Existence and Uniqueness of Recursive Equilibria with Aggregate and Idiosyncratic Risk

Elisabeth Pröhl

February 28, 2023

Abstract

In this paper, I study the existence and uniqueness of recursive equilibria in economies with aggregate and idiosyncratic risk. Rather than relying on compactness to establish existence, I exploit the monotonicity property of the equilibrium model and rely on arguments from convex analysis. This methodology does not only give rise to a convergent iterative procedure, but more strikingly, it also yields uniqueness. To illustrate my theoretical results, I establish sufficient conditions for the existence and uniqueness of solutions to the stochastic growth model as in Krusell and Smith (1998) and the heterogeneous-agent exchange economy as in Huggett (1993) with aggregate risk.

Keywords: Existence, Uniqueness, Dynamic stochastic general equilibrium, Incomplete markets, Heterogeneous agents, Aggregate uncertainty, Convergence

JEL Classification: C61, C62, D51, D52, E21

University of Amsterdam and Tinbergen Institute, Roetersstraat 11, 1018 WB Amsterdam, Netherlands, e.proehl@uva.nl.
I thank Fernando Alvarez, Adrien Bilal, Rajna Gibson Brandon, Pierre Cardaliaguet, John Conlon, Felix Kübler, Kurt Mitman, Benjamin Moll, Marcelo Pedroni, Jean-Charles Rochet, Olivier Scaillet, Petr Sedláček, Robert Shimer, Christian Stoltenberg, and, my anonymous referees for fruitful suggestions and discussions. Furthermore, I thank seminar participants at Cornell, Wharton, the University of Amsterdam and the University of Freiburg, and, participants at the Aggregate Dynamics in Models with Heterogeneous Agents Workshop 2021, SED 2021, SAET 2021, VMACS Junior Conference 2020, the EEA 2020, the ES World Congress 2020, the Global Macro Workshop 2019, the ES European Winter Meeting 2018, and, the Midwest Economic Theory Meeting 2018 for helpful comments. I would like to acknowledge that I started this project when receiving support by the Macro Financial Modeling Initiative which included a grant funded by the Alfred P. Sloan Foundation, the CME Group Foundation and Fidelity Management and Research.

1 Introduction

The renewed interest in inequality in recent years has sparked a wealth of novel research based on economic models where heterogeneity across agents arises due to idiosyncratic risk. Such models go back to a dynamic stochastic general equilibrium model by Bewley (1977) where agents face idiosyncratic income shocks affecting their wealth which was extended by Aiyagari (1994) to include a production technology. Aggregate risk resulting in business cycles was first added by Krusell and Smith (1998). Similarly, an asset pricing model with idiosyncratic risk was investigated by Huggett (1993). Despite the importance of these models in economics, many theoretical questions surrounding existence and uniqueness of solutions to models with both aggregate and idiosyncratic risk remain open. The challenge lies in handling the cross-sectional distribution of the agents' idiosyncratic variables, which becomes an infinite-dimensional component of the state space. In particular, this distribution changes stochastically over time depending on the realization of the aggregate shocks. The aggregate variables, in turn, evolve depending on how the cross-sectional distribution changes.

It lies in the nature of such models that they have to be solved numerically in practice which is why simple recursive equilibria, i.e., equilibria where the set of policy and price functions solely depend on the exogenous shocks, the agent-specific endogenous variables and the cross-sectional distribution of those variables, are of particular importance for applied research. Even though existence of sequential equilibria has been shown[1], existence of simple recursive equilibria in a heterogeneous-agent model with a continuum of agents has yet to be established. Recently, Cao (2020) and Brumm, Kryczka, and Kubler (2017) made advances in that direction by showing the existence of extended recursive equilibria which additionally dependent on the value function and a sunspot, respectively. However, the existence of simple recursive equilibria used in practice remains elusive. Moreover, whether such equilibria are unique is an open question.

This paper contributes to closing that gap. I consider an equilibrium model with aggregate and idiosyncratic risk and a continuum of agents who maximize their CRRA utility when trading in two types of assets, capital and a one-period bond, while facing borrowing constraints. I am able to establish both existence and uniqueness of a simple recursive equilibrium because my methodology differs from the existing literature in two aspects.

First, a simple recursive equilibrium in a heterogeneous agent model is typically

[1]See for instance Cao (2020), Cheridito and Sagredo (2016) and Miao (2006).

defined in terms of the cross-sectional distribution of agent-specific variables. I develop an equivalent representation of the equilibrium functions which feature a random variable instead of the distribution as an argument. Sets of random variables are usually well behaved, especially the set of square-integrable random variables. The advantage of this approach fully enfolds when considering the Euler equation of the equilibrium problem. As I work with the random variable of asset holdings instead of their distribution, I can substitute this random variable into the Euler equations of the individual agents. This transforms the continuum of individual Euler equations which are linked by the market-clearing condition into one generalized Euler equation on random variables. This significantly simplifies the problem at hand.

The second aspect in which I depart from the existing literature lies in the type of fixed point argument I use. In contrast to the existing literature, which predominantly relies on fixed-point theory requiring compactness of the state space, this paper exploits the monotonicity properties of the model. In particular, results from a series of papers by Rockafellar (1969, 1970, 1976a,b) on monotone operator theory constitute the backbone of this paper. I show that the equilibrium equation, i.e., the generalized Euler equation on random variables together with the market clearing equation, is a maximal monotone operator. This implies that there exists a convex Lagrangian which has the equilibrium equation as its first-order condition. In other words, I show that there exists a social planner who solves the heterogeneous-agent model by optimizing. Furthermore, there exists a root of the social planner's first-order condition if one can find a candidate policy at which the equilibrium equation has a negative value and another candidate policy at which it has a positive value. Furthermore, as this equilibrium problem can be solved using arguments from convex analysis, uniqueness of the solution can be examined in a straightforward manner. When using fixed-point theory relying on compactness instead as is prevalent in the existing literature, it is much more difficult to investigate the uniqueness of a solution. A nice additional side-effect of exploiting the monotonicity properties of the equilibrium model is that there also exists an iterative procedure which is guaranteed to converge to the equilibrium solution.

To illustrate the applicability of my methodology, I derive sufficient conditions for existence and uniqueness of solutions to two standard heterogeneous-agent models with aggregate and idiosyncratic risk. First, I consider a stochastic growth model as in Krusell and Smith (1998). The sufficient condition for this model allows for risk aversion parameters larger than one. It places an implicit bound on the subjective discount factor which decreases with capital depreciation and risk aversion. Thus, I extend results of the existing literature on this growth model solely featuring idiosyncratic risk and not aggregate risk,[2] which shows uniqueness for risk aversion parameters smaller than or equal to one. Second, I consider a heterogeneous-agent asset pricing model as in Huggett (1993) with aggregate risk, i.e., an endowment economy with a one-period bond. The sufficient condition for this model solely requires the average endowment at any given time point to be finite and positive.

In terms of the existing literature on models with both aggregate and idiosyncratic risk, the paper at hand is most closely related to Cao (2020) and Brumm et al. (2017). Cao (2020) shows existence of extended recursive equilibria for the stochastic growth model with a continuum of heterogeneous agents and unbounded utility. The recursive equilibrium in that work consists of policy and price functions depending on the value function in addition to the minimal state space of the exogenous shocks, agent-specific endogenous variables and the distribution thereof. Cao (2020), thus, extends earlier work on the existence of solutions to the Aiyagari-Bewley growth model with aggregate risk by Miao (2006) and Cheridito and Sagredo (2016). Brumm et al. (2017), on the other hand, show existence of recursive equilibria dependent on sunspots as they incorporate a transitory shock which does not affect fundamentals. Compared to the distributional approach in Cao (2020), however, they consider models with finitely many agents and bounded utility. In doing so, they are able to show existence of both a stochastic growth model and an exchange economy. The manuscript at hand goes beyond those existing results by reducing the recursive equilibrium to the minimal state space and, moreover, investigating uniqueness.

This paper is also related to the strand of literature on models with idiosyncratic risk but without aggregate risk. Results on uniqueness for the stochastic growth model have been established by Light (2020) in discrete time and Achdou, Han, Lasry, Lions, and Moll (2017) in continuous time. However, in both cases uniqueness is only shown for a risk aversion parameter smaller than or equal to one. In contrast to those results, my uniqueness result for the Aiyagari-Bewley economy includes risk aversion parameters greater than one. Existence for various model specifications of the growth model has been shown by Acemoglu and Jensen (2015) and Açıkgöz (2018). The Huggett economy solely featuring idiosyncratic risk has been considered by Wang (2003) who proved existence and Toda (2017) who added uniqueness for AR(1) shocks. Both, however, use CARA utility instead

[2]See Light (2020) and Achdou et al. (2017).

of CRRA utility as in this manuscript.

The paper proceeds as follows. I first introduce a generic model framework encompassing both the production and exchange economy. Second, I characterize the recursive equilibrium by functions depending on random variables which results in a generalized Euler equation substituting the continuum of individual Euler equations. In Section 4, I establish the monotonicity properties leading to existence and uniqueness of equilibria. Then, I introduce the corresponding convergent iterative procedure which can be used to compute the equilibrium numerically. The last section applies this general framework to the Aiyagari-Bewley and Huggett economies both with aggregate risk. Appendix A contains all proofs. Additionally, I provide an online appendix where I introduce the more technical concepts which underlie some of my arguments in detail for the interested reader. The relevant sections are referenced in the main text.

2 A Generic Model

Consider a discrete-time infinite-horizon model with a continuum of agents of measure one. There are two kinds of exogenous shocks, an aggregate shock and an idiosyncratic shock. The aggregate shock characterizes the state of the economy with outcomes in $\mathcal{Z}^{ag} \subset \mathbb{R}$. It follows a first-order Markov process with transition probability $\mathbb{P}(.|z) : \mathcal{B}(\mathcal{Z}^{ag}) \times \mathcal{Z}^{ag} \to [0,1]$ defined on the generating Borel σ-algebra. The idiosyncratic shock with outcomes in $\mathcal{Z}^{id} \subset \mathbb{R}$ represents the agent-specific risk. It is a first-order Markov process which is i.i.d. across agents and whose transition probability at any point in time t is conditional on the aggregate shocks $\mathbb{P}(.|\varepsilon_{t-1}, z_{t-1}, z_t) : \mathcal{B}(\mathcal{Z}^{id}) \times \mathcal{Z}^{id} \times \mathcal{Z}^{ag} \times \mathcal{Z}^{ag} \to [0,1]$. I denote the compound exogenous process by $(z_t, \varepsilon_t) \in \mathcal{Z}$ with $\mathcal{Z} = \mathcal{Z}^{ag} \times \mathcal{Z}^{id}$. The only requirement I impose on the exogenous stochastic processes is square integrability.

Assumption 1 (Square integrability)**.** Let $\mathcal{Z}$ be a complete separable metric space. The aggregate and idiosyncratic exogenous processes $(z_t)_{t\geq 0}$ and $(\varepsilon_t)_{t\geq 0}$ with $(z_t, \varepsilon_t) \in \mathcal{Z}$ are square integrable, i.e., $\mathbb{E}[z_t^2] < \infty$ and $\mathbb{E}[\varepsilon_t^2] < \infty$ at any time point $t \in \mathbb{N}$.

This specification of the aggregate and idiosyncratic shock is fairly flexible. It does include finite state Markov chains as well as continuous Markov processes in discrete time. Linear growth ensures square integrability in the latter case.

Example. Examples for both exogenous processes include the following.

(i) Finite Markov chain: Define a finite state space $\mathcal{S} = \{s_1, \ldots, s_N\}$. Then, $a_t \in \mathcal{S}$ with the transition probabilities being given by $\pi_{ij} = \mathbb{P}(a_t = s_i | a_{t-1} = s_j)$ if it is an aggregate process, i.e., $z_t = a_t$, or $\pi_{ij} = \mathbb{P}(a_t = s_i | a_{t-1} = s_j, z_{t-1}, z_t)$ if it is idiosyncratic, i.e., $\varepsilon_t = a_t$.

(ii) AR(1) process: Assume a normally distributed innovation $\eta \sim N(0, \sigma^2)$ and define $a_{t+1} = c + b a_t + \eta$ with c constant and $b \in [0, 1)$. The dependency of the idiosyncratic shock on the aggregate shock can be achieved by letting the mean and/or volatility of η vary depending on the current aggregate outcome.

Agents earn an endowment and wage for their labor and they can invest in 2 assets, risky capital and a risk-free one-period bond. Labor supply is assumed exogenous and the wage is denoted by $(W_t)_{t \geq 0}$. An agent's share of capital and bond holdings is denoted by $(k_t)_{t \geq 0}$ and $(b_t)_{t \geq 0}$, respectively. After one holding period, capital pays a risky rate of return $(R_t^k)_{t \geq 0}$, whereas, the bond pays a risk-free rate of return $(R_t^b)_{t \geq 0}$. Each agent chooses her share of the assets and consumption such that they satisfy certain constraints. First, individual consumption must be positive at all times $c_t > 0$, $t \geq 0$, and asset holdings are subject to borrowing constraints $k_t \geq \bar{k}$ and $b_t \geq \bar{b}$, $t \geq 0$, where $\bar{k}, \bar{b} \leq 0$. Second, given the initial holdings $k_{-1} \geq \bar{k}$ and $b_{-1} \geq \bar{b}$, each agent adheres to a budget constraint, which equates individual consumption and current asset holdings to current endowment, income and the return on previous holdings

$$k_t + b_t + c_t = e\left(z_t, \varepsilon_t\right) + W_t l\left(z_t, \varepsilon_t\right) + \left(1 + R_t^k\right) k_{t-1} + \left(1 + R_t^b\right) b_{t-1} \, \forall \, t \geq 0. \quad (1)$$

The endowment process e and labor supply process l are given exogenously. The wage, return and bond price are aggregate endogenous variables. They are defined through the market clearing conditions which aggregate over labor and the asset holdings to equalize demand and supply. The bond return is implicitly defined by a zero-net supply condition for the households' bond holdings. The wage and return for physical capital on the other hand are explicitly set by a perfectly competitive representative firm producing according to a production function F. Thus, the wage and return depend on the firm's aggregate capital and labor demand which has to be met by the households' labor supply and capital savings in equilibrium.

Assumption 2. The production function $F : \mathcal{Z}^{ag} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is measurable in the first argument and strictly increasing, strictly concave and twice continuously differentiable in the second and third argument. Aggregating the exogenous individual labor supply l over the cross-section of agents defines the aggregate labor demand of the firm $(L_t)_{t\geq 0} > 0$. The wage and return on capital are given by

$$
\begin{aligned}
R_t^k &= \frac{\partial}{\partial K} F(z_t, L_t, K_t) - \delta \\
W_t &= \frac{\partial}{\partial L} F(z_t, L_t, K_t),
\end{aligned}
$$

where $(K_t)_{t\geq 0}$ denotes the aggregate capital demand of the firm and $\delta \in [0,1]$ denotes capital depreciation. Lastly, assume for the production function F that $\log(1 + R_t^k)$ is convex in K_t.

Agents optimize their utility. I assume that all agents have a time-separable CRRA utility with a risk aversion coefficient $\gamma > 0$ or logarithmic utility when $\gamma = 1$. Then, given an agent's initial asset holdings $k_{-1} \geq \bar{k}$ and $b_{-1} \geq \bar{b}$, the individual optimization problem reads

$$
\begin{aligned}
\max_{\{c_t, k_t, b_t\}_{t\geq 0}} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}\right] \qquad (2) \\
\text{s.t.} \quad & k_t + b_t + c_t = e\left(z_t, \varepsilon_t\right) + W_t l\left(z_t, \varepsilon_t\right) + \left(1 + R_t^k\right) k_{t-1} \\
& \qquad + \left(1 + R_t^b\right) b_{t-1} \, \forall \, t \geq 0 \\
& c_t > 0, \; k_t \geq \bar{k}, \; b_t \geq \bar{b}, \; \forall \, t \geq 0,
\end{aligned}
$$

where $\beta \in (0,1)$ is the time preference parameter.

Before I define the necessary conditions for this model, let me clarify the time line with Figure 1. Note that I specify the time line slightly differently from existing papers. Often, (k_t, b_t) is substituted with (k_{t+1}, b_{t+1}) in the budget constraint (1) because this is the asset holding with a payout at $t+1$. In contrast to that notation, however, I want to emphasize the time period, at which the agent optimally chooses the magnitude of her asset holdings. Taking this view, the optimal consumption and asset holdings choices have the same time subscript. My time line, therefore, indicates which information the agent's choices are adapted to.

Necessary conditions for individual optimality are the Euler equations which I state using prime-notation, where a prime denotes variables in the current period and variables with no prime refer to the previous period. The Euler equations

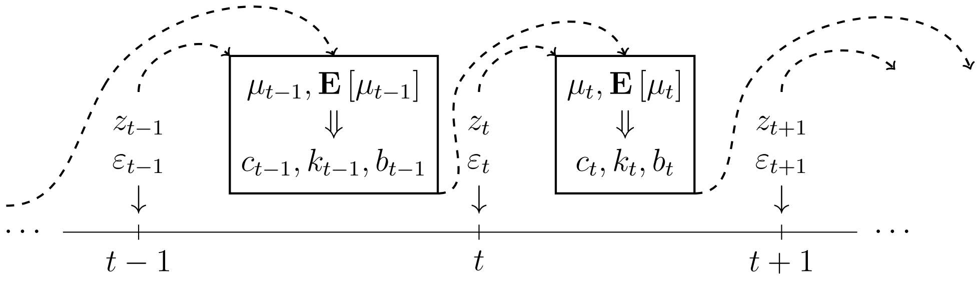

Figure 1: **Time line of events.** Before period t, the agent observes how much assets everybody decided to hold in the previous period. At period t, the agent observes the exogenous shocks (z_t, ε_t), and therefore, knows the beginning-of-period cross-sectional distribution μ_t and the aggregated quantities $\mathbf{E}[\mu_t]$. The agent then decides how much to consume c_t and how much to invest, i.e., (k_t, b_t).

read

$$\begin{aligned}
&(c')^{-\gamma} = \beta \mathbb{E}\left[\left(1+R_k''\right)(c'')^{-\gamma}\right] + y_k \\
&(c')^{-\gamma} = \beta \mathbb{E}\left[\left(1+R_b''\right)(c'')^{-\gamma}\right] + y_b \\
&\text{s.t. } c' = e\left(z', \varepsilon'\right) + W' l\left(z', \varepsilon'\right) + \left(1+R_k'\right) k - k' + \left(1+R_b'\right) b - b' \\
&\qquad k' \geq \bar{k}, y_k \geq 0, (k' - \bar{k}) \perp y_k \\
&\qquad b' \geq \bar{b}, y_b \geq 0, (b' - \bar{b}) \perp y_b.
\end{aligned} \tag{3}$$

I show in Section 4.3 that the Euler equations are sufficient for optimality given a suitable transversality condition.

Let me now introduce the cross-sectional distribution of the model. I use the methodology of Fubini extension by Sun (2006) to ensure the validity of the law of large numbers when aggregating over the continuum of agents with measure one. In particular, denote the atomless measure space of agents by $(I, \mathcal{I}, \lambda)$ with $\lambda(I) = 1$ and the sample probability space at time t by $(\mathcal{Z}^{id}, \mathcal{B}(\mathcal{Z}^{id}), P^{id})$ with $P^{id} = \mathbb{P}(.|\varepsilon_{t-1}, z_{t-1}, z_t)$. Let f be a measurable function mapping the Fubini extension $(I \times \mathcal{Z}^{id}, \mathcal{I} \boxtimes \mathcal{B}(\mathcal{Z}^{id}), \lambda \boxtimes P^{id})$ into $\mathbb{R}$. If the random variables $f(i, .)$ are essentially pairwise independent, then $f(i, .)$ have a common distribution μ for λ-almost all $i \in I$. The same holds for the samples $f(., \varepsilon)$. When f represents individual asset holdings, we have that $(k_t, b_t) = f(i, \varepsilon_t)$ for agent i and, thus, (k_t, b_t) is distributed according to the c.d.f. $\mu_t : \mathcal{Z}^{id} \times [\bar{k}, \infty) \times [\bar{b}, \infty) \to [0, 1]$. Hence, I denote the cross-sectional c.d.f. of agent-specific variables at the beginning of period t by μ_t. Note that the aggregate shocks cause the cross-sectional distribution to vary over time, which is indicated by the time subscript of μ_t. In equilibrium, the cross-sectional distribution's evolution over time needs to be consistent with the agent's optimal asset holdings. To write down the consistency condition, I switch to prime notation. Given a fixed distribution μ' over the cross-section of individual asset holdings at the beginning of the current period, the distribution changes in two steps $\mu' \to \tilde{\mu}' \to \mu''$. In the first step, the agents implement their optimal current-period asset holdings satisfying the Euler equations (3), which leads to the end-of-current period distribution

$$\tilde{\mu}'\left(\hat{\varepsilon}, \hat{k}, \hat{b}\right) = \int_{\varepsilon' \in \mathcal{Z}^{id} \cap \{\varepsilon' \leq \hat{\varepsilon}\}} \int_{\bar{k}}^{\infty} \int_{\bar{b}}^{\infty} \mathbb{1}_{\left\{\left\{k' \leq \hat{k} \middle| z', \varepsilon', k, b\right\} \cap \left\{b' \leq \hat{b} \middle| z', \varepsilon', k, b\right\}\right\}} d\mu'\left(\varepsilon', k, b\right).$$

In the second step, the next-period shocks (z'', ε'') realize for all agents and shift the quantities of the agents with a specific idiosyncratic shock according to the exogenous transition probabilities conditional on the aggregate shock outcome. The beginning-of-next period distribution is, hence, computed by integrating over the transition probabilities that the idiosyncratic state changes from ε' to ε'' given the observed trajectory of z' to z''. The consistent law of motion for the cross-sectional distribution is then given by

$$\begin{aligned} \mu''\left(\varepsilon'', \hat{k}, \hat{b}\right) &= \int_{\hat{\varepsilon} \in \mathcal{Z}^{id}} \tilde{\mu}\left(\hat{\varepsilon}, \hat{k}, \hat{b}\right) \mathbb{P}\left(\varepsilon'' \middle| d\hat{\varepsilon}, z', z''\right) \\ &= \int_{\hat{\varepsilon} \in \mathcal{Z}^{id}} \int_{\varepsilon' \in \mathcal{Z}^{id} \cap \{\varepsilon' \leq \hat{\varepsilon}\}} \int_{\bar{k}}^{\infty} \int_{\bar{b}}^{\infty} \mathbb{1}_{\left\{\left\{k' \leq \hat{k} \middle| z', \varepsilon', k, b\right\} \cap \left\{b' \leq \hat{b} \middle| z', \varepsilon', k, b\right\}\right\}} \\ &\quad d\mu'\left(\varepsilon', k, b\right) \mathbb{P}\left(\varepsilon'' \middle| d\hat{\varepsilon}, z', z''\right). \end{aligned} \tag{4}$$

for all $\varepsilon'' \in \mathcal{Z}^{id}$ and $(\hat{k}, \hat{b}) \in [\bar{k}, \infty) \times [\bar{b}, \infty)$.

