On the (Im-)Possibility of Representing Probability Distributions as a Difference of I.I.D. Noise Terms

Christian Ewerhart† Marco Serena‡

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Abstract. A random variable is *difference-form decomposable (DFD)* if it may be written as the difference of two i.i.d. random terms. We show that densities of such variables exhibit a remarkable degree of structure. Specifically, a DFD density can be neither approximately uniform, nor quasiconvex, nor strictly concave. On the other hand, a DFD density need, in general, be neither unimodal nor logconcave. Regarding smoothness, we show that a compactly supported DFD density cannot be analytic and will often exhibit a kink even if its components are smooth. The analysis highlights the risks for model consistency resulting from the strategy widely adopted in the economics literature of imposing assumptions directly on a difference of noise terms rather than on its components.

Keywords. Differences of random variables · Density functions · Characteristic function · Uniform distribution

JEL-Codes. C46 – Specific Distributions, Specific Statistics; C6 – Mathematical Methods, Programming Models, Mathematical and Simulation Modeling

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†) (corresponding) Department of Economics, University of Zurich, Schönberggasse 1, 8001 Zurich, Switzerland; christian.ewerhart@econ.uzh.ch.

‡) Max Planck Institute for Tax Law and Public Finance, Marstallplatz 1, 80539 Munich, Germany; marco.serena@tax.mpg.

1. Preliminaries

1.1 Introduction

In a large variety of economic models, uncertainty enters the framework in the form of the difference of two i.i.d. random variables, say ε_1 and ε_2. For instance, in a rank-order tournament á la Lazear and Rosen (1981), individual performance is the sum of input and some randomness, and the winner is determined by comparing levels of individual performance across contestants. The economic prediction (or power of statistical test) may then crucially depend on the distribution of the difference of the noise terms, $\varepsilon_1 - \varepsilon_2$. This is similarly the case in a variety of other settings, including contests (Hirshleifer, 1989; Che and Gale, 2000), models of location choice (Rosen, 1979; Roback, 1982), vertical differentiation (Lin, 1988), probabilistic voting (Lindbeck and Weibull, 1987), random utility (Becker et al., 1963; Goeree et al., 2005), and paired comparisons (Thurstone, 1927; Bradley and Terry, 1952). For the applied economist entrusted with the analysis of such models, it may appear natural and convenient to impose assumptions directly on the distribution of the difference rather than on the distribution of the components ε_1 and ε_2. This approach, however, is not entirely innocuous. Indeed, as will be discussed, certain familiar probability distributions, such as the uniform distribution, simply cannot be represented as the difference of two i.i.d. random variables. This fact does not seem to be widely known and is sometimes neglected in economic modeling. Thus, there is the risk of ending up with an inconsistent set of assumptions. At the same time, there does not seem to be a single reference that offers help on this issue.[1]

The present paper aims at providing an initial, systematic, and accessible study of the class of random variables that correspond to the difference of two i.i.d. random variables. We refer to such random variables as *difference-form decomposable (DFD)*. It turns out that density functions of DFD random variables exhibit a remarkable degree of structure. Starting from the important special case of the uniform

[1] The related literature will be reviewed later in this section.

distribution, the analysis identifies several broad classes of distributions that have the property of not being DFD. We show that random variables with an approximately uniform, quasiconvex, or strictly concave density function cannot be DFD. The observation that a strictly concave density never corresponds to a DFD random variable may be of particular interest. To prove this *necessary* condition, we recycle the argument underlying Pólya's (1949) *sufficient* condition for a function to be the characteristic function of a real-valued random variable. To illustrate the applicability of this result, we point out that the strictly concave beta density is never DFD. We also show that other properties that may be expected from a DFD density, such as unimodality or logconcavity, need not hold in general (but do so if the components possess these properties).

To study the smoothness properties of DFD densities, we analyze the limit behavior of characteristic functions using a theorem of Erdélyi (1955). Results are obtained for compactly supported densities. Specifically, it turns out that the continuous differentiability of a DFD density implies that it necessarily vanishes at the lower and upper boundaries of its support interval. A similar relationship holds for the higher derivatives. Going to the limit, we find that a compactly supported DFD density is never analytic, i.e., there must be at least one point in its support interval where the density cannot be approximated arbitrarily well by its Taylor series. This necessary condition is easy to apply. E.g., we use it to show that neither the beta distribution with integer parameter nor the raised cosine distribution can be DFD. We also point out that a DFD density will often exhibit kinks even if its difference-form components are smooth.

We go on and explore sufficient conditions for difference-form decomposability. We note that any random variable with the property that the square root of its characteristic function is a *positive definite* function is DFD. An example are families of infinitely divisible distributions. This sufficient condition is strong enough to allow the computation of the (identical) densities of the components ε_1 and ε_2 from a given distribution of the difference $\varepsilon_1 - \varepsilon_2$. Several extensions will be offered as well. Specifically, we will discuss distributions with finite support, functional inequalities (i.e., inequalities that restrict the values of DFD densities), and ratio-form decomposability. The last extension regarding ratio-form decomposability sheds light on Jia's (2008) characterization of the Tullock contest technology.

Our conditions will be formulated without any reference to the complex numbers. In particular, the characteristic function will be generally expressed as a cosine transform, which is an ordinary integral of a real-valued integrand. As we believe, this helps intuition but also simplifies the application of our results.

1.2 Economic motivation

Despite Lazear and Rosen (1981) having noted the impossibility of an i.i.d. difference being uniform,[2] this fact does not seem to be widely known. Screening the economics literature on the before-mentioned applications where difference-form decomposability emerges quite naturally, we found numerous papers assuming uniform differences. Very few of those papers, however, are explicit about the problem.

Working with an inconsistent set of assumptions is risky not only because the conclusions may become shaky but also because the intuitive interpretation of the model may become difficult. In a standard tournament, for instance, assuming that $\varepsilon_1 - \varepsilon_2$ is uniform would render the marginal probability of winning independent of the opponent's effort, and therefore blur the borderline between relative performance evaluation and individual contracts.[3] Similarly, in a model of location choice or in a random utility model, assuming a non-DFD stochastic difference of the utility of two options would be at odds with the usual understanding that individual choices have a well-defined utility level after the resolution of uncertainty. Finally, a statistical test might be biased if the difference of two noise terms is assumed to follow a distribution that is not DFD.

[2] Lazear and Rosen (1981, p. 860) wrote "Since g is symmetric and nonuniform [...]", where g is the distribution of the difference between the two (worker-idiosyncratic) i.i.d. random components.

[3] A number of notable recent contributions has stressed the salience of the shape of noise for economic predictions (e.g., Morgan et al., 2018; Drugov and Ryvkin, 2020; Morgan et al., 2022).

In those and other economic models, the question of whether a density is DFD naturally emerges. It thus seems important to understand which random variables can be represented as the difference of two i.i.d. terms, and how stochastic properties of the components relate to properties of the difference term.

1.3 Related literature

We are not aware of any work that tried characterizing the set of DFD distributions. Notwithstanding, almost any treatment of the topic of characteristic functions, our main tool of investigation, touches upon the matter (see, e.g., Linnik, 1964, or Lukacs, 1970). Feller (1970) explains how *symmetrization* helps avoiding messy arguments in probability theory, e.g., because the tails of the distribution functions of a random variable and its symmetrization are of comparable magnitude. The celebrated 123 Theorem by Alon and Yuster (1995) offers inequalities satisfied by DFD distribution functions and is, therefore, complementary to our investigation of DFD densities.

The indecomposability of the continuous uniform distribution dates back to M. Puri and Sen (1968, p. 970), who thank Basu and P. Puri for the short proof. We have not seen attempts to generalize this result, however. Our criterion that strict concavity of the density of incompatible with being DFD is related to Pólya's (1949) sufficient condition for being a characteristic function (see also Tuck, 2006).

In optics and crystallography, the task of recovering a measure from its symmetrization or, equivalently, from the modulus of its characteristic function, is known as *phase retrieval*. This problem arises in optics because the measurement of a diffracted wavefront (e.g., resulting from a beam of laser light sent through a gap) gives only the intensity of the wave form rather than its complex amplitude (Patterson, 1935; Walther, 1963; Rosenblatt, 1984). In related work, Giraud and Peschanski (2006) and Gori (2017) studied nonnegative functions whose Fourier transform is likewise nonnegative.[4]

[4] Our sufficient conditions loosely relate to the *Wiener-Khintchine-Kolmogorov criterion* and *convolution roots* studied by Boas and Kac (1945). See also Ehm et al. (2004) and Akopyan and

1.4 Overview of the paper

The remainder of this paper is structured as follows. In Section 2, we introduce the notion of difference-form decomposability. Section 3 deals with conditions on the shape of DFD densities. Section 4 discusses smoothness conditions. Section 5 derives sufficient conditions for difference-form decomposability and a formula for the construction of the difference-form component. Section 6 offers extensions. Section 7 concludes. Technical proofs have been relegated to an Appendix.

