Distributionally Robust Auction Design*

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Abstract

A single unit of a good is sold to one of a group risk-neutral bidders whose privately-known values are drawn independently from an identical distribution. The seller only has limited information about the value distribution and believes that the value distribution is designed by Nature adversarially to minimize revenue. In addition, the seller knows that bidders play undominated strategies. For the two-bidder case, we construct a *strong maxmin solution*, consisting of a mechanism, a value distribution, and an equilibrium in undominated strategies, such that neither the seller nor Nature can move revenue in their respective preferred directions, even if the deviator can select the new equilibrium in undominated strategies. The mechanism and value distribution solve a family of maxmin mechanism design and minmax information design problems, regardless of how an equilibrium in undominated strategies is selected. The maxmin mechanism is *a second-price auction with a random reserve price*. For arbitrary number of bidders, a second-price auction with a random reserve price remains a maxmin mechanism among a subclass of dominant-strategy mechanisms.

Keywords: Robust mechanism design, information design, independent private value, second-price auctions, random reserve price, undominated strategies.

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1 Introduction

The classic auction theory assumes that the seller knows bidders' information structure and derives optimal (revenue-maximizing) mechanisms. A useful benchmark in auction theory is the symmetric independent private value (IPV) model in which bidders' privately-known values are drawn independently from an identical distribution. In this model, Myerson (1981) assumes that the seller knows a generic bidder's value distribution, and finds that a second-price auction with a deterministic reserve price is an optimal mechanism, provided that the known value distribution satisfies certain regularity conditions. This result is one of the "all-time greatest hits" of mechanism design. However, the seller may not know whether a bidder's value distribution satisfies Myerson's regularity conditions in practice. If Myerson's regularity conditions fail, then the optimal mechanism no long admits a simple implementation. Even if those conditions are known to hold, the optimal mechanism is sensitive to the details of the value distribution: The characterization of the deterministic reserve price requires fine details of the value distribution.

To address these issues, we assume that the seller does not know every aspect of a generic bidder's value distribution and evaluates a mechanism using the worst-case expected revenue over uncertainties about value distributions. More specifically, we assume that the seller only knows the expectation of a generic bidder's value. In contrast, the value distribution is common knowledge among the bidders. A value distribution that is consistent with the known expectation is referred to as a *possible* value distribution. The seller seeks a mechanism from a vast class of mechanisms with the only requirement that the mechanism "secures" bidders' participation: There is a message for each payoff type of each bidder that guarantees a non-negative payoff regardless of the other bidder' messages. In addition, the seller knows that bidders play undominated strategies[1] in a mechanism. The seller believes that an adversarial Nature chooses a possible value distribution and an equilibrium in undominated strategies to minimize the expected revenue.

The joint mechanism design and information design problem is not a standard zero-sum game, as a given mechanism and a given information structure need not have a unique equilibrium in undominated strategies. Moreover, an equilibrium in undominated strategies may not exist at all. We address the issues of equilibrium multiplicity and existence by using

[1]The assumption that bidders play undominated strategies (or "admissible strategies") is often considered as a reasonable assumption for an individual's "rationality" in the literature on decision theory and game theory. See, for example, Kohlberg and Mertens (1986). In the literature on implementation theory, Palfrey and Srivastava (1991) use the concept of undominated Nash equilibrium, which is the same refinement on the set of equilibria considered in this paper. Yamashita (2015) studies robust mechanism design problems assuming that bidders play undominated strategies. In contrast to our model, bidders may not play a Bayesian equilibrium in his model.

a new solution concept, with a flavour similar to the one used in Brooks and Du (2021b): A *strong maxmin solution* is a triple of a mechanism, a possible value distribution, and an equilibrium in undominated strategies such that, i) fixing the mechanism, the possible value distribution and the equilibrium in undominated strategies minimize the expected revenue, and ii) fixing the possible value distribution, the mechanism and the equilibrium in undominated strategies maximize the expected revenue. Indeed, these statements remain true regardless of which equilibrium in undominated strategies is played. The strong maxmin solution has an associated *revenue guarantee*, which is the expected revenue in the constituent equilibrium under the constituent value distribution. The revenue guarantee is both a tight lower bound on the expected revenue for the mechanism across all possible value distributions and all equilibria in undominated strategies and a tight upper bound on the expected revenue for the value distribution across all mechanisms and all equilibria in undominated strategies.

The main result (Theorem 1) constructs a strong maxmin solution for the two-bidder case. First, the maxmin mechanism is a *second-price auction with a random reserve price*. The distribution of the random reserve price is atomless and admits a density function everywhere on $[0, 1]$. Remarkably, a simple dominant-strategy mechanism arises as a robustly optimal mechanism across all participation-securing mechanisms, which include all dominant-strategy mechanisms as well as nondominant-strategy mechanisms used in practice, e.g., first-price auctions and all-pay auctions. Therefore, even if nondominant-stategy mechanisms are allowed, the seller would use a dominant-strategy mechanism to maximize the revenue guarantee in our setting, which is a priori unanticipated. The result is consistent with the prevalence of second-price auctions used in practice for selling a good. Indeed, the result provides a rationale for using a second-price auction from the perspective of robustness. Moreover, the rationale is strong, as it is established across a vast class of mechanisms that incorporates arguably any conceivable practical mechanisms. Second, the minmax value distribution is an *equal-revenue distribution*, defined by the property of a unit-elastic demand: In the monopoly pricing problem, the monopoly's revenue from charging any price in the support of this distribution is the same. Equal-revenue distributions are familiar in several literatures: they emerge endogenously in many robust mechanism design environments and information design environment, e.g., Bergemann and Schlag (2008), Carrasco et al. (2018), Zhang (2022b), Roesler and Szentes (2017), Condorelli and Szentes (2020), Chen and Yang (2020), etc. Finally, the constituent equilibrium is the truth-telling equilibrium in which bidders always truthfully report their true values.

Let us give a heuristic illustration of the result. We start with the minmax value distribution. This value distribution has the property that each bidder's "virtual value" (this is the standard Myerson's virtual value) is zero except when the bidder observes the highest possible value. Therefore, the seller is *indifferent* between a wide range of mechanisms under this value distribution. Indeed, the seller is indifferent between all Bayesian incentive compatible and Bayesian individually rational mechanisms in which 1) the participation constraint is *binding* for a bidder with the lowest possible value, and 2) the good is *fully* allocated to the bidder(s) with the highest possible value(s), given that the truth-telling equilibrium is played. This is because the seller is indifferent between allocating and not allocating the good when both bidders' virtual values are zeros.

Given that the adversarial Nature chooses the worst-case equilibrium in undominated strategies, second-price auctions would be natural candidates for a maxmin mechanism, as the truth-telling equilibrium is the unique equilibrium in undominated strategies if the mechanism is a second-price auction (with or without a reserve price). In addition, given that the adversarial Nature chooses the worst-case value distribution, certain randomization device is expected to be employed in a maxmin mechanism as it would hedge against uncertainties over the value distributions. Hence, we propose a second-price auction with a random reserve price as a candidate for a maxmin mechanism. The distribution of the random reserve price is then constructed so that the minmax value distribution minimizes the expected revenue in the truth-telling equilibrium across all possible value distributions. More elaborately, given a second-price auction with a random reserve price, the adversarial Nature solves an expected revenue minimization problem subject to the constraint that the value distribution is possible. We construct a Lagrangian and use the first-order condition to derive a differential equation that the distribution of the random reserve price satisfies so that the minmax value distribution is a solution to the Nature's constrained minimization problem.