The market clearing conditions of the model aggregate over the cross-sectional distribution to equate the assets' demand and supply. Let $\mathbf{E}$ denote a linear aggregation operator $\mathbf{E} : \mathcal{P}(\mathcal{Z}^{id} \times [\bar{k}, \infty) \times [\bar{b}, \infty)) \to \mathbb{R}^2$ on the space of cross-sectional distributions which computes the vector of aggregate asset holdings. Then, the market clearing conditions read

$$\mathbf{E}\left[\mu'\right] = \begin{bmatrix} \int_{\mathcal{Z}^{id}} \int_{\bar{k}}^{\infty} \int_{\bar{b}}^{\infty} k d\mu'(\varepsilon', k, b) \\ \int_{\mathcal{Z}^{id}} \int_{\bar{k}}^{\infty} \int_{\bar{b}}^{\infty} b d\mu'(\varepsilon', k, b) \end{bmatrix} = \begin{bmatrix} K' \\ 0 \end{bmatrix}. \tag{5}$$

In contrast, aggregate labor supply is exogenously defined as

$$L' = \int_{\mathcal{Z}^{id}} \int_{\bar{k}}^{\infty} \int_{\bar{b}}^{\infty} l(z', \varepsilon') d\mu'(\varepsilon', k, b)$$

by Assumption 2.

In a competitive equilibrium, the individual problems are solved such that markets clear and the cross-sectional distribution's law of motion is consistent with the agents' optimal choices. In this paper, I consider a competitive equilibrium of recursive form.

Definition 3 (Recursive equilibrium)**.** *Consider the measurable functions*[3]

$$\begin{aligned} g_c, g_k, g_b &: \mathcal{Z} \times \mathbb{R}^2 \times \mathcal{P}\left(\mathcal{Z}^{id} \times [\bar{k}, \infty) \times [\bar{b}, \infty)\right) \to \mathbb{R} \\ g_{R_k} &: \mathcal{Z}^{ag} \times \mathcal{P}\left(\mathcal{Z}^{id} \times [\bar{k}, \infty) \times [\bar{b}, \infty)\right) \to \mathbb{R} \\ g_{R_b} &: \mathcal{P}\left(\mathcal{Z}^{id} \times [\bar{k}, \infty) \times [\bar{b}, \infty)\right) \to \mathbb{R}. \end{aligned}$$

Given Assumption 2 and an initial cross-sectional distribution of individual asset holdings μ_0 with $K_0 > 0$ and zero aggregate initial bond holdings, a competitive equilibrium consists of returns (R'_k, R'_b), the agents' choices (c', k', b') and the cross-sectional distribution μ' such that

1. *given (R'_k, R'_b), the individual choices (c', k', b') solve the Euler equations (3),*
2. *the returns (R'_k, R'_b) ensure that the market clearing conditions (5) hold,*
3. *the law of motion of the cross-sectional distribution μ' is consistent, i.e., it follows (4).*

Furthermore, the equilibrium is called recursive if the returns and optimal choices for any agent with previous-period asset holdings (k, b) who observes the current-period exogenous shocks (z', ε') and the beginning-of-current period cross-sectional

[3] I assume that for any fixed $\mu' \in \mathcal{P}(\mathcal{Z}^{id} \times [\bar{k}, \infty) \times [\bar{b}, \infty))$, the equilibrium functions $g_.(.,.,.,.,\mu')$ are measurable w.r.t. the probability triple $(\mathcal{Z} \times [\bar{k}, \infty) \times [\bar{b}, \infty), \mathcal{B}(\mathcal{Z} \times [\bar{k}, \infty) \times [\bar{b}, \infty)), \nu')$ where $\nu' = \mathbb{P}^{z'}\mu'$. Note that the functions $g_.(.,.,.,.,\mu')$ are, in fact, random variables on the Fubini extension $(I \times \mathcal{Z}^{id}, \mathcal{I} \boxtimes \mathcal{B}(\mathcal{Z}^{id}), \lambda \boxtimes P^{id})$. As those random variables have common distribution μ' for λ-almost every agent $i \in I$, I write the agents' optimal choices directly in terms of the beginning-of-period distribution μ'. Hence, the equilibrium functions g are the solution functions for λ-almost every agent $i \in I$. Null sets of agents are not considered further as they do not effect the aggregator **E**.

distribution μ' are given by functions

$$
\begin{aligned}
R_k' &= g_{R_k}\left(z',\mu'\right) = \frac{\partial}{\partial K} F\left(z', L', \mathbf{E}^1[\mu']\right) - \delta \\
R_b' &= g_{R_b}\left(\mu'\right) \\
c' &= g_c\left(z',\varepsilon',k,b,\mu'\right) \\
k' &= g_k\left(z',\varepsilon',k,b,\mu'\right) \\
b' &= g_b\left(z',\varepsilon',k,b,\mu'\right).
\end{aligned}
$$

Note that I use $x = [k, b]$ and $g_x = [g_k, g_b]$ in the following to simplify notation. Using the recursive equilibrium functions, the cross-sectional distribution's consistent law of motion can be rewritten as

$$
\begin{aligned}
\mu'\left(\varepsilon', x\right) = \int_{\varepsilon\in\mathcal{Z}^{id}} \int_{\xi\in\mathcal{Z}^{id}\cap\{\xi\leq\varepsilon\}} \int_{\bar{k}}^{\infty} \int_{\bar{b}}^{\infty} & \ _{\{g_x(z,\xi,\chi,\mu)\leq x\}} \\
& d\mu\left(\xi,\chi\right) \mathbb{P}\left(\varepsilon' \middle| d\varepsilon, z, z'\right).
\end{aligned}
\tag{6}
$$

for all $\varepsilon' \in \mathcal{Z}^{id}$ and $x \in [\bar{k}, \infty) \times [\bar{b}, \infty)$. Given the previous-period distribution, the rates of return follow immediately from this definition of the current-period distribution and are, thus, observed at the beginning of the period.

3 Characterizing the Incomplete Markets Equilibrium

Now that the model and its equilibrium are defined in a general manner, I show that we can rewrite the recursive equilibrium in terms of random variables and I explain how that leads to a set of operators characterizing the equilibrium.

3.1 Rewriting the Recursive Equilibrium

As we consider a heterogeneous agent model with a continuum of agents, the equilibrium defined in Definition 3 consists of policy functions which depend on the cross-sectional distribution. Since functions on distributions are typically difficult to handle, I will show in this section that the equilibrium functions can be restated in a more tractable form. To do so, I rely on an idea which has been used in the mean field game literature. It is well known from measure theory that for any distribution on a complete separable metric space, we can find a random variable whose law equals that distribution (see e.g. Bogachev, 2007, Theorem

9.1.5). Furthermore, weak convergence of distributions translates into almost sure convergence of random variables (see Dudley, 2002, Theorem 11.7.2). The literature on mean field games harnesses those results to develop a differential calculus for functions of distributions by rewriting those functions as functions of random variables and applying standard Fréchet differentiation. My argument to establish the existence result of the recursive equilibrium is based on monotonicity which can be expressed in terms of the derivative w.r.t. the cross-sectional distribution. Thus, I switch from distributions to random variables and use the differential calculus developed in the theory of mean field games for the existence argument. I provide a more detailed summary of the relevant results from mean field games in Section 1 of the technical online appendix.

To show that we can state the recursive equilibrium as a function of a random variable rather than the cross-sectional distribution, I consider an alternative way of defining the consistent law of motion of the cross-sectional distribution. Instead of using c.d.f.s as in the previous section, I define it in terms of random variables. Let me start by specifying further notation for the exogenous processes. I assume that the process $\{z_t, \varepsilon_t\}_{t\geq 0}$ lives on a filtered probability space $(\mathcal{Z}, \Sigma, \{\mathcal{F}_t^{z,\varepsilon}\}_{t\geq 0}, \mathbb{P})$ where $\mathcal{F}^{z,\varepsilon}$ denotes the process' natural filtration with $\mathcal{F}_0^{z,\varepsilon} \subseteq \ldots \subseteq \mathcal{F}_\infty^{z,\varepsilon} \subseteq \Sigma$. Another exogenous element is the initial distribution of asset holdings per agent group. I denote this by the random variable χ_{-1} which lives on the probability space $(\prod_{j=1}^2 [\bar{x}^j, \infty), \mathcal{B}(\prod_{j=1}^2 [\bar{x}^j, \infty)), \mathbb{P})$. By denoting $\mathbf{P} = \mathcal{Z} \times \prod_{j=1}^2 [\bar{x}^j, \infty)$ and defining the filtered product space $(\mathbf{P}, \Sigma \otimes \mathcal{B}(\prod_{j=1}^2 [\bar{x}^j, \infty)), \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the filtration $\mathcal{F}_t = \mathcal{F}_t^{z,\varepsilon} \otimes \mathcal{B}(\prod_{j=1}^2 [\bar{x}^j, \infty))$, we can represent the initial cross-sectional distribution by the random variable $(\varepsilon_0, \chi_{-1}) \in \mathcal{F}_0$ with $(\varepsilon_0, \chi_{-1}) \sim \mu_0$. Given the aggregate shock trajectory $z^1 = (z_0, z_1)$ and the optimal response of the agents, the next beginning-of-period distribution μ_1 can be represented by

$$(\varepsilon_1, \chi_0) = (\varepsilon_1, g_x(z_0, \varepsilon_0, \chi_{-1}, \mu_0)) \in \mathcal{F}_1.$$

Denoting the history of shocks by $z^t = (z_0, \ldots, z_t)$ and $\varepsilon^t = (\varepsilon_0, \ldots, \varepsilon_t)$, it follows by induction that the cross-sectional distribution μ_t has a representation

$$(\varepsilon_t, \chi_{t-1}) = \left(\varepsilon_t, f\left(z^{t-1}, \varepsilon^{t-1}, \chi_{-1}\right)\right) \in \mathcal{F}_t, \tag{7}$$

where f is a composition of policy functions g_x. Note that the process $(\varepsilon_t, \chi_{t-1})$ implicitly depends on z_t as well due to the idiosyncratic shock distribution being conditional on the aggregate shock outcome. It follows that the asset holdings at any time point are measurable w.r.t. the product space

$$(\varepsilon_t, \chi_{t-1}) \in L^{\text{full}}_{\mathbf{P}} = L\left(\mathbf{P}, \Sigma \otimes \mathcal{B}\left(\prod_{j=1}^{2}[\bar{x}^j, \infty)\right), \mathbb{P}\right) \; \forall\, t \geq 0.$$

Hence, we can write any cross-sectional distribution μ_t as a function of z_t and $(\varepsilon_t, \chi_{t-1}) \in L^{\text{full}}_{\mathbf{P}}$ by

$$\mu_t\left(\varepsilon, x\right) = \int_{\varepsilon_t \in \mathcal{Z}^{id}} \int_{\bar{k}}^{\infty} \int_{\bar{b}}^{\infty} {}_{\{\{\varepsilon_t \leq \varepsilon\} \cap \{\chi_{t-1} \leq x\}\}} d\mathbb{P}\left(\varepsilon_t, \chi_{t-1} \middle| \, z_t\right), \tag{8}$$

which implies that the beginning-of-period cross-sectional distribution at any time t is a measurable function w.r.t. the product space $L^{\text{full}}_{\mathbf{P}}$.

It is important to note that we do not necessarily know the full history of exogenous shocks for an arbitrary random variable $(\varepsilon', \chi) \in L^{\text{full}}_{\mathbf{P}}$. This depends on the sub-σ-algebra w.r.t. which the random variable (ε', χ) is measurable. For instance $(\varepsilon', \chi) \in \mathcal{F}_t$ tells us the history z^t and ε^t and which initial values the distribution started from as in (7). However, an arbitrary $(\varepsilon', \chi) \in L^{\text{full}}_{\mathbf{P}}$ may also be measurable w.r.t. a much smaller sub-σ-algebra of $\Sigma \otimes \mathcal{B}\left(\prod_{j=1}^{2}[\bar{x}^j, \infty)\right)$. Such a smaller sub-$\sigma$-algebra $\mathcal{G}_t$ can be constructed as the product σ-algebra $\mathcal{G}^{z,\varepsilon}_t \otimes \mathcal{B}(\prod_{j=1}^{2}[\bar{x}^j, \infty))$, where $\mathcal{G}^{z,\varepsilon}_t$ contains the sets of complete histories z^∞ and ε^∞ with $z_t = z'$ and $\varepsilon_t = \varepsilon'$, and, their complements and countable unions. Note that the collection of σ-algebras $\{\mathcal{G}_t\}_{t \geq 0}$ is not a filtration as $\mathcal{G}_t \nsubseteq \mathcal{G}_{t+1}$ but we have $\mathcal{G}_0 \subseteq \mathcal{F}_0$ and $\mathcal{G}_t \subset \mathcal{F}_t$. Furthermore, all $\mathcal{G}_t$ have the same size in terms of the number of sets they contain. Thus, $(\varepsilon', \chi) \in \mathcal{G}_t$ for any $t \geq 0$ only tells us what the current shocks and the current asset holdings are. Denote the set of $\mathcal{G}_t$-measurable random variables for arbitrary t, which are spanned by a Gaussian orthogonal basis,[4] by $G \subset L^{\text{full}}_{\mathbf{P}}$. Let $\mathcal{B}(G)$ be the smallest Borel σ-algebra containing the pre-images of the random variables in G and denote the corresponding subspace by

$$L_{\mathbf{P}} = L\left(\mathbf{P}, \mathcal{B}(G), \mathbb{P}\right).$$

Note that $L_{\mathbf{P}} \subset L^{\text{full}}_{\mathbf{P}}$.

Due to the previous arguments, we can rewrite the recursive equilibrium in

[4]For a rigorous definition of this condition, I refer to Section 2.2 and, in particular, Assumption 9 in the technical online appendix.

terms of an arbitrary random variable $(\varepsilon', \chi) \in L_{\mathbf{P}}$. Note that the conditional random variable of the beginning-of-current period asset holdings is actually a function $\chi : \mathbf{P} \to \mathbb{R}^2$. Accordingly, we can identify an agent-specific state using this random variable as there exists an $\omega \in \mathbf{P}$ with $x = \chi(\omega)$. The same holds for the specific idiosyncratic outcome, i.e., ω determines the current idiosyncratic shock. By inserting ω into the equilibrium functions instead of the individual state of the agent corresponding to the random sample ω and using $m' = (\varepsilon', \chi)$ instead of the cross-sectional distribution μ', the equilibrium functions can be written as functions $h(z', \omega, m') := g(\varepsilon'(\omega), \chi(\omega), \mu')$.

Proposition 4 (Recursive equilibrium with random variables)**.** *Consider the recursive equilibrium in Definition 3. According to (8), we can rewrite the equilibrium functions in terms of a random variable $m' = (\varepsilon', \chi) \in L_{\mathbf{P}}$ with law $m' \sim \mu'$ by*

$$
\begin{aligned}
h_c, h_k, h_b :& \mathbf{P} \times L_{\mathbf{P}} \to \mathbb{R} \\
h_{R_k} :& \mathcal{Z}^{ag} \times L_{\mathbf{P}} \to \mathbb{R} \\
h_{R_b} :& L_{\mathbf{P}} \to \mathbb{R},
\end{aligned}
$$

such that

$$
\begin{aligned}
R_k' &= h_{R_k}(z', m') := \frac{\partial}{\partial K} F\left(z', L', \mathbf{E}^1[m']\right) - \delta \\
R_b' &= h_{R_b}(m') := g_{R_b}(\mu') \\
c' &= h_c(z', \omega, m') := g_c(z', \varepsilon'(\omega), \chi(\omega), \mu') \\
k' &= h_k(z', \omega, m') := g_k(z', \varepsilon'(\omega), \chi(\omega), \mu') \\
b' &= h_b(z', \omega, m') := g_b(z', \varepsilon'(\omega), \chi(\omega), \mu').
\end{aligned}
$$

Remark. The identification between $m' \sim \mu'$ is not unique. There may be multiple random variables m' with the same law μ'. However, these random variables can only differ on the null set of μ', i.e., outside the support of μ'. Thus, this multiplicity is limited to states which are impossible to obtain for any household. This is true even for off-equilibrium considerations as μ' describes the currently observed distribution and thus, summarizes the starting conditions of all agents. Hence, for any $(\varepsilon, x) \in \operatorname{supp} \mu'$, the correspondence between h and g is bijective. This implies that existence and uniqueness results for h also hold for g where g is defined on the support of μ' for any $\mu' \in \mathcal{P}\left(\mathcal{Z}^{id} \times [\bar{k}, \infty) \times [\bar{b}, \infty)\right)$.

When analyzing the policy functions h, an important question is which space those functions live on. First, note that we can interpret the policy functions $h(z', \omega, m')$[5] as a collection of random variables $\{h(.,.,m')\}_{m' \in L_{\mathbf{P}}}$ indexed by $m' \in L_{\mathbf{P}}$ representing the cross-sectional distributions.[6] This is possible because it follows from the measurability of g that the policies h, when keeping m' fixed, are measurable functions w.r.t. the product probability space $(\mathbf{P}, \mathcal{B}(G), \mathbb{P})$. Hence, the policies are stochastic processes with a general index set instead of the standard time index. I refrain from using the short-hand $h_{m'}$ though to keep notational clarity. These general stochastic processes are called random fields. To put more structure on this set of random fields, let me first make an assumption on some model preliminaries and then introduce the concept of continuity of a random field w.r.t. the index m'.

Assumption 5 (Square integrability)**.**

(*i*) The initial conditional random variable of asset holdings χ_{-1} distributed according to the initial conditional cross-sectional distribution $(\varepsilon_0, \chi_{-1}) \sim \mu_0$ is square integrable, i.e., $\chi_{-1} \in L^2_{\mathbf{P}}$.

(*ii*) The endowment function e, the labor supply function l and the marginal productivity of capital $\frac{\partial}{\partial K}F$ and labor $\frac{\partial}{\partial L}F$ are square integrable w.r.t. the exogenous variables, i.e., $e, l, \frac{\partial}{\partial K}F, \frac{\partial}{\partial L}F \in L^2_{\mathbf{P}}$.

Definition 6 (Sample Path Continuity)**.** *A random field given by $h : \mathbf{P} \times L^2_{\mathbf{P}} \to \mathbb{R}$ with index $m' = (\varepsilon', \chi) \in L^2_{\mathbf{P}}$ is called continuous if for any series of random variables $m'_n \in L^2_{\mathbf{P}}, n \in \{1, 2, \ldots\}$, which converges almost surely to $m' \in L^2_{\mathbf{P}}$, it holds that*

$$\mathbb{P}\left(\lim_{n \to \infty} h(z', \omega, m'_n) = h(z', \omega, m')\right) = 1.$$

Denote the set of all such continuous random fields by $C(L^2_{\mathbf{P}})$. Furthermore, denote the set of all continuous random fields which are of second order themselves, i.e., $\mathbb{E}[h(.,.,m')^2] < \infty$ for any index $m' \in L^2_{\mathbf{P}}$, by $C^2(L^2_{\mathbf{P}})$.

Remark. Note that this notion of continuity implies continuity of the policy functions in both the individual asset holdings as well as the cross-sectional distribution.

The random fields in $C^2(L^2_{\mathbf{P}})$ generate a Hilbert space which does, in fact, equal $L^2_{\mathbf{P}}$. A brief introduction to random fields and details on the Hilbert space

[5] This notation abstracts from the super- and subscripts, i.e., h stands for h_k, h_b, etc.

[6] Note that this interpretation also applies to the aggregate policies h_{R_k} and h_{R_b} as those are constant functions w.r.t. ω.

generated by them can be found in Section 2 of the technical online appendix. Overall, I show in the appendix that $C^2(L^2_{\mathbf{P}}) \subseteq L^2_{\mathbf{P}}$.

3.2 The Euler Equation and Bond Market Clearing Operators

In this section, I derive a set of operators characterizing the equilibrium. I exploit the fact that the optimal policy functions of the recursive equilibrium solve the Euler equations which, if a suitable transversality condition holds, are necessary and sufficient for optimality. The set of Euler equations corresponding to the model from Section 2 reads

$$0 = -\left(c'\right)^{-\gamma} + \mathbb{E}^{(z'',\varepsilon''|z',\varepsilon')}\left[\beta(1+R''_j)\left(c''\right)^{-\gamma}\right] + y'_j, j \in \{k, b\}.$$

Note that I attach the borrowing constraints of the asset holdings $x' = [k', b']$ with Lagrange multipliers y'. The set of Euler equations has to hold at any exogenous state for any agent in the economy which means that it has to hold for a.e. (z', ε') and μ'-a.e. $x = [k, b]$. Inserting the recursive equilibrium functions yields the Euler equation operator.

