximately rationalisable for a concave utility u.*[11]

Proof. Define function h^t as in the proof of Proposition B.1, for all $t \in T$. Take any $x, x' \in X$, $\alpha \in [0,1]$, and $z \in \Gamma_{\triangleright}\big(\alpha x + (1-\alpha)x'\big)$. By assumption, there is $z \in \Gamma_{\triangleright}(x)$, $z' \in \Gamma_{\triangleright}(x')$ such that $y \geq \alpha z + (1-\alpha)z'$. By monotonicity and concavity of f^t,

$$f^t(y) \;\geq\; f^t\big(\alpha z + (1-\alpha)z'\big) \;\geq\; \alpha f^t(z) + (1-\alpha)f^t(z') \;\geq\; \alpha h^t(x) + (1-\alpha)h^t(x').$$

Once we take the infimum over the left hand-side of the inequality, we conclude that the function h^t is concave, for all $t \in T$. Since the *min* operator preserves concavity, this suffices to show that the utility u specified in Theorem B.1 is concave. □

[10] A function $f : X \to \mathbb{R}$, defined over a convex domain X, is *quasiconcave* if, for any $x, x' \in X$ and $\alpha \in [0,1]$, we have $f\big(\alpha x + (1-\alpha)x'\big) \geq \min\big\{f(x), f(x')\big\}$.

[11] A function $f : X \to \mathbb{R}$, defined over a convex domain X, is *concave* if, for any $x, x' \in X$ and $\alpha \in [0,1]$, we have $f\big(\alpha x + (1-\alpha)x'\big) \geq \alpha f(x) + (1-\alpha) f(x')$.

The correspondence $\Gamma_{\triangleright}(x) := \{y \in X : y > x\}$, the mapping introduced in Example 1, and the one studied in Dziewulski (2020) are all concave.[12] However, since the first one violates Assumption B.2, rationalising the data with a concave utility may be impossible in such a case. This is because concavity implies continuity, which is not guaranteed for this correspondence, as shown in Section 3 of the main paper. Nevertheless, by Proposition B.3, one can rationalise such datasets with a quasiconcave utility.

Propositions B.3 and B.4 can be applied directly to the setup of Afriat (1967), Diewert (1973), and Varian (1982). Since the original framework assumes that the budget set A^t can be represented with the function $f^t(y) := p^t \cdot y - m^t$, for some prices p^t and income m^t, for all $t \in T$, the requirements of the two results are satisfied.

B.2 State-contingent consumption under risk

Here we revisit the results in Section 6.1 of the main paper regarding choice over state-contingent consumption under risk. First, we state the proof of Proposition 7. Then we discuss some additional properties of these models.

B.2.1 Proof of Proposition 7

We prove implication (i) ⇒ (ii). Take any strictly increasing Bernoulli function v such that $u(y) := F\big(v(y_1), v(y_2), \dots, v(y_\ell)\big)$ approximately rationalises the data. By Proposition 1 in the main paper, if xP^*z then $u(x) > u(z)$, for any $x, z \in X$. Moreover, since u is strictly increasing, we have $u(x) > u(y)$, for any $x, y, z \in X$ such that xP^*z and $z \geq y$. In particular, the latter must be true for any $x, y \in \mathcal{X}^\ell$.

To show the converse, let $\mathcal{X} = \{z_1, z_2, \dots, z_K\}$, where $0 = z_1 < z_2 < \dots < z_K$. Take any strictly increasing function $\bar{v} : \mathcal{X} \to \mathbb{R}_+$ specified in statement (ii) and any strictly positive number $a \leq [\bar{v}(z_{k+1}) - \bar{v}(z_k)]/(z_{k+1} - z_k)$, for all $k = 1, \dots, (K-1)$, define an upper semi-continuous and strictly increasing extension $v_a : \mathbb{R}_+ \to \mathbb{R}_+$ of $\bar{v}$ by

$$v_a(z) \;:=\; \sum_{k=1}^{K} \big[\bar{v}(z_k) + a(z - z_k)\big]\chi_{B_k}(y),$$

where $B_k = [z_k, z_{k+1})$, for all $k = 1, \dots (K-1)$, and $B_K = [z_K, \infty)$.

[12] The correspondence $\Gamma_{\triangleright}(x) := \{y \in X : y \gg x\}$ is also concave.