We partially extend the main result to the $n-$bidder case ($n \geq 3$): A second-price auction with a random reserve price is a maxmin mechanism among a subclass of dominant-strategy mechanisms which we call *highest-bidder lotteries*, whose defining property is that only a bidder with the highest bid (breaking ties randomly) could win the good, and he wins the good if and only if his bid is higher than a outcome drawn from a lottery that can depend on the bidder's identity and the other bidders' bids. The minmax value distribution is a combination of an atom and an equal-revenue distribution.

The remainder of the introduction discusses related work. Section 2 presents the model. Section 3 characterizes the main result. Section 4 characterizes the maxmin mechanism among highest-bidder lotteries. Section 5 discovers a reversed Bulow-Klemperer result. Section 6 extends and discusses the main result. Section 7 is a conclusion.

1.1 Related Work

This paper lies at the intersection of several different literatures. The first literature is the classic auction design literature, initiated by the seminal work of Myerson (1981) who characterizes optimal auctions in the independent private value environment. Crémer and McLean (1988) characterize optimal auctions in the private value environment for generic correlation structures. Strikingly, the seller is able to extract the full surplus by carefully design side bets.

While these results are of significant theoretical interest, they rely on the common knowledge assumption. The "Wilson doctrine" (Wilson, 1987) motivates the robust mechanism design literature that searches for economic institutions not sensitive to unrealistic assumptions about the information structure.

This paper is in sharp contrast to Myerson (1981). First, our problem is different: Myerson studies optimal auction design assuming the seller knows the a generic bidder's value distribution, whereas we study robust auction design assuming the seller only has partial information about a generic bidder's value distribution. Second, our result is different: Myerson finds that (under Myerson's regularity conditions) a second-price auction with a deterministic reserve price is an optimal auction, whereas we finds a second-price auction with a random reserve price is a robustly optimal auction. This paper contributes to the robust mechanism design literature.

The closest related work is Suzdaltsev (2020), who considers a model of auction design assuming the seller only knows the expectation of a generic bidder's value, and characterizes the optimal deterministic reserve price for a second-price auction. He shows that it is optimal to set the reserve price to seller's own valuation, which is zero in our setting. In contrast, we do not place any restrictions on the mechanism except for a participation security constraint, and characterize a maxmin mechanism for the two-bidder case. For arbitrary number of bidders, we characterize a maxmin mechanism among a subclass of dominant-strategy mechanism including a second-price auction as a special case. Importantly, for any known expectations, the revenue guarantee under our proposed mechanism is strictly higher than that under his mechanism for the two-bidder case; for a range of known expectations, the revenue guarantee under our proposed mechanism is strictly higher than that under his mechanism for the n-bidder ($n \geq 3$) case.

This paper is closely related to Che (2019), Brooks and Du (2021a), He and Li (2022) and Zhang (2022b). Che (2019) considers a model of auction design in the private value environment and assumes that the seller only knows the mean of bidders' valuation distribution. He characterizes a second-price auction with a random reserve price as a maxmin mechanism within a subclass of mechanisms termed as competitive mechanisms.

Interestingly, the formats of maxmin mechanisms in both papers are second-price auctions, albeit with different distributions of the random reserve price. In this regard, this paper provides further support on using second-price auctions for selling a good. In addition, we both assume that bidders play undominated strategies. However, there are several differences. First, the seller in his model does not know how bidders' valuations are correlated, whereas the seller in mine knows that bidders' valuations are independently and identically distributed. That is, the seller knows more in our model, and therefore the revenue guarantee in our model is an upper bound of the one in his model. Second, our solution concept is stronger in that we allow the seller to choose a mechanism from the class of all participation-securing mechanisms, which is a strict superset of the class of competitive mechanisms.

Brooks and Du (2021a) consider a model of auction design in the interdependent value environment and assume that the seller knows only the expectation of bidders' prior valuation. They finds, among others, that a proportional auction, in which the aggregate allocation is equal to the minimum of the sum of bidders' value and 1, and each bidder's individual allocation is proportional to their value, is a maxmin mechanism across all participation-securing mechanisms for the symmetric case. In their model, the seller knows less: The set of possible information structures is much larger, including but not limited to independent private value environment. Therefore, the revenue guarantee in our model is an upper bound of the one in their model. Indeed, they find that in the minmax information structure, bidders' prior valuation are perfectly correlated, so essentially a common value model emerges endogenously. Hence, our methodologies differs. In their model, the Nature's minimization problem is a linear program using a powerful tool of Bayes correlated equilibrium in Bergemann and Morris (2013), whereas the Nature's minimization problem is a non-linear program in ours. In addition, the solution concept in their paper is stronger than the one in ours, as there are no restrictions on the set of equilibria that the Nature can choose.

He and Li (2022) and Zhang (2022b) both consider a model of auction design in the private value environment and assume that the seller knows the marginal distribution of a generic bidder's valuation but does not knows the correlation structure between bidders' valuations. Under different conditions on the marginal distributions, they characterize, among others, that second-price auctions with random reserve prices are maxmin mechanisms within (standard) dominant-strategy mechanisms. One of the main differences is that the sets over which the seller evaluates worst-case expected revenue are different: the seller knows the marginal distribution but not the correlation structure in those two papers, whereas the seller knows the correlation structure but not the marginal distribution (except for the mean) in this paper. In addition, the solution concept in this paper is stronger: the class of mechanisms considered in this paper is much wider than the class of dominant-strategy mechanisms.

This paper is also related to Carrasco et al. (2018), Zhang (2022d) and Zhang (2022e). Carrasco et al. (2018) consider a model of monopoly selling in which the seller sells a good to a single buyer when the seller knows an arbitrary number of moment conditions. When there is only one bidder, our model is reduced to to their special case in which the seller knows only the expectation. However, adding a second bidder with independently and identically distributed valuations, there is strictly more competition in our model. Therefore, the seller guarantees a higher revenue in our model. Zhang (2022d) considers a model of bilateral trade and characterizes optimal dominant-strategy mechanisms when the profit-maximizing intermediary knows only the expectations of each trader's valuations. Zhang (2022e) considers a model of public-good provision and characterizes optimal dominant-strategy mechanisms when the profit-maximizing principal knows only the expectations of each agent's valuation. One of the main differences from those two papers is that our paper studies auction designs. Moreover, the designers in those two papers knows less: the designers do not know how agents' valuations are correlated.

Finally, this paper is related to the information design literature: See Kamenica (2019) and Bergemann and Morris (2019) for recent surveys. The closest related paper in this literature is Chen and Yang (2020) who study information design problems in the auction model. The model is equivalent to the minmax counterpart of our maxmin problem: Nature chooses a possible value distribution for a generic bidder. The seller, after observing the choice of the distribution but not the realization of the value profile, designs a revenue-maximizing mechanism. They find, among others, that the seller-worst value distribution is an equal-revenue distribution for the two-bidder case. It has been an open question whether strong duality holds and what a maxmin mechanism looks like. This paper provides a positive answer to this question for the two-bidder case. Indeed, the result in this paper implies that the seller-worst value distribution is an equal-revenue distribution for the two-bidder case. In this regard, this paper complements their work.

2 Model

2.1 Auction environment

A seller sells a single unit of good. For exposition, we assume that the supply cost of the good is zero. There are n ($n \geq 2$) bidders, indexed by $i \in \{1, 2, \cdots, n\}$, competing for the good. Each bidder privately knows her value for the good, which is modeled as a random variable v_i. We assume their values have the same range, denoted by V. We assume that V is bounded. As a normalization, we assume that $V = [0, 1]$. The set of all possible value profiles is $V^n = [0, 1]^n$ with a typical value profile v. Bidders' values are drawn independently from the same distribution, denoted by F. That is, we consider the canonical single-unit auction environment under symmetric independent private values (IPV). Importantly, there is no technical assumption on F. That is, F can be continuous, discrete, or any mixtures. The set of all probability distributions on V is denoted by $\Delta(V)$. We denote by $\mu = E[v_i]$ the expectation of each bidder's values. To rule out trivial cases, we assume that $\mu \in (0, 1)$.