Definition 7 (Law of motion and Euler equation operator)**.** *Suppose that Assumption 2 holds. Then, the law of motion operator of the random variable of asset holdings corresponding to the model from Section 2 is defined by* $\mathbf{m}'' : L^2_{\mathbf{P}} \longrightarrow L_{\mathbf{P}}, [h_k, h_c, h_{R_b}] \mapsto (\varepsilon'', h_k, \mathbf{h_b}[h_k, h_c, h_{R_b}])$ *with*

$$\begin{aligned}\mathbf{h_b}[h_k, h_c, h_{R_b}](z', \omega, m') =& e\left(z', \varepsilon'(\omega)\right) + \frac{\partial F}{\partial L}\left(z', L', \mathbf{E}^1[m']\right) l\left(z', \varepsilon'(\omega)\right) \\ &+ \left(1 + h_{R_k}\left(z', m'\right)\right)\chi^k(\omega) + \left(1 + h_{R_b}\left(m'\right)\right)\chi^b(\omega) \\ &- h_k\left(z', \omega, m'\right) - h_c\left(z', \omega, m'\right), \end{aligned} \tag{9}$$

where $m' = (\varepsilon', \chi)$. *Furthermore, the Euler equation operator is defined by* $\mathbf{T} : L^2_{\mathbf{P}} \longrightarrow L_{\mathbf{P}}, [h_k, h_c, h_{R_b}] \mapsto [\mathbf{T}^k[h_k, h_c, h_{R_b}], \mathbf{T}^b[h_k, h_c, h_{R_b}]]$ *with*

$$\begin{aligned}\mathbf{T}^j[h_k, h_c, h_{R_b}]\left(z', \omega, m'\right) =& -h_c\left(z', \omega, m'\right)^{-\gamma} \\ &+ \mathbb{E}^{(z'',\varepsilon''|z',\varepsilon')}\left[\beta\left(1 + R''_j\right) h_c\left(z'', \omega, \mathbf{m}''[h_k, h_c, h_{R_b}]\right)^{-\gamma}\right],\end{aligned}$$

$j \in \{k, b\}$, where

$$R''_j = \begin{cases} \frac{\partial}{\partial K} F\left(z', L', \mathbf{E}^1\left[\mathbf{m}''[h_k, h_c, h_{R_b}]\right]\right) - \delta & j = k \\ h_{R_b}\left(\mathbf{m}''[h_k, h_c, h_{R_b}]\right) & j = b \end{cases}. \tag{10}$$

The Euler equation operator summarizes the Euler equations of all agents which is possible by switching to the random variables m'. Thus, the agents are indexed by the random outcome ω. Note that the law of motion operator directly incorporates the budget constraint to define the law of motion of the random variable of bond holdings. Furthermore, both operators directly incorporate the capital market clearing condition of (5) by incorporating Assumption 2. However, the bond market clearing condition needs to be handled separately.

Definition 8 (Bond market clearing operator)**.** *The bond market clearing operator corresponding to the model from Section 2 is defined by $\mathbf{B} : L^2_{\mathbf{P}} \longrightarrow L_{\mathbf{P}}, [h_k, h_c, h_{R_b}] \mapsto \mathbf{B}[h_k, h_c, h_{R_b}]$ with*

$$\mathbf{B}[h_k, h_c, h_{R_b}]\left(z', m'\right) = \int_{\mathcal{Z} \times \prod_{j=1}^{2} [\bar{x}^j, \infty)} \mathbf{h_b}[h_k, h_c, h_{R_b}](z', \omega, m') d\mathbb{P}(\omega)$$

with $m' = (\varepsilon', \chi)$ and $\mathbf{h_b}$ as in (9).

To summarize, we obtain a candidate equilibrium solution by finding functions $h_k, h_c, h_{R_b}, y \in C(L^2_{\mathbf{P}})$ which solve the following equations

$$\begin{bmatrix} \mathbf{T}[h_k, h_c, h_{R_b}]\left(z', \omega, m'\right) + y\left(z', \omega, m'\right) \\ \mathbf{B}[h_k, h_c, h_{R_b}]\left(z', m'\right) \end{bmatrix} = 0, \ (h_k - \bar{k}, \mathbf{h_b} - \bar{b}) \perp y \geq 0, \tag{11}$$

for $\mathbb{P}$-almost every $\omega \in \mathbf{P}$ and for every $m' = (\varepsilon', \chi) \in L^2_{\mathbf{P}}$. If a solution to equation (11) additionally satisfies a suitable transversality condition, it is indeed an equilibrium solution. I explain in the next section how to ensure that a solution to (11) exists and leads to a unique equilibrium solution.

4 Existence and Uniqueness of an Equilibrium Solution

As is shown in Stokey, Lucas, and Prescott (1989), the extension of existence results with bounded utility functions to unbounded utility functions like the case of CRRA utility is typically done via constant returns to scale. However, due to the idiosyncratic shocks, there is a disjunction between individual asset holdings and their rates of returns which aggregate over the individual holdings. Each agent in the continuum has zero weight and cannot influence aggregates. Therefore, it can happen that the individual asset holdings grow substantially for an agent, but the rate of return does not change significantly to counteract this growth. According to Stokey et al. (1989), this model, thus, falls into the category of unbounded returns. To establish existence, I rely on arguments of monotonicity because compactness cannot be proven without further restrictions.

As I do not rely on a standard fixed-point theorem, let me first state the main mathematical result which I use to establish existence.

Corollary 9 (Rockafellar (1969, Corollary 1.4))**.** *Let $\mathcal{C}$ be a Hilbert space over $\mathbb{R}$, and let $\mathbf{M} : \mathcal{C} \to \mathcal{C}^*$ be a maximal monotone operator.*[7] *Suppose that there exists a subset $B \subset \mathcal{C}$ such that $0 \in \operatorname{int}(\operatorname{conv}(\mathbf{M}(B)))$. Then, there exists a $c \in \mathcal{C}$ such that $0 \in \mathbf{M}(c)$.*

Remark. This corollary essentially generalizes the intermediate value theorem, which states that there exists a root for a continuous real function $f : \mathbb{R} \to \mathbb{R}$ if there exist two points $a, b \in \mathbb{R}$ with $f(a) > 0$ and $f(b) < 0$, to higher-dimensional spaces. Note that requiring continuity is not enough for mappings on multidimensional spaces.[8] Instead, the operator needs to be maximal monotone. If this property is satisfied, the corollary requires a subset B in the domain of the operator such that the interior of the convex hull of the subset's image contains zero. In some particular cases, it is sufficient to find two elements c_- and c_+ such that the image $\mathbf{M}(c_-)$ is negative and the image $\mathbf{M}(c_+)$ is positive. If a convex combination of the two results is zero, Rockafellar's Corollary applies.

The goal is to apply this corollary to the left-hand side of equation (11)

$$\begin{aligned} &\mathbf{M}[h_k, h_c, h_{R_b}, y](z', \omega, m') \\ &= \left\{ \begin{bmatrix} \mathbf{T}[h_k, h_c, h_{R_b}]\,(z', \omega, m') + y\,(z', \omega, m') \\ \mathbf{B}[h_k, h_c, h_{R_b}]\,(z', m') \end{bmatrix}, \ (h_k - \bar{k}, \mathbf{h_b} - \bar{b}) \perp y \geq 0 \right\}. \end{aligned} \tag{12}$$

Note that the operator $\mathbf{M}$ is defined on the Hilbert space $L^2_{\mathbf{P}}$. The dual of which is $L^2_{\mathbf{P}}$ itself. Thus, the first step to apply Rockafellar's corollary is to show that

[7] Monotonicity (see e.g., Phelps, 1997; Bauschke and Combettes, 2010): Let $\mathcal{E}$ be a Hilbert space. An operator $\mathbf{M} : \mathcal{E} \to \mathcal{E}$ is called a monotone operator if for any two elements of its graph $(e, f), (\tilde{e}, \tilde{f}) \in G(\mathbf{M}) = \{(e, f) \in \mathcal{E}^2 | f \in \mathbf{M}(e)\}$ it holds that $\langle e - \tilde{e}, f - \tilde{f} \rangle \geq 0$. It is, additionally, called maximal monotone if any $(\tilde{e}, \tilde{f}) \in \mathcal{E}^2$ with $\langle e - \tilde{e}, f - \tilde{f} \rangle \geq 0 \, \forall \, (e, f) \in G(\mathbf{M})$ is necessarily also an element of the graph $(\tilde{e}, \tilde{f}) \in G(\mathbf{M})$.

[8] A simple counterexample is $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x, y) = [\log(x + y), (x + y)^3]$.

the definition of $\mathbf{M}$ can be refined in such a way to ensure that it maps into $L^2_{\mathbf{P}}$. Second, I have to establish that the operator is maximal monotone. To ensure the square-integrable range and maximal monotonicity, I proceed in two steps. First, I consider the unconstrained case where, by definition, $y = 0$. From the square-integrability and maximal monotonicity of $[\mathbf{T}, \mathbf{B}] : L^2_{\mathbf{P}} \to L^2_{\mathbf{P}}$, I then derive the same properties for $\mathbf{M}$ in the constrained case.

4.1 Maximal Monotonicity in the Unconstrained Case

Let me first define an admissible set $\mathcal{H}_\epsilon$ and show subsequently that the operator $[\mathbf{T}, \mathbf{B}]$ restricted to the admissible set maps into the square-integrable random variables and is maximal monotone. The proofs can be found in Appendix A.

Proposition 10 (Admissible set). *Consider the model from Section 2 and suppose that Assumptions 1, 2 and 5 hold. For some $\epsilon > 0$, define the subset $\mathcal{H}_\epsilon \subset C^2(L^2_{\mathbf{P}}) \subseteq L^2_{\mathbf{P}}$ as the set of continuous random fields $[h_k, h_c, h_{R_b}]$ for which the following conditions hold for any index $m' = (\varepsilon', \chi) \in L^2_{\mathbf{P}}$ with non-negative aggregate capital $\mathbf{E}^1[m'] \geq 0$.*

(i) Gâteaux differentiability[9] in m': $h_c(z', \omega, m')$ and $h_{R_b}(m')$ are twice Gâteaux differentiable in m' for $\mathbb{P}$-a.e. (z', ω), the derivatives are square-integrable, i.e., $\mathbb{E}^{\mathbf{P}}\left[(d^i_G h_c(z', \omega, m', \tilde{m}'))^2\right] < \infty$ for any $\tilde{m}' \in L^2_{\mathbf{P}}$, $i \in \{1, 2\}$, and, they are sample-path continuous in m'

(ii) Consumption policy:

- *a) Nonnegative individual consumption: $h_c(z', \omega, m') \geq 0$*
- *b) Curvature in m': $h_c(z', \omega, m')$ is concave, i.e., $d^2_G h_c(z', \omega, m'; \tilde{m}') \leq 0$ for any $\tilde{m}' \in L^2_{\mathbf{P}}$*
- *c) Slope in m': $h_c(z', \omega, m')$ co-moves with total savings which can be expressed by $d_G h_c(z', \omega, m'; \tilde{m}') \left(\tilde{\chi}^k(\omega) + \tilde{\chi}^b(\omega)\right) \geq 0$, and, its slope is larger than its curvature, i.e, $d_G h_c(z', \omega, m'; \tilde{m}') \geq \frac{1}{2} d^2_G h_c(z', \omega, m'; \tilde{m}')$ for any $\tilde{m}' \in L^2_{\mathbf{P}}$*

where the inequalities hold for $\mathbb{P}$-a.e. (z', ω)

(iii) Aggregate policies:

[9] Gâteaux derivative and differential (see e.g., Zeidler, 1985): Let $\mathcal{E}$ be a Hilbert space. The Gâteaux differential of an operator $\mathbf{M} : \mathcal{E} \to \mathcal{E}$ at a point $e \in \mathcal{E}$ in the direction $\tilde{e} \in \mathcal{E}$ is defined by $d_G \mathbf{M}[e; \tilde{e}] = d/dt \mathbf{M}(e + t\tilde{e})|_{t=0}$. The linear operator $\mathbf{M}'(e)$ with $\mathbf{M}'(e)\tilde{e} = d_G \mathbf{M}[e; \tilde{e}]$ is the Gâteaux derivative at $e \in \mathcal{E}$.

a) *Positive aggregate capital:* $\langle h_k, 1\rangle \geq \epsilon$, *where the inner product denotes the conditional expectation of the random field for an arbitrary but fixed index* m' *and is given by*

$$\langle h_1, h_2\rangle = \mathbb{E}^{\mathbb{P}}\left(h_1(z',\omega,m')h_2(z',\omega,m') \middle| \, z'\right) \tag{13}$$

for any $h_1, h_2 \in L^2_{\mathbf{P}}$

b) *Positive bond price:* $1 + h_{R_b}(m') \geq \epsilon$

c) *Curvature in* m'*:* $\log\left(1 + h_{R_b}(m')\right)$ *is convex, i.e., for any* $\tilde{m}' \in L^2_{\mathbf{P}}$*, it holds that* $d^2_G\left(\log\left(1 + h_{R_b}(m')\right); \tilde{m}'\right) \geq 0$

(*iv*) *Bounded aggregated Gâteaux differentials of the Euler equations:*

$$\begin{aligned}
&\mathbf{E}^b[m']\tilde{h}^2_{R_b} - \left\langle \tilde{h}_{R_b}, \tilde{h}_k + \tilde{h}_c\right\rangle - \left\langle d_G\left(h_c^{-\gamma}; \tilde{h}\right), \tilde{h}_k + \tilde{h}_c\right\rangle \\
&+ \mathbb{E}^{(z'',\varepsilon''|z',\varepsilon')}\beta\left\langle d_G\left(\left(1+R''_k\right) h_c\left(z'',\omega,m''\right)^{-\gamma}; \tilde{h}\right), \tilde{h}_k\right\rangle \\
&+ \mathbb{E}^{(z'',\varepsilon''|z',\varepsilon')}\beta\left\langle d_G\left(\left(1+R''_b\right) h_c\left(z'',\omega,m''\right)^{-\gamma}; \tilde{h}\right), \tilde{h}_c\right\rangle \geq 0
\end{aligned}$$

for $\mathbb{P}$*-a.e.* z' *and any* $\tilde{h} = [\tilde{h}_k, \tilde{h}_c, \tilde{h}_{R^b}] \in L^2_{\mathbf{P}}$*,* R'' *as in (10), and* $m'' = \mathbf{m}''[h_k, h_c, h_{R_b}]$ *is given by the law of motion operator*

(*v*) *Transversality condition:*

$$\mathbb{E}^{(z',\varepsilon'|z,\varepsilon)}\left[h_c\left(z',\omega,m'\right)^{-\gamma}\left(1 + h_{R_j}(z',m')\right)\left(\chi^j(\omega) - \bar{x}^j\right)\right] < \infty$$

for $\mathbb{P}$*-a.e.* (z',ω) *and any* $j \in \{k, b\}$.

Then, $\mathcal{H}_\epsilon \subset L^2_{\mathbf{P}}$ *is a closed, convex subset of the Hilbert space.*

Remark. (*i*) Condition (*i.c*) implies that the consumption policy is concave in m' for any agent. In combination with condition (*i.b*), this statement can be made even stronger. I show in Proposition 21 of the appendix that $(i.b - c)$ ensure concavity of future consumption. Thus, also the consumption stream $\{c_t\}_{t\geq 0}$ which results from reapplying an admissible recursive consumption policy h_c is concave in this policy choice.

(*ii*) Note that conditions (*ii*) and (*iii*) are aggregate conditions meaning that individual agents may deviate, e.g., hold capital less than ϵ, as long as the majority of agents behaves in line with the aggregate condition.

The first step in applying Rockefellar's corollary is to ensure that the equilibrium operator has a suitable domain and range.

Proposition 11 (Square-integrable range)**.** *Suppose that the assumptions of Proposition 10 hold. Then, the equilibrium operator* $[\mathbf{T}, \mathbf{B}]$ *specified in Definition 7 and 8 maps the admissible set* $\mathcal{H}_\epsilon$ *into the space of continuous random fields of second order* $C^2(L^2_{\mathbf{P}}) \subseteq L^2_{\mathbf{P}}$.

In the next step, I establish monotonicity for the operator $\mathbf{M}$ on the subset $\mathcal{H}_\epsilon \subset C^2(L^2_{\mathbf{P}})$. Note that the operator can be extended to the whole domain $C^2(L^2_{\mathbf{P}})$ in a way which preserves monotonicity, and moreover, yields maximality.

Proposition 12 (Maximal monotone extension)**.** *Suppose that the equilibrium operator* $\mathbf{M} = [\mathbf{T}, \mathbf{B}]$ *specified in Definition 7 and 8 is monotone on the admissible set* $\mathcal{H}_\epsilon$ *defined in Proposition 10. Define the extended equilibrium operator* $\overline{\mathbf{M}} : C^2(L^2_{\mathbf{P}}) \to C^2(L^2_{\mathbf{P}})$ *by* $\overline{\mathbf{M}} = \mathbf{M} \circ \mathbf{P}$ *where the projection operator* $\mathbf{P} : C^2(L^2_{\mathbf{P}}) \to \mathcal{H}_\epsilon$ *is defined by the closest admissible point*

$$
\begin{aligned}
\mathbf{P}[a_k, a_c, a_{R^b}] = \{ & [h_k, h_c, h_{R_b}] \in \mathcal{H}_\epsilon \,|\, \forall [\bar{y}_k, \bar{y}_c, \bar{y}_{R^b}] \in \mathcal{H}_\epsilon \\
& \langle \bar{y}_k - h_k, a_k - h_k \rangle + \langle \bar{y}_c - h_c, a_c - h_c \rangle + \langle \bar{y}_{R^b} - h_{R_b}, a_{R^b} - h_{R_b} \rangle \leq 0 \} \,.
\end{aligned}
$$

Then, $\overline{\mathbf{M}}$ *is maximal monotone.*

Lemma 13 (Maximal monotone equilibrium operator)**.** *Suppose that the assumptions of Proposition 10 hold. Then, the extended equilibrium operator* $\overline{\mathbf{M}} : C^2(L^2_{\mathbf{P}}) \to C^2(L^2_{\mathbf{P}})$ *defined as in Proposition 12 is maximal monotone.*

Remark. (i) The proof of Lemma 13 exploits that monotonicity of a Gâteaux differentiable operator $\mathbf{M}$ is equivalent to

$$
\begin{aligned}
& \left\langle d_G \mathbf{M} \left[h; \tilde{h} \right], \tilde{h} \right\rangle \geq 0 \\
\Leftrightarrow & \left\langle d_G \mathbf{T}^k \left[h_k, h_c, h_{R_b}; \tilde{h}_k, \tilde{h}_c, \tilde{h}_{R^b} \right], \tilde{h}_k \right\rangle \\
& + \left\langle d_G \mathbf{T}^b \left[h_k, h_c, h_{R_b}; \tilde{h}_k, \tilde{h}_c, \tilde{h}_{R^b} \right], \tilde{h}_c \right\rangle \\
& + \left\langle d_G \mathbf{B} \left[h_k, h_c; \tilde{h}_k, \tilde{h}_c \right], \tilde{h}_{R^b} \right\rangle \geq 0
\end{aligned} \tag{14}
$$

for any $m' \in L^2_{\mathbf{P}}$.[10]

[10] A proof of this claim can be found in Section A.4 of the appendix. The inner product here denotes the $L^2_{\mathbf{P}}$-inner product as $h \in C^2(L^2_{\mathbf{P}}) \subseteq L^2_{\mathbf{P}}$. Hence, this equation has to hold for any fixed index $m' \in L^2_{\mathbf{P}}$ of the random field. Note that this pointwise requirement is stronger

(*ii*) The proof for the general model with both capital and a bond can be easily adapted to models featuring only one of the two assets, i.e., either capital or the bond. Hence, the result also holds for the Aiyagari-Bewley economy and the Huggett economy, respectively.[11]

4.2 Maximal Monotonicity in the Constrained Case

To show that the extension of the operator $\mathbf{M}$ is maximal monotone in the constrained case where the Lagrange multiplier is not necessarily zero, I introduce a convex objective function $F_{\mathbf{M}} : C^2(L^2_{\mathbf{P}}) \to [-\infty, \infty]$ which has its optima exactly at the roots of the unconstrained equilibrium operator. I show in Section 3 of the technical appendix that this objective function is given by

$$\begin{aligned} F_{\mathbf{M}}[h_k, h_c, h_{R_b}] &= \sup_{a \in \mathcal{H}_\epsilon} \langle [h_k, h_c, h_{R_b}] - a, \mathbf{M}[a] \rangle \\ &= \sup_{a \in \mathcal{H}_\epsilon} \left\{ \left\langle \mathbf{T}^k \left[a_k, a_c, a_{R^b}\right], h_k - a_k \right\rangle + \left\langle \mathbf{T}^b \left[a_k, a_c, a_{R^b}\right], h_c - a_c \right\rangle \right. \\ &\qquad \left. + \left\langle \mathbf{B}\left[a_k, a_c\right], h_{R_b} - a_{R^b} \right\rangle \right\}. \end{aligned} \tag{15}$$

and it satisfies

$$\mathbf{M}[h_k, h_c, h_{R_b}] = 0 \Leftrightarrow [h_k, h_c, h_{R_b}] = \arg \inf_{h \in \mathcal{H}_\epsilon} F_{\mathbf{M}}[h].$$

The objective $F_{\mathbf{M}}$ aggregates over two things. The first two terms represent an aggregated first-order Taylor approximation of the agents' utility over two time points, whereas, the last term represents a regularization to achieve market clearing. Therefore, we can interpret this function as the objective function of a benevolent social planner. For each agent, the social planner uses a linearization of the agent's utility at two time points. As we are looking for a recursive equilibrium, summing the utility over two time points suffices to optimize in the infinite horizon. The social planner weighs each agent equally since aggregation over the cross-sectional distribution evaluates each state by the amount of agents which currently observe that same state.

than if one would additionally aggregate over the index set. Although not necessary here, it is nevertheless possible to construct an inner product which additionally aggregates over the indexes of the random field. To do so, one can resort to the inner product of a Hilbert space of functions defined on the index set which is isomorphic to the Hilbert space generated by the random fields $L^2_{\mathbf{P}}$. More details on this isomorphic Hilbert space can be found in Section 2.3 of the technical online appendix.

[11]The details of this adaptation are spelled out in Section A.4.1 of the appendix.