2. Difference-form decomposability

In this section, we introduce the class of DFD distributions. We will provide basic definitions in Subsection 2.1, review the necessary background on characteristic functions in Subsection 2.2, survey examples of DFD distributions in Subsection 2.3, and similarly survey examples of distributions that are not DFD in Subsection 2.4.

2.1 Basic definitions

All random variables considered in this paper are assumed to be real-valued. The following concept is central to our analysis.

Definition 1. *Let Z be a random variable. We will say that Z is **difference-form decomposable (DFD)** if there are two i.i.d. random variables X and Y such that $Z \overset{\mathrm{d}}{=} X - Y$.*

The equation $Z \overset{\mathrm{d}}{=} X - Y$ says that Z and $X - Y$ follow the same probability law. When a random variable Z is DFD as specified in Definition 1, then the random variable X is referred to as a *difference-form component* of Z.[5]

To understand the specific nature of the analysis pursued in the present paper, it is important to acknowledge that the components X and Y are required be *iden-*

Efimov (2017). However, while that literature admits such roots to be complex-valued, we are seeking convolution roots that are probability densities, i.e., that are real-valued and nonnegative.

[5] Note that we do not take the absolute value of the difference. Puri and Rubin (1970) and Stadje (1994) studied distributions with the property that the absolute difference $Z = |X - Y|$ is identically distributed as its two i.i.d. components X and Y. Interestingly, if the distribution admits a density, then this property characterizes the exponential distribution.

tically distributed. Indeed, dropping that requirement would lead to an entirely different research question, easily seen to be equivalent to the problem of additive decomposability. That problem, however, has been well-studied (e.g., Linnik, 1964; Linnik et al., 1977). Similarly important is the requirement that X and Y are *independent*. For example, if X and Y are uniform and perfectly negatively correlated, then Z is again uniform (Meyer, 1991; Bagnoli et al., 2001). Thus, the i.i.d. requirement is crucial to all what follows.[6]

Definition 2. *A random variable Z is called **symmetric** (about zero) if $Z \overset{\mathrm{d}}{=} -Z$.*

The following observation is simple but important.

Lemma 1. *Any DFD random variable Z is symmetric.*

Proof. Suppose that Z is DFD. Then, there exist i.i.d. random variables X and Y such that $Z \overset{\mathrm{d}}{=} X - Y$. But then, $-Z \overset{\mathrm{d}}{=} Y - X$, which proves the claim. $\square$

In view of Lemma 1, a DFD random-variable Z with component X is called the *symmetrization* of X.

Suppose that X is a difference-form component of Z. Then the distribution functions of Z and X will be denoted by $G = G(z)$ and $F = F(x)$, respectively. In view of our applications, we will mostly focus on continuous distributions, i.e., on distributions that admit a density. If densities exist, these will correspondingly be denoted by $g = g(z)$ and $f = f(x)$. Moreover, the convolution relationship

$$g(z) = \int_{-\infty}^{\infty} f(x+z) f(x) dx \tag{1}$$

admits an interpretation as an autocorrelation function. To avoid clumsy language, the concepts introduced above, i.e., DFD, difference-form component, and symmetry, will be informally extended to distributions and density functions.

[6] It is immediate to see that if X is a difference-form component of Z, then so is $-X$. In fact, this is even the case for $c + X$ and $c - X$, for any constant $c \in \mathbb{R}$. Thus, any difference-form decomposition, provided it exists, can be unique at most up to reflection at the origin and arbitrary translations.

2.2 Characteristic functions

In general, the *characteristic function* of a random variable Z is defined as $\varphi_Z(t) = E[e^{itZ}] \equiv \int_{-\infty}^{\infty} e^{itz} dG(z)$, where $i = \sqrt{-1}$, and the parameter t is real (e.g., Lukacs, 1970). The relevance of the characteristic function for the problem at hand is the following important observation.

Lemma 2. *Suppose that Z is DFD with component X. Then, $\varphi_Z(t) = |\varphi_X(t)|^2 \geq 0$ for all $t \in \mathbb{R}$, where $\varphi_X(t)$ denotes the characteristic function of X.*

Proof. The characteristic function of $-X$ is given by $\varphi_{-X}(t) = E[e^{-itX}]$. Since $t \in \mathbb{R}$ and X is real-valued, this implies $\varphi_{-X}(t) = \overline{\varphi_X(t)}$, where the upper bar denotes complex conjugation. Hence, by the multiplication theorem for characteristic functions, $\varphi_Z(t) = \varphi_X(t)\varphi_{-X}(t) = \varphi_X(t)\overline{\varphi_X(t)} = |\varphi_X(t)|^2$. In particular, $\varphi_Z(t) \geq 0$ for any $t \in \mathbb{R}$. The claim follows. □

Thus, a necessary condition for Z to be DFD is that its characteristic function is real-valued and nonnegative. Using Lemma 2 as a first numerical check of decomposability of a given density is feasible. However, the reader is cautioned that the computation of Fourier transforms is not trivial (e.g., Ahmed et al., 1974). The nonnegativity condition on the integral transform is, however, not sufficient for Z to be DFD. A counterexample due to Lukacs (1970) is replicated in Table II below. In fact, we will exhibit a more elementary counterexample in our discussion of distributions with finite support.

While the characteristic function is, in general, complex-valued, there is an intuitive representation of $\varphi_Z(t)$ as a *cosine transform* if Z is symmetric (e.g., DFD).

Lemma 3. *If Z is symmetric, then $\varphi_Z(t) = \int_{-\infty}^{\infty} \cos(tz) dG(z)$, for any $t \in \mathbb{R}$.*

Proof. Since Z is symmetric, $E[e^{itZ}] = \frac{1}{2}E[e^{itZ} + e^{-itZ}] = E[\cos(tZ)]$. □

The advantage of expressing the characteristic function in that way is that no reference to complex numbers is needed. In addition, there is an intuitive interpretation now that is not so immediate otherwise. Specifically, the cosine factor works like an *amplitude modulation (AM)*. Amplitude modulation is used in electronic communication, radio transmission, computer modems, etc. to transmit a low-frequency audio signal via a high-frequency radio signal (see, e.g., Carson, 1915). Similarly, the density of Z may be understood to modulate the cosine signal of a given frequency t, so that some information about the distribution of Z is captured in $\varphi_Z(t)$.[7]

Table I. Examples of difference-form decomposable distributions.

Distribution, parameter	**Support**	**Density function** $g(z)$	**Characteristic function** $\varphi_Z(t)$	**Component**	**Support**	**Density function** $f(x)$	**Characteristic function** $\varphi_X(t)$
Normal ($\sigma_Z > 0$)	$\mathbb{R}$	$\frac{1}{\sqrt{2\pi}\sigma_Z}\exp(-\frac{z^2}{2\sigma_Z^2})$	$\exp(-\frac{\sigma_Z^2 t^2}{2})$	Normal ($\sigma_X^2 = \sigma_Z^2/2$)	$\mathbb{R}$	$\frac{1}{\sqrt{2\pi}\sigma_X}\exp(-\frac{x^2}{2\sigma_X^2})$	$\exp(-\frac{\sigma_X^2 t^2}{2})$
Cauchy ($\gamma_Z > 0$)	$\mathbb{R}$	$\frac{\gamma_Z}{\pi(\gamma_Z^2 + z^2)}$	$\exp(-\gamma_Z \lvert t \rvert)$	Cauchy ($\gamma_X = \gamma_Z/2$)	$\mathbb{R}$	$\frac{\gamma_X}{\pi(\gamma_X^2 + x^2)}$	$\exp(-\gamma_X \lvert t \rvert)$
Logistic ($\beta > 0$)	$\mathbb{R}$	$\frac{\exp(-z/\beta)}{\beta(1+\exp(-z/\beta))^2}$	$\frac{\pi\beta t}{\sinh(\pi\beta t)}$	Gumbel	$\mathbb{R}$	$\frac{1}{\beta}\exp\left(-\frac{x}{\beta} - \exp(-\frac{x}{\beta})\right)$	$\Gamma(1 - i\beta t)$
Laplace ($\lambda > 0$)	$\mathbb{R}$	$\frac{\lambda}{2}\exp(-\lambda \lvert z \rvert)$	$\frac{\lambda^2}{\lambda^2 + t^2}$	Bessel	$\mathbb{R}$	$\frac{\lambda}{\pi}K_0(\lambda \lvert x \rvert)$	$\frac{\lambda}{\sqrt{\lambda^2 + t^2}}$
				Exponential	$\mathbb{R}_{\geq 0}$	$\lambda\exp(-\lambda x)$	$\frac{\lambda}{\lambda - it}$
Triangular ($c > 0$)	$[-2c, 2c]$	$\max\left\{\frac{1}{2c} - \frac{\lvert z \rvert}{4c^2}, 0\right\}$	$\frac{\sin^2(ct)}{c^2t^2}$	Uniform	$[-c, c]$	$\begin{cases} 1/(2c) & \text{if } x \in [-c, c] \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin(ct)}{ct}$

2.3 Examples of distributions that are DFD

Table I provides details on the difference-form decomposability of several standard families of probability distributions. See also Figure 1 for an illustration of the corresponding density functions. The list starts with the examples of the normal and Cauchy distributions, both of which are infinitely divisible. As these examples suggest, any infinitely divisible distribution that is symmetric about zero is DFD. Particularly strong implications are feasible for normal distributions. Indeed, by Cramér's (1936) theorem, any non-degenerate component of a normal distribution

[7] The moment-generating function $E[e^{tZ}]$ has the same advantage of being real-valued, but its use would obscure the idea of the Fourier analysis. In addition, the moment-generating function does not exist for all distributions of interest (e.g., for the Cauchy distribution).

is normal. Thus, the difference-form decomposition shown in Table I is essentially unique in the normal case. Conversely, as shown by Carnal and Dozzi (1989), this uniqueness property regarding difference-form decomposability is shared by no other infinitely divisible distribution.