2.2 Information

The seller has limited information about the value distribution. Specifically, the seller only knows the expectation μ of each bidder's value, but does not know the value distribution F. A value distribution is *possible* if and only if its mean is μ. We denote the set of possible value distributions by $\mathcal{F}(\mu)$. Formally,

$$\mathcal{F}(\mu) = \left\{ F \in \Delta V | \int_0^1 x dF(x) = \mu \right\}.$$

2.3 Mechanism

A *mechanism* $\mathcal{M}$ consists of measurable sets of messages M_i for each i and measurable allocation rules $q_i : M \to [0, 1]$ and measurable payment rules: $t_i : M \to \mathbb{R}$ for each i, where $M = \times_{i=1}^n M_i$ is the set of message profiles, such that $\sum_{i=1}^n q_i(m) \leq 1$ for each $m \in M$. Bidders' preferences are quasilinear. Given a mechanism $\mathcal{M}$ and a simultaneously submitted message profile m, bidder i with a value of v_i has a utility

$$U_i(v_i, m) = v_i \cdot q_i(m) - t_i(m). \tag{1}$$

We require the mechanism to satisfy a *participation security* constraint: For each i, there exists a message $m_i \equiv 0 \in M_i$ such that for each $v_i \in V$ and each $m_{-i} \in M_{-i}$,

$$U_i(v_i, (0, m_{-i})) \geq 0. \tag{PS}$$

Bidder i with a value v_i can guarantee a non-negative utility by sending this message, regardless of messages sent by the other bidder.

2.4 Equilibrium

Given a mechanism $\mathcal{M}$ and a value distribution F, we have a game of incomplete information. A *Bayes Nash Equilibrium* (BNE) of the game is a strategy profile $\sigma = (\sigma_i)$, $\sigma_i : V \to \Delta(M_i)$, such that σ_i is best response to σ_{-i}: let $U_i(v_i, \mathcal{M}, F, \sigma) = \int_{v_{-i}} U_i(v_i, (\sigma_i(v_i), \sigma_{-i}(v_{-i})))dF(v_{-i})$ where $U_i(v_i, (\sigma_i(v_i), \sigma_{-i}(v_{-i})))$ is the multilinear extension of U_i in Equation (1), then for any i, v_i, σ_i',

$$U_i(v_i, \mathcal{M}, F, \sigma) \geq U_i(v_i, \mathcal{M}, F, (\sigma_i', \sigma_{-i})). \tag{BR}$$

The set of all Bayes Nash Equilibria in undominated strategies for a given mechanism $\mathcal{M}$ and a given value distribution F is denoted by $\Sigma(\mathcal{M}, F)$.

Given a mechanism $\mathcal{M}$, the expected revenue at a value distribution F and an equilibrium σ is

$$R(\mathcal{M}, F, \sigma) = \int_{v_1} \cdots \int_{v_n} \left[\sum_{i=1}^{n} t_i(\sigma(v))\right] dF(v_1) \cdots dF(v_n).$$

We refer to the integrand as the *ex-post revenue* given the value profile v at the equilibrium σ. If $\Sigma(M, F)$ is empty, we define the revenue of the mechanism M under the value distribution F to be minus infinity.

2.5 Solution concept

We adopt the following solution concept:

A *strong maxmin solution* is a triple $(\mathcal{M}, F, \sigma)$ of a mechanism, a possible value distribution, and a strategy profile, with revenue $R = R(\mathcal{M}, F, \sigma)$, such that the following conditions are satisfied:

C1. For any possible value distribution F' and any equilibrium in undominated strategies σ' in $\Sigma(\mathcal{M}, F')$, $R \leq R(\mathcal{M}, F', \sigma')$.

C2. For any mechanism $\mathcal{M}'$ and any equilibrium in undominated strategies σ' in $\Sigma(\mathcal{M}', F)$, $R \geq R(\mathcal{M}', F, \sigma')$.

C3. Strategy profile σ is in $\Sigma(\mathcal{M}, F)$.

We refer to R as the revenue guarantee of the solution. Note that if $\Sigma(\mathcal{M}, F) = \emptyset$, then the condition C1 holds trivially; similarly, if $\Sigma(\mathcal{M}', F) = \emptyset$, then the condition C2 holds trivially.

Our solution concept is similar to but slightly weaker than one in Brooks and Du (2021b): On the one hand, we both allow general participation-securing mechanisms. On the other hand, we restrict attention to equilibria in undominated strategies, whereas they do not place any restrictions on equilibria.

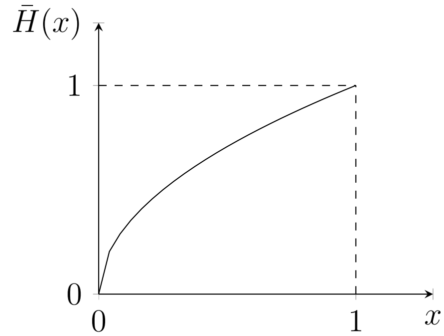

Figure 1: CDF of the Random Reserve Price in $\bar{\mathcal{M}}$

3 Main Result: Two bidders

In this section, we first formally define a mechanism $\bar{\mathcal{M}}$ (Section 3.1), a possible value distribution $\bar{F}$ (Section 3.2) and a strategy profile $\bar{\sigma}$ (Section 3.3), then we present the formal statement of the result (Section 3.4) that the proposed mechanism, the proposed value distribution together with the proposed strategy profile constitute a strong maxmin solution. Finally, we prove the formal statement (Section 3.5).

3.1 Mechanism $\bar{\mathcal{M}}$

The mechanism $\bar{\mathcal{M}}$ is a second-price auction with a random reserve price whose cumulative distribution function $\bar{H}$ is as follows:

$$\bar{H}(x) = \begin{cases} -\frac{x(1-a)(\ln x - \ln a)}{(x-a)\ln a} & \text{if } x \in (0,a) \cup (a,1], \\ -\frac{1-a}{\ln a} & \text{if } x = a, \\ 0 & \text{if } x = 0, \end{cases}$$

where $a \in (0,1)$ is the unique solution to

$$\tilde{a}(1 - \ln \tilde{a}) = \mu. \tag{2}$$

To see that $\bar{H}$ is a distribution on V, note first that $\bar{H}(x)$ is continuous, following from $\lim_{x\to a} \bar{H}(x) = \bar{H}(a)$ and $\lim_{x\to 0} \bar{H}(x) = \bar{H}(0) = 0$ using L'Hôpital's rule. Second, it can be shown that $\bar{H}(x)$ is strictly increasing (Lemma 1). Third, $\bar{H}(1) = 1$. In addition, it is straightforward to show that $\bar{H}(x)$ is differentiable at any $x \neq 0$ using L'Hôpital's rule, i.e., $\bar{H}$ is a continuous distribution (Lemma 1). See Figure 1 for an illustration.

Lemma 1. *$\bar{H}(x)$ is strictly increasing. In addition, $\bar{H}(x)$ is differentiable at any $x \neq 0$. Moreover,* $\lim_{x\to 0} x\bar{H}'(x) = 0$.