For any set $Z \subseteq \mathbb{R}^\ell_+$, let $\underline{Z} := \{y' \in \mathbb{R}^\ell_+ : y' \leq y,\ \text{for some } y \in Z\}$ be its downward comprehensive hull. Take any $\bar{z} \in \mathbb{R}_+$ such that $\bar{\mathbf{z}} := (\bar{z}, \bar{z}, \ldots, \bar{z}) \geq y$, for all $y \in \bigcup_{t \in T} \underline{A}^t$. Since the menus A^t are bounded, for all $t \in T$, such a number exists and $\bar{z} \geq z_K$. Without loss of generality, suppose that $\bar{z} - z_K \geq z_{k+1} - z_k$, for all $k = 1, \ldots, (K-1)$. By construction of the function v_a, for any $\epsilon > 0$ there is a sufficiently small $a > 0$ such that $\epsilon \geq v_a(z) - \bar{v}(z_k) \geq 0$, for any $z \in [0, \bar{z}]$, where $z_k = \max\{z' \in \mathcal{X} : z' \leq z\}$.[13]

Recall that $x^t P^* y$ if and only if $\Gamma_\triangleright(y) \cap A^t \neq \emptyset$, for any $t \in T$. Equivalently, this is to say that y belongs to the lower inverse $\Gamma^\ell_\triangleright(A^t)$. Since $\Gamma_\triangleright(x) \subseteq \{y \in \mathbb{R}^\ell_+ : y \geq x\}$, for $x \in X$, we have $\underline{\Gamma^\ell_\triangleright(A^t)} \subseteq \bigcup_{t \in T} \underline{A}^t$, and so $\bar{\mathbf{z}} \geq y$, for all $y \in \underline{\Gamma^\ell_\triangleright(A^t)}$. Moreover, for any $y \in \underline{\Gamma^\ell_\triangleright(A^t)}$, there is some $x \in \mathcal{X}^\ell \cap \underline{\Gamma^\ell_\triangleright(A^t)}$ such that $x_i = \max\{z \in \mathcal{X} : z \leq y_i\}$, for all $i = 1, \ldots, \ell$. By by our previous observation, there are numbers $\epsilon, a > 0$ such that

$$F\big(\mathbf{v}_a(x^t)\big) \ = \ F\big(\bar{\mathbf{v}}(x^t)\big) \ > \ F\big(\bar{\mathbf{v}}(y') + \epsilon\mathbf{1}\big) \ \geq \ F\big(\mathbf{v}_a(y)\big),$$

for any $y \in \underline{\Gamma^\ell_\triangleright(A^t)}$ and some $y' \in \mathcal{X}^\ell \cap \underline{\Gamma^\ell_\triangleright(A^t)}$, where $\mathbf{v}(y) := \big(v(y_1), v(y_2), \ldots, v(y_\ell)\big)$, for any function v, and $\mathbf{1}$ is the ℓ-dimensional unit vector.

For each $t \in T$, take any such a and denote it a_t. Define an upper semi-continuous and strictly increasing function $v : \mathbb{R}_+ \to \mathbb{R}_+$ by $v(z) := \min\{v_{a_t}(z) : t \in T\}$. Moreover, let $u(y) := F\big(\mathbf{v}(y)\big)$, which is strictly increasing and satisfies $u(x^t) > u(y)$, for all $y \in \Gamma^\ell_\triangleright(A^t)$ and $t \in T$. Since $\Gamma_\triangleright(x) \subseteq \{y \in \mathbb{R}^\ell_+ : y > x\}$, for all $x \in X$, this suffices to show that both $x \in \Gamma_\triangleright(y)$ and xP^*y imply $u(x) > u(y)$. By Proposition 1, there is a threshold function δ such that u approximately rationalises the data.

B.2.2 Related results

Continuity First, we address the question of continuity of the Bernoulli function v. Suppose that the menu A^t is compact, for all $t \in T$, and the relation $\triangleright$ satisfies Assumption 2(ii). Specifically, this means that the lower inverse of the correspondence $\Gamma_\triangleright$ is compact-valued over the space of compact menus. In other words, the set $\Gamma^\ell_\triangleright(Z) := \{x \in X : \Gamma_\triangleright(x) \cap Z \neq \emptyset\}$ is compact, for any compact $Z \subseteq X$. We claim that this suffices for the Bernoulli function v specified in Proposition 7 to be continuous.

Indeed, in such a case, the lower inverse $\Gamma^\ell_\triangleright(A^t)$ is compact, for all $t \in T$, as is its

[13] It suffices to take any strictly positive $a \leq \epsilon/(\bar{z} - z_K)$

downward comprehensive hull $\underline{\Gamma^{\ell}_{\triangleright}(A^t)}$.[14] Since $\mathcal{X}^{\ell}$ is finite, there is a closed neighbourhood V of $\underline{\Gamma^{\ell}_{\triangleright}(A^t)}$ such that $\mathcal{X}^{\ell} \cap \underline{\Gamma_{\triangleright}}^{\ell}(A^t) = \mathcal{X}^{\ell} \cap V$. Denote $\underline{B}^t := V \cup \{y \in \mathbb{R}^{\ell}_+ : y \leq x^t\}$, which is compact and contains $\underline{\Gamma^{\ell}_{\triangleright}(A^t)}$ in its interior. Moreover, for any strictly increasing function $\bar{v}$ specified as in statement (ii) of Proposition 7, we have $F\big(\bar{\mathbf{v}}(x^t)\big) > F\big(\bar{\mathbf{v}}(y)\big)$, for all $y \in (\underline{B}^t \cap \mathcal{X}^{\ell}) \setminus \{x^t\}$. By Theorem 1 in Polisson et al. (2020), there is a continuous and strictly increasing extension v of $\bar{v}$ such that $F\big(\mathbf{v}(x^t)\big) > F\big(\mathbf{v}(y)\big)$, for all $y \in \underline{\Gamma^{\ell}_{\triangleright}(A^t)}$. The rest of the result follows from Proposition 1 in the main paper.