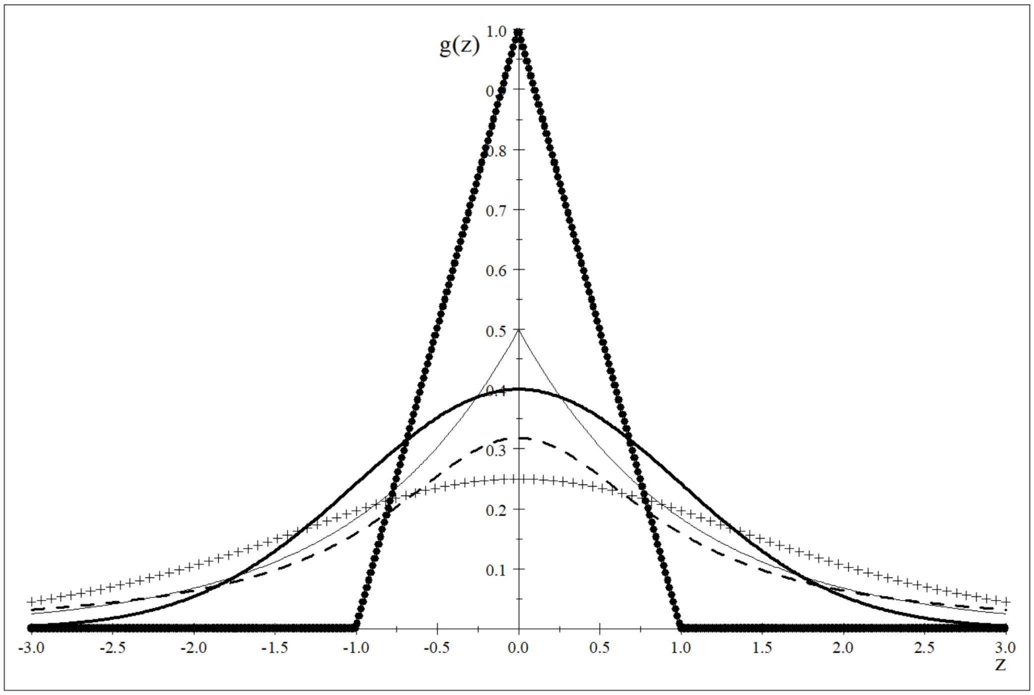

Figure 1. Examples of difference-form decomposable distributions. Shown are densities of the normal (solid), Cauchy (dashed), logistic (crossed), Laplace (thin), and triangular (dotted) distributions.

In the general case (i.e., if the distribution is not necessarily infinitely divisible), one can still show that any representation of a random variable as the difference of two *symmetric* i.i.d. noise terms is essentially unique. However, the difference-form decomposition is not unique in general if one allows for components that are not symmetric. For instance, the Laplace distribution may be represented either as the difference of two symmetric Bessel distributions or as the difference of two exponential distributions (which are not symmetric).

An example of a DFD distribution with compact support is the *triangular* distribution.[8] Apart from this example, we do not know of any DFD distribution with compact support that has been (correctly) used in applications.

[8]The fact that the difference of two i.i.d. uniform random terms follows a triangular density has found numerous applications. See, e.g., Bull et al. (1987), Schotter and Weigelt (1992), Prendergast (2002), Altmann et al. (2012), Moldovanu et al. (2012), and Ewerhart (2016).

Table II. Examples of distributions that are *not* difference-form decomposable. Only distributions that are symmetric with respect to the origin are listed (cf. Lemma 1).

Distribution, parameter	**Support**	**Density function** $g(z)$	**Characteristic function** $\varphi_Z(t)$	**Reference**
Uniform ($c > 0$)	$[-c, c]$	$\begin{cases} 1/(2c) & \text{if } z \in [-c, c] \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin(ct)}{ct}$	Puri and Sen (1968); this paper (Ex. 1)
n/a		*Any approximately uniform*	*n/a*	This paper (Prop. 1)
Beta ($0 < \alpha \leq 2$ or $\alpha \in \mathbb{N}$)	$[-1, 1]$	$\frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha)}(1 - z^2)^{\alpha - 1}$	*Confluent hypergeometric*	This paper (Ex. 2)
n/a		*Any quasiconvex*	*n/a*	This paper (Prop. 2)
n/a		*Any strictly concave*	*n/a*	This paper (Thm. 1)
Raised cosine ($c > 0$)	$[-c, c]$	$\begin{cases} \frac{1 + \cos(z\pi/c)}{2c} & \text{if } z \in [-c, c] \\ 0 & \text{otherwise} \end{cases}$	$\frac{\pi^2 \sin(ct)}{ct(\pi^2 - c^2t^2)}$	This paper (Ex. 3)
n/a		*Any compactly supported analytic*	*n/a*	This paper (Prop. 3)
n/a	$\mathbb{R}$	$\frac{1}{\sqrt{2\pi}} z^2 \exp\left(-\frac{z^2}{2}\right)$	$(1 - t^2)\exp\left(-\frac{t^2}{2}\right)$	Dugué (1957, p. 38); this paper (Lemma 2)
n/a	$\mathbb{R}$	$\sqrt{\frac{2}{3\pi}}\left(\frac{8}{9}z^2 - 1\right)^2 \exp\left(-\frac{2}{3}z^2\right)$	$\left(1 - \frac{t^2}{2}\right)^2 \exp\left(-\frac{3}{8}t^2\right)$	Lukacs (1970, p. 127)

2.4 Examples of distributions that are not DFD

Table II lists a number of distributions that are not DFD. Particularly prominent is the example of the uniform distribution. But as will be shown below, the beta distribution and the raised cosine distribution likewise fall in this class. All these examples have compact support. Dugué (1957) constructed a distribution with full support that is not DFD. While that example is still covered by Lemma 2 above, the more complicated example of Lukacs (1970) even has a nonnegative characteristic function, and therefore illustrates the fact that the conclusion of Lemma 2 is only necessary, but not sufficient for a random variable to be DFD. In fact, both distributions are entirely indecomposable (i.e., even allowing for nontrivial heterogeneous factors). The density functions of these examples are illustrated in Figure 2.

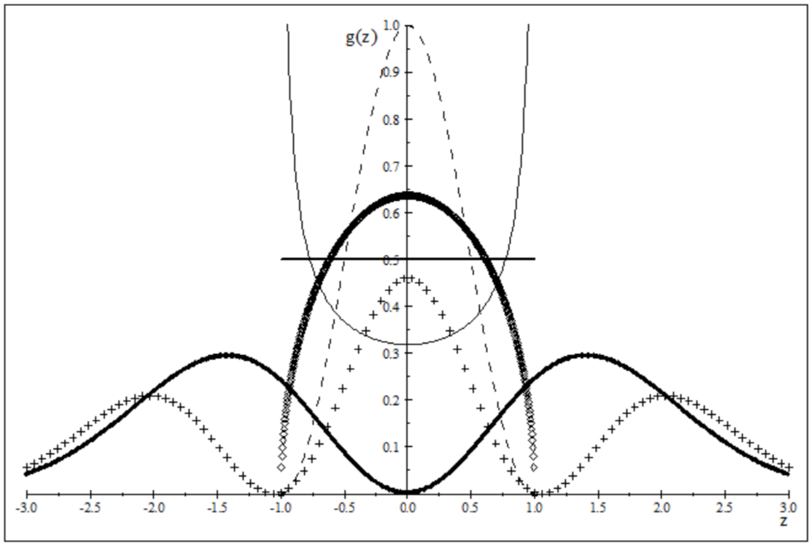

Figure 2. Examples of distributions that are not difference-form decomposable. Shown are the densities of the uniform (solid), raised cosine (dashed), concave beta (diamonds), convex beta (thin), Dugué (bold), and Lukacs (crosses) distributions.