Now that a suitable objective function associated with $\mathbf{M}$ is defined, I can attach the borrowing constraints $h_k \geq \bar{k}$ and $h_b \geq \bar{b}$. Therefore, I obtain a Lagrangian for the constrained problem $L : \mathcal{H}_\epsilon \times C^2(L^2_{\mathbf{P}}) \to [-\infty, \infty]$ given by

$$L(h_k, h_c, h_{R_b}, y) = F_{\mathbf{M}}[h_k, h_c, h_{R_b}] + \langle h_k - \bar{k}, y_k \rangle + \langle h_{pb} - h_{R_b}\bar{b}, y_b \rangle . \tag{16}$$

Similarly to the equilibrium operator, this Lagrangian can be extended to the whole space $\bar{L} : C^2(L^2_{\mathbf{P}}) \times C^2(L^2_{\mathbf{P}}) \to [-\infty, \infty]$ with the projection operator from Proposition 12, i.e., setting $\bar{L}(h_k, h_c, h_{R_b}, y) = L(\mathbf{P}[h_k, h_c, h_{R_b}], y)$. Note that the subdifferential[12] of this Lagrangian coincides by construction with the extension of the constrained equilibrium operator $\mathbf{M}$. The following result shows that maximal monotonicity continues to hold in this case.

Lemma 14 (Maximal monotone $\mathbf{M}$)**.** *Consider the model from Section 2 and suppose that Assumptions 2 and 5 hold. Then, the extended operator associated with the constrained problem* $\overline{\mathbf{M}} : C^2(L^2_{\mathbf{P}}) \times C^2(L^2_{\mathbf{P}}) \to C^2(L^2_{\mathbf{P}})$, $[h_k, h_c, h_{R_b}, y] \mapsto \mathbf{M}[\mathbf{P}[h_k, h_c, h_{R_b}], y]$, *where* $\mathbf{M}$ *as in (12) and the projection* $\mathbf{P}$ *as in Proposition 12, is maximal monotone.*

Now that the property of maximal monotonicity is established for the extended constrained equilibrium operator, one can apply Corollary 9 to show that there exist random fields $h_k(z', \omega, m')$, $h_c(z', \omega, m')$ and $h_{R_b}(z', m')$ for which the Euler equation operator and the bond market clearing operator equal zero for $\mathbb{P}$-a.e. (z', ω) and for any $m' = (\varepsilon', \chi) \in L^2_{\mathbf{P}}$. The strategy to do so is finding a set of points containing $(h_k, h_c, h_{R_b}, y), (\hat{h}_k, \hat{h}_c, \hat{h}_{R^b}, \hat{y}) \in \mathcal{H}_\epsilon \times C^2(L^2_{\mathbf{P}})$ such that $\mathbf{M}[h_k, h_c, h_{R_b}, y] > 0$ and $\mathbf{M}[\hat{h}_k, \hat{h}_c, \hat{h}_{R^b}, \hat{y}] < 0$ enabling the construction of a suitable convex hull of the corresponding image set in Corollary 9. I tailor this last step to the concrete example models in Section 6 where I consider the Aiyagari-Bewley model and the Huggett economy both with aggregate risk.

4.3 Sufficiency and Uniqueness

Setting the equilibrium operator which contains the Euler equations to zero is normally only a necessary condition for equilibrium. It needs to be verified that such a candidate solution indeed maximizes individual utility.

Lemma 15 (Sufficiency)**.** *Consider the model from Section 2 and suppose that Assumptions 2 and 5 hold. Then, the equilibrium equation in (11), where the Euler*

[12] A rigorous definition of the term can be found in the online appendix, Definition 17.

equation and bond market clearing operator are defined on $\mathcal{H}_\epsilon$ as in Proposition 10, is necessary and sufficient for an equilibrium solution.

The sufficiency is mainly due to the fact that monotone operators are concepts from convex analysis. As such, it is well known that the Euler equation is necessary and sufficient for optimality if a suitable transversality condition holds. This is ensured by condition (v) of the admissible set in Proposition 10.

Another property from standard convex analysis carries over as well which is uniqueness. A strictly convex optimization problem has a unique solution. I obtain an equivalent result due to strict monotonicity of the equilibrium operator.

Lemma 16 (Uniqueness). *Consider the model from Section 2 and suppose that Assumptions 2 and 5 hold. Suppose that a solution to the equilibrium equation in (11), where the Euler equation and bond market clearing operator are defined on $\mathcal{H}_\epsilon$ as in Proposition 10, exists. Then, this solution is unique.*

The uniqueness result refers to recursive equilibrium solutions in the admissible set. Even though there might exist other sequential equilibria, I argue that recursive equilibria are the most important type of sequential equilibria for practical purposes. It is striking that the recursive equilibrium is unique for this fairly elaborate class of models with aggregate and idiosyncratic risk, especially given the wealth of literature on multiplicity of equilibria. It is well known that multiplicity can occur, for instance, in overlapping generations models, in the Arrow-Debreu setup or in bank run models. The main difference between these setups and the one in this paper lies in the specification of risk and the type of equilibrium solution considered. In these aforementioned models, one typically solves for a steady-state equilibrium where large populations have to coordinate on finitely many possible actions. The coordination problem, i.e., the requirement to know which exact action the other agents choose, results in multiplicity. Morris and Shin (2000) show that this coordination problem is resolved and uniqueness obtained by introducing even a small amount of uncertainty about the other agent's behavior. The model investigated in this paper features the exact same remedy in form of idiosyncratic risk.

5 An Iterative Solution Procedure

Due to the fact that I do not rely on compactness to establish existence, the convergent iterative procedure of the contraction mapping theorem does not apply here.

Hence, I cannot compute the equilibrium using value function iteration. However, the monotonicity approach leads to another convergent iterative procedure which is similar. This procedure is explained subsequently.

We can construct an iterative procedure P where $h^{n+1} = P[h^n]$ with h^n converging to a solution of (11) by exploiting the monotonicity of the equilibrium operator $\mathbf{M}$. To illustrate the idea, I will first look at the simplified problem without borrowing constraint. We can rewrite the equilibrium equation by

$$\mathbf{M}[h] = 0 \Leftrightarrow \mathbf{M}[h] + h = h \Leftrightarrow (\mathbf{M} + \mathbf{Id})[h] = h \Leftrightarrow h = (\mathbf{M} + \mathbf{Id})^{-1}[h],$$

where $\mathbf{Id}$ is the identity operator. The last equality contains the resolvent of the equilibrium equation $(\mathbf{M} + \mathbf{Id})^{-1}$. This operator has a desirable property. If the equilibrium operator is maximal monotone, its resolvent is firmly nonexpansive,[13] a property slightly stronger than Lipschitz continuity with coefficient one (see e.g. Bauschke and Combettes, 2010, Proposition 23.8). It is well known that any firmly nonexpansive operator is equivalent to a mixture $(1/2)\mathbf{Id} + (1/2)\mathbf{R}$ of the identity operator $\mathbf{Id}$ and a nonexpansive operator $\mathbf{R}$ (see e.g., Bauschke and Combettes, 2010, Remark 4.34 (iii)). Weak convergence of the iteration of such a mixture to its fixed point is well established (see e.g., Zeidler, 1986, Proposition 10.16). This procedure is also known as damped fixed-point iteration. Iterating on the resolvent of a maximal monotone operator yields the proximal point algorithm. Therefore, iterating as in

$$h^{n+1} = (\mathbf{M} + \mathbf{Id})^{-1}[h^n],$$

where n is the iteration count, converges to the optimal policy, i.e. the root of the equilibrium operator.[14] To understand how the resolvent is constructed, let us look at a simplified example first.

[13] Nonexpansiveness (see e.g., Bauschke and Combettes, 2010): Let $\mathcal{E}$ be a Hilbert space. An operator $\mathbf{M} : \mathcal{E} \to \mathcal{E}$ is called nonexpansive if it is Lipschitz continuous with constant 1. It is called firmly nonexpansive if for all $e, \tilde{e} \in \mathcal{E}$ it holds that $\|\mathbf{M}[e] - \mathbf{M}[\tilde{e}]\|^2 \leq \langle e - \tilde{e}, \mathbf{M}[e] - \mathbf{M}[\tilde{e}] \rangle$.

[14] As a root, the optimal policy of the equilibrium operator represents an eigenfunction of the equilibrium operator's eigenvalue zero. This set of eigenfunctions is the same set which corresponds to the eigenvalue problem of the resolvent $\lambda \mathbf{Id} - (\mathbf{Id} + \mathbf{M})^{-1} = 0$ for the eigenvalue $\lambda = 1$. As the resolvent is Lipschitz continuous with coefficient one, which follows from the maximal monotonicity of the equilibrium operator, we can, in fact, characterize the resolvent's spectrum. The spectrum for nonlinear operators is not uniquely defined as the corresponding spectral theory is much more complex than for linear operators (see e.g., Appell et al., 2004). However, due to the Lipschitz property, we can use the definition by Kachurovskij leading to a compact spectrum with spectral radius of one. Hence, the optimal policy represents the eigenfunction corresponding to the resolvent's maximal eigenvalue.

Example (Resolvent of a subdifferential)**.** Let $\mathcal{E}$ be a Hilbert space. Consider a proper, lower semicontinuous, convex function $F : \mathcal{E} \to [-\infty, \infty]$. It is well known that its subdifferential ∂F is maximal monotone (see e.g., Bauschke and Combettes, 2010, Theorem 20.48). We are looking for a fixed point $e^* \in \mathcal{E}$ of the resolvent of F, which can be computed by simple iteration with iteration count n,

$$e_n \overset{n\to\infty}{\longrightarrow} e^* \text{ with } e_{n+1} = (\partial F + \mathbf{Id})^{-1}[e_n].$$

The resolvent $(\partial F + \mathbf{Id})^{-1}$ can be represented by

$$\begin{aligned} & e_{n+1} = (\partial F + \mathbf{Id})^{-1}[e_n] \Leftrightarrow e_n = (\partial F + \mathbf{Id})[e_{n+1}] \\ & \Leftrightarrow 0 = (\partial F + \mathbf{Id})[e_{n+1}] - \mathbf{Id}[e_n] \Leftrightarrow e_{n+1} = \arg\min_{e\in\mathcal{E}} F[e] + \frac{1}{2}\|e - e_n\|^2. \end{aligned}$$

The latter is the update of the proximal point algorithm.[15]

This example shows that the proximal point algorithm in our case translates into an algorithm on augmented Lagrangians. To ensure convergence, a regularization term containing the previous iterate has to be added to the Lagrangian (16) associated with the equilibrium operator. I follow Rockafellar (1976b) for defining the proximal point algorithm's update. The augmented Lagrangian is an operator $L^A : \mathcal{H}_\epsilon \times \mathcal{H}_\epsilon \times C^2(L^2_{\mathbf{P}}) \to [-\infty, \infty]$ given by

$$\begin{aligned} & L^A\left[[h_k, h_c, h_{R_b}], [h^n_k, h^n_c, h^n_{R_b}], y\right](z', \omega, m') \\ = & F_{\mathbf{M}}[h_k, h_c, h_{R_b}](z', \omega, m') + \frac{1}{2\lambda}\sum_{x\in\{k,b\}} \left\|h_x(z', \omega, m') - h^n_x(z', \omega, m')\right\|^2 \\ & + \frac{1}{2\lambda}\left\|h_{R_b}(z', m') - h^n_{R_b}(z', m')\right\|^2 \\ & + \sum_{x\in\{k,b\}} \begin{cases} \{-y_x(z', \omega, m')q_x(z', \omega, m') \\ +\frac{\lambda}{2}\|q_x(z', \omega, m')\|^2\} & , q_x(z', \omega, m') \leq \frac{y_x(z',\omega,m')}{\lambda}, \\ -\frac{1}{2\lambda}\|y_x(z', \omega, m')\|^2 & , q_x(z', \omega, m') > \frac{y_x(z',\omega,m')}{\lambda} \end{cases} \end{aligned} \tag{17}$$

where $F_{\mathbf{M}}$ as in (15), $\lambda > 0$ is the step size parameter of the proximal point algorithm, $\|.\|^2 = \langle ., . \rangle$ with the inner product as in (13) and

$$q(z', \omega, m') = \begin{bmatrix} h_k(z', \omega, m') - \bar{k} \\ h_{pb}(z', \omega, m') - h_{R_b}(z', m')\bar{b} \end{bmatrix}. \tag{18}$$

[15]The proximal point update presented here is a simplified version. Rockafellar (1976a) proves convergence for the generalized resolvent $\lambda^n(\mathbf{Id} + 1/\lambda^n\mathbf{M})^{-1}$, also called Yosida approximation, where $\{\lambda^n\}^\infty_{n=1}$ is either constant and bounded away from zero or a series $0 < \lambda^n \nearrow \lambda^\infty \leq \infty$.

The first term of the augmented Lagrangian features the objective function corresponding to the equilibrium operator in (11). The second and third term consist of the objective's proximal point augmentation, which transforms the equilibrium operator into its resolvent. The last term corresponds to the inequality constraint. It also consists of the Lagrange term and the augmentation, but it is defined piecewise to account for the case of a binding constraint.

Remark. The augmentation term in the Lagrangian (17) of the proximal point algorithm does, in fact, represent a Tikhonov regularization.[16]

With the augmented Lagrangian as above, I now state the algorithm to approximate the recursive equilibrium of the model in Algorithm 1.

Algorithm 1 Proximal point algorithm

▷ A INITIALIZATION

1: Set $n = 0$. Initialize the agents' choices and prices $h^n = [h_k^n, h_c^n, h_{R_b}^n]$ and the Lagrange multiplier y^n.

2: Set the step size parameter $\lambda > 0$.

3: Set the termination criterion small $\tau > 0$ and the initial distance larger $d > \tau$.

▷ B ITERATIVE PROCEDURE

4: **while** $d > \tau$ **do**

5: Update

$$h^{n+1}(z', \omega, m') = \arg\min_{h \in \mathcal{H}_\epsilon} L^A[h, h^n, y^n](z', \omega, m')$$
$$y^{n+1}(z', \omega, m') = \max\left\{0, y^n(z', \omega, m') - \lambda q^{n+1}(z', \omega, m')\right\}$$

for every $m' = (\varepsilon, \chi) \in L^2_{\mathbf{P}}$ and almost every $(z', \omega) \in \mathbb{P}$. L^A is defined as in (17) and q as in (18).

6: Compute the distance $d = \max_{m' \in L^2_{\mathbf{P}}} \{\|h^{n+1} - h^n\| + \|y^{n+1} - y^n\|\}$ where $\|.\| = \sqrt{\langle ., . \rangle}$ with the inner product as in (13).

7: Set $n = n + 1$.

8: **end while**

Remark. Since the augmented Lagrangian L^A can be interpreted as the objective of a social planner optimizing the whole heterogeneous-agent economy, the proximal point algorithm is comparable to the value function iteration of said social planner.

[16]It has been shown in (Bauschke and Combettes, 2010, Theorem 27.23) that regularizations other than Tikhonov are admissible as well as long as the regularization function is uniformly convex in the policy (h_k, h_c, h_{R_b}). An avenue for future research might, therefore, be to explore alternatives like the Sobolev regularization. However, one should keep in mind that the policies will not be differentiable everywhere when there are borrowing constraints.

6 Examples

In the following, I illustrate my existence and uniqueness results with two standard incomplete market models. First, I consider the Aiyagari-Bewley economy with aggregate risk and second, I examine the Huggett economy with aggregate risk.

6.1 The Aiyagari-Bewley Growth Model

I use the same growth model with aggregate shocks as in Krusell and Smith (1998) and den Haan, Judd, and Juillard (2010). It is an Aiyagari-Bewley economy which fits the framework of this paper. The aggregate shock characterizes the state of the economy with outcomes in $\mathcal{Z}^{ag} = \{0, 1\}$ standing for a bad and a good state, respectively. The idiosyncratic shock with outcomes in $\mathcal{Z}^{id} = \{0, 1\}$ indicates that an agent is unemployed or employed, respectively. Hence, the transition probabilities of the compound process $p^{z',\varepsilon'|z,\varepsilon}$ are exogenously given by a four-by-four matrix.

The asset market consists of a claim to aggregate capital $(K_t)_{t\geq 0}$ and does not contain the bond. Each agent chooses her share of physical capital and consumption such that they satisfy certain constraints. First, individual consumption must be positive at all times $c_t > 0$, $t \geq 0$, and capital holdings are subject to a hard borrowing constraint $k_t \geq \bar{k} := 0$, $t \geq 0$. Second, given an initial cross-sectional distribution μ_0 with non-negative support, each agent adheres to a budget constraint, which equates individual consumption and current capital stock to productive income and saved capital stock[17]

$$k_t + c_t = I\left(z_t, \varepsilon_t, k_{t-1}, K_t\right) + k_{t-1}\, \forall\, t > 0, \tag{19}$$

where k_{-1} is distributed according to $\mu_0/\mathbb{P}(\varepsilon_0|z_0)$. Note that, due to the timing convention of this paper, aggregate capital here aggregates over the beginning-of-period distribution $(\varepsilon_t, k_{t-1}) \sim \mu_t$, i.e., K_t is the cross-sectional mean of k_{t-1}. The parameters in the budget constraint are defined as follows. The productive

[17] Note that I specify the time line slightly differently than den Haan et al. (2010) and Krusell and Smith (1998). These authors substitute k_t with k_{t+1} in the budget constraint (19) because this is the capital, which is put forward as start capital to period $t + 1$. In contrast to that notation, however, I want to emphasize the time period, at which the agent optimally chooses the magnitude of her capital savings. Taking this view, the optimal consumption and capital savings choice have the same time subscript. My time line, therefore, indicates which filtration the endogenous variables are adapted to.

income is given by

$$
\begin{aligned}
I\left(z_t, \varepsilon_t, k_{t-1}, K_t\right) &= R\left(z_t, K_t\right) k_{t-1} \\
&\quad + \varepsilon_t \pi \left[1-\tau_t\right] W\left(z_t, K_t\right) + \left[1-\varepsilon_t\right] \nu W\left(z_t, K_t\right).
\end{aligned} \tag{20}
$$

It is composed of, first, the return on capital stock, and second, labor income, which equals the individual's wage W when the agent is employed and a proportional unemployment benefit νW otherwise. The agent's wage is subject to a tax at rate $\tau_t = \nu(1-p_t^e)/(\pi p_t^e)$ whose sole purpose it is to redistribute money from the employed to the unemployed. The parameter $\nu \in [0,1)$ denotes the unemployment benefit rate, whereas, $p_t^e = \mathbb{P}(\varepsilon_t = 1|z_t)$ is the employment rate at time t and $\pi > 0$ is a time endowment factor. It is reasonable to assume $\nu/\pi < 1 - \tau_t \Leftrightarrow \nu < \pi p_t^e$ for all $t \geq 0$. The wage W and the rental rate R are derived from a Cobb-Douglas production function for the consumption good

$$
W\left(z_t, K_t\right) = (1-\alpha)\left(1 + z_t a - (1-z_t)a\right)\left[\frac{K_t}{\pi p_t^e}\right]^{\alpha} \tag{21}
$$

$$
R\left(z_t, K_t\right) = \alpha\left(1 + z_t a - (1-z_t)a\right)\left[\frac{K_t}{\pi p_t^e}\right]^{\alpha-1} - \delta, \tag{22}
$$

where $a \in (0,1)$ is the absolute aggregate productivity rate and $\alpha \in (0,1)$ is the output elasticity parameter. The capital stock brought forward from period $t-1$ depreciates by a rate $\delta \in (0,1)$. Labor supply is defined by the employment rate p_t^e scaled by the time endowment factor π.

All agents have time-separable CRRA utility with a risk aversion coefficient $\gamma > 0$ or log-utility if $\gamma = 1$ and time preference parameter $\beta \in (0,1)$. Then, given the initial cross-sectional distribution μ_0 with non-negative support, the Euler equation reads

$$
\begin{aligned}
&\left(c'\right)^{-\gamma} = \beta \mathbb{E}\left[\left(1+R'\right)\left(c''\right)^{-\gamma}\right] + y \\
&\text{s.t. } c' = I\left(z', \varepsilon', k, K'\right) + k - k' \\
&\quad k' \geq 0, y \geq 0, k' \perp y.
\end{aligned} \tag{23}
$$

where the productive income I is defined as in (20).

The question is how this model fits the framework introduced earlier. Assumption 2 on the production function is obviously fulfilled with a Cobb-Douglas production function. Furthermore, the aggregation of labor supply is given by

$$L_t = \int_{\mathcal{Z}^{id}} \int_{\bar{k}}^{\infty} l(z_t, \varepsilon) d\mu_t(\varepsilon, k) = \pi p_t^e$$

with

$$l(z_t, \varepsilon_t) = \varepsilon_t \pi \left[1 - \frac{\nu(1 - p_t^e)}{\pi p_t^e} \right] + [1 - \varepsilon_t] \, \nu,$$

whereas, aggregate capital in equilibrium has to clear the market

$$K_t = \int_{\mathcal{Z}^{id}} \int_0^{\infty} k d\mu_t(\varepsilon, k). \tag{24}$$

Furthermore, with the specification of exogenous shocks at hand, Assumption 5 (ii) is also satisfied. In contrast to the general formulation, this example model does not feature the bond market. The conditions on the admissible set from Proposition 10 can be adjusted[18] such that the general results on maximal monotonicity for the now simplified equilibrium operator

$$\mathbf{M}[h_c, y](z', \omega, m') = \left\{ \mathbf{T}^k[h_c] \, (z', \omega, m') + y \, (z', \omega, m') \, , \, h_k \perp y \geq 0 \right\}$$

with h_k as in Corollary 23 continue to hold.

Lastly, to assess existence and uniqueness for this example model in particular, the condition of Corollary 9 has to be verified. To do so, I make the following technical assumption on the model parameters.

Assumption 17. Suppose that one of the following condition holds

$$\beta(1 - \delta)^{1-\gamma} < 1.$$

Remark. Note that this condition holds automatically whenever $\gamma \leq 1$. For larger risk aversion, the condition essentially prevents exploding returns. Agents are risk-averse enough to always save at least a little in the aggregate as long as capital depreciation is not too excessive.