Proof. For any $x \neq 0$ or a, $\bar{H}'(x) = -\frac{(1-a)(x-a\ln x-\mu)}{(x-a)^2\ln a}$. Define $J(x) \equiv x - a\ln x$. Because $J'(x) = 1 - \frac{a}{x}$ and $J''(x) = \frac{a}{x^2} > 0$, $J(x)$ is minimized at $x = a$ and the minimized value is equal to $a - a\ln a = \mu$. Therefore for any $x \neq 0$ or a, $\bar{H}'(x) > 0$. In addition, $\lim_{x\to a}\bar{H}'(x) = -\frac{1-a}{2a\ln a} > 0$ using L'Hôpital's rule (twice). Moreover, $\lim_{x\to 0} x\bar{H}'(x) = -\frac{(1-a)(x^2-ax\ln x-x\mu)}{(x-a)^2\ln a} = 0$ using L'Hôpital's rule. $\square$

More formally, the mechanism $\bar{\mathcal{M}} = (\bar{M}, \bar{q}_i, \bar{t}_i)_{i\in\{1,2\}}$ is defined as follows. It is a direct mechanism, i.e., $\bar{M} = V^2$. With slight abuse of notations, we denote by (v_1, v_2) the reported message profile. If $v_1 > v_2$, then $\bar{q}_1(v_1,v_2) = \bar{H}(v_1)$, $\bar{q}_2(v_1,v_2) = 0$, $\bar{t}_1(v_1,v_2) = v_1\bar{H}(v_1) - \int_{v_2}^{v_1}\bar{H}(x)dx$, $\bar{t}_2(v_1,v_2) = 0$; if $v_1 < v_2$, then $\bar{q}_1(v_1,v_2) = 0$, $\bar{q}_2(v_1,v_2) = \bar{H}(v_2)$, $\bar{t}_1(v_1,v_2) = 0, \bar{t}_2(v_1,v_2) = v_2\bar{H}(v_2) - \int_{v_1}^{v_2}\bar{H}(x)dx$; if $v_1 = v_2 = x$, then $\bar{q}_1(v_1,v_2) = \bar{q}_2(v_1,v_2) = \frac{\bar{H}(x)}{2}$, $\bar{t}_1(v_1,v_2) = \bar{t}_2(v_1,v_2) = \frac{x\bar{H}(x)}{2}$.

3.2 Value distribution $\bar{F}$

$\bar{F}$ is an equal-revenue distribution as follows:

$$\bar{F}(x) = \begin{cases} 1 - \frac{a}{x} & \text{if } a \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Note that $\bar{F}$ is a possible value distribution because a is a solution to Equation (2). Define $\bar{R} \equiv 2a - a^2$. As will be shown (Section 3.5.2), $\bar{R}$ is an upper bound of the expected revenue for any participation-securing mechanism and any equilibrium under the value distribution $\bar{F}$.

3.3 Strategy profile $\bar{\sigma}$

Finally, let $\bar{\sigma}$ be the truth-telling strategy profile in the mechanism $\bar{\mathcal{M}}$ under the value distribution $\bar{F}$: for all i and v_i, $\bar{\sigma}_i(v_i)$ puts probability one on v_i. This completes the construction of the solution.

3.4 Formal statement: Theorem 1

Theorem 1. *$(\bar{\mathcal{M}}, \bar{F}, \bar{\sigma})$ is a strong maxmin solution with a revenue guarantee of $\bar{R}$.*

3.5 Proof of Theorem 1

3.5.1 Lower Bound on Revenue for $\bar{\mathcal{M}}$

We first establish C1 in the definition of a strong maxmin solution.

Proposition 1. *For any possible value distribution F and any equilibrium in undominated strategies σ in $\Sigma(\bar{\mathcal{M}}, F)$, $R(\bar{\mathcal{M}}, F, \sigma) \geq \bar{R}$.*

Proof. Given the mechanism $\bar{\mathcal{M}}$, truth-telling is the unique equilibrium in undominated strategies under any value distribution. Therefore, we focus on the truth-telling equilibrium and show that $\bar{F}$ minimizes the expected revenue across possible value distributions. Given a value profile (v_1, v_2), let $v(1)$ be $\max\{v_1, v_2\}$ and $v(2)$ be $\min\{v_1, v_2\}$. Then in the truth-telling equilibrium, the expected revenue given the mechanism $\bar{\mathcal{M}}$ and an arbitrary value distribution F can be expressed as follows:

$$\begin{aligned} E[t_1(v_1,v_2)+t_2(v_1,v_2)] &= \int_0^1\int_0^1 [v(1)\bar{H}(v(1)) - \int_{v(2)}^{v(1)} \bar{H}(x)dx]dF(v_1)dF(v_2) \\ &= \int_0^1\int_0^1 v(1)\bar{H}(v(1))dF(v_1)dF(v_2) - \int_0^1\int_0^1\int_0^1 \bar{H}(x)\mathbb{1}_{v(2)\leq x\leq v(1)}dxdF(v_1)dF(v_2) \\ &= \int_0^1 \{\underbrace{(1-F^2(x))[x\bar{H}'(x)+\bar{H}(x)]}_{\text{the first term}} - \underbrace{\bar{H}(x)[2F(x)(1-F(x))]}_{\text{the second term}}\}dx, \end{aligned}$$

where the first term of the last line is obtained using integration by parts[2] and the fact that $v(1)$'s cumulative distribution function is F^2, and the second term of the last line is obtained using Fubini's theorem and the fact that $v(2)$'s cumulative distribution function is $F^2 + 2F(1-F)$. Note that $\int_0^1 (1-F(x))dx = \mu$ holds for any possible value distribution, using integration by parts. Then to show that $\bar{F}$ minimizes

$$\mathcal{L}(F, \bar{H}) \equiv \int_0^1 \left\{(1-F^2(x))[x\bar{H}'(x)+\bar{H}(x)] - \bar{H}(x)[2F(x)(1-F(x))]\right\} dx,$$

it suffices to show that there exists a real number λ such that $\bar{F}$ minimizes

$$\mathcal{L}(F, \bar{H}, \lambda) \equiv \int_0^1 \left\{(1-F^2(x))[x\bar{H}'(x)+\bar{H}(x)] - \bar{H}(x)[2F(x)(1-F(x))] - \lambda(1-F(x))\right\} dx.$$

This is because $\mathcal{L}(\bar{F}, \bar{H}, \lambda) \leq \mathcal{L}(F, \bar{H}, \lambda)$ implies $\mathcal{L}(\bar{F}, \bar{H}) \leq \mathcal{L}(F, \bar{H})$ for any possible value distribution F by adding $\lambda\mu$ to both sides. Now take $\lambda = -\frac{2(1-a)}{\ln a}$. We are going to pointwise minimize the integrand of $\mathcal{L}(F, \bar{H}, \lambda)$ for any $x \neq 0, a$ or 1. First, with slight rewriting,

[2] $x\bar{H}(x)$ is differentiable everywhere by Lemma 1.

the integrand of $\mathcal{L}(F, \bar{H}, \lambda)$ becomes

$$\mathcal{I}(F, \bar{H}, \lambda) \equiv [\bar{H}(x) - x\bar{H}'(x)]F^2(x) - 2\left[\bar{H}(x) + \frac{1-a}{\ln a}\right] F(x) + \bar{H}(x) + x\bar{H}'(x) + \frac{2(1-a)}{\ln a}. \quad (3)$$

This is a simple quadratic function of $F(x)$. By simple calculation, $\bar{H}(x) - x\bar{H}'(x) = \frac{x\left[\bar{H}(x)+\frac{1-a}{\ln a}\right]}{x-a} > 0$ for any $x \neq 0$ or a, following from $\bar{H}(x)$ being strictly increasing and $\bar{H}(a) = -\frac{1-a}{\ln a}$. Then, for any $a < x < 1$, $F(x) = 1 - \frac{a}{x}$ is the unique minimizer of (3). For any $0 < x < a$, $F(x) = 0$ is the unique minimizer of (3) as $1 - \frac{a}{x} < 0$ if $0 < x < a$. This implies that the value distribution $\bar{F}$ minimizes $\mathcal{L}(F, \bar{H}, \lambda)$ and therefore minimizes the expected revenue across possible value distributions. By simple calculation, the minimized expected revenue is equal to $2a - a^2$. The details about the construction of the distribution $\bar{H}$ as well as the Lagrangian multiplier λ are provided below. □