3. Shape conditions

This section starts the formal analysis by deriving a variety of necessary conditions on the shape of DFD distributions. We will discuss uniform and approximately uniform distributions (see Subsection 3.1), quasiconvex densities (see Subsection 3.2), and strictly concave densities (see Subsection 3.3). The section concludes with a discussion of unimodality of the density (see Subsection 3.4).

3.1 Uniform and approximately uniform distributions

The following classic result sets the stage for our analysis.

Example 1. *$Z \sim U[-1, 1]$ is not DFD.*

This observation, a formal proof of which is included in the Appendix, is stated in Puri and Sen (1968, p. 970), for instance.[9] Example 1 also shows that symmetry is not sufficient for a random variable to be DFD. To understand how this observation follows from Lemma 2 (the nonnegativity of the cosine transform), see Figure 3. The

[9] Lewis (1967) characterized the complete set of decompositions of the uniform distribution into arbitrarily many independent, but not necessarily identically distributed, components. See also Tortrat (1969), cited by Rusza (1982-1983), and Topolyan (2014).

point to note is that the signed area between the plotted graph and the horizontal axis is negative. Intuitively, *the uniform density declines too slowly*, so that the integral $\varphi_Z(t)$ turns negative for a suitably chosen value of t, which is inconsistent with difference-form decomposability. For $t = \frac{3\pi}{2}$, for instance, one obtains

$$\varphi_Z(t) = \frac{1}{2}\int_{-1}^{1} \cos(\frac{3\pi z}{2})dz = -\frac{2}{3\pi} < 0. \tag{2}$$

This, however, is in conflict with Lemma 2.[10]

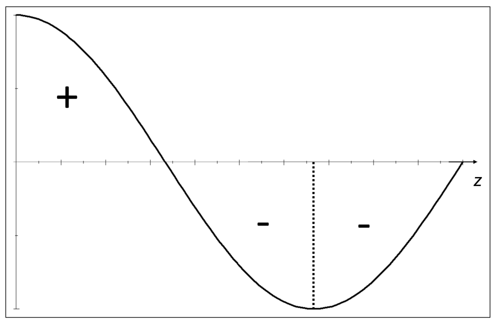

Figure 3. Plot of the function $z \mapsto \cos(\frac{3}{2}\pi z)$ over the interval $[0, 1]$.

As discussed in the Introduction, such an impossibility result is relevant for economic applications. Below, we will see that broad classes of distributions that either contain the uniform distribution as a special case or feature it as a limit case are likewise not DFD.

To start with, one might ask if, despite a uniform distribution not being DFD, there exist two i.i.d. random variables whose difference can at least *approximate* a uniform distribution. The answer is negative.

Proposition 1. *Any density function on the interval* $[-1, 1]$ *that has values in the interval* $(\frac{1}{2} - \delta, \frac{1}{2} + \delta)$, *where* $\delta > 0$ *is small, is not DFD.*

Proof. See the Appendix. □

[10] Further intuition may be gained from considering a discrete setting. Suppose that the components X and Y independently realize as $x = 0$ with probability q_0 and as $x = 1$ with probability q_1. Then, $Z = X - Y$ realizes as $z = 0$ with probability $p_0 = q_0^2 + q_1^2$, and as $z = 1$ with probability $p_1 = q_0 q_1$. Hence, $p_0 - 2p_1 \geq 0$ if Z is DFD. Thus, $p_0 \geq \frac{1}{2}$ and $p_1 \leq \frac{1}{4}$, i.e., the uniform distribution on $\{-1, 0, 1\}$ is not DFD.

The proof shows that $\delta = \frac{1}{6} \simeq 0.166$. To understand why Proposition 1 holds true, one takes another look at Figure 3 and note that the area with the negative weight, where $g(z) > \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$, is precisely twice as large as the area with the positive weight, where $g(z) < \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$. Thus, again, the cosine transform has a negative sign.

3.2 Quasiconvex densities

While Proposition 1 is useful, it cannot deal with the following example.

Example 2. The density of the symmetric beta distribution is given as

$$g(z) = \frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha)}(1 - z^2)^{\alpha - 1} \qquad (z \in [-1, 1]), \tag{3}$$

where $\alpha > 0$ is a shape parameter.[11] For $\alpha \in (0, 1]$, this density is convex, where the boundary case $\alpha = 1$ corresponds to the uniform distribution. For $\alpha \in (1, 2]$, the density is strictly concave. For $\alpha > 2$, however, g is neither convex nor concave.

In the example, the conditions of Proposition 1 are not satisfied. However, the intuition underlying the uniform case admits another generalization, viz. to the class of quasiconvex density functions. We say that a density function g of a symmetric distribution with compact support $[-c, c]$, where $c > 0$, is *quasiconvex* if and only if g is weakly decreasing on $(-c, 0]$ and weakly increasing on $[0, c)$ (i.e., we disregard the boundary points of the support for convenience). Clearly, the uniform distribution satisfies this definition, as does the convex beta density.

Proposition 2. *A quasiconvex density cannot be DFD.*

Proof. See the Appendix. □

This result is again derived by extending the graphical proof of the uniform case. If the symmetric density is strictly declining on $[0, 1]$, the signed integral outlined

[11] See Figure 2 for illustration. Details on this example can be found in the Appendix.

in Figure 3 is smaller than for the uniform distribution, in which case it is already negative.

To see Proposition 2 at work, it suffices to briefly go back to the example of the beta distribution. Specifically, if the density is convex (i.e., if $\alpha \in (0,1]$), then the beta distribution cannot be DFD.

Further below, we will obtain a variation of Proposition 2 saying that a random variable represented by a continuous density function g on $[-1,1]$ does not admit a difference-form component with continuously differentiable density unless g assumes its maximum at the origin.

3.3 Strictly concave densities

For $\alpha \in (1,2]$, the density of the beta distribution is strictly concave. Our previous observations do not apply. This case is, however, covered by the following result, which for us was the least expected finding of the present analysis.

Proposition 3. *A density that is strictly concave on its support cannot be DFD.*

Proof. See the Appendix. $\square$

The proof of Proposition 3 is inspired by Pólya's (1949) sufficient condition for characteristic functions.[12] The first point to note is that a strictly concave density must be compactly supported, say on $[-1,1]$. Next, one notes that the evaluation of the cosine transform at the special value $t = 2\pi$ decomposes the interval $[0,1]$ into four subintervals of length $\frac{1}{4}$. Moreover, for any $z \in [0, \frac{1}{4})$, we know that $\cos(2\pi z) = -\cos(2\pi(\frac{1}{2} - z)) = -\cos(2\pi(\frac{1}{2} + z)) = \cos(2\pi(1 - z))$, as illustrated in

[12] Pólya's sufficient criterion says that a function ϕ is a characteristic function if ϕ is continuous, convex on the positive real line, and satisfies $\phi(0) = 1$. For an account of Pólya's sufficient condition and other sufficient conditions, we refer the reader to Lukacs' (1972) survey.

Figure 4. Hence, the integrand of the cosine transform satisfies

$$\begin{aligned} & g(z)\cos(2\pi z) + g(\tfrac{1}{2}-z)\cos(2\pi(\tfrac{1}{2}-z)) \\ & \quad + g(\tfrac{1}{2}+z)\cos(2\pi(\tfrac{1}{2}+z)) + g(1-z)\cos(2\pi(1-z)) \\ = \; & \underbrace{\left(g(z) - g(\tfrac{1}{2}-z) - g(\tfrac{1}{2}+z) + g(1-z)\right)}_{<0} \underbrace{\cos(2\pi z)}_{>0} < 0, \end{aligned} \tag{4}$$

as a consequence of strict concavity of g. Integrating over $[0, \frac{1}{4})$, the cosine transform is seen to be negative, i.e., $\varphi_Z(2\pi) < 0$, in conflict with Lemma 2. The proof given in the Appendix works with partial integration like Pólya's original proof but captures the very same intuition.

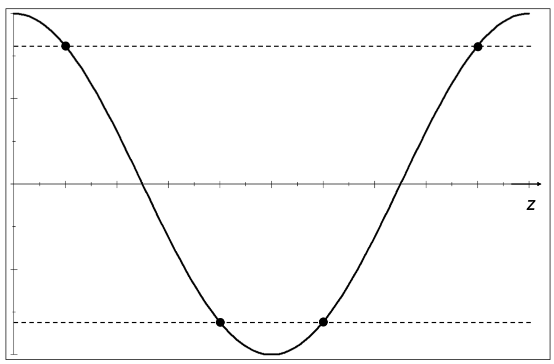

Figure 4. Intuition underlying the proof of Proposition 3.