Variable aggregator Proposition 7 can be extended to the case where the aggregator function varies across observations. Formally, consider a collection of continuous and strictly increasing functions $F_t : \mathbb{R}^{\ell}_+ \to \mathbb{R}$, for all $t \in T$. We claim that there is a strictly increasing Bernoulli function $v : \mathbb{R}_+ \to \mathbb{R}_+$ and a threshold δ_t such that

$$y \in A^t \text{ implies } F_t\big(\mathbf{v}(x^t)\big) + \delta_t(y) \geq F_t\big(\mathbf{v}(y)\big),$$

for all $t \in T$, if and only if there is a function $\bar{v} : \mathcal{X} \to \mathbb{R}_+$ such that $F_t\big(\bar{\mathbf{v}}(x^t)\big) > F_t\big(\bar{\mathbf{v}}(y)\big)$, for any $t \in T$ and $y \in \mathcal{X}$ satisfying $x^t P^* z$ and $z \geq y$, for some $z \in X$.

Indeed, partition the set T into disjoint subsets $T_1, T_2, \ldots, T_K$ such that $t, t' \in T_k$ implies $F_t = F_{t'}$, for all $k = 1, \ldots, K$. By Proposition 7, our claim is true for any sub-dataset $\mathcal{O}_k = \{(A^t, x^t) : t \in T_k\}$, for all k. One can show that this holds for the entire dataset $\mathcal{O}$ for the Bernoulli function $v(z) := \min\{v_k(z) : k = 1, \ldots, K\}$.

Preference symmetry In some cases, the utility u of the agent may depend only on the distribution of consumption in a portfolio x, rather than the precise allocation of consumption to each state. Formally, we say that such a utility function is *symmetric*. That is, for any bundle $x \in X$ and permutation σ on $\{1, 2, \ldots, \ell\}$, we have $u(x) = u(x_\sigma)$, where we denote $x_\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(\ell)})$. For example, this is true when u takes the expected utility formulation when all states $s \in S$ are equally probable, i.e., we have $u(x) = \sum_{s=1}^{\ell} (1/\ell) v(x_s)$, for some Bernoulli function v.

Whenever a dataset $\mathcal{O}$ is approximately rationalisable with a symmetric utility u, one would expect the corresponding threshold function δ to be symmetric as well. That is, the agent should be equally imprecise regarding a bundle x as with its permutation x_σ. This is indeed true, without loss of generality.

[14] The *lower inverse* of $\Gamma_{\triangleright}$ is given by $\underline{\Gamma^{\ell}_{\triangleright}}(A) := \{x \in X : \Gamma(x) \cap A \neq \emptyset\}$. By Assumption 2 in the main body of the paper, values of the correspondence are compact, for any compact set A.

Proposition B.5. *Suppose that* $\Gamma_{\triangleright}(x_\sigma) = \{y_\sigma : y \in \Gamma_{\triangleright}(x)\}$, *for any* $x \in X$ *and any permutation* σ. *If the dataset* $\mathcal{O}$ *is approximately rationalisable for a symmetric utility* u *and some threshold* δ, *then the function* δ *is symmetric, without loss of generality.*

Proof. Suppose that the dataset $\mathcal{O}$ is rationalisable with a symmetric utility u and some threshold δ', and define $\delta(y) := \max\{\delta'(y_\sigma) : \text{for some } \sigma\}$, which is well-defined and symmetric. We claim that u, δ approximately rationalise $\mathcal{O}$. First, we show that the model is $\triangleright$-monotone. Take any $y \in \Gamma_{\triangleright}(x)$. By assumption, we have $y_\sigma \in \Gamma_{\triangleright}(x_\sigma)$. Since u, δ' rationalise the data, there is some permutation σ such that $u(y) - \delta(y) = u(y_\sigma) - \delta'(y_\sigma) > u(x_\sigma) = u(x)$. To show that the model rationalises the data, take any $t \in T$ and $y \in A^t$. Then, $u(x^t) \geq u(y) - \delta'(y) \geq u(y) - \delta(y)$. □

The additional restriction on the correspondence $\Gamma_{\triangleright}$ imposes symmetry on the monotonicity of choice. Clearly, the condition holds for $\triangleright = >$. Similarly, so does the relation in Example 1, as long as $\lambda_s = \lambda_{s'}$, for all $s, s' = 1, \ldots, \ell$, and the mapping $\Gamma_{\triangleright}(x) = \{\lambda' x : \lambda' \geq \lambda\}$, for some $\lambda > 1$, in Dziewulski (2020).

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