I can now apply the general results from the previous sections. I show that the two points which result in the left-hand side of the equilibrium operator being greater and smaller than zero correspond to the save almost everything/consume almost nothing and the save almost nothing/consume almost everything strategies. From these two polar strategies, I construct a set which contains zero in the convex

[18] See Corollary 23 in the appendix.

hull of its image so that Corollary 9 applies.

Theorem 18 (Existence of a unique recursive equilibrium)**.** *Consider the Aiyagari-Bewley growth model with aggregate risk together with Assumptions 17 and 5 (i). Define the admissible set $\mathcal{H}_{\epsilon,D}^{KS}$ as in Corollary 23. Then, the recursive equilibrium is unique and consists of a continuous, square-integrable function $h_c \in \mathcal{H}_{\epsilon,D}^{KS}$, $h_c : \mathcal{Z} \times [0,\infty) \times L_{\mathbf{P}} \to \mathbb{R}$, which solves the Euler equation (23) and satisfies market clearing (24).*

It may seem surprising that I obtain uniqueness for the Aiyagari-Bewley economy with aggregate risk considering existing results in the literature on the Aiyagari-Bewley economy without aggregate risk. Light (2020) and Achdou et al. (2017) find uniqueness in discrete and continuous time, respectively, under the restriction that the risk aversion parameter $\gamma \leq 1$ which is in line with my result. However, Açıkgöz (2018) hints at potential multiplicity of equilibria for larger risk aversion. In contrast, I show uniqueness for a fairly general joint condition on the risk aversion parameter, the subjective discount factor and the depreciation rate, see Assumption 17. Risk aversion may be greater than one under this assumption. In fact, numerical experiments for the model without aggregate risk lead me to conjecture that a unique equilibrium may even exist if condition 17 is violated. Thus, it seems that this condition is too strong to be necessary as well. The question is how these findings can be reconciled with the multiplicity example given in Açıkgöz (2018). In that example, there are two equilibrating points. However, this example keeps the wage rate fixed and, thus, looks at the model from a partial equilibrium perspective disregarding the optimizing firm. Furthermore, the two equilibrating points are the intersections of the firm's equilibrium capital demand and an upper bound on the household's capital supply. Thus, my results rather suggest that this upper bound is not always tight and the true capital supply intersects demand only once.

6.2 The Huggett Model

This section modifies the Huggett (1993) model in order to accommodate aggregate risk. This model solely includes a one-period bond as an asset. Rather than earning a rental rate on capital and a wage for labor, the agents receive an exogenously given endowment. In line with the general model, bond holdings and prices are denoted by $(b_t)_{t\geq 0}$ and $(P_t)_{t\geq 0}$, respectively. Prices are determined through market clearing

$$\int_{\mathcal{Z}^{id}}\int_{\bar{b}}^{\infty} b d\mu_t(\varepsilon, b) = 0, \tag{25}$$

where $\bar{b} < 0$ denotes the borrowing constraint.

All agents have time-separable CRRA utility with a risk aversion coefficient $\gamma > 0$ or log-utility if $\gamma = 1$ and time preference parameter $\beta \in (0, 1)$. Then, given the initial cross-sectional distribution $(\varepsilon_0, b_{-1}) \sim \mu_0$ with support $[\bar{b}, \infty)$, the Euler equation reads

$$\begin{aligned} (c')^{-\gamma} &= \beta \mathbb{E}\left[\frac{1}{P'}(c'')^{-\gamma}\right] + y \\ \text{s.t. } c' &= e(z', \varepsilon') + b - P'b' \\ b' &\geq \bar{b}, y \geq 0, (b' - \bar{b}) \perp y. \end{aligned} \tag{26}$$

As there is no capital in this model, the conditions of the admissible set from Proposition 10 can be adjusted[19] such that the general results on maximal monotonicity for the now simplified equilibrium operator

$$\mathbf{M}[h_c, h_{R_b}, y](z', \omega, m') = \left\{ \begin{bmatrix} \mathbf{T}^b[h_c, h_{R_b}](z', \omega, m') \\ \mathbf{B}[h_c](z', \omega, m') \end{bmatrix} + y(z', \omega, m'), (h_{pb} - h_{R_b}\bar{b}) \perp y \geq 0 \right\}$$

with h_{pb} as in Corollary 25 continue to hold.

To show existence and uniqueness of this particular model, I make the following assumptions on the model parameters and the endowment function to verify the condition of Corollary 9.

Assumption 19. Assume that

$$0 < \min_{z \in \mathcal{Z}^{ag}} \mathbb{E}[e(\varepsilon, z)|z] \leq \max_{z \in \mathcal{Z}^{ag}} \mathbb{E}[e(\varepsilon, z)|z] < \infty$$

holds.

The strategy of proving existence and uniqueness is the same as before. I find two points in the admissible set such that the Euler equation and bond market clearing operator result in a positive and a negative value such that Corollary 9 can be applied. Those two polar strategies correspond to a high price/low consumption and a low price/high consumption strategy.

Theorem 20 (Existence of a unique recursive equilibrium)**.** *Consider the Huggett model together with Assumptions 19, 2 and 5. Define the admissible set* $\mathcal{H}^{H}_{\epsilon,D}$ *as*

[19] See Corollary 25 in the appendix.

in Proposition 25. Then, there exists a unique recursive equilibrium consisting of a continuous, square-integrable function $h_c \in \mathcal{H}^H_{\epsilon,D}$, $h_c : \mathcal{Z} \times [\bar{b}, \infty) \times L_{\mathbf{P}} \to \mathbb{R}$ and a continuous function $h_{R_b} \in \mathcal{H}^H_{\epsilon,D}$, $h_{R_b} : \mathcal{Z} \times L_{\mathbf{P}} \to \mathbb{R}$, which solve the Euler equation (26) and satisfy market clearing (25).

7 Conclusions

In this paper, I establish existence and uniqueness of simple recursive equilibria for economies with a continuum of agents facing idiosyncratic shocks in combination with aggregate risk. Instead of relying on compactness arguments to establish the fixed point, I use the monotonicity of the equilibrium problem and arguments from convex analysis. An advantage of this approach is that it is easy to examine whether the equilibrium is unique. In particular, I establish sufficient conditions for existence and uniqueness of a simple recursive equilibrium for the Aiyagari-Bewley economy with aggregate risk not limited to a risk aversion parameter of $\gamma \leq 1$. It allows for risk aversion $\gamma > 1$ satisfying the condition $\beta(1-\delta)^{1-\gamma} < 1$ where β is the subjective discount factor and δ denotes the depreciation rate of capital. Discounting needs to be sufficiently small when depreciation or risk aversion is high. Furthermore, I establish existence and uniqueness for an exchange economy with a one-period bond by restricting the endowment process to be positive and finite on average.

A Proofs

A.1 Proof of Proposition 10

Before proving that the admissible set is closed and convex, let me first prove some implications of conditions $(i)-(iii)$ in Proposition 10. These results will be needed to verify the convexity of condition (iv).

Proposition 21 (Curvature of future consumption and marginal utility)**.** *Consider the model from Section 2 and suppose that Assumptions 1, 2 and 5 hold. Define future consumption of a policy $h = [h_k, h_c, h_{R_b}] \in C^2(L^2_{\mathbf{P}})$ as the composition of the consumption policy h_c and the law of motion operator $\mathbf{m}''[h]$. Denote the set of policies for which the consumption policy h_c satisfies condition $(i)-(iii)$ of Proposition 10 by $\mathcal{C}$. Assume that $\mathcal{C}$ is a closed, convex set.*[20] *Then,*

[20] I prove closedness and convexity of $\mathcal{C}$ in the proof of Proposition 10.

a) future consumption is concave on $\mathcal{C}$,

b) future marginal utility is convex on $\mathcal{C}$,

c) the return-weighted future marginal utility is convex on $\mathcal{C}$.

Proof. **a)** : Note that $h_c(z'', \omega, \mathbf{m}''[h])$ maps into $\mathbb{R}$ for any fixed (z'', z', ω) on the support of $\mathbf{P}$. Thus, we can use Proposition 15 of the online appendix which states that concavity of a functional is equivalent to a nonpositive second-order Gâteaux differential. The proposition is stated w.r.t. to a Hilbert space rather than a closed, convex subset of that Hilbert space. However, we can apply the proposition to the composition of future consumption and the projection of the elements of $C^2(L^2_{\mathbf{P}})$ onto $\mathcal{C}$ to obtain the corresponding results for the subset. Hence, it suffices to show that the second-order Gâteaux differential of future consumption at $h \in \mathcal{C}$ in any direction $\tilde{h} \in \mathcal{C} - h$ is nonpositive. The first-order Gâteaux differential of future consumption equals

$$d_G\left(h_c(z'', \omega, \mathbf{m}''[h]); \tilde{h}\right) = d_G h_c\left(z'', \omega, \mathbf{m}''[h]; d_G\mathbf{m}''\left(h; \tilde{h}\right)\right) + \tilde{h}_c(z'', \omega, \mathbf{m}''[h]).$$

The second-order differential reads

$$\begin{aligned}
&d_G^2\left(h_c(z'', \omega, \mathbf{m}''[h]); \tilde{h}\right)\\
=&d_G^2 h_c\left(z'', \omega, \mathbf{m}''[h]; d_G\mathbf{m}''\left(h; \tilde{h}\right)\right) + d_G h_c\left(z'', \omega, \mathbf{m}''[h]; d_G^2\mathbf{m}''\left(h; \tilde{h}\right)\right)\\
&+ 2d_G\tilde{h}_c\left(z'', \omega, \mathbf{m}''[h]; d_G\mathbf{m}''\left(h; \tilde{h}\right)\right)\\
=&d_G^2 h_c\left(z'', \omega, \mathbf{m}''[h]; d_G\mathbf{m}''\left(h; \tilde{h}\right)\right) + d_G h_c\left(z'', \omega, \mathbf{m}''[h]; d_G^2\mathbf{m}''\left(h; \tilde{h}\right)\right)\\
&+ 2d_G\hat{h}_c\left(z'', \omega, \mathbf{m}''[h]; d_G\mathbf{m}''\left(h; \tilde{h}\right)\right) - 2d_G h_c\left(z'', \omega, \mathbf{m}''[h]; d_G\mathbf{m}''\left(h; \tilde{h}\right)\right)
\end{aligned}$$

where $\hat{h} = \tilde{h} + h \in \mathcal{C}$. The first term is negative due to condition $(ii.b)$. Regarding the second term, note that $d_G\mathbf{m}''(h; \tilde{h}) = (\tilde{h}_k, d_G\mathbf{h_b}(h; \tilde{h}))$ with

$$d_G\mathbf{h_b}\left(h; \tilde{h}\right) = \tilde{h}_{R_b}\chi^b - \tilde{h}_k - \tilde{h}_c$$

resulting in $d_G^2\mathbf{m}''(h; \tilde{h}) = 0$. Condition $(ii.c)$ ensures that neither of the last two terms switch the sign of the second derivative. The fourth term is bounded by the first term so that the sum of both remains negative. Regarding the third term, assume w.l.o.g. that total savings decrease, i.e., $\tilde{h}_k + d_G\mathbf{h_b}(h; \tilde{h}) < 0$. Then, the third term of the derivative of future consumption is negative. The opposite case for total savings can be shown by considering $-\tilde{h}$ and the fact that the second Gâteaux differential is quadratic in the sign of the direction, i.e., $d_G^2(h_c(z'',\omega,\mathbf{m}''[h]);-\tilde{h}) = d_G^2(h_c(z'',\omega,\mathbf{m}''[h]);\tilde{h})$. Thus, future consumption is concave in the policy choice.

b) : We again check the second derivative

$$d_G^2\left(h_c(z'',\omega,\mathbf{m}''[h])^{-\gamma};\tilde{h}\right) = \gamma h_c(z'',\omega,\mathbf{m}''[h])^{-\gamma-1}\left\{-d_G^2\left(h_c(z'',\omega,\mathbf{m}''[h]);\tilde{h}\right)\right.$$
$$\left. +(\gamma+1)\frac{\left(d_G\left(h_c(z'',\omega,\mathbf{m}''[h]);\tilde{h}\right)\right)^2}{h_c(z'',\omega,\mathbf{m}''[h])}\right\} \geq 0 \text{ a.s.}$$

which, due to condition $(ii.a)$ and the concavity of future consumption, results in convexity.

c) : Note that we can rewrite the return weighted future marginal utility by

$$(1+R''_j)h_c(z'',\omega,\mathbf{m}''[h])^{-\gamma} = \exp\left(\log\left(1+R''_j\right) - \gamma\log\left(h_c(z'',\omega,\mathbf{m}''[h])\right)\right),$$

where $j \in \{k,b\}$, with R'' as in (10). The exponential of a convex function is convex. Thus, it remains to show that $\log(1+R''_j)$ and $-\log(h_c(z'',\omega,\mathbf{m}''[h]))$ are convex. Assumption 2 ensures convexity of $\log(1+R''_k)$, whereas, condition $(iii.c)$ ensures convexity of $\log(1+R''_b)$. The logarithm of a concave function is concave. Thus, a) delivers the convexity of $-\log(h_c(z'',\omega,\mathbf{m}''[h]))$. This concludes the proof. □

Proof of Proposition 10. **Closedness:** It is easy to see that any limiting element h^* of a Cauchy sequence $h^n \in \mathcal{H}_\epsilon$, $n \in \{1,2,\dots\}$, satisfies conditions $(i)-(v)$ as well. The subset $\mathcal{H}_\epsilon$ is, therefore, closed.

Convexity: It remains to show that for any $[h_k^1,h_c^1,h_{R^b}^1],[h_k^2,h_c^2,h_{R^b}^2] \in \mathcal{H}_\epsilon$ it follows that $[h_k^\lambda,h_c^\lambda,h_{R^b}^\lambda] = \lambda[h_k^1,h_c^1,h_{R^b}^1] + (1-\lambda)[h_k^2,h_c^2,h_{R^b}^2] \in \mathcal{H}_\epsilon$ for any $\lambda \in (0,1)$. Let me check each condition of the admissible set for $[h_k^\lambda,h_c^\lambda,h_{R^b}^\lambda]$.

(i) Condition (i) holds due to the linearity of the Gâteaux derivative and the convexity of the space of square-integrable random variables.

(ii) Condition $(ii.a)$ is trivial due to its linearity. Conditions $(ii.b)$ and $(i.c)$ hold due to the linearity of the Gâteaux derivative.

(iii) Conditions $(iii.a)$ and $(iii.b)$ are trivial due to their linearity. Condition $(iii.c)$ requires a little more care. The fact that $\log(1+h_{R_b}^j(m'))$, $j \in \{1,2\}$, is convex yields

$$d_G^2\left(\log\left(1+h^j_{R_b}(m')\right);\tilde{m}'\right)=\frac{d_G^2 h^j_{R_b}(m';\tilde{m}')}{1+h^j_{R_b}(m')}-\left(\frac{d_G h^j_{R_b}(m';\tilde{m}')}{1+h^j_{R_b}(m')}\right)^2\geq 0.$$

For the convex combination, we obtain

$$\begin{aligned}
&\left(1+h^\lambda_{R_b}(m')\right)^2 d_G^2\left(\log\left(1+h^\lambda_{R_b}(m')\right);\tilde{m}'\right)\\
=&\left(1+h^\lambda_{R_b}(m')\right) d_G^2 h^\lambda_{R_b}(m';\tilde{m}')-\left(d_G h^\lambda_{R_b}(m';\tilde{m}')\right)^2\\
=&\lambda^2\left(1+h^1_{R_b}(m')\right)^2 d_G^2\left(\log\left(1+h^1_{R_b}(m')\right);\tilde{m'}\right)\\
&+(1-\lambda)^2\left(1+h^2_{R_b}(m')\right)^2 d_G^2\left(\log\left(1+h^2_{R_b}(m')\right);\tilde{m'}\right)\\
&+\lambda(1-\lambda)\left[\left(1+h^1_{R_b}(m')\right) d_G^2 h^2_{R_b}(m';\tilde{m}')+\left(1+h^2_{R_b}(m')\right) d_G^2 h^1_{R_b}(m';\tilde{m}')\right.\\
&\left.-2d_G h^1_{R_b}(m';\tilde{m}')d_G h^2_{R_b}(m';\tilde{m}')\right]\\
\geq&\lambda(1-\lambda)\left[\frac{\left(1+h^1_{R_b}(m')\right)}{\left(1+h^2_{R_b}(m')\right)}\left(d_G h^2_{R_b}(m';\tilde{m}')\right)^2\right.\\
&\left.+\frac{\left(1+h^2_{R_b}(m')\right)}{\left(1+h^1_{R_b}(m')\right)}\left(d_G h^1_{R_b}(m';\tilde{m}')\right)^2-2d_G h^1_{R_b}(m';\tilde{m}')d_G h^2_{R_b}(m';\tilde{m}')\right]\\
=&\frac{\lambda(1-\lambda)}{\left(1+h^1_{R_b}(m')\right)\left(1+h^2_{R_b}(m')\right)}\left[\left(1+h^1_{R_b}(m')\right)d_G h^2_{R_b}(m';\tilde{m}')\right.\\
&\left.-\left(1+h^2_{R_b}(m')\right)d_G h^1_{R_b}(m';\tilde{m}')\right]^2\geq 0,
\end{aligned}$$

which proves convexity of condition $(iii.c)$.

(iv) The first two terms of this condition are constant across h^1, h^2 and h^λ. The remaining terms contain the Gâteaux derivative of the Euler equation operator $\mathbf{T}=[\mathbf{T}^k,\mathbf{T}^b]$. Thus, let me characterize these Gâteaux derivatives in further detail. Note that the following statements about curvature and Gâteaux derivatives hold for $\mathbb{P}$-a.e. (z'',z',ω) and any $m'\in L^2_{\mathbf{P}}$.
The current marginal utility $(h_c(z',\omega,m'))^{-\gamma}$ is convex in h_c. Furthermore, Proposition 21 shows that the return-weighted future marginal utility $(1+R''_j)(h_c(z'',\omega,\mathbf{m}''[h]))^{-\gamma}$ is jointly convex in $[h_k,h_c,h_{R_b}]$ for $j\in\{k,b\}$ and R'' as in (10). Thus, the expectations in the Euler equation operator are convex overall. To sum up, the first term of the Euler equation is concave, whereas, the second term is convex.

These curvature properties now allow me to analyze condition (iv) in more detail. This condition aggregates over $\mathbb{P}$-a.e. (ω). Let me consider a single summand of this inner product, i.e., fix the argument ω. Each summand is the derivative of

$$
\begin{aligned}
\mathbf{F}[h_k, h_c, h_{R_b}](\omega) = &- \left[\tilde{h}_k + \tilde{h}_c\right] \left(h_{R_b} + h_c^{-\gamma}\right) \\
&+ \mathbb{E}^{(z'',\varepsilon'')|(z',\varepsilon')} \left[\tilde{h}_k \beta (1 + r^{1''}) h_c(z'', \omega, m'')^{-\gamma}\right] \\
&+ \mathbb{E}^{(z'',\varepsilon'')|(z',\varepsilon')} \left[\tilde{h}_c \beta \frac{1}{h_{R_b}} h_c(z'', \omega, m'')^{-\gamma}\right],
\end{aligned} \tag{27}
$$

i.e., a linear combination of convex functions which in itself is either convex or concave. The sum over these functions $\bar{\mathbf{F}}[h] = \mathbb{E}^{\mathbf{P}}[\mathbf{F}[h](\omega)]$ is, hence, also either convex or concave and the right-hand side of condition (iv) equals $d_G\bar{\mathbf{F}}[h; \tilde{h}]$. If the expectation is convex (concave), Proposition 15 implies that its (negative) Gâteaux derivative is monotone. Hence, the Gâteaux derivative is increasing (decreasing) in some given direction and vice versa. Suppose that h^1 and h^2 lie both in a subset on which $\bar{\mathbf{F}}[h]$ is either convex or concave. Then, w.l.o.g.

$$
d_G\bar{\mathbf{F}}[h^1; \tilde{h}] \geq d_G\bar{\mathbf{F}}[h^\lambda; \tilde{h}] \geq d_G\bar{\mathbf{F}}[h^2; \tilde{h}] \geq 0
$$

and condition (iv) holds for h^λ as well. Now, suppose that h^1 is in a subset on which $\bar{\mathbf{F}}[h]$ is convex and h^2 is in an adjacent subset on which $\bar{\mathbf{F}}[h]$ is concave. Then, due to continuity there exists a $\kappa \in (0, 1)$ such that h^κ is on the boundary of the two subsets. If $d_G\bar{\mathbf{F}}[h; \tilde{h}]$ is increasing (decreasing) from h^1 to h^κ, then it must be decreasing (increasing) from h^κ to h^2 due to the monotonicity of the Gâteaux derivative for convex functions. In case of a $\cap$-shape, condition (iv) caries over to h^λ. In case of a $\cup$-shape, condition (iv) caries over to any h^λ if it holds for h^κ. We can rule out that condition (iv) fails for h^κ by contradiction because $\bar{\mathbf{F}}[h]$ would be first concave decreasing as $d_G\bar{\mathbf{F}}[h^\kappa; \tilde{h}] < 0$ when moving from h^κ to h^2 and then concave increasing. However, this implies the existence of positive second-order derivatives along the path from h^κ to h^2 which contradicts the assumption of concavity. Thus, condition (iv) carries over to any h^λ even if the curvature of the aggregated Euler equations differs for h^1 and h^2.

(v) Transversality holds for h_c^1 and h_c^2 implying that $h_c^1, h_c^2 > 0$ a.s. Thus, also $c^\lambda > 0$ a.s. and transversality holds for c^λ as well.

This concludes the proof. □

A.2 Proof of Proposition 11

Proof. **Sample path continuity:** Note that h_{pb} as defined in (9) is continuous in h_k, h_c, aggregate capital and aggregate labor and thus, preserves sample path continuity for the bond market clearing operator $\mathbf{B}$. The Euler equation is continuous in c' and c''. The composition $c'' = h_c \circ (h_k, h_{pb}/h_{R_b})$ preserves sample path continuity since $[h_k, h_c, h_{R_b}] \in \mathcal{H}_\epsilon$ and h_{pb} is sample path continuous. Hence, $\mathbf{T}$ is sample path continuous as well.