3.4 Unimodal densities

Unimodal densities are widely used in economics. It is known that the difference (but not necessarily the sum, see Chung, 1953) of two i.i.d. unimodal random variables is necessarily unimodal (Hodges and Lehmann, 1954; Vogt, 1983; see also Barlevy and Neal, 2012, for application). The same is true for strongly unimodal (i.e., logconcave) densities (Ibragimov, 1956; An, 1998). As mentioned before, we will later show that any DFD density with well-behaved components necessarily assumes its maximum at the origin. However, even though unimodality with mode at zero seems to be a common feature of many DFD densities, neither unimodality nor logconcavity are necessary for a density to be DFD. E.g., let $f(x) = \frac{3}{2}$ if $x \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and

$f(x) = 0$ otherwise. Clearly, f is a density. Let X, Y be i.i.d. according to f. Then, the random variable $Z = X - Y$ does not admit a unimodal density g. Unimodality and, similarly, logconcavity are neither sufficient for, say, a symmetric density to be DFD, as follows immediately from Proposition 3.

4. Smoothness conditions

In this section, we will derive necessary conditions that rely on smoothness properties of the DFD density. We first discuss boundary conditions (see Subsection 4.1), then analyticity (see Subsection 4.2), and finally kinks (see Subsection 4.3).

4.1 Boundary conditions

So far, we have evaluated the characteristic function $\varphi_Z = \varphi_Z(t)$ at specific values for t. Additional necessary conditions can be deduced by considering the asymptotic behavior of the characteristic function $\varphi_Z(t)$ for $t \to \infty$. Intuitively, large values for t correspond to the case where the cosine term is changing sign very frequently, so that the integral approaches zero over intervals where g is smooth. And indeed, as discussed in Erdélyi (1955), the asymptotics of φ_Z depend entirely on the behavior of the integrand in the neighborhood of certain distinguished points, called *critical points*. These critical points are either the endpoints of the interval of integration or the points at which the integrand (or some derivative thereof) exhibits a discontinuity. Given this intuition, it should not be too surprising that one may obtain the following auxiliary result for compactly supported DFD densities.

Lemma 4. *Suppose that $g : [-1, 1] \to \mathbb{R}_+$ is N-times continuously differentiable as well as DFD. Then, $g^{(M)}(1) = g^{(M)}(-1) = 0$ for any $M \in \{0, \ldots, N-1\}$.*[13]

Proof. See the Appendix. □

4.2 Analyticity

[13] As usual, $g^{(M)}(z)$ denotes the M-th derivative of g at z, provided it exists. In particular, $g^{(0)}(z) = g(z)$.

A function is *analytic* if it admits derivatives of any finite order and can be extended into a Taylor series at each point of its domain of definition. Examples for analytic functions are polynomials and the exponential function. Sums, differences, and products of analytic functions are likewise analytic. On the other hand, the analyticity condition is violated, e.g., when higher-order differentiability fails or when, even though derivatives of all orders exist, the Taylor series does not converge to the density.

Letting $N \to \infty$ in Lemma 4 and subsequently exploiting the fact that an analytic function is identified by its derivatives at any single point of its domain of definition, we arrive at the following useful observation.

Proposition 4. *Suppose that g is the density of a compactly supported DFD distribution. Then, g is not equal on its support to some analytic function.*

Proof. See the Appendix. □

Proposition 4 may be used to extend our earlier observations regarding the beta distribution. Indeed, the beta density with integer parameter $\alpha \in \mathbb{N} = \{1, 2, \ldots\}$ is a polynomial on $[-1, 1]$, hence analytic. Therefore, the beta density is not DFD for any integer value $\alpha > 0$. In particular, this includes cases (viz., for $\alpha = 3, 4, \ldots$) where the density is neither convex nor concave. A similar example is the raised cosine, which likewise admits an analytic density and consequently is not DFD.

4.3 Kinks

As illustrated by the examples of the triangular and the Laplace distribution, DFD densities may exhibit a kink at the origin (cf. Figure 2). In fact, the triangular density, if considered as a function on the real line, has two additional kinks, viz. at the boundary of its support. The following result shows that such kinks are, under smoothness conditions on the component densities, a quite typical feature of a DFD density with bounded support.

Proposition 5. *Let X and Y be i.i.d. random variables admitting a continuously differentiable density f on $[0, 1]$.*

(i) If $\max\{f(0), f(1)\} > 0$, then g has a kink at the origin.

(ii) If $\min\{f(0), f(1)\} > 0$, then g has kinks also at ± 1.

Proof. See the Appendix. □

Thus, if the component density is positive at *at least one* boundary point of its support interval, then the density g of the DFD distribution necessarily exhibits a kink at the center of its support. If the component density is positive even at *both* boundary points of its support interval (as in the case of the uniform density), then g exhibits additional kinks at the boundary points of its own support interval.

5. Sufficient conditions and the construction of components

This section explores conditions sufficient for a density to be DFD and derives a formula for the construction of the difference-form component under those conditions.

Proposition 6. *Suppose that $\varphi_Z \geq 0$ and that $\sqrt{\varphi_Z}$ is positive definite. Then, Z is DFD, and a difference-form component of Z is given by the density function*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\varphi_Z(t)} \cos(tx) dt. \tag{5}$$

Proof. See the Appendix. □

Positive definiteness of the square root of $\sqrt{\varphi_Z}$ intuitively imposes restrictions on the shape of φ_Z. We will discuss the positive definiteness condition in more detail further below. In principle, any of the sufficient conditions known for characteristic functions may be used to check the conditions of Proposition 6. E.g., in a straightforward application of Pólya's condition, if $\sqrt{\varphi_Z}$ is convex on $\mathbb{R}_{\geq 0}$, then Z is DFD. In applications, however, the often most convenient way to verify positive definiteness of a function is by checking that the inverse Fourier transform (5) is globally nonnegative. For illustration of this approach, we reconstruct the uniform component from the triangular distribution.

Example 3. The characteristic function of the triangular density on $[-1,1]$ is given as $\varphi_Z(t) = \frac{4\sin^2(t/2)}{t^2} \geq 0$. To find a difference-form component, we apply formula (5). This yields

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \sin\left(\tfrac{t}{2}\right)\cos(tx)\frac{dt}{t} && (6)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{\sin\left(t\left(\tfrac{1}{2}+x\right)\right) + \sin\left(t\left(\tfrac{1}{2}-x\right)\right)\right\}\frac{dt}{t} && (7)\\ &= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin t}{t}dt && (8)\\ &= 1. && (9) \end{aligned}$$

Thus, $\sqrt{\varphi_Z(t)}$ is positive definite, and by Proposition 6, the uniform distribution on the unit interval has been retrieved as a difference-form component of the triangular distribution.[14]

Similar calculations are feasible for any of the relevant examples listed in Table I (i.e., with the exceptions of the Gumbel, exponential, and Poisson components).

The proof of Proposition 6 is not deep but abstract. Technically, the assumptions of Proposition 6 ensure that g admits a convolution root that is a symmetric density. This automatically leads to a condition sufficient for difference-form decomposability, since any sum of two symmetric i.i.d. random variables is obviously DFD.[15]

Finally, recall that a characteristic function φ is called *infinitely divisible* if, for every positive integer n, there is a characteristic function ϕ such that $\varphi = \phi^n$. A probability distribution is called infinitely divisible if its characteristic function is

[14] In fact, this decomposition is unique (cf. O'Neill and Walther, 1963).

[15] Notably, the converse statement is not generally true. I.e., there are DFD densities that do not admit a symmetric difference-form component. E.g., the component X given by the density $f(x) = 2x$ on $[0,1]$ cannot be replaced by any symmetric component (Carnal and Dozzi, 1989, p. 168). In fact, the same is true for any strictly monotone component density (Gushchin and Küchler, 2005).

infinitely divisible (e.g., Lukacs, 1972, p. 19).

Corollary 1. *Any symmetric infinitely divisible distribution is DFD.*

Proof. See the Appendix. □

6. Extensions

6.1 Distributions with finite support

One might wonder if a consideration of distributions with finite support might help to shed light on the class of DFD distributions. The insights from such exercise are limited, however. To understand why, consider the simplest case of an equidistant grid. Suppose given a vector of probabilities $(p_0, p_1, \ldots, p_N)$, for $N \geq 1$, such that $p_0 + 2\sum_{n=1}^{N} p_n = 1$. The interpretation is that p_n corresponds to the probability that the symmetric random variable Z realizes as $n \in \{0, \ldots, N\}$, and for any $n > 0$, likewise to the probability that Z realizes as $-n$. A difference-form component X, if it exists, may then be represented by a vector of probabilities $(q_0, \ldots, q_N)$, where q_n denotes the probability that X realizes as $n \in \{0, \ldots, N\}$. The system of equations to be solved is the following (cf. Hodges and Lehmann, 1954):

$$p_0 = q_0^2 + \ldots + q_N^2 \tag{10}$$

$$p_1 = q_0 q_1 + \ldots + q_{N-1} q_N \tag{11}$$

$$\vdots$$

$$p_{N-1} = q_0 q_{N-1} + q_1 q_N \tag{12}$$

$$p_N = q_0 q_N \tag{13}$$

The set of vectors $(p_0, p_1, \ldots, p_N)$ for which a solution $(q_0, \ldots, q_N)$ exists may, in principle, be characterized in explicit terms.