Construction of $\bar{H}$ and λ. Consider a second-price auction with a random reserve price whose cumulative distribution function is H. Assuming that $H(x)$ is differentiable at any $x \in (0, 1]$ and $\lim_{x\to 0} xH(x)$ exists, then the expected revenue under a value distribution F can be expressed as $\mathcal{L}(F, H)$. We subtract $\lambda\mu$ from $\mathcal{L}(F, H)$ where λ is some real number, and obtain $\mathcal{L}(F, H, \lambda)$. Then a sufficient condition for $\bar{F}$ to be a minimizer of $\mathcal{L}(F, H)$ is that $\bar{F}(x)$ point-wise minimizes the integrand of $\mathcal{L}(F, H, \lambda)$. The first order condition with respect to $F(x)$ is as follows:

$$-2F(x)[xH'(x) + H(x)] - 2H(x)(1 - 2F(x)) + \lambda = 0. \quad (4)$$

Plugging $F(x) = 1 - \frac{a}{x}$ to (4),

$$(x - a)H'(x) + \frac{a}{x} \cdot H(x) = \frac{\lambda}{2}. \quad (5)$$

Solving this differential equation: for any $x \neq 0$ or a,

$$H(x) = \frac{x(\frac{\lambda}{2}\ln x + c)}{x - a}, \quad (6)$$

where c is some constant. Using $H(1) = 1$, we obtain that $c = 1 - a$.

By the standard envelope theorem, the Lagrangian multiplier should be the marginal contribution of the mean parameter μ to the expected revenue. Suppose the strong duality holds, then we have that

$$\lambda = \frac{\partial \bar{R}}{\partial \mu} = \frac{\partial}{\partial \mu}\left(2a - a^2\right) = 2(1-a) \cdot \frac{\mathrm{d}a}{\mathrm{d}\mu} = \frac{2\,(1-a)}{-\ln a}.$$

Remark 1. The assumption that bidders play undominated strategies is important for this result. Without this assumption, there indeed exist a possible value distribution and an equilibrium such that the expected revenue in the mechanism $\bar{\mathcal{M}}$ is lower than $\bar{R}$. Take the value distribution to be a point mass on μ, i.e., bidders observe no value at all, and consider the strategy profile where one bidder reports μ and the other bidder reports 0. It is straightforward to verify that this strategy profile is an equilibrium (in which one of the bidders uses a dominated strategy). The expected revenue in this equilibrium under this value distribution is $\mu \cdot \bar{H}(\mu) - \int_0^{\mu} \bar{H}(x)dx$, which is lower than $2a - a^2$ for any μ. For a parametric example, when $\mu = 0.5$, the former is about 0.1223, whereas the latter is about 0.3385.

3.5.2 Upper Bound on Revenue for $\bar{F}$

Next, we establish C2 in the definition of a strong maxmin solution.

Proposition 2. *For any participation-securing mechanism $\mathcal{M}$ and any equilibrium in undominated strategies σ in $\Sigma(\mathcal{M}, \bar{F})$, $R(\mathcal{M}, \bar{F}, \sigma) \leq \bar{R}$.*

Proof. Indeed, we will establish a slightly stronger statement: $\bar{R}$ is an upper bound on the expected revenue for any participation-securing mechanism and any equilibrium.

First, to identify an upper bound on the expected given a value distribution, it is without loss to restrict attention to direct mechanisms, i.e., $M = V^2$, as the revelation principle holds. Next, for exposition, we parameterize each type v_i by its quantile z_i.[3] Formally, we define the inverse quantile function as follows:

$$\bar{v}(z_i) = \min\{\tilde{v}_i | \bar{F}(\tilde{v}_i) \geq z_i\} = \begin{cases} \frac{a}{1-z_i} & \text{if } 0 \leq z_i < 1-a, \\ 1 & \text{if } z_i \geq 1-a. \end{cases}$$

We denote the value profile by $z = (z_1, z_2) \in V^2$. Note that z_1 and z_2 follow independently and identically distributed uniform distributions on V. Then, for any direct mechanism $(q_i(z), t_i(z))_{i \in \{1,2\}}$ where $q_i(z) \in [0,1]$ is the allocation probability to bidder i given the parameterized value profile z and $t_i(z) \in \mathrm{R}$ is the payment made by bidder i given z, (BR)

[3] See Carroll (2017) and Zhang (2022a) for a similar method.

and (PS) together imply that for all i, all z_i, and all z_i',

$$U_i(z_i) \equiv \bar{v}(z_i)Q_i(z_i) - T_i(z_i) \geq \bar{v}(z_i)Q_i(z_i') - T_i(z_i'), \tag{BIC}$$

$$\bar{v}(z_i)Q_i(z_i) - T_i(z_i) \geq 0, \tag{BIR}$$

where $Q_i(z_i) = \int_{z_{-i}} q_i(z_i, z_{-i})dz_{-i}$ and $T_i(z_i) = \int_{z_{-i}} t_i(z_i, z_{-i})dz_{-i}$ are the expected allocation to type z_i of bidder i and the expected payment made by type z_i of bidder i, respectively.

For $z_i' \geq z_i$, (BIC) implies that

$$(\bar{v}(z_i') - \bar{v}(z_i))Q_i(z_i') \geq U_i(z_i') - U_i(z_i) \geq (\bar{v}(z_i') - \bar{v}(z_i))Q_i(z_i). \tag{7}$$

Then $U_i(z_i)$ is Lipschitz, thus absolutely continuous w.r.t. z_i, and so equal to the integral of its derivative. In addition, note that $\bar{v}(z_i)$ is differentiable for all z_i except for $z_i = 1 - a$. Then applying the envelope theorem to (7) at each point of differentiability, we obtain that

$$\frac{\partial U_i(z_i)}{\partial z_i} = \frac{\partial \bar{v}(z_i)}{\partial z_i} Q_i(z_i) = \begin{cases} \frac{a}{(1-z_i)^2} Q_i(z_i) & \text{if } 0 \leq z_i < 1 - a, \\ 0 & \text{if } z_i > 1 - a. \end{cases}$$

Thus,

$$U_i(z_i) = \begin{cases} U_i(0) + \int_0^{z_i} \left[\frac{a}{(1-\tilde{z}_i)^2} Q_i(\tilde{z}_i) \right] d\tilde{z}_i & \text{if } 0 \leq z_i < 1 - a, \\ U_i(0) + \int_0^{1-a} \left[\frac{a}{(1-\tilde{z}_i)^2} Q_i(\tilde{z}_i) \right] d\tilde{z}_i & \text{if } z_i \geq 1 - a. \end{cases}$$