Second order: Next, let me check that both $\mathbf{T}[h_k, h_c, h_{R_b}](z', \omega, m')$ and $\mathbf{B}[h_k, h_c](z', m')$ are of second order for any $[h_k, h_c, h_{R_b}] \in \mathcal{H}_\epsilon$. First, consider the bond market clearing operator. We have that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\mathbf{B}[h_k, h_c](z', m')^2\right] =& \mathbb{E}^{\mathbb{P}}\left[\langle h_{pb}(z', \omega, m'), 1\rangle^2\right] \\
\leq& \mathbb{E}^{\mathbb{P}}\Bigg[\langle e(z', \varepsilon'(\omega)), 1\rangle^2 + \frac{\partial F}{\partial L}\left(L', \mathbf{E}^1[m']\right)^2 L'^2 \\
&+ \left(1 + h_{R_k}(z', m')\right)^2 \mathbf{E}^1[m']^2 + \mathbf{E}^2[m']^2 \\
&+ \langle h_k(z', \omega, m'), 1\rangle^2 + \langle h_c(z', \omega, m'), 1\rangle^2\Bigg].
\end{aligned}$$

Due to Assumptions 1, 2 and 5 and the fact that $m' \in L^2_{\mathbf{P}}$ with $\mathbf{E}^1[m'] > 0$, the first four terms are finite. The last two terms are finite as well due to $[h_k, h_c] \in \mathcal{H}_\epsilon \subset C^2(L^2_{\mathbf{P}})$. Hence, the bond market clearing operator is of second order. For the same reasons, h_{pb} itself (without aggregation) is square integrable.

Let us consider the Euler equation operator, i.e.,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\mathbf{T}^j[h_k, h_c, h_{R_b}](z', \omega, m')^2\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\left(h_c(z', \omega, m')^{-\gamma} - \mathbb{E}^{(z'', \varepsilon''|z', \varepsilon')}\left[\beta\left(1 + r''_j\right) h_c(z'', \omega, m'')^{-\gamma}\right]\right)^2\right] \qquad (28)
\end{aligned}$$

where $j \in \{1, 2\}$, r''_j as in (10) and $m'' = (\varepsilon'', \chi')$ with

$$\chi' = \left(h_k(z', \omega, m'), \frac{h_{pb}(z', \omega, m')}{h_{R_b}(z', m')}\right).$$

Proposition 10 $(ii.a)$ and Jensen's inequality yield that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[h_c(z', \omega, m')\right] \geq \epsilon > 0 \\
\Rightarrow\ &\mathbb{E}^{\mathbb{P}}\left[c(z', \omega, m')^{-2\gamma}\right] < \infty
\end{aligned}$$

Note that the inequalities of Proposition 10 hold for any index $m' \in L^2_{\mathbf{P}}$ with positive aggregate capital. Also, $m'' \in L^2_{\mathbf{P}}$ due to h_k and h_{pb} being square integrable and Proposition 10 (*iii.b*). Furthermore, it has positive aggregate capital due to Proposition 10 (*iii.a*). Thus, Proposition 10 (*ii.a*) also applies to $h_c(z'', \omega, m'')$ such that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[h_c(z'', \omega, m'')\right] &\geq \epsilon > 0 \\ \Rightarrow \mathbb{E}^{\mathbb{P}}\left[h_c(z'', \omega, m'')^{-2\gamma}\right] &< \infty \\ \Rightarrow \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{(z'', \varepsilon''|z', \varepsilon')}\left[h_c(z'', \omega, m'')^{-2\gamma}\right]\right] &< \infty. \end{aligned}$$

Similarly, Proposition 10 (*iii*) ensures that $r''_j < \infty$ which implies

$$\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{(z'', \varepsilon''|z', \varepsilon')}\left[\beta^2\left(1 + r''_j\right)^2 c(z'', \omega, m'')^{-2\gamma}\right]\right] < \infty.$$

Jensen's inequality then delivers that (28) is finite. $\square$

A.3 Proof of Proposition 12

Proof. First of all, the projection operator is a well defined bijection because the admissible set is closed, convex and nonempty (see Bauschke and Combettes, 2010, Theorem 3.16). The latter is true since one can set $[h_k, h_c, h_{R_b}]$ constant. Furthermore, the projection is continuous (see Bauschke and Combettes, 2010, Proposition 4.16). The equilibrium operator $\mathbf{M} = [\mathbf{T}, \mathbf{B}]$ is also continuous as already discussed in the proof of Proposition 11. Therefore, the extended equilibrium operator $\overline{\mathbf{M}} = \mathbf{M} \circ \mathbf{P}$ is continuous. By assumption of this proposition, it is also monotone. Bauschke and Combettes (2010, Corollary 20.28) shows that continuous monotone operators are maximal monotone which concludes the proof. $\square$

A.4 Proof of Lemma 13 and Corollaries

To prove that the extended equilibrium operator is maximal monotone, I solely need to show that it is monotone on the admissible set due to Proposition 12. To do so, I first characterize monotonicity in terms of the operator's Gâteaux derivative below.

Proposition 22 (Equivalence of Monotonicity and nonnegative Gâteaux differentials)**.** *Let $\mathcal{E}$ be a Hilbert space. Consider a continuously Gâteaux differentiable operator $\mathbf{M} : \mathcal{E} \to \mathcal{E}$ and a closed and convex subset $\mathcal{H} \subseteq \mathcal{E}$. Then,*

$$\langle e - \tilde{e}, \mathbf{M}[e] - \mathbf{M}[\tilde{e}]\rangle \geq 0 \,\forall\, e, \tilde{e} \in \mathcal{H} \Leftrightarrow \langle d_G\mathbf{M}[e](\hat{e} - e), \hat{e} - e\rangle \geq 0 \,\forall\, e \in \mathcal{H}, \hat{e} \in \mathcal{E}.$$

Proof. **Necessity:** Note that

$$\begin{aligned}\langle d_G\mathbf{M}[e](\hat{e}-e),\hat{e}-e\rangle &= \left\langle \frac{d}{dt}\mathbf{M}[e+t(\hat{e}-e)]\bigg|_{t=0},\hat{e}-e\right\rangle \\ &= \left\langle \lim_{s\to 0}\frac{\mathbf{M}[e+s(\hat{e}-e)]-\mathbf{M}[e]}{s},\hat{e}-e\right\rangle. \end{aligned} \tag{29}$$

Monotonicity implies for $\tilde{e}=e+s(\hat{e}-e)$, $s\in\mathbb{R}$, that

$$\langle s(\hat{e}-e),\mathbf{M}[e+s(\hat{e}-e)]-\mathbf{M}[e]\rangle \geq 0.$$

Dividing both sides of the inequality by $s^2>0$ yields

$$\left\langle \hat{e}-e,\frac{\mathbf{M}[e+s(\hat{e}-e)]-\mathbf{M}[e]}{s}\right\rangle \geq 0.$$

For e in the interior of $\mathcal{H}$, this inequality holds for any s in the neighborhood of zero and necessity follows for any $\hat{e}\in\mathcal{E}$. For e on the boundary of $\mathcal{H}$, necessity follows for any $\hat{e}\in\mathcal{E}$ due to continuity of $\mathbf{M}$.

Sufficiency: Due to the continuity of the inner product, we obtain from (29) that

$$\langle d_G\mathbf{M}[e](\hat{e}-e),\hat{e}-e\rangle = \lim_{s\to 0}\left\langle \frac{\mathbf{M}[e+s(\hat{e}-e)]-\mathbf{M}[e]}{s},\hat{e}-e\right\rangle \geq 0\,\forall\, e\in\mathcal{H},\hat{e}\in\mathcal{E}.$$

Thus, we can find an $\epsilon_{e,\hat{e}}>0$ for any $e\in\mathcal{H},\hat{e}\in\mathcal{E}$ such that for any $0<s<\epsilon_{e,\hat{e}}$

$$\left\langle \frac{\mathbf{M}[e+s(\hat{e}-e)]-\mathbf{M}[e]}{s},\hat{e}-e\right\rangle \geq 0.$$

For any particular $0<s<\epsilon_{e,\hat{e}}$, define $\tilde{e}=e+s(\hat{e}-e)$ and multiply by s^2 resulting in

$$\langle \mathbf{M}[\tilde{e}]-\mathbf{M}[e],\tilde{e}-e\rangle \geq 0.$$

As $\hat{e}\in\mathcal{E}$ is arbitrary, we obtain that the equation holds for arbitrary $\tilde{e}\in\mathcal{H}$ in the neighborhood of $e\in\mathcal{H}$. As the monotonicity condition holds in the neighborhood of any point in $\mathcal{H}$, it holds globally on $\mathcal{H}$ due to convexity of $\mathcal{H}$. $\square$

Proof of Lemma 13. Proposition 22 shows that monotonicity on the admissible set is equivalent to condition (14) for any $[h_k,h_c,h_{R_b}]\in\mathcal{H}_\epsilon$ and $[h_k,h_c,h_{R_b}]\in L^2_{\mathbf{P}}$. Let me compute the Gâteaux derivative of the bond market clearing operator. It reads

$$
\begin{aligned}
d_G\mathbf{B}[h_k, h_c; \tilde{h}_k, \tilde{h}_c] &= \left\langle d_G \left(h_{pb}; \tilde{h}_k, \tilde{h}_c \right), 1 \right\rangle \\
&= - \left\langle \tilde{h}_k + \tilde{h}_c, 1 \right\rangle .
\end{aligned}
$$

such that

$$
\begin{aligned}
\left\langle d_G\mathbf{B}[h_k, h_c; \tilde{h}_k, \tilde{h}_c], \tilde{h}_{R^b} \right\rangle &= - \left\langle \tilde{h}_k + \tilde{h}_c, 1 \right\rangle \left\langle \tilde{h}_{R^b}, 1 \right\rangle \\
&= - \left\langle \tilde{h}_k + \tilde{h}_c, \tilde{h}_{R^b} \right\rangle
\end{aligned}
$$

as $\tilde{h}_{R^b}$ is an aggregate quantity and thus, constant over ω. This implies that condition (14) is equal to condition *(iv)* of the admissible set in Proposition 10. This concludes the proof. □

A.4.1 Corollaries

In this subsection, I adapt the monotonicity results of the general model of Section 2 to the example models in Section 6. These examples each omit one of the markets of the general model. However, maximal monotonicity of the respective extended equilibrium operators continues to hold.

First, consider the Krusell-Smith model in which agents solely invest in capital. There is no bond trading. Therefore, the equilibrium operator consists of the Euler equation for capital only

$$
\mathbf{M}^{\mathrm{KS}}[h_c] = \mathbf{T}^k[h_c]
$$

where $\mathbf{T}^k$ as in Definition 7. As a result, the admissible set, on which this operator is monotone, has to be modified as well.

Corollary 23 (Admissible set - Krusell Smith)**.** *Consider the model from Section 6.1 and suppose that Assumptions 1, 2 and 5 hold. For some $\epsilon > 0$ and $D \geq 0$, define the subset $\mathcal{H}^{KS}_{\epsilon,D} \subset C^2(L^2_{\mathbf{P}}) \subseteq L^2_{\mathbf{P}}$ as the set of continuous random fields h_c for which the following conditions hold for any index $m' = (\varepsilon', \chi^k) \in L^2_{\mathbf{P}}$ with positive aggregate capital $\mathbf{E}^1[m'] > 0$. In addition to the conditions listed below, conditions (ii) with $\chi^b(\omega) = 0$ a.s., (iii.a) with*

$$
\begin{aligned}
h_k(z', \omega, m') =& \frac{\partial F}{\partial L} \left(L', \mathbf{E}^1[m'] \right) l \left(z', \varepsilon'(\omega) \right) \\
&+ \left(1 + h_{R_k} \left(z', m' \right) \right) \chi^k(\omega) - h_c \left(z', \omega, m' \right) \;\; a.s.,
\end{aligned}
$$

and (v) of Proposition 10 are assumed to hold.

(i) Gâteaux differentiability in m': The policy $h_c(z', \omega, m')$ is Gâteaux differentiable in m' for $\mathbb{P}$-a.e. (z', ω), the derivative is square-integrable, i.e., $\mathbb{E}^{\mathbf{P}}\left[d_G\left(h_c; \tilde{m}'\right)^2\right] < \infty$ for any $\tilde{m}' \in L^2_{\mathbf{P}}$, and, the second-order Gâteaux differential exists

(iv) Bounded aggregated Gâteaux differential of the Euler equation:

$$0 \leq -\left\langle d_G\left(h_c^{-\gamma}; \tilde{h}_c\right), \tilde{h}_c\right\rangle + \mathbb{E}^{(z'', \varepsilon''|z', \varepsilon')}\beta\left\langle d_G\left(\left(1 + h_{R_k}\left(z'', m''\right)\right) h_c\left(z'', \omega, m''\right)^{-\gamma}; \tilde{h}_c\right), \tilde{h}_c\right\rangle$$

for $\mathbb{P}$-a.e. z', any $\tilde{h}_c \in L^2_{\mathbf{P}}$ and $m'' = (\varepsilon'', \chi^{k'})$ with $\chi^{k'} = h_k$.

Then, $\mathcal{H}^{KS}_{\epsilon,D} \subset L^2_{\mathbf{P}}$ is a closed, convex subset of the Hilbert space.

I omit the proof as it closely follows the steps in the proof of Proposition 10.

Corollary 24 (Maximal monotone equilibrium operator - Krusell Smith)**.** *Suppose that the assumptions of Corollary 23 hold. Define the extended equilibrium operator $\overline{\mathbf{M}}^{KS} : L^2_{\mathbf{P}} \to L^2_{\mathbf{P}}$ by $\overline{\mathbf{M}}^{KS} = \mathbf{M}^{KS} \circ \mathbf{P}$ where the projection operator $\mathbf{P} : L^2_{\mathbf{P}} \to \mathcal{H}^{KS}_{\epsilon,D}$ is defined by the closest admissible point*

$$\mathbf{P}[a_c] = \left\{h_c \in \mathcal{H}^{KS}_{\epsilon,D} \,\middle|\, \langle \bar{y}_c - h_c, a_c - h_c\rangle \leq 0 \,\forall \bar{y}_c \in \mathcal{H}^{KS}_{\epsilon,D}\right\}.$$

Then, this extended equilibrium operator is maximal monotone.

I omit the proof as it closely follows the steps in the proof of Lemma 13.

Second, consider the Huggett economy, which considers agents with some endowment who trade in a bond in zero-net supply. There is no labor income and no capital investment. Thus, the equilibrium operator consists of

$$\mathbf{M}^{\mathrm{H}}[h_c, h_{R_b}] = \begin{bmatrix} \mathbf{T}^b[h_c, h_{R_b}] \\ \mathbf{B}[h_c] \end{bmatrix}$$

where $\mathbf{T}^b$ as in Definition 7 and $\mathbf{B}$ as in Definition 8. The corresponding admissible set is then given as follows.

Corollary 25 (Admissible set - Huggett)**.** *Consider the model from Section 6.2 and suppose that Assumptions 1, 2 and 5 hold. For some $\epsilon > 0$ and $D \geq 0$, define* *the subset $\mathcal{H}^H_{\epsilon,D} \subset C^2(L^2_{\mathbf{P}}) \subseteq L^2_{\mathbf{P}}$ as the set of continuous random fields $[h_c, h_{R_b}]$ for which the following conditions hold for any index $m' = (\varepsilon', \chi^b) \in L^2_{\mathbf{P}}$. In addition to the conditions listed below, conditions (ii) with $\chi^k(\omega) = 0$ a.s., $(iii.b)$ and (v) of Proposition 10 are assumed to hold.*

(i) *Gâteaux differentiability in m': The policy $h_c(z', \omega, m')$ and the bond price $h_{R_b}(z', m')$ are Gâteaux differentiable in m' for $\mathbb{P}$-a.e. (z', ω), the derivatives are square-integrable, i.e., $\mathbb{E}^{\mathbf{P}}\left[d_G\left(h; \tilde{m}'\right)^2\right] < \infty$ for any $\tilde{m}' \in L^2_{\mathbf{P}}$, and, the second-order Gâteaux differential of $h_c(z', \omega, m')$ exists*

(iv) *Bounded aggregated Gâteaux differential of the Euler equation:*

$$\begin{aligned} 0 \leq &- \left\langle \tilde{h}_{R^b}, \tilde{h}_c \right\rangle - \left\langle d_G\left(h_c^{-\gamma}; \tilde{h}_c, \tilde{h}_{R^b}\right), \tilde{h}_c \right\rangle \\ &+ \mathbb{E}^{(z'',\varepsilon''|z',\varepsilon')} \beta \left\langle d_G\left(\frac{1}{h_{R_b}} h_c\left(z'', \omega, m''\right)^{-\gamma}; \tilde{h}_c, \tilde{h}_{R^b}\right), \tilde{h}_c \right\rangle \end{aligned}$$

for $\mathbb{P}$-a.e. z' and any $\tilde{h}_c, \tilde{h}_{R^b} \in L^2_{\mathbf{P}}$, and $m'' = (\varepsilon'', \chi^{b'})$ with $\chi^{b'} = h_{pb}/h_{R_b}$ and

$$h_{pb}\left(z', \omega, m'\right) = e\left(z', \varepsilon'(\omega)\right) + \chi^b(\omega) - h_c\left(z', \omega, m'\right).$$

Then, $\mathcal{H}_\epsilon \subset L^2_{\mathbf{P}}$ is a closed, convex subset of the Hilbert space.

I omit the proof as it closely follows the steps in the proof of Proposition 10.

Corollary 26 (Maximal monotone equilibrium operator - Huggett)**.** *Suppose that the assumptions of Corollary 25 hold. Define the extended equilibrium operator $\overline{\mathbf{M}}^H : L^2_{\mathbf{P}} \to L^2_{\mathbf{P}}$ by $\overline{\mathbf{M}}^H = \mathbf{M}^H \circ \mathbf{P}$ where the projection operator $\mathbf{P} : L^2_{\mathbf{P}} \to \mathcal{H}^H_{\epsilon,D}$ is defined by the closest admissible point*

$$\begin{aligned} \mathbf{P}[a_c, a_{R^b}] = \big\{ [h_c, h_{R_b}] \in \mathcal{H}^H_{\epsilon,D} \,\big|\, &\forall [\bar{y}_c, \bar{y}_{R^b}] \in \mathcal{H}^H_{\epsilon,D} \\ &\langle \bar{y}_c - h_c, a_c - h_c \rangle + \langle \bar{y}_{R^b} - h_{R_b}, a_{R^b} - h_{R_b} \rangle \leq 0 \big\}. \end{aligned}$$

Then, this extended equilibrium operator is maximal monotone.

I omit the proof as it closely follows the steps in the proof of Lemma 13.

A.5 Proof of Lemma 14

Before I start the proof, let me state some preliminaries. To prove Lemma 14, I need to show that the Lagrangian in (16) is a saddle function. Let me first define what a saddle function is in this context.

Definition 27 (Saddle function (see Rockafellar, 1970)). *(i) Let $\mathcal{C}$ and $\mathcal{D}$ be Hilbert spaces over $\mathbb{R}$. A saddle-function is an everywhere-defined function $L : \mathcal{C} \times \mathcal{D} \to [-\infty, \infty]$ such that $L(c, d)$ is a convex function of $c \in \mathcal{C}$ for any $d \in \mathcal{D}$ and a concave function of $d \in \mathcal{D}$ for any $c \in \mathcal{C}$.*

(ii) A saddle function is called proper if there exists a point $(c, d) \in \mathcal{C} \times \mathcal{D}$ with $L(c, \tilde{d}) < +\infty$ for any $\tilde{d} \in \mathcal{D}$ and $L(\tilde{c}, d) > -\infty$ for any $\tilde{c} \in \mathcal{C}$.

(iii) The operator associated with the saddle function L is defined as the set-valued mapping

$$\begin{aligned}\mathbf{M}_L(c, d) = \Big\{(v, w) | L(\tilde{c}, d) &- \langle \tilde{c}, v\rangle + \langle d, w\rangle \\ &\geq L(c, d) - \langle c, v\rangle + \langle d, w\rangle \\ &\geq L(c, \tilde{d}) - \langle c, v\rangle + \langle \tilde{d}, w\rangle \, \forall (\tilde{c}, \tilde{d}) \in \mathcal{C} \times \mathcal{D}\Big\},\end{aligned}$$

where $\langle ., .\rangle$ denotes the Hilbert space inner product. A saddle point is a point $(c^*, d^*) \in \mathcal{C} \times \mathcal{D}$ such that $0 \in \mathbf{M}_L(c^*, d^*)$.[21]

Note that if our Lagrangian satisfies all properties of a saddle function, then the equilibrium operator $\overline{\mathbf{M}}$ coincides with the operator $\mathbf{M}_L$. This operator can be further characterized by the following Corollary.

Corollary 28 (Rockafellar (1970, Corollary 1)). *Let $\mathcal{C}$ and $\mathcal{D}$ be Hilbert spaces over $\mathbb{R}$. If $L(c, d)$ is a proper saddle function on $\mathcal{C} \times \mathcal{D}$, which is lower semicontinuous in its convex element $c \in \mathcal{C}$ and upper semicontinuous in its concave element $d \in \mathcal{D}$, then the operator $\mathbf{M}_L$ associated with L is maximal monotone.*

Proof of Proposition 14. According to Ghoussoub (2009, Lemma 5.1), the objective $F_{\mathbf{M}}$ in (15) is convex and lower semicontinuous in $(h_k, h_c, h_{R_b}) \in \mathcal{H}_\epsilon$. It follows that the Lagrangian of the constrained problem L in (16) and its extension $\bar{L}$ is convex and lower semicontinuous in $(h_k, h_c, h_{R_b}) \in \mathcal{H}_\epsilon$ and concave and upper semicontinuous in y. The Lagrangian and its extension are also a proper functions because $L(h_x, h_{R_b}, y) > -\infty$ for any $(h_k, h_c, h_{R_b}) \in \mathcal{H}_\epsilon$ when $y = ({}_{\{h_k<\bar{k}\}}, {}_{\{h_{pb}<h_{R_b}\bar{b}\}})$. Conversely, the policies h can be constructed such that $L(h_x, h_{R_b}, y) < \infty$ for all $y \in C^2(L^2_{\mathbf{P}})$. We need $\mathbf{T}[h_k, h_c, h_{R_b}] < \infty$ and $\mathbf{B}[h_k, h_c] < \infty$ to bound $F_{\mathbf{M}}$. The boundedness of the last two terms of the Lagrangian accommodating the borrowing constraints follows from square-integrability. This is achieved by setting consumption to a small positive number

[21] The operator $\mathbf{M}_L$ is closely related to the subdifferential of the saddle function L as v equals the subgradient of $L(., d)$ at $c \in \mathcal{C}$ and w is the subgradient of $-L(c, .)$ at $d \in \mathcal{D}$.