Proposition 7. *For any fixed $N \geq 2$, the set of DFD discrete distributions forms a semi-algebraic set, i.e., it may be described by a finite number of algebraic identities* *and inequalities in the variables* $(p_0, p_1, \ldots, p_N)$.

Proof. See the Appendix. $\square$

We illustrate this general result with the help of a tractable example, the details of which may be found in the Appendix.

Example 4. For $N = 2$, a distribution given by (p_0, p_1, p_2) is DFD if and only if $p_1 \leq \frac{1}{4}$ and $p_2 \leq \frac{\left(1+\sqrt{1-4p_1}\right)^2}{16}$. In this case, the set of difference-form components may be described in explicit terms.

Figure 5 illustrates the set of DFD distributions for $N = 2$ as the area below the thick curve. The straight lines correspond to three nonnegativity constraints of the discrete cosine transform (defined in analogy to the continuous case), which are $p_1 \leq \frac{1}{4}$, $p_1 + p_2 \leq \frac{1}{3}$, and $p_1 + 2p_2 \leq \frac{1}{2}$. As can be seen, the set of DFD distributions is a *strict* subset of the distributions with nonnegative cosine transform. E.g., $(p_0, p_1, p_2) = (\frac{1}{3}, \frac{1}{6}, \frac{1}{6})$, satisfies the nonnegativity constraints but is not DFD. This observation is in line with the corresponding fact for the continuous case, i.e., that a nonnegative cosine transform is a necessary, but not a sufficient condition for difference-form decomposability.

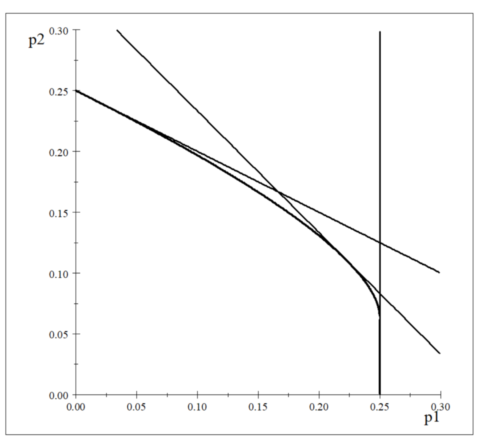

Figure 5. Illustration of the set of DFD distributions.

For $N \geq 3$, however, running the Tarski-Seidenberg algorithm that leads to a characterization of the set of DFD distributions through a finite number of algebraic identities and inequalities, while theoretically feasible, becomes substantially more involved. Moreover, determining a difference-form component in explicit form ceases to be tractable for $N \geq 4$.[16]

6.2 Functional inequalities

Additional necessary conditions on the shape of a DFD density may be derived if the corresponding characteristic function is *integrable*. The following lemma provides a simple condition sufficient for this to be the case.

Lemma 5. *Suppose that the difference-form component X of some random variable Z is distributed according to some continuously differentiable density function f_X with compact support. Then, φ_Z is integrable.*

Proof. See the Appendix. $\square$

We are now all set to state our result regarding functional inequalities.

Proposition 8. *Suppose that a DFD random variable Z is distributed according to some density function g. Assume also that φ_Z is integrable. Then, g is positive definite. In particular, the following inequalities hold:*

(i) $g(z) \leq g(0)$ for any $z \in \mathbb{R}$;

(ii) if $g(z) = 0$ outside of $[-1, 1]$, then $g(z) \leq g(0)\cos(\frac{\pi}{1+\lfloor 1/z \rfloor})$ for any $z \in (0, \frac{1}{2}]$.[17]

Proof. See the Appendix. $\square$

Thus, a DFD density with integrable characteristic function is positive definite, which is a fairly strong property. E.g., it follows that any continuous DFD density assumes its maximum at the origin (even if the component is not unimodal). This observation is intuitively in line with the interpretation of g as an autocorrelation

[16] However, in analogy to Proposition 6, a *symmetric* difference-form component (i.e., satisfying $q_n = q_{N-n}$ for $n \in \{0, \ldots, \lceil \frac{N}{2} \rceil - 1\}$), can be computed (provided it exists) from the system (10)-(13).

[17] Here, $\lfloor 1/z \rfloor$ denotes the largest integer weakly smaller than the ratio $1/z$.

function (cf. Section 2) and admits a simple direct proof.[18] The less obvious inequality in part (ii) says that $g(z)/g(0)$ remains weakly below the staircase function displayed in Figure 6. This inequality was shown to be an implication of positive definiteness by Boas and Kac (1945). It should be noted that Proposition 8 is in a sense equivalent to Lemma 2 and, hence, equally strong in its implications. Thus, the two results may be seen as complementary methods for testing for difference-form decomposability.

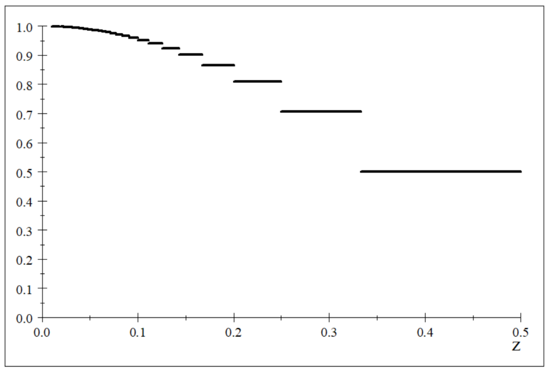

Figure 6. Illustration of the inequality in Proposition 8(ii).

To prove that g is positive definite under the conditions of Proposition 8, one combines two powerful theorems in the literature on characteristic functions, Bochner's theorem, and the Fourier Inversion Theorem. Bochner's theorem says that a function ϕ is a characteristic function if and only if ϕ is continuous, positive definite, and satisfies $\phi(0) = 1$.[19] The proof then proceeds as follows. By the Fourier Inversion Theorem, we may reconstruct a density from its characteristic function. In the case of a symmetric density, however, the cosine transform of the characteristic function coincides (up to a constant factor) with the transform that generates the characteristic function from the density. We may therefore interpret φ_Z, provided it is integrable, as a density $\widehat{g}$ (after suitable normalization) of some "dual" random

[18] Specifically, it suffices to note that

$$g(0) - g(z) = \frac{1}{2} \int_{-\infty}^{\infty} (f(x+z) - f(x))^2 \, dx \geq 0.$$

[19] A real-valued function ϕ is *positive definite* if, for every $n \in \{1, 2, \ldots\}$ and $x_1, \ldots, x_n \in \mathbb{R}$, the matrix $[\phi(x_i - x_j)]_{i,j=1}^n$ is positive semidefinite.

variable $\widehat{Z}$ and apply Bochner's theorem to $\widehat{Z}$ to derive necessary properties of its characteristic function $\widehat{\varphi}$ that happens to coincide with g (again, up to a constant factor).

Proposition 8 may be used to obtain another partial result for the beta distribution in the left-over case where $\alpha > 2$ (and α not an integer). Indeed,

$$\frac{\partial^2}{\partial z^2} \cos(\frac{\pi}{1+1/z})\Big|_{z=0} = -\pi^2, \tag{14}$$

$$\frac{\partial^2}{\partial z^2}(1-z^2)^{\alpha-1}\Big|_{z=0} = -2\,(\alpha-1)\,. \tag{15}$$

Thus, for $\alpha < \alpha^*$, where $\alpha^* = \frac{\pi^2}{2} + 1 \simeq 5.93$, making use of Lemma 5, the density of the beta distribution cannot be represented as the difference-form convolution of two i.i.d. continuously differentiable densities.[20]

6.3 Ratio-form decomposability

Our results have equivalent formulations for distributions that may be represented as the *ratio* $\boldsymbol{Z} = \boldsymbol{X}/\boldsymbol{Y}$ of two i.i.d. random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$, following Huntington (1939) and Curtiss (1941). Given our focus on applications in economic theory, we focus on the case that the components assume positive values only. Then, however, it is *always* feasible to transform the argument of the distribution function using the logarithm. E.g., any lognormal distribution may be expressed as the ratio of two i.i.d. lognormal distributions. This observation also has some implications for contest success functions of the ratio-form, such as Tullock's (1980). If individual, multiplicative noise is distributed according to the inverse exponential distribution (Jia, 2008; Fu and Lu, 2012), then the density is given as $\mathbf{f}(\boldsymbol{x}) = \alpha m \mathbf{x}^{-(m+1)} \exp(-\alpha \boldsymbol{x}^{-m})$ on $[0, \infty)$, where $\alpha > 0$ and $m > 0$ are parameters. Transforming the corresponding distribution function $\boldsymbol{F}(\boldsymbol{x}) = \exp(-\alpha \boldsymbol{x}^{-m})$ via the transform $\boldsymbol{x} = \exp(x)$ leads to the distribution function of the corresponding additive noise term, which is

[20] The 123 Theorem (Alon and Yuster, 1995), suitably reformulated, says that if G is the distribution function of a DFD random variable, then $G(b) - G(a) \leq 2(\lceil b/a \rceil - 1)(G(a) - G(0))$, for any $b > a > 0$, where $\lceil b/a \rceil$ denotes the lowest integer weakly larger than the ratio b/a. That result, however, cannot be used to easily derive Proposition 8 or any other result of the present paper.