Therefore, the expected revenue from bidder i satisfies

$$
\begin{aligned}
\int_0^1 T_i(z_i)dz_i &= \int_0^1 \left[\bar{v}(z_i)Q_i(z_i) - U_i(z_i)\right] dz_i \\
&= \int_0^{1-a} \left\{\bar{v}(z_i)Q_i(z_i) - U_i(0) - \int_0^{z_i} \left[\frac{a}{(1-\tilde{z}_i)^2}Q_i(\tilde{z}_i)\right] d\tilde{z}_i\right\} dz_i + \\
&\quad \int_{1-a}^1 \left\{\bar{v}(z_i)Q_i(z_i) - U_i(0) - \int_0^{1-a} \left[\frac{a}{(1-\tilde{z}_i)^2}Q_i(\tilde{z}_i)\right] d\tilde{z}_i\right\} dz_i \\
&\leq \int_0^{1-a} \left\{\bar{v}(z_i)Q_i(z_i) - \int_0^{z_i} \left[\frac{a}{(1-\tilde{z}_i)^2}Q_i(\tilde{z}_i)\right] d\tilde{z}_i\right\} dz_i + \\
&\quad \int_{1-a}^1 \left\{\bar{v}(z_i)Q_i(z_i) - \int_0^{1-a} \left[\frac{a}{(1-\tilde{z}_i)^2}Q_i(\tilde{z}_i)\right] d\tilde{z}_i\right\} dz_i \\
&= \int_0^{1-a} \left[\left(\bar{v}(z_i) - (1-a-z_i)\,\frac{a}{(1-z_i)^2}\right) Q_i(z_i)\right] dz_i + \\
&\quad \int_{1-a}^1 \left[\bar{v}(z_i)Q_i(z_i) - \int_0^{1-a} \left[\frac{a}{(1-\tilde{z}_i)^2}Q_i(\tilde{z}_i)\right] d\tilde{z}_i\right] dz_i \\
&= \int_0^{1-a} \left[\left(\bar{v}(z_i) - (1-z_i)\frac{a}{(1-z_i)^2}\right) Q_i(z_i)\right] dz_i + \int_{1-a}^1 \left[(\bar{v}(z_i)Q_i(z_i)\right] dz_i \\
&= \int_{1-a}^1 Q_i(z_i)dz_i,
\end{aligned}
$$

where the first inequality holds because (BIR) implies that $U_i(0) \geq 0$, the third equality is obtained via integration by parts, the last equality holds because $\bar{v}(z_i) - (1-z_i)\frac{a}{(1-z_i)^2} = 0$ for $0 \leq z_i < 1-a$ and $\bar{v}(z_i) = 1$ for $z_i > 1-a$.

Then, the expected revenue from all the bidders satisfies

$$
\begin{aligned}
\sum_{i=1}^2 \int_0^1 T_i(z_i)dz_i &\leq \int_0^{1-a}\int_{1-a}^1 q_1(z_1,z_2)dz_1dz_2 + \int_0^{1-a}\int_{1-a}^1 q_2(z_1,z_2)dz_2dz_1 \\
&\quad + \int_{1-a}^1\int_{1-a}^1 [q_1(z_1,z_2) + q_2(z_1,z_2)]dz_1dz_2 \\
&\leq 2a(1-a) + a^2 = 2a - a^2,
\end{aligned}
$$

where the inequality holds because $q_i(z) \leq 1$ and $q_1(z) + q_2(z) \leq 1$ for all i and z. This finishes the proof. □

This argument is standard (Myerson, 1981). The parameterization makes the proof clean.

Finally, recall that the truth-telling strategy profile $\bar{\sigma}$ is an equilibrium in undominated strategies in the mechanism $\bar{\mathcal{M}}$ under the value distribution $\bar{F}$, as the mechanism $\bar{\mathcal{M}}$ is a dominant-strategy mechanism. Then, Theorem 1 follows immediately from Proposition 1 and 2.

4 n bidders

In this section, we partially extend the main result to $n-$bidder case where $n \geq 3$. Specifically, we restrict attention to a subclass of dominant-strategy mechanisms. The revelation principle holds, and we can restrict attention to direct mechanisms. A direct mechanism $\mathcal{M} = (M, q_i, t_i)_{i \in I}$ is a dominant-strategy mechanism if $M = V^n$ and for all $i \in I$, all $v \in V^n$, and all $v_i' \in V$,

$$U_i(v_i, v_{-i}) := v_i q_i(v) - t_i(v) \geq v_i q_i(v_i', v_{-i}) - t_i(v_i', v_{-i}), \tag{DSIC}$$

$$v_i q_i(v) - t_i(v) \geq 0. \tag{EPIR}$$

We say a dominant-strategy mechanism is a *highest-bidder lottery* if the following conditions hold:

L1. Bidders who do not place the highest bid never win the good;

L2. When there are multiple bidders with the highest bid, select one randomly. Only the selected bidder is possible to win the good;

L3. The selected bidder wins the good if and only if his bid is higher than the outcome drawn from a lottery that can depend on the bidder's identity and the other bidders' bids.

Indeed, the three conditions impose a restriction on the allocation rule: if the probability that bidder i wins the good is α when all bidders' bids are x's, then the probability that bidder i wins the good is at least $n\alpha$ if bidder i bids higher than x while the others' bids remain x's. This is because bidder i becomes the unique highest bidder by bidding higher than x and therefore the probability of being selected is n times as large as that before. We denote by $\mathcal{L}$ the set of all highest-bidder lotteries. It includes any second-price auction with or without a (random or deterministic) reserve price as a special case[4].

We are interested in the expected revenue in the dominant-strategy equilibrium in which each bidder truthfully reports his value. Let $R(\mathcal{M}, F) = \int_{V^n} \sum_{i \in I} t_i(v) dF(v_1) \cdots dF(v_n)$. That is, we denote by $R(\mathcal{M}, F)$ the expected revenue under the mechanism $\mathcal{M}$ at the value distribution F.

We adopt a weaker solution concept:

A *maxmin solution* is a pair $(\mathcal{M}, F)$ of a mechanism and a possible value distribution, with revenue $R = R(\mathcal{M}, F)$, such that the following conditions are satisfied:

[4] It also includes any second-price auction with bidder-specific reserve prices that can depend on the other bidder's bids.

C1'. For any possible value distribution F', $R \geq R(\mathcal{M}, F')$.

C2'. For any mechanism $\mathcal{M}' \in \mathcal{L}$, $R \geq R(\mathcal{M}', F)$.

We refer to R as the revenue guarantee of the solution.

In the remainder of this section, we first formally define a mechanism (Section 4.1) and a possible value distribution, then we present the formal statement of the result (Section 4.3) that the proposed mechanism and the proposed value distribution constitute a maxmin solution. Finally, We prove the formal statement (Section 4.4).

4.1 Mechanism $\bar{\bar{\mathcal{M}}}$ and $\bar{\bar{\bar{\mathcal{M}}}}$

The mechanism $\bar{\bar{\mathcal{M}}}$ is a second-price auction with a random reserve price whose cumulative distribution function $\bar{\bar{H}}$ is as follows:

$$\bar{\bar{H}}(x) = \begin{cases} \left(\frac{x_1}{x_1-a}\right)^{n-1} \cdot \left((1-a)^{n-1} + \frac{\lambda \ln x_1}{n}\right) & \text{if } x \in [0, x_1), \\ \left(\frac{x}{x-a}\right)^{n-1} \cdot \left((1-a)^{n-1} + \frac{\lambda \ln x}{n}\right) & \text{if } x \in [x_1, 1], \end{cases}$$

where a, x_1 and λ satisfy the following equations:

$$(n-1)a + a\left(1 - \frac{1}{n-1} - \ln\left((n-1)^2 a\right)\right) = \mu, \tag{8}$$

$$x_1 = (n-1)^2 a, \tag{9}$$

$$\lambda = \frac{n(1-a)^{n-1}}{(n-1) - \frac{1}{n-1} - \ln x_1}. \tag{10}$$

A solution $x \in (0,1)$ exists if and only if

$$\mu < \frac{n(n-1)-1}{(n-1)^3}. \tag{LM}$$

Lemma 2. *$\bar{\bar{H}}(x)$ is a strictly increasing for $x \in (x_1, 1)$.*

Proof. The proof is in Appendix A.1. □

The mechanism $\bar{\bar{\bar{\mathcal{M}}}}$ is a second-price auction without a reserve price.