$h_c = \epsilon$ which is less than the endowment e. For any cross-sectional distribution with positive aggregate capital, we have $h_{R_k} < \infty$ and $w > 0$. Setting $h_{R_b} = 1$ and $h_k = h_b = \frac{1}{2}(e + wl + (1 + h_{R_k})k + b - \epsilon)$ results in finite $\mathbf{M}$. Thus, Corollary 28 holds such that the first-order conditions summarized by $\mathbf{M}_L$ define a maximal monotone operator. The unconstrained equilibrium operator $\mathbf{M}$ is the subgradient of $L(., y)$ at a Lagrange multiplier $y \in C^2(L^2_{\mathbf{P}})$ and, therefore, equals v in the definition of $\mathbf{M}_L$. Hence, maximal monotonicity of $\overline{\mathbf{M}}$ follows. □

A.6 Proof of Lemma 15

Proof. Given that $[h_k, h_c, h_{R_b}] \in \mathcal{H}_\epsilon$ solves the equilibrium equation (11), we obtain a sequence of asset holdings, consumption and bond prices, which by construction satisfies the first-order condition of the individual optimization problem and the market clearing conditions, by

$$\begin{aligned}
c^*_{t+1} =& h_c\left(z_{t+1}, \varepsilon_{t+1}, k^*_t, b^*_t, m_{t+1}\right)\\
k^*_{t+1} =& h_k\left(z_{t+1}, \varepsilon_{t+1}, k^*_t, b^*_t, m_{t+1}\right)\\
b^*_{t+1} =& \frac{1}{P^*_{t+1}}\left(e(z_{t+1}, \varepsilon_{t+1}) + W_{t+1}l(z_{t+1}, \varepsilon_{t+1}) + \left(1 + h_{R_k}\left(z_{t+1}, m_{t+1}\right)\right)k^*_t + b^*_t\right.\\
P^*_{t+1} =& h_{R_b}\left(z_{t+1}, m_{t+1}\right)\\
&\left.-k^*_{t+1} - c^*_{t+1}\right)\\
y^*_{t+1} =& y\left(z_{t+1}, \varepsilon_{t+1}, k^*_t, b^*_t, m_{t+1}\right),
\end{aligned}$$

where $m_{t+1} = \mathbf{m}''[h_k, h_c, h_{R_b}] \in L^2_{\mathbf{P}}$ is defined by the law of motion operator. To simplify notation, let me denote $x = (k, b)$ in the following. Suppose that there is another arbitrary feasible series of asset holdings, consumption and prices in $L^2_{\mathbf{P}}$ satisfying $x_t \geq \bar{x}, c_t, P_t > 0$ for all $t \geq 0$ and the budget constraint. Suppose that there exists a corresponding Lagrange multiplier $y_t \geq 0$, $t \geq 0$. By comparing the value functions, we obtain

$$\begin{aligned}
&\lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T} \beta^t \left(u(c^*_t) - u(c_t) - \sum_{j=1}^{2} y^{j^*}_t\left(x^{j^*}_t - \bar{x}\right) + \sum_{j=1}^{2} y^j_t\left(x^j_t - \bar{x}\right)\right)\right]\\
\geq& \lim_{T\to\infty}\left\{\mathbb{E}\left[\sum_{t=1}^{T} \beta^t \sum_{j=1}^{2} \frac{d}{dx^{j^*}_{t-1}} u(c^*_t)\left(x^{j^*}_{t-1} - x^j_{t-1}\right)\right]\right.\\
&\left.+\mathbb{E}\left[\sum_{t=0}^{T} \beta^t \sum_{j=1}^{2}\left(\frac{d}{dx^{j^*}_t} u(c^*_t) - y^{j^*}_t\right)\left(x^{j^*}_t - x^j_t\right)\right]\right\}
\end{aligned}$$

due to concavity of u and linearity of c_t in x_{t-1} and x_t. Rearranging the sums on the right-hand side yields

$$\lim_{T\to\infty}\left\{\mathbb{E}\left[\sum_{t=0}^{T-1}\beta^t\sum_{j=1}^{2}\left(\frac{d}{dx_t^{j^*}}\left(u(c_t^*)+\beta u(c_{t+1}^*)\right)-y_t^{j^*}\right)\left(x_t^{j^*}-x_t^j\right)\right]\right.$$
$$\left.+\beta^T\mathbb{E}\left[\sum_{j=1}^{2}\left(\frac{d}{dx_T^{j^*}}u(c_T^*)-y_T^{j^*}\right)\left(x_T^{j^*}-x_T^j\right)\right]\right\}.$$

where the first term equals zero because the first-order condition holds $\mathbb{P}$-a.e. For the same reason, I rewrite the second term by

$$-\lim_{T\to\infty}\beta^{T+1}\mathbb{E}\left[\sum_{j=1}^{2}\frac{d}{dx_T^{j^*}}u(c_{T+1}^*)\left(x_T^{j^*}-x_T^j\right)\right]$$
$$\geq-\lim_{T\to\infty}\beta^{T+1}\mathbb{E}\left[\sum_{j=1}^{2}\frac{d}{dx_T^{j^*}}u(c_{T+1}^*)\left(x_T^{j^*}-\bar{x}\right)\right].$$

Setting this last term equal to zero results in the transversality condition which ensures optimality of the series x_t^* when satisfied. Thus, it remains to show that the transversality condition holds. We can compute the transversality condition by

$$-\lim_{T\to\infty}\beta^{T+1}\mathbb{E}\left[\sum_{j=1}^{2}\frac{d}{dx_T^{j^*}}u(c_{T+1}^*)\left(x_T^{j^*}-\bar{x}\right)\right]$$
$$=-\lim_{T\to\infty}\beta^{T+1}\mathbb{E}\left[\sum_{j=1}^{2}(c_{T+1}^*)^{-\gamma}\left(1+r_{T+1}^j\right)\left(x_T^{j^*}-\bar{x}\right)\right]$$
$$=0,$$

where r_{T+1}^j as in (10). It follows from condition (v) imposed on the admissible set $\mathcal{H}_\epsilon$ in Proposition 10. This condition ensures that the expectation in the transversality condition is finite for any admissible policy. This concludes the proof. □

A.7 Proof of Lemma 16

Proof. It is well known from convex analysis that the solution to a strictly convex optimization problem is unique (see e.g. Bauschke and Combettes, 2010, Corollary 11.9). The Lagrangian in (16) is indeed strictly convex in h if the equilibrium operator is strictly monotone. This can be proven by contradiction. Assume that the equilibrium operator is not strictly monotone. Then, there may be multiple equilibria, one interior solution where condition (*iv*) of Proposition 10 holds strictly[22] and potentially multiple solutions at the boundary where condition (*iv*) holds with equality. These boundary solutions, thus, satisfy

$$\left\langle d_G\mathbf{M}[h,\tilde{h}],\tilde{h}\right\rangle = 0 = \mathbf{M}[h].$$

The left hand side of this equation needs to hold for any $\tilde{h} \in L^2_{\mathbf{P}}$ which implies that the Gâteaux derivative of $\mathbf{M}[h]$ must be equal to zero. This leads to a contradiction as we compute the derivative of the Bond holding operator to be nonzero

$$d_G\mathbf{B}[h;\tilde{h}] = \left\langle \underbrace{[-1,-1,0]}_{\neq 0}, [\tilde{h}_k,\tilde{h}_c,\tilde{h}_{R^b}] \right\rangle.$$

It can be checked that the derivative of the Euler equation operator $\mathbf{T}$ is nonzero as well. Hence, if a solution exists, it must be in the interior of the admissible set and it is unique. $\square$

A.8 Proof of Theorem 18

Proof of Theorem 18. I construct two polar policies h_c at which the left-hand side of the Euler equation is positive and negative, respectively. The idea is to use the strategies save almost everything/consume almost nothing and save almost nothing/consume almost everything.

Let me first define the candidate policy

$$\begin{aligned} h_c\left(z',\varepsilon',k,m'\right) =& \psi\left(W(z',K')L' + \left(1+R(z',K')\right)K'\right) \\ =& \psi\left(1-\delta+\frac{1}{\alpha}F_K(z',K')\right)K', \end{aligned} \tag{30}$$

where wage and the rental rate are as in (21) and (22). The parameter $\psi \in [0,1]$ allows to vary the strategy. Current period individual capital and next-period

[22]Recall that I showed in the proof of Lemma 13 that this condition is equivalent to the monotonicity condition (14).

aggregate capital and consumption are then given by

$$
\begin{aligned}
h_k\left(z', \varepsilon', k, m'\right) =& W\left(z', K'\right)\left(l(z', \varepsilon') - \psi L'\right) + \left(1 + R(z', K')\right)\left(k - \psi K'\right) \\
K'' =& \left(1 - \psi\right)\left(W(z', K')L' + \left(1 + R(z', K')\right) K'\right) \\
=& \left(1 - \psi\right)\left(1 - \delta + \frac{1}{\alpha} F_K(z', K')\right) K' \\
h_c\left(z'', \varepsilon'', k, m''\right) =& \psi'\left(1 - \delta + \frac{1}{\alpha} F_K(z'', K'')\right) K'' \\
=& \psi'(1 - \psi)\left(1 - \delta + \frac{1}{\alpha} F_K(z'', K'')\right) \cdot \\
& \left(1 - \delta + \frac{1}{\alpha} F_K(z', K')\right) K',
\end{aligned}
$$

where F denotes the Cobb-Douglas production function and its subscripts denote the corresponding derivative. Note that, in general, I do not require the parameter ψ to be constant over time. It may follow a square-integrable process with $\psi \in [0, 1]$.

Admissibility: It remains to check that this strategy is admissible, i.e. $h_c \in \mathcal{H}^{\mathrm{KS}}_{\epsilon,D}$ as in Corollary 23. The policies h_c, h_k are continuous in k and K and, thus, also in the distribution. Square-integrability is ensured due to linearity in k. Let me check the rest of the conditions one by one.

(i) Gâteaux differentiability: The differentials for $m' = (0, \tilde{\chi}^K)$ read

$$
\begin{aligned}
d_G h_c(z', \omega, m'; \tilde{m}') =& \psi\left(1 - \delta + \frac{1}{\alpha} F_K(z', K')\right)\tilde{K}' - \psi\frac{1}{\alpha} F_{KK}(z', K')\tilde{K}'K' \\
=& \psi\left(1 + R(z', K')\right)\tilde{K}' \\
d_G^2 h_c(z', \omega, m'; \tilde{m}') =& \psi F_{KK}(z', K')(\tilde{K}')^2 \\
d_G h_k(z', \omega, m'; \tilde{m}') =& \left(1 + R(z', K')\right)\left(\tilde{k} - \psi\tilde{K}'\right) + \alpha\frac{\tilde{K}'}{K'} W(z', K')\left(l(z', \varepsilon') - \psi L'\right) \\
& - (1 - \alpha)\frac{\tilde{K}'}{K'} F_K(z', K')\left(k - \psi K'\right).
\end{aligned}
$$

Square-integrability follows due to their linearity in k and K'.

(ii) Individual consumption is clearly nonnegative.

Condition (i) of Corollary 23 is fulfilled as long as $\psi > 0$ and the initial distribution has positive mean $K' > 0$. In particular, when restricting $\psi \in [\psi_0, 1]$ with $\psi_0 > 0$, we can define $\mathcal{H}_\epsilon$ with $0 < \epsilon < \psi_0(1 - \delta)$. condition (ii) is satisfied whenever $\psi < 1$ and the initial distribution has positive mean $K' > 0$. Thus, we restrict

$\psi \in [\psi_0, \psi_1]$ with $0 < \psi_0 < \psi_1 < 1$. condition *(iii)* requires slightly more care. The Gâteaux differential of consumption yields

$$\begin{aligned}\langle d_G c'(z', \omega, m'; \tilde{\chi}), \tilde{\chi}\rangle =& \left\langle \psi \left(F_K(z', K') + 1 - \delta\right) \tilde{K}, \tilde{\chi} \right\rangle \\ =& \psi \left(F_K(z', K') + 1 - \delta\right) \tilde{K}^2 \geq 0.\end{aligned}$$

Hence, condition *(iii)* of Proposition 23 is fulfilled. Lastly, the transversality condition *(v)* for admissible policies $\mathcal{H}_\epsilon$

$$\mathbb{E}^{(z''|z')}\left[(c'')^{-\gamma} \left(1 + R(z'', K'')\right) h_x\left(z', \varepsilon', k, m'\right)\right] < \infty \text{ a.s.}$$

is fulfilled whenever $\psi \in [\psi_0, \psi_1]$ as this ensures $c'' > 0$ and $R'' < \infty$. Note that due to the construction of the law of motion of aggregate capital, returns are finite at any time point whenever $K_0 > 0$.

I investigate the limiting strategies $\psi \to \psi_0$ and $\psi \to \psi_1$ in the following. As usual, the first-order condition of this model is given by

$$\frac{\partial}{\partial c} u(c') - \beta \sum_{(z'', \varepsilon'') \in \mathcal{Z}} p^{(z'', \varepsilon'')|(z', \varepsilon')} \left(1 + R(z'', K'')\right) \frac{\partial}{\partial c} u(c'').$$

Positive FOC: This outcome is equivalent to

$$1 > \beta \sum_{(z'', \varepsilon'') \in \mathcal{Z}} p^{(z'', \varepsilon'')|(z', \varepsilon')} \left(1 + R(z'', K'')\right) \left(\frac{c'}{c''}\right)^\gamma, \tag{31}$$

where

$$\frac{c'}{c''} = \frac{\psi}{\psi' (1 - \psi) \left(1 - \delta + \frac{1}{\alpha} F_K(z'', K'')\right)},$$

which is increasing in α. As $\alpha \leq 1$, we have that

$$\frac{c'}{c''} \leq \frac{1}{(1 - \delta + F_K(z'', K''))} \max_{\psi' \in [\psi_0, \psi_1]} \frac{\psi}{(1 - \psi)\psi'}.$$

I let ψ go to ψ_0 which is equivalent to the save everything/consume nothing strategy. Using the upper bound of $\frac{c'}{c''}$, we get

$$\begin{aligned}&\beta \sum_{(z'', \varepsilon'') \in \mathcal{Z}} p^{(z'', \varepsilon'')|(z', \varepsilon')} \left(1 + R(z'', K'')\right) \left(\frac{c'}{c''}\right)^\gamma \\ \leq& \frac{\beta}{(1 - \psi_0)^\gamma} \sum_{(z'', \varepsilon'') \in \mathcal{Z}} p^{(z'', \varepsilon'')|(z', \varepsilon')} \left(1 + R(z'', K'')\right)^{1-\gamma}.\end{aligned}$$

Next, I show that this upper bound is less than one. If $\gamma = 1$, it equals $\frac{\beta}{(1-\psi_0)^\gamma}$. We know that $\beta < 1$. As $\psi_0 > 0$ is arbitrarily chosen, we can find one such that $\frac{\beta}{(1-\psi_0)^\gamma} < 1$ which results in the positive value of the first-order condition. When $\gamma \geq 1$, the right hand side is an increasing function of K''. It goes to zero when $K'' \to 0$ and to $\frac{\beta}{(1-\psi_0)^\gamma}(1-\delta)^{1-\gamma}$ when $K'' \to \infty$. This also results in a positive value of the first-order condition by assumption. Lastly, let us consider the case of $\gamma < 1$. We can rewrite

$$K'' = \left(1 + \frac{1-\alpha}{\alpha}\delta + \frac{1}{\alpha}R(z', K')\right) K',$$

which, using $L' = \pi p^{e'}$, implies that

$$\begin{aligned} R(z'', K'') = & \underbrace{\frac{1 + z''a - (1-z'')a}{1 + z'a - (1-z')a}}_{\leq \frac{1+a}{1-a}} \underbrace{\left(\frac{p^{e''}}{p^{e'}}\right)^{1-\alpha}}_{\leq \frac{\max p^e}{\min p^e}} (R(z', K') + \delta) \\ & \cdot \left(1 + \frac{1-\alpha}{\alpha}\delta + \frac{1}{\alpha}R(z', K')\right)^{\alpha - 1} - \delta. \end{aligned}$$

From this, we can derive (31) by induction using the model assumptions for $\gamma < 1$. Assume that $R(z', K') < \beta^{-\frac{1}{1-\gamma}} - 1$, then, by assumption, it also holds for the next time-period's return $R(z'', K'') < \beta^{-\frac{1}{1-\gamma}} - 1$ because whenever the return R approaches this bound, it decreases due to

$$\frac{(1+a)\max p^e}{(1-a)\min p^e}\left(\left(1 - \frac{1}{\alpha}\right)(1-\delta) + \frac{1}{\alpha\beta^{\frac{1}{1-\gamma}}}\right)^{\alpha-1} < 1.$$

Furthermore, for any return satisfying $R < \beta^{-\frac{1}{1-\gamma}} - 1$, equation (31) is satisfied due to Jensen's inequality given that ψ_0 is chosen small enough. Thus, as we start with an initial return satisfying this condition, the save everything/consume nothing strategy for $\gamma < 1$ delivers a positive value for the first-order condition as well.

Negative FOC: This outcome is equivalent to

$$1 < \beta \sum_{(z'', \varepsilon'') \in \mathcal{Z}} p^{(z'', \varepsilon'')|(z', \varepsilon')} \left(1 + R\left(z'', K''\right)\right) \left(\frac{c'}{c''}\right)^\gamma.$$

I now let $\psi \to \psi_1$ which corresponds to the save nothing/consume everything strategy. Considering

$$\frac{c'}{c''} \geq \frac{1}{\left(1 - \delta + \frac{1}{\alpha} F_K(z'', K'')\right)} \min_{\psi' \in [\psi_0, \psi_1]} \frac{\psi_1}{\psi' \left(1 - \psi_1\right)}$$

and $K'' \to 0$ when $\psi_1 \to 1$ yields that c'/c'' becomes sufficiently large to ensure a negative value of the first-order condition when choosing ψ_1 sufficiently close to one.

Convex Hull: In order to apply Corollary 9 to ensure existence, we first have to construct a subset $B \subset \mathcal{H}_\epsilon \subset C^2(L^2_{\mathbf{P}})$ such that $0 \in \mathrm{int}(\mathrm{conv}(\mathbf{M}(B)))$. Note that our candidate policy in (30) is constant across the agent-specific variables (ε', k). It only varies with the aggregate variables (z', m'). Thus, the value of the equilibrium operator is constant across (ε', k), but it varies with (z', m'). However, since we can vary the candidate policy continuously with the parameter ψ, we can define ψ as a function of (z', K') such that the Euler equation for the candidate policies with $\psi_0(z', K')$ in the neighborhood of ψ_0 results in a constant positive value across $(z', \varepsilon', k, m')$, i.e., $\mathbf{M}[h^{\psi_0}](z', \varepsilon', k, m') = x_0 > 0$.
Note that we have to ensure that admissibility still holds when using a function $\psi_0(z', K')$ with $0 < \psi_0 \leq \psi_0(z', K') \leq \psi_0 + \Delta_0 < \psi_1 - \Delta_1$ and $\Delta_0, \Delta_1 > 0$ instead of a constant ψ. The only condition affected is the one on the Gâteaux differential of consumption which remains positive when requiring $\frac{\partial}{\partial K'} \psi_0(z', K') \geq 0$. Thus, we choose $\psi_0(z', K')$ to be continuous and once differentiable in K'. Furthermore, we have to ensure that such a function $\psi_0(z', K')$ exists resulting in a constant first-order condition, i.e.,

$$x_0 = (\psi_0(z', K') Y(z', K'))^{-\gamma} - \beta \sum_{z'' \in \mathcal{Z}} p^{z''|z'} R(z'', K'') c(z'', K'')^{-\gamma},$$

where

$$\begin{aligned}
Y(z', K') =& W(z', K') L' + (1 + R(z', K'))\, K' \\
R(z'', K'') =& 1 + R\left(z'', \left(1 - \psi_0(z', K')\right) Y(z', K')\right) \\
c(z'', K'') =& \psi_0\left(z'', \left(1 - \psi_0(z', K')\right) Y(z', K')\right) Y\left(z'', \left(1 - \psi_0(z', K')\right) Y(z', K')\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
0 =& - x_0 \psi_0(z', K')^{\gamma+1} + Y(z', K')^{-\gamma} \psi_0(z', K') \\
& - \beta \sum_{z'' \in \mathcal{Z}} p^{z''|z'} \psi_0(z', K')^{\gamma+1} \left(1 + R\left(z'', K''\right)\right) c(z'', K'')^{-\gamma}.
\end{aligned}$$