$F(x) = \exp(-\alpha \exp(-mx))$, i.e., a Gumbel distribution (cf. Table I). Therefore, the stochastic foundation of the Tullock contest is a direct consequence of the fact that the logistic distribution is the symmetrization of the Gumbel distribution.

7. Concluding remarks

In numerous economic models, uncertainty enters through a noise term that corresponds to the difference of two i.i.d. random variables. Our results allow to decide in many cases which distributions admit an i.i.d. difference-form decomposition and which do not. This sheds some light on the elusive class of DFD distributions.

Our analysis shows that imposing distributional assumptions on the difference term is far from innocuous. Even intuitively plausible assumptions on the density of the difference term, such as approximate uniformity, quasiconvexity, strict concavity, or compact support combined with analyticity are *always* inconsistent, and therefore entail the risk of ending up with incorrect economic conclusions. We were also able to show that the lack of decomposability of the uniform distribution, which is particularly relevant for economic applications, is a robust problem, rather than an isolated pathological case of limited relevance.

To avoid the numerous pitfalls identified by the present studies, the applied economist has essentially three choices. First, if a specific distribution with compact support is desired, the triangular distribution, with its uniform components, seems to be the most natural assumption. If instead the uniform difference must be chosen (e.g., to ensure tractability), then this would require both a rationale (like perfect negative correlation, see Meyer, 1991, or Bagnoli et al., 2005) and a discussion of how a change in the assumptions about the distribution of noise would likely affect the results. Second, if a specific distribution with full support is desired, then any infinitely divisible distribution, with components taken from the same family, will do the job. Common examples are the normal and the Cauchy distributions, but it may be kept in mind that any infinitely divisible distribution will work. Finally, if the researcher aims at keeping distributional assumptions at a minimum, then properties automatically fulfilled, like symmetry, positive definiteness, and functional inequalities should be used. Any additional assumptions, such as unimodality (Hodges and Lehmann, 1954) or logconcavity (Ibragimov, 1989), should be imposed on the components, and the fact that such properties are inherited should be used to obtain conclusions on the difference term.

There are several dimensions in which the present study could be extended. Of interest, for instance, might be the consideration of multivariate distributions, i.e., random variables with values in Banach spaces. More interesting, albeit also more challenging, might be the question of what happens if the number N of contestants is larger than two. In that case, one would have to study the joint distribution of $\binom{N}{2}$ correlated difference terms, each of which would be DFD. We will, in any case, leave such investigations to future work.

Appendix. Proofs

This appendix contains technical material omitted from the body of the paper.

Details on Example 1. Suppose that $Z \sim U[-1,1]$. The characteristic function of Z is

$$\varphi_Z(t) = \frac{1}{2}\int_{-1}^{1} \cos(tz)dz = \frac{\sin t}{t}. \tag{16}$$

It is then clear that $\varphi_Z(t) < 0$ for selected values of t, in conflict with Lemma 2. □

Proof of Proposition 1. Evaluating the characteristic function at $t = \frac{3\pi}{2}$ yields

$$\begin{aligned}
\varphi_Z(\tfrac{3\pi}{2}) &= \int_{-1/3}^{1/3} \underbrace{\cos(\frac{3\pi}{2}x)}_{\geq 0} g(x)dx + 2\int_{1/3}^{1} \underbrace{\cos(\frac{3\pi}{2}x)}_{\leq 0} g(x)dx && (17)\\
&< \int_{-1/3}^{1/3} \cos(\frac{3\pi}{2}x)\left(\frac{1}{2}+\frac{1}{6}\right)dx + 2\int_{1/3}^{1} \cos(\frac{3\pi}{2}x)\left(\frac{1}{2}-\frac{1}{6}\right)dx && (18)\\
&= 0. && (19)
\end{aligned}$$

This proves the claim. □

Proof of Proposition 2. By contradiction. Suppose that Z is DFD. Then, by Lemma 2, $\varphi_Z(t) \geq 0$ for any $t > 0$. However, evaluating $\varphi_Z(t)$ at $t = 2\pi$, we see that

$$0 \leq \varphi_Z(2\pi) = \int_{-1}^{1} \cos(2\pi z) g(z) dz = \frac{1}{2\pi} \int_{0}^{2\pi} \cos(\widehat{z}) g\left(\frac{\widehat{z}}{2\pi}\right) d\widehat{z}, \tag{20}$$

with $\widehat{z} = 2\pi z$. Since $\cos(\widehat{z} + \pi) = -\cos(\widehat{z})$, it follows that

$$\begin{aligned} \varphi_Z(2\pi) &= \frac{1}{2\pi} \int_{0}^{\pi} \cos(\widehat{z}) g\left(\frac{\widehat{z}}{2\pi}\right) d\widehat{z} + \cos(\widehat{z} + \pi) g\left(\frac{\widehat{z} + \pi}{2\pi}\right) d\widehat{z}. & (21) \\ &= \frac{1}{2\pi} \int_{0}^{\pi} \cos(\widehat{z}) \left\{ g\left(\frac{\widehat{z}}{2\pi}\right) - g\left(\frac{\widehat{z} + \pi}{2\pi}\right) \right\} d\widehat{z} & (22) \\ &< 0, & (23) \end{aligned}$$

where the inequality is strict because g is not uniform by Example 1. The contradiction proves the claim. □

Proof of Proposition 3. Suppose that $g(z)$ is DFD and strictly concave on its support. Then, $g(z)$ is symmetric by Lemma 1. Clearly, therefore, Z has compact support, say $[-1, 1]$. Moreover, by standard results on concave functions (e.g., Royden and Fitzpatrick, 1988, p. 117), the derivative g' is well-defined except possibly at kinks that form a set of measure zero. Moreover, g' is strictly declining. Hence, evaluating the integral transform at $t = 2\pi$, integration by parts delivers

$$\begin{aligned} \varphi(2\pi) &= \int_{-1}^{1} \cos(2\pi z) g(z) dz & (24) \\ &= \underbrace{\frac{1}{2\pi} \sin(2\pi z) g(z) \Big|_{-1}^{1}}_{=0} - \frac{1}{\pi} \int_{0}^{1} \sin(2\pi z) g'(z) dz & (25) \\ &< \frac{1}{\pi} \int_{0}^{1/2} \underbrace{\left(\sin(2\pi z) + \sin(2\pi (z + \frac{1}{2})) \right)}_{=0} g'(z) dz, & (26) \end{aligned}$$

but this is in conflict with Lemma 2. □

Details on Example 2. On the unit interval, the density of the beta distribution with shape parameters $\alpha > 0$, $\beta > 0$ is commonly defined as

$$g_{[0,1]}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad (x \in [0,1]). \tag{27}$$

A stretched variant with support $[-1, 1]$ is given as

$$g(z) \equiv \frac{1}{2} g_{[0,1]}\left(\frac{z+1}{2}\right) = \frac{\Gamma(\alpha+\beta)}{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)}(1+z)^{\alpha-1}(1-z)^{\beta-1}. \tag{28}$$

Assuming $\alpha = \beta$, and exploiting the duplication rule for the gamma function, i.e., $\Gamma(2\alpha) = 2^{2\alpha-1}\pi^{-1/2}\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})$, leads to the parametric form considered in the body of the paper. The shape of the beta density is determined by its second derivative

$$\frac{\partial^2(1-x^2)^{\alpha-1}}{\partial x^2} = 2\,(1-\alpha)\left(1-x^2\right)^{\alpha-3}\left(2x^2(2-\alpha)+\left(1-x^2\right)\right). \tag{29}$$

Hence, the beta density g is convex for $\alpha \in (0, 1]$, strictly concave for $\alpha \in (1, 2]$, and neither convex nor concave for $\alpha > 2$.

The following auxiliary result will be used in the proofs of Lemmas 4 and 6. As usual, $o(t^{-N})$ denotes a function that goes to zero more quickly than t^{-N} (i.e., $\lim_{t\to\infty} t^N o(t^{-N}) = 0$).