4.2 Value distribution $\bar{\bar{F}}$ and $\bar{\bar{\bar{F}}}$

$\bar{\bar{F}}$ is a combination of an atom and an equal-revenue distribution as follows:

$$\bar{\bar{F}}(x) = \begin{cases} 0 & \text{if } x \in [0, x_0), \\ 1 - \frac{a}{x_1} & \text{if } x \in [x_0, x_1), \\ 1 - \frac{a}{x} & \text{if } x \in [x_1, 1), \\ 1 & \text{if } x = 1, \end{cases}$$

where

$$x_1 = (n-1)x_0. \tag{11}$$

Note that the value distribution $\bar{\bar{F}}$ is a possible value distribution if (LM) holds, implied by Equations (8) and (11). Define $\bar{\bar{R}} \equiv 1 - (1-a)^n$. As will be shown (Section 4.4.2), $\bar{\bar{R}}$ is an upper bound of the expected revenue for any highest-bidder lotteries under the value distribution $\bar{\bar{F}}$ if (LM) holds.

$\bar{\bar{\bar{F}}}$ is a two-point discrete distribution as follows:

$$\bar{\bar{\bar{F}}}(x) = \begin{cases} 0 & \text{if } x \in [0, x_0), \\ 1 - \frac{1}{(n-1)^2} & \text{if } x \in [x_0, 1), \\ 1 & \text{if } x = 1, \end{cases}$$

where

$$\left(1 - \frac{1}{(n-1)^2}\right) x_0 + \frac{1}{(n-1)^2} = \mu. \tag{8'}$$

Define $\bar{\bar{\bar{R}}} \equiv 1 - (1-x_0) \cdot \left[\left(1 - \frac{1}{(n-1)^2}\right)^n - \frac{n}{(n-1)^2}\left(1 - \frac{1}{(n-1)^2}\right)^{n-1}\right]$. As will be shown (Section 4.4.2), $\bar{\bar{\bar{R}}}$ is an upper bound of the expected revenue for any highest-bidder lotteries under the value distribution $\bar{\bar{\bar{F}}}$ if (LM) does not hold.

4.3 Formal statement: Theorem 2

Theorem 2. *$(\bar{\bar{\mathcal{M}}}, \bar{\bar{F}})$ (resp, $(\bar{\bar{\bar{\mathcal{M}}}}, \bar{\bar{\bar{F}}})$) is a maxmin solution with a revenue guarantee of $\bar{\bar{R}}$ (resp, $\bar{\bar{\bar{R}}}$) if (LM) holds (resp, fails).*

4.4 Proof of Theorem 2

4.4.1 Lower Bound on Revenue for $\bar{\bar{\mathcal{M}}}$ and $\bar{\bar{\bar{\mathcal{M}}}}$

We first establish C1' in the definition of a maxmin solution.

Proposition 3. *For any possible value distribution F, $R(\bar{\bar{M}}, F) \geq \bar{\bar{R}}$ (resp, $R(\bar{\bar{\bar{M}}}, F) \geq \bar{\bar{\bar{R}}}$) if* (LM) *holds (resp, fails) .*

Proof. The statement when (LM) fails has been established by Suzdaltsev (2020), so we focus on the case when (LM) holds. Given a value profile $(v_1, v_2, \cdots, v_n)$, let $v(1)$ be the highest value and $v(2)$ be the second highest value. Then, similar to the derivation for the two-bidder case, the expected revenue given the mechanism $\bar{\bar{M}}$ and an arbitrary value distribution F for the $n-$bidder case can be expressed as follows:

$$
\begin{aligned}
E[\sum_{i \in I} t_i(v)] &= \int_{V^n} \left[v(1)\bar{\bar{H}}(v(1)) - \int_{v(2)}^{v(1)} \bar{\bar{H}}(x)dx \right] dF(v_1) \cdots dF(v_n) \\
&= \int_{V^n} v(1)\bar{\bar{H}}(v(1))dF(v_1) \cdots dF(v_n) - \int_{V^n} \int_0^1 \bar{\bar{H}}(x)\mathbb{1}_{v(2) \leq x \leq v(1)} dx dF(v_1) \cdots dF(v_n) \\
&= \int_0^1 \left\{ (1 - F^n(x)) \left[x\bar{\bar{H}}'(x) + \bar{\bar{H}}(x) \right] - \bar{\bar{H}}(x) \left[nG^{n-1}(x)(1 - F(x)) \right] \right\} dx.
\end{aligned}
$$

To show that $\bar{\bar{F}}$ minimizes

$$
\mathcal{L}(F, \bar{\bar{H}}) \equiv \int_0^1 \left\{ (1 - F^n(x)) \left[x\bar{\bar{H}}'(x) + \bar{\bar{H}}(x) \right] - \bar{\bar{H}}(x) \left[nF^{n-1}(x)(1 - F(x)) \right] \right\} dx,
$$

it suffices to show that there exists a real number λ such that $\bar{\bar{F}}$ minimizes

$$
\mathcal{L}(F, \bar{\bar{H}}, \lambda) \equiv \int_0^1 \left\{ (1 - F^n(x)) \left[x\bar{\bar{H}}'(x) + \bar{\bar{H}}(x) \right] - \bar{\bar{H}}(x) \left[nF^{n-1}(x)(1 - F(x)) \right] - \lambda(1 - F(x)) \right\} dx.
$$

Now take $\lambda = \frac{n(1-a)^{n-1}}{(n-1) - \frac{1}{n-1} - \ln x_1}$. We are going to point-wise minimize the integrand of $\mathcal{L}(F, \bar{\bar{H}}, \lambda)$ for any $x \neq 1$, denoted by $\mathcal{I}(F, \bar{\bar{H}}, \lambda)$. The first order derivative with respect to F is

$$
\mathcal{I}_F(F, \bar{\bar{H}}, \lambda) = -nxF^{n-1}(x)\bar{\bar{H}}'(x) + n(n-1)\bar{\bar{H}}(x) \left(F^{n-1}(x) - F^{n-2}(x) \right) + \lambda.
$$

When $x_1 \leq x < 1$, it is straightforward to show that $F(x) = 1 - \frac{a}{x}$ is a solution to $\mathcal{I}_F(F, \bar{\bar{H}}, \lambda) = 0$.

Lemma 3. *For $x_1 \leq x < 1$, $F(x) = 1 - \frac{a}{x}$ minimizes $\mathcal{I}(F, \bar{\bar{H}}, \lambda)$.*

Proof. The proof is in Appendix A.2. ☐

When $x < x_1$, note that $\bar{\bar{H}}'(x) = 0$, then it is straightforward to show that $F(x) = 1 - \frac{1}{(n-1)^2}$ is a solution to $\mathcal{I}_F(F, \bar{\bar{H}}, \lambda) = 0$. In addition, it is straightforward to show that

$\mathcal{I}(0, \bar{\bar{H}}, \lambda) = \mathcal{I}\left(1 - \frac{1}{(n-1)^2}, \bar{\bar{H}}, \lambda\right)$. Define $y(x) \equiv x^{n-1} - x^{n-2}$. Then $y(0) = y(1) = 0$, and $y(x)$ is decreasing when $x \in \left(0, 1 - \frac{1}{n-1}\right)$ and increasing when $x \in \left(1 - \frac{1}{n-1}, 1\right)$. As $1 > 1 - \frac{1}{(n-1)^2} > 1 - \frac{1}{n-1}$, there exists $\tilde{x} \in \left(0, 1 - \frac{1}{n-1}\right)$ such that $y(\tilde{x}) = y\left(1 - \frac{1}{(n-1)^2}\right)$, so $\mathcal{I}_F(F, \bar{\bar{H}}, \lambda)$ is increasing in $(0, \tilde{x})$, decreasing in $\left(\tilde{x}, 1 - \frac{1}{(n-1)^2}\right)$, and increasing in $\left(1 - \frac{1}{(n-1)^2}, 1\right)$. Thus, $F(x) = 0$ and $F(x) = 1 - \frac{1}{(n-1)^2}$ are both minimizers when $x < x_1$. Therefore, the value distribution $\bar{\bar{F}}$ minimizes $\mathcal{I}_F(F, \bar{\bar{H}}, \lambda)$ and therefore minimizes the expected revenue across possible value distributions. By simple calculation, the minimized expected revenue is equal to $1-(1-a)^n$. The details about the construction of the distribution $\bar{\bar{H}}$ as well as the Lagrangian multiplier λ are provided below. $\square$