The right-hand side defines the operator $\mathbf{T}[\psi](z', K')$ which maps real-valued bounded continuous functions $C_b(\mathcal{Z}^{ag} \times \mathbb{R}_+)$ into $C_b(\mathcal{Z}^{ag} \times \mathbb{R}_+)$. It is well-known that this space is complete under the sup-norm. As $\mathbf{T}$ is Fréchet-differentiable and the subset

$$W = \left\{ \psi_0 \in C_b(\mathcal{Z}^{ag} \times \mathbb{R}_+) \middle| \underline{\psi}_0 < \psi_0(z', K') < \underline{\psi}_0 + \Delta_0, \frac{\partial}{\partial K'} \psi_0(z', K') > 0 \right\}$$

is open and convex, Hefti (2015, Proposition 1) yields that $\mathbf{T}+\mathbf{Id}$ with $\mathbf{Id}$ denoting the identity operator is a contraction iff

$$\sup_{\psi_0 \in W} \| d_G \mathbf{T}[\psi_0] + 1 \|_\infty < 1. \tag{32}$$

The derivative reads

$$\begin{aligned} d_G\mathbf{T}[\psi_0](z', K') =& Y(z', K')^{-\gamma} - (\gamma+1)\psi_0(z', K')^{\gamma} \Big(x_0 \\ & + \beta \sum_{z'' \in \mathcal{Z}} p^{z''|z'} \Big[\left(1 + R\left(z'', K''\right)\right) c(z'', K'')^{-\gamma} \\ & - \frac{1}{\gamma+1} \psi_0(z', K') Y(z', K') \underbrace{\frac{\partial}{\partial K''} \left(\left(1 + R\left(z'', K''\right)\right) c(z'', K'')^{-\gamma} \right)}_{\leq 0} \Big] \Big). \end{aligned}$$

Equation (32) is satisfied if $-2 \leq d_G\mathbf{T}[\psi_0](z', K') \leq 0$ for all functions $\psi_0 \in W$ and all $(z', K') \in \mathcal{Z}^{ag} \times \mathbb{R}_+$. Note that

$$\begin{aligned} & Y(\max z', K_{max})^{-\gamma} - (\gamma+1)(\underline{\psi}_0 + \Delta_0)^{\gamma} \Big(x_0 \\ & \quad + \beta \max_{z' \in \mathcal{Z}^{ag}} \sum_{z'' \in \mathcal{Z}} p^{z''|z'} \Big[\left(1 + R(z'', K_{min})\right) \left(\underline{\psi}_0 Y(z'', K_{min})\right)^{-\gamma} \\ & \quad - \frac{1}{\gamma+1} (\underline{\psi}_0 + \Delta_0) Y(z', K_{max}) \frac{\partial}{\partial K''} \left(\left(1 + R\left(z'', K_{min}\right)\right) \left(\underline{\psi}_0 Y(z'', K_{min})\right)^{-\gamma} \right) \Big] \Big) \\ \leq & d_G\mathbf{T}[\psi_0](z', K') \\ \leq & Y(\min z', K_{min})^{-\gamma} - (\gamma+1)\underline{\psi}_0^{\gamma} \Big(x_0 \\ & \quad + \beta \min_{z' \in \mathcal{Z}^{ag}} \sum_{z'' \in \mathcal{Z}} p^{z''|z'} \Big[\left(1 + R(z'', K_{max})\right) \left((\underline{\psi}_0 + \Delta_0) Y(z'', K_{max})\right)^{-\gamma} \\ & \quad - \frac{1}{\gamma+1} \underline{\psi}_0 Y(z', K_{min}) \frac{\partial}{\partial K''} \left(\left(1 + R\left(z'', K_{max}\right)\right) \left((\underline{\psi}_0 + \Delta_0) Y(z'', K_{max})\right)^{-\gamma} \right) \Big] \Big), \end{aligned}$$

where $K_{min} = \min(K_0, K^*_{min})$ and $K^*_{min} > 0$ is the unique solution to

$$K^*_{min} = \left(1 - \underline{\psi}_0 - \Delta_0\right) \left((1-\delta) K^*_{min} + F\left(\min z', K^*_{min}\right) \right).$$

K_{min} is indeed the minimal capital as the law of motion leads to increasing capital when we start with $K_0 < K^*_{min}$. Furthermore, $K_{max} = \max(K_0, K^*_{max})$ where $K^*_{max} > 0$ is the unique solution to

$$K^*_{max} = (1 - \psi_0)\left((1-\delta)K^*_{max} + F\left(\max z', K^*_{max}\right)\right).$$

Similarly here, capital decreases when we start with $K_0 > K^*_{max}$. As the derivative's bounds depend continuously on the free parameters $\psi_0, \Delta_0, x_0 > 0$, we can choose them such that the bounds ensure (32). Therefore, there exists a unique $\psi_0(z', K')$ such that $\mathbf{M}[h^{\psi_0}](z', \varepsilon', k, m') = x_0 > 0$.
We can similarly define the candidate policies with $\psi_1(z', K') \in [\psi_1 - \Delta_1, \psi_1]$ such that the Euler equation results in a constant negative value across $(z', \varepsilon', k, m')$, i.e., $\mathbf{M}[h^{\psi_1}](z', \varepsilon', k, m') = x_1 < 0$. I omit the proof as it follows the same steps as for the positive value. Hence, by defining the subset B as all candidate policies with the functions $\psi_0(z', K')$ and $\psi_1(z', K')$, the convex hull of the equilibrium operator contains zero. Therefore, Corollary 9 ensures existence. Applying Lemma 16 yields uniqueness and concludes the proof. $\square$

A.9 Proof of Theorem 20

Proof of Theorem 20. Similar to the previous proof, I construct two bond holding strategies and corresponding price functions such that the Euler equation and the bond market clearing equation yield positive and negative values, respectively. The candidate policy is defined as follows

$$\begin{aligned}
b' = h_x(z', \varepsilon', b, \mu') &= (b - B' + e(z', \varepsilon') - \bar{e}(z'))\,\frac{\psi}{\eta(1-\psi)} + \varphi\,(1 - 2\psi) \\
P' = h_A(z', \mu') &= \eta\frac{1-\psi}{\psi} \\
c'(z', \varepsilon', b, \mu') &= \bar{e}(z') + B' - \eta\frac{1-\psi}{\psi}\varphi\,(1-2\psi)\,,
\end{aligned}$$

where $\bar{e}(z') = \mathbb{E}[e(z', \varepsilon')|z']$ and the parameters $\psi, \eta \in [0, 1]$ allow to vary the policy. Aggregation yields

$$\begin{aligned}
B'' &= \varphi\,(1 - 2\psi) \\
P'B'' &= \eta\frac{1-\psi}{\psi}\varphi\,(1-2\psi)\,.
\end{aligned}$$

Admissibility: Let me first check that the policy functions are admissible.

They are continuous in b and B' and, thus, also in the distribution. Square-integrability is ensured by linearity in b. condition (i) of Proposition 25 is fulfilled for prices as long as $\psi \in [0, \psi_1]$ with $\psi_1 < 1$ and $\eta \in [\eta_0, 1]$ with $\eta_0 > 0$. Consumption

$$c'(z', \varepsilon', b, \mu') \geq \min \bar{e}(z') - \varphi \left(1 + \eta \frac{1-\psi}{\psi} (1 - 2\psi)\right)$$

is positive when we set

$$\begin{aligned} \varphi <& \frac{1}{2} \min \bar{e}(z') \frac{\psi_0}{\psi_0 + (1-\psi_0)(1-2\psi_0)} \\ \epsilon <& \frac{1}{2} \min \bar{e}(z') \end{aligned} \tag{33}$$

where $\psi_0 \in (0, 0.5)$ and $\psi_0 < \psi_1$ and we restrict $\psi \in [\psi_0, \psi_1]$. We can find such an admissible set $\mathcal{H}_\epsilon$ due to the assumption that $\min \bar{e}(z') > 0$. condition (ii) is ensured by construction as B'' is a strictly decreasing function of $\psi \in [0, 1]$ ranging from $[\varphi, -\varphi]$. Note that $B'' = 0$ at 0.5 such that $B'' > 0$ at ψ_0. condition (iii) holds as the Gâteaux differential of consumption satisfies

$$\langle d_G c(z', \omega, m'; \tilde{\chi}), \tilde{\chi} \rangle = \left\langle \tilde{B}, \tilde{\chi} \right\rangle = \tilde{B}^2 \geq 0.$$

condition (iv) requires a little more care. To show that the parameterized policies belong to the admissible set for any $\psi \in [\psi_0, \psi_1]$, note that B'' and P' are functions of ψ. It is easy to see that P' and B'' are decreasing in ψ. This verifies the first inequality of condition (iv). The second inequality holds as well when $P'B''$ is also a decreasing function of ψ. This holds true for any $\psi \in [0, 1]$. It remains to check the third inequality of condition (iv)

$$\langle P_1' b_1' - P_2' b_2', b_1' - b_2' \rangle = \langle P_1' B_1'' - P_2' B_2'', B_1'' - B_2'' \rangle \geq 0$$

which holds because B'' and $P'B''$ are both decreasing on $\psi \in [0, 1]$. Lastly, the fact that $\psi_0, \eta_0 > 0$ ensures that the transversality condition holds.

I will now investigate the limiting strategies $\psi \to \psi_0$ and $\psi \to \psi_1$ in the following. Note that we can bound consumption for any $\psi \in [\psi_0, \psi_1]$ by

$$\frac{1}{2} \min \bar{e}(z') \leq c' \leq \max \bar{e}(z') + \varphi - \varphi \underbrace{\min_{\psi, \eta} \eta \frac{(1-2\psi)(1-\psi)}{\psi}}_{=(2\sqrt{2}-3)\eta_0}.$$

Furthermore, $\varphi \to 0$ whenever $\psi_0 \to 0$ due to (33). The first-order conditions of this model are given by

$$\frac{\partial}{\partial c} u\left(c'\right) - \beta \sum_{(z'',\varepsilon'')\in\mathcal{Z}} p^{(z'',\varepsilon'')|(z',\varepsilon')} \frac{1}{P'} \frac{\partial}{\partial c} u\left(c''\right)$$

$$P'B''.$$

Positive FOC: This outcome is equivalent to

$$1 > \beta \sum_{(z'',\varepsilon'')\in\mathcal{Z}} p^{(z'',\varepsilon'')|(z',\varepsilon')} \frac{1}{P'} \left(\frac{c'}{c''}\right)^{\gamma} \tag{34}$$

$$P'B'' > 0.$$

Letting $\psi \to \psi_0$ yields

$$\frac{c'}{c''} \le \frac{\max \bar{e}(z') - \frac{\varphi}{\psi_0}\left(\eta_0(1-2\psi_0)(1-\psi_0) - \psi_0\right)}{\frac{1}{2}\min \bar{e}(z')}$$

which for $\psi_0 \le \frac{3\eta_0+1-\sqrt{\eta_0^2+6\eta_0+1}}{4\eta_0}$ results in

$$\frac{c'}{c''} \le \frac{2\max \bar{e}(z')}{\min \bar{e}(z')} < \infty.$$

Thus, we obtain

$$\beta \sum_{(z'',\varepsilon'')\in\mathcal{Z}} p^{(z'',\varepsilon'')|(z',\varepsilon')} \frac{1}{P'} \left(\frac{c'}{c''}\right)^{\gamma} \le \beta \frac{1}{P'} \left(\frac{2\max \bar{e}(z')}{\min \bar{e}(z')}\right)^{\gamma}$$

Choosing ψ_0 close to zero ensures the first inequality of (34) as $P' \to \infty$ when $\psi_0 \to 0$ and $\frac{c'}{c''} < \infty$. Furthermore, the second inequality is satisfied due to

$$P'(\psi_0)B''(\psi_0) > 0$$

as B'' is positive for $\psi_0 < 0.5$.

Negative FOC: This outcome is equivalent to

$$1 < \beta \sum_{(z'',\varepsilon'')\in\mathcal{Z}} p^{(z'',\varepsilon'')|(z',\varepsilon')} \frac{1}{P'} \left(\frac{c'}{c''}\right)^{\gamma} \tag{35}$$

$$P'B'' < 0.$$

Letting $\psi \to \psi_1$ yields $B'' = (1-2\psi_1)\varphi < 0$ whenever $\psi_1 > 0.5$. Thus, the second inequality of (35) is satisfied. Furthermore, we can bound

$$\begin{aligned}\frac{c'}{c''} \geq & \frac{\frac{1}{2}\min \bar{e}(z')}{\max \bar{e}(z') + (1-2\psi_1)\varphi - \min_{\psi',\eta'} P''B'''} \\ = & \frac{\frac{1}{2}\min \bar{e}(z')}{\max \bar{e}(z') - \left(1 + (3-2\sqrt{2})\eta_0 - 2\psi_1\right)\varphi}.\end{aligned}$$

For ψ_1 large and η_0 small enough, this results in

$$\beta \sum_{(z'',\varepsilon'')\in\mathcal{Z}} p^{(z'',\varepsilon'')|(z',\varepsilon')} \frac{1}{P'} \left(\frac{c'}{c''}\right)^{\gamma} \geq \beta \frac{1}{P'} \underbrace{\left(\frac{\min \bar{e}(z')}{2\max \bar{e}(z')}\right)^{\gamma}}_{>0}.$$

Letting $\psi_1 \to 1$ yields $P' \to 0$ such that one can find a ψ_1 which satisfies the first inequality of (35).

Convex Hull: The last step again consists of checking whether the interior of the convex hull of the image of our candidate strategies contains zero. This step is similar to its counterpart in the proof for the Krusell-Smith model. Consumption resulting from our candidate strategies is constant across agents, i.e., across (ε', b). To make it constant across $(z', \varepsilon', b, \mu')$, I define the parameters ψ and η as functions of (z', B'). As the candidate policies are continuous in those parameters, we define the candidate policies with $\psi(z', B')$ and $\eta(z', B')$ such that both first-order equations result in the same small positive and negative value which is constant across $(z', \varepsilon', b, \mu')$, respectively.

Let us start with $\psi_0(z', B')$ defined in the neighborhood of ψ_0, i.e., $\psi_0(z', B') \in [\psi_0, \psi_0 + \Delta_0]$, and the corresponding $\eta_0(z', B')$. I define the latter by

$$\eta_0(z', B') = \frac{x_0}{\frac{1-\psi_0(z',B')}{\psi_0(z',B')}\varphi\left(1 - 2\psi_0(z', B')\right)}$$

with $0 < x_0 < \frac{1-\psi_0-\Delta_0}{\psi_0+\Delta_0}\varphi\left(1 - 2\psi_0 - 2\Delta_0\right)$ such that $0 < \eta_0(z', B') < 1$. This ensures that the second first-order condition is constant and it leads to

$$\begin{aligned}c' = & \bar{e}(z') + B' - x_0 \\ c'' = & \bar{e}(z'') + \varphi\left(1 - 2\psi_0(z', B')\right) - x_0.\end{aligned}$$

As in the previous proof, we choose $\psi_0(z', K')$ to be continuous. Admissibility is easily confirmed. Next, we have to ensure that a function $\psi_0(z', K')$ exists resulting in a constant first-order condition, i.e.,

$$\begin{aligned}x_0 =& \left(\bar{e}(z') + B' - x_0\right)^{-\gamma} \\ &- \frac{\varphi\left(1 - 2\psi_0(z', B')\right)}{x_0}\beta \sum_{z''\in\mathcal{Z}} p^{z''|z'} \left(\bar{e}(z'') + \varphi\left(1 - 2\psi_0(z', B')\right) - x_0\right)^{-\gamma}\end{aligned}$$

The right-hand side defines the operator $-\mathbf{T}[\psi](z', K')$ which maps real-valued bounded continuous functions $C_b(\mathcal{Z}^{ag} \times \mathbb{R})$ into $C_b(\mathcal{Z}^{ag} \times \mathbb{R})$. As $\mathbf{T}$ is Fréchet-differentiable and the subset

$$W = \{\psi_0 \in C_b(\mathcal{Z}^{ag} \times \mathbb{R}) |\, \psi_0 < \psi_0(z', K') < \psi_0 + \Delta_0\}$$

is open and convex, we can again use Hefti (2015, Proposition 1) that yields that $\mathbf{T} + \mathbf{Id}$ is a contraction iff $\sup_{\psi_0 \in W} \|d_G\mathbf{T}[\psi_0] + 1\|_\infty < 1$. The derivative reads

$$\begin{aligned}d_G\mathbf{T}[\psi_0] =& -2\frac{\varphi}{x_0}\beta \sum_{z''\in\mathcal{Z}} p^{z''|z'} \left(c''\right)^{-\gamma} \\ &+ 2\varphi\frac{\varphi\left(1 - 2\psi_0(z', B')\right)}{x_0}\beta\gamma \sum_{z''\in\mathcal{Z}} p^{z''|z'} \left(c''\right)^{-\gamma-1} \\ =& -2\frac{\varphi}{x_0}\beta \sum_{z''\in\mathcal{Z}} p^{z''|z'} \left(c''\right)^{-\gamma} \left(1 - \gamma \underbrace{\frac{\varphi\left(1 - 2\psi_0(z', B')\right)}{\bar{e}(z'') + \varphi\left(1 - 2\psi_0(z', B')\right) - x_0}}_{\leq 1}\right).\end{aligned}$$

The derivative is bounded from below by $-2 \leq d_G\mathbf{T}[\psi_0] \leq 0$ for small enough φ and small enough γ. For larger γ, consider $d_G(-\mathbf{T}[\psi_0])$ instead of $d_G\mathbf{T}[\psi_0]$. Hence, there exists a unique $\psi_0(z', B')$ and thus, also a unique $\eta_0(z', B')$ such that $\mathbf{M}[h^{\psi_0,\eta_0}](z', \varepsilon', b, m') = x_0 > 0$.
We can similarly define the candidate policies with $\psi_1(z', B') \in [\psi_1 - \Delta_1, \psi_1]$ and corresponding $\eta_1(z', B')$ such that the Euler equation results in a constant negative value across $(z', \varepsilon', b, m')$, i.e., $\mathbf{M}[h^{\psi_1,\eta_1}](z', \varepsilon', k, m') = x_1 < 0$. I omit the proof as it follows the same steps as for the positive value. Hence, by defining the subset B as all candidate policies with the functions $\psi_0(z', B')$, $\eta_0(z', B')$, $\psi_1(z', B')$ and $\eta_1(z', B')$, the convex hull of the equilibrium operator contains zero. Therefore, Corollary 9 ensures existence. Applying Lemma 16 yields uniqueness and concludes the proof. $\square$

References

Ömer T. Açıkgöz. On the existence and uniqueness of stationary equilibrium in Bewley economies with production. *Journal of Economic Theory*, 173:18–55, 2018.

Daron Acemoglu and Martin Kaae Jensen. Robust comparative statics in large dynamic economies. *Journal of Political Economy*, 123(3):587–640, 2015.

Yves Achdou, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. Income and wealth distribution in macroeconomics: A continuous-time approach. *Working Paper*, 2017.

S. Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684, 1994.

Jürgen Appell, Espedito De Pascale, and Alfonso Vignoli. *Nonlinear Spectral Theory*. De Gruyter, Berlin, 2004.

Heinz H. Bauschke and Patrick L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. CMS Books in Mathematics. Springer, 2^{nd} edition, 2010.

Truman Bewley. The permanent income hypothesis: A theoretical formulation. *Journal of Economic Theory*, 16(2):252–292, 1977.

V.I. Bogachev. *Measure Theory*. Springer Berlin Heidelberg, 2007.

Johannes Brumm, Dominika Kryczka, and Felix Kubler. Recursive equilibria in dynamic economies with stochastic production. *Econometrica*, 85(5):1467–1499, 2017.

Dan Cao. Recursive equilibrium in Krusell and Smith (1998). *Journal of Economic Theory*, 186:104978, 2020.

Patrick Cheridito and Juan Sagredo. Existence of sequential competitive equilibrium in Krusell–Smith type economies. *Working Paper, Available at SSRN 2794629*, 2016.

Wouter J. den Haan, Kenneth L. Judd, and Michel Juillard. Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. *Journal of Economic Dynamics and Control*, 34(1):1–3, 2010.

R. M. Dudley. *Real Analysis and Probability.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 2002.

Nassif Ghoussoub. *Self-dual partial differential systems and their variational principles.* Springer Science & Business Media, 2009.

Andreas Hefti. A differentiable characterization of local contractions on Banach spaces. *Fixed Point Theory and Applications*, 2015(1):1–5, 2015.

Mark Huggett. The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5):953–969, 1993.

Per Krusell and Anthony A. Smith. Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896, 1998.

Bar Light. Uniqueness of equilibrium in a Bewley-Aiyagari model. *Economic Theory*, 69:435–450, 2020.

Jianjun Miao. Competitive equilibria of economies with a continuum of consumers and aggregate shocks. *Journal of Economic Theory*, 128(1):274–298, 2006.

Stephen Morris and Hyun Song Shin. Rethinking multiple equilibria in macroeconomic modeling. *NBER Macroeconomics Annual*, 15:139–161, 2000.

Robert R. Phelps. Lectures on maximal monotone operators. *Extracta Mathematicae*, 12(3):193–230, 1997.

R. Tyrrell Rockafellar. Local boundedness of nonlinear, monotone operators. *The Michigan Mathematical Journal*, 16(4):397–407, 1969.

R. Tyrrell Rockafellar. *Monotone operators associated with saddle-functions and minimax problems*, volume 18 of *Proceedings of Symposia in Pure Mathematics: Nonlinear Functional Analysis, Part 1*, pages 248–257. American Mathematical Society, 1970.

R. Tyrrell Rockafellar. Monotone operators and the proximal point algorithm. *SIAM Journal on Control and Optimization*, 14(5):877–898, 1976a.

R. Tyrrell Rockafellar. Augmented Lagrangians and applications of the proximal point algorithm in convex programming. *Mathematics of Operations Research*, 1(2):97–116, 1976b.

N. L. Stokey, R. E. Lucas, and E. C. Prescott. *Recursive Methods in Economic Dynamics*. Harvard University Press, 1989.

Yeneng Sun. The exact law of large numbers via Fubini extension and characterization of insurable risks. *Journal of Economic Theory*, 126(1):31–69, January 2006.

Alexis Akira Toda. Huggett economies with multiple stationary equilibria. *Journal of Economic Dynamics and Control*, 84:77 – 90, 2017.

Neng Wang. Caballero meets Bewley: The permanent-income hypothesis in general equilibrium. *American Economic Review*, 93(3):927–936, June 2003.

Eberhard Zeidler. *Nonlinear Functional Analysis and its Applications III. (Variational Methods and Optimization)*. Springer, 1985.

Eberhard Zeidler. *Nonlinear Functional Analysis and its Applications I. (Fixed Point Theorems)*. Springer, 1986.