Lemma A.1 (A. Erdélyi) *Suppose that $g(z)$ is N-times continuously differentiable on the interval $[\alpha, \beta]$, where $-\infty < \alpha < \beta < \infty$. Then,*

$$\int_\alpha^\beta g(z)\exp(itz)dz = \Phi_N(t,\beta) - \Phi_N(t,\alpha) + o(t^{-N}), \tag{30}$$

where $i = \sqrt{-1}$, and

$$\Phi_N(t,z) = \sum_{M=0}^{N-1} i^{M-1} g^{(M)}(z)\frac{\exp(itz)}{t^{M+1}}. \tag{31}$$

Proof. See Erdélyi (1955, Thm. 1). □

Proof of Lemma 4. By induction. (*Induction basis*) Suppose that g is continuously differentiable on $[-1, 1]$ as well as DFD. Since g is continuously differentiable, $\Phi_1(t, 1) = g(1)\frac{\sin(t)}{t} = -\Phi_1(t, -1)$. Moreover, since g is DFD, Lemma 2 implies

$$0 \leq \varphi_Z(t) = \int_1^1 g(z)\cos(zt)dz = 2g(1)\frac{\sin t}{t} + o(\frac{1}{t}). \tag{32}$$

For t large, the $\frac{\sin t}{t}$ term dominates, so that necessarily $g(1) = g(-1) = 0$. (*Induction step*) Let $N \geq 1$, and assume that the claim has been shown for N. Suppose that g is $(N+1)$-times continuously differentiable. Then, g is N-times continuously differentiable so that, from the induction hypothesis, $g^{(M)}(1) = g^{(M)}(-1) = 0$ for any $M \in \{0, \dots, N-1\}$. Now, from the definition of $\Phi_{N+1}(t, 1)$ and $\exp(it) = \cos(t) + i\sin(t)$,

$$\Phi_{N+1}(t,1) = \sin(z)\left(\sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \frac{(-1)^k}{t^{2k+1}} g^{(2k)}(1)\right) + \cos(z)\left(\sum_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} \frac{(-1)^{k+1}}{t^{2k}} g^{(2k-1)}(1)\right). \tag{33}$$

If $N = 2K$ is even, then

$$\Phi_{N+1}(t, 1) = \frac{(-1)^K \sin(z)}{t^{N+1}} g^{(N)}(1) = -\Phi_{N+1}(t, -1), \tag{34}$$

so that $g^{(N)}(1) = g^{(N)}(-1) = 0$. If $N = 2K - 1$ is odd, then

$$\Phi_{N+1}(t, 1) = \frac{(-1)^{K+1} \cos(z)}{t^{N+1}} g^{(N)}(1) = -\Phi_{N+1}(t, -1), \tag{35}$$

and we find $g^{(N)}(1) = g^{(N)}(-1) = 0$, as before. This proves the claim. □

Proof of Proposition 4. Suppose that g is DFD. Since g is analytic at -1 and at 1, it is infinitely differentiable there, so that all derivatives at -1 and at 1 are zero by Lemma 4. By the identity theorem for analytic functions, this implies that

g vanishes on $[-1,1]$, which is impossible. The contradiction shows that g cannot be DFD. □

Proof of Proposition 5. For $z \in [0,1]$, we have that

$$g(z) = \int_0^{1-z} f(z+x)f(x)dx. \tag{36}$$

Hence, using Leibniz' rule,

$$\begin{aligned}
\lim_{z\searrow 0} \frac{g(z)-g(0)}{z} &= \lim_{z\searrow 0} \frac{1}{z}\left\{\int_0^{1-z} f(z+x)f(x)dx - \int_0^1 f(z+x)f(x)dx\right\} && (37)\\
&= -\lim_{z\searrow 0} \frac{1}{z}\int_{1-z}^1 f(z+x)f(x)dx && (38)\\
&= -f(1)^2 + \int_0^1 f'(x)f(x)dx && (39)\\
&= -f(1)^2 + \frac{f(1)^2 - f(0)^2}{2} && (40)\\
&= -\frac{f(0)^2 + f(1)^2}{2} && (41)\\
&< 0. && (42)
\end{aligned}$$

On the other hand, by symmetry, $g(z) = g(-z)$, so that

$$\lim_{z\nearrow 0} \frac{g(z)-g(0)}{z} > 0. \tag{43}$$

Similarly, one finds

$$\lim_{z\nearrow 1} \frac{g(z)-g(1)}{z} = \lim_{z\nearrow 1} \frac{1}{z}\left\{\int_0^{1-z} f(z+x)f(x)dx\right\} = -f(1)f(0), \tag{44}$$

and a corresponding expression at $z=-1$. This proves the lemma. □

Proof of Proposition 6. Immediate from Bochner's theorem and the Fourier inversion theorem. □

Proof of Corollary 1. Let the distribution of Z be infinitely divisible and symmetric. Then, $\varphi_Z \geq 0$ is continuous (by Bochner's theorem), and does not possess any real zeros (Lukacs, 1972). Let ϕ be a characteristic function such that $\varphi = \phi^2$. Then, $\phi > 0$ or $\phi < 0$, but the second alternative is not feasible because $\phi(0) = 1$. Hence, $\phi > 0$, and $\varphi = |\phi|^2$. The claim follows. □

Proof of Proposition 7. Immediate from the Tarski-Seidenberg theorem (e.g., Neyman and Sorin, 1999, p. 65). □

Details on Example 4. Suppose that $N = 2$ and fix a vector of probabilities (p_0, p_1, p_2) such that $p_0 + 2p_1 + 2p_2 = 1$. Taking account of redundancies, a vector of probabilities (q_0, q_1, q_2) is sought such that

$$p_1 = q_0 q_1 + q_1 q_2 \tag{45}$$

$$p_2 = q_0 q_2 \tag{46}$$

$$1 = q_0 + q_1 + q_2. \tag{47}$$

Combining (45) and (47), we see that $p_1 = q_1(1 - q_1)$, hence necessarily $p_1 \leq \frac{1}{4}$ and $q_1 \in \{q_1^+, q_1^-\}$, where $q_1^+ = \frac{1}{2} + \sqrt{\frac{1}{4} - p_1}$ and $q_1^- = \frac{1}{2} - \sqrt{\frac{1}{4} - p_1}$. From (46) and (47), one notes that $q_0 + q_2 = 1 - q_1$ and $q_0 q_2 = p_2$. Therefore, a difference-form component with $q_1 \in \{q_1^+, q_1^-\}$ exists if and only if $4p_2 \leq (1 - q_1)^2$, and the corresponding solution, or pair of solutions, is given by

$$\{q_0, q_2\} = \frac{1 - q_1}{2} \pm \sqrt{\frac{(1 - q_1)^2}{4} - p_2}. \tag{48}$$

Next, we note that $q_1^- \leq q_1^+ \leq 1$, so that $\left(1 - q_1^-\right)^2 \geq \left(1 - q_1^+\right)^2$. Thus, a necessary and sufficient condition for a solution to exist is $p_1 \leq \frac{1}{4}$ and $p_2 \leq \frac{1}{16}\left(1 + \sqrt{1 - 4p_1}\right)^2$, as claimed. Moreover, there are at most four solutions.[21]

[21] For the special case $N = 2$, this confirms a conjecture of Carnal and Dozzi (1989, p. 172), according to which the number of difference-form decompositions is at most 2^N.

Proof of Proposition 8. Suppose that Z is DFD. Then, by Lemma 2, $\varphi_Z(t) \geq 0$ for all $t \in \mathbb{R}$. Moreover, φ_Z is continuous by Bochner's theorem, with $\varphi_Z(0) = \int_{-\infty}^{\infty} g(z)dz = 1$. Hence, using that φ_Z is integrable, we see that $\int_{-\infty}^{\infty} \varphi_Z(s)ds > 0$. Moreover, the Fourier inversion theorem, $g(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_Z(s)ds$. Let

$$\widehat{g}(z) = \frac{\varphi_Z(z)}{\int_{-\infty}^{\infty} \varphi_Z(s)ds} = \frac{\varphi_Z(z)}{2\pi g(0)}. \tag{49}$$

Then, $\widehat{g}$ is a density of some random variable $\widehat{Z}$. The characteristic function of $\widehat{Z}$ is, therefore, given by

$$\varphi_{\widehat{Z}}(t) = \frac{1}{g(0)} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_Z(z) \cos(tz) dt \right) = \frac{g(t)}{g(0)}, \tag{50}$$

where we applied again the Fourier inversion theorem. But, by Bochner's theorem, the characteristic function of $\widehat{Z}$ is positive definite. Hence, g is positive definite as well. Inequality (i) is an immediate consequence of the definition of positive definiteness. Inequality (ii) was shown to follow from positive definiteness by Boas and Kac (1945). □

Proof of Lemma 5. Since f is continuous differentiable, Lemma A.1 implies

$$\varphi_X(t) = \int_0^c f(z) \exp(izt) dz = \frac{1}{t} \left(\sin(ct) + i(f(0) - \cos(ct)f(c))\right) + o(t^{-1}). \tag{51}$$

By Lemma 2,

$$\varphi_Z(t) = |\varphi_X(t)|^2 = \frac{(\sin(ct)f(c))^2 + (f(0) - \cos(ct)f(c))^2}{t^2} + o(t^{-2}). \tag{52}$$

Thus, φ_Z is integrable, as claimed. □

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