Construction of $\bar{\bar{H}}$ and λ. Consider a second-price auction with a random reserve price whose cumulative distribution function is H. A sufficient condition for $\bar{\bar{F}}$ to be a minimizer of $\mathcal{L}(F, H)$ is that $\bar{\bar{F}}(x)$ point-wise minimizes the integrand of $\mathcal{L}(F, H, \lambda)$. The first order condition with respect to $F(x)$ is as follows:

$$-nxF^{n-1}(x)H'(x) + n(n-1)H(x)\left(F^{n-1}(x) - F^{n-2}(x)\right) + \lambda = 0. \tag{12}$$

Plugging $F(x) = 1 - \frac{a}{x}$ to (12),

$$x\left(1 - \frac{a}{x}\right)^{n-1} H'(x) + (n-1)\bar{H}(x)\left(1 - \frac{a}{x}\right)^{n-1} \cdot \frac{a}{x} = \frac{\lambda}{n}. \tag{13}$$

Solving this differential equation: for any $x \geq x_1$,

$$H(x) = \left(\frac{x}{x-a}\right)^{n-1}\left(c + \frac{\lambda \ln x}{n}\right),$$

where c is some constant. Using $H(1) = 1$, we obtain that $c = (1-a)^n$.

Similar to the case that $n = 2$, by the standard envelope theorem, the Lagrangian multiplier should be the marginal contribution of the mean parameter μ to the expected revenue. Suppose the strong duality holds, then we have that

$$\lambda = \frac{\partial \bar{\bar{R}}}{\partial \mu} = \frac{\partial}{\partial \mu}\left(1 - (1-a)^n\right) = n(1-a)^{n-1} \cdot \frac{\mathrm{d}a}{\mathrm{d}\mu} = \frac{n\left(1-a\right)^{n-1}}{(n-1) - \frac{1}{n-1} - \ln x_1}.$$

Finally, we conjecture that H is a constant when $x < x_1$, i.e., $H'(x) = 0$ if $x \leq x_1$.

Plugging in $F(x) = 1 - \frac{a}{x_1}$ to (12), we obtain that for $x < x_1$,

$$H(x) = \left(\frac{x_1}{x_1 - a}\right)^{n-1} \cdot \left((1-a)^{n-1} + \frac{\lambda \ln x}{n}\right).$$

4.4.2 Upper Bound on Revenue for $\bar{\bar{F}}$ and $\bar{\bar{\bar{F}}}$

Next, we establish C2' in the definition of a maxmin solution.

Proposition 4. *For any highest-bidder lottery $\mathcal{M}$, $R(\mathcal{M}, \bar{\bar{F}}) \leq \bar{\bar{R}}$ (resp, $R(\mathcal{M}, \bar{\bar{\bar{F}}}) \leq \bar{\bar{\bar{R}}}$) if* (LM) *holds (resp, fails).*

Proof. First, by standard argument, dominant-strategy incentive compatibility (DSIC) implies that the allocation rule is monotone, i.e., $q_i(v_i, v_{-i})$ is non-decreasing in v_i for all $i \in I$ and all $v_{-i} \in V^{n-1}$; by the envelope theorem, $t_i(v_i, v_{-i}) = v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(v_i, v_{-i}) - U_i(0, v_{-i}) \leq v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(v_i, v_{-i})$ where the inequality holds by (EPIR). Without loss of generality, we evaluate the expected revenue from a generic bidder i. By L1, the expected revenue from bidder i is zero if his value is not the highest. Therefore, we can restrict attention to value profiles in which bidder i's value is the highest. For a given v_{-i}, we use $\tilde{q}_i(v_i)$ (resp, $\tilde{t}_i(v_i)$) to represent $q_i(v_i, v_{-i})$ (resp, $t_i(v_i, v_{-i})$). If (LM) holds, we discuss three cases.

Case 1: $v(2) = x_0$. The conditional expected revenue from bidder i

$$\begin{aligned}
E[\tilde{t}_i(v_i)] &\leq \left(1 - \frac{a}{x_1}\right) x_0 \tilde{q}_i(x_0) + \int_{x_1}^{1^-} \left[x\tilde{q}_i(x) - \int_{x_0}^{x} \tilde{q}_i(t)dt\right] d\bar{\bar{F}}(x) + a\left(1 \cdot \tilde{q}_i(1) - \int_{x_0}^{1} \tilde{q}_i(x)dx\right) \\
&= \left(1 - \frac{a}{x_1}\right) x_0 \tilde{q}_i(x_0) - \frac{a}{x_1} \int_{x_0}^{x_1} \tilde{q}_i(x)dx + \int_{x_1}^{1^-} \left[x\tilde{q}_i(x) - \int_{x_1}^{x} \tilde{q}_i(t)dt\right] d\bar{\bar{F}}(x) + \\
&\quad a\left(\tilde{q}_i(1) - \int_{x_1}^{1} \tilde{q}_i(x)dx\right) \\
&= \left(1 - \frac{a}{x_1}\right) x_0 \tilde{q}_i(x_0) - \frac{a}{x_1} \int_{x_0}^{x_1} \tilde{q}_i(x)dx + \int_{x_1}^{1^-} \tilde{q}_i(x)[x\bar{\bar{F}}'(x) - (\bar{\bar{F}}(1^-) - \bar{\bar{F}}(x)]dx \\
&\quad + a\left(\tilde{q}_i(1) - \int_{x_1}^{1} \tilde{q}_i(x)dx\right) \\
&= \left(1 - \frac{a}{x_1}\right) x_0 \tilde{q}_i(x_0) - \frac{a}{x_1} \int_{x_0}^{x_1} \tilde{q}_i(x)dx + \int_{x_1}^{1^-} \tilde{q}_i(x)[x\bar{\bar{F}}'(x) - (\bar{\bar{F}}(1) - \bar{\bar{F}}(x))]dx + a\tilde{q}_i(1) \\
&\leq \int_{x_1}^{1^-} \tilde{q}_i(x)\left[x\bar{\bar{F}}'(x) - (\bar{\bar{F}}(1) - \bar{\bar{F}}(x))\right]dx + a\tilde{q}_i(1) \\
&\leq a,
\end{aligned}$$

where the second equality uses integration by parts, the second inequality holds because $\tilde{q}_i(x) \geq n\tilde{q}_i(x_0)$ when $x > x_0$ and $\left(1 - \frac{a}{x_1}\right) x_0 - \frac{a}{x_1} \cdot n \cdot (x_1 - x_0) = 0$ by Equations (9) and (11), and the last inequality holds because $x\bar{\bar{F}}'(x) - (\bar{\bar{F}}(1) - \bar{\bar{F}}(x)) = 0$ when $x_1 < x < 1$ and $\tilde{q}_i(1) \leq 1$.

Case 2: $v(2) \in [x_1, 1)$. The conditional expected revenue from bidder i

$$\begin{aligned}
E[\tilde{t}_i(v)] &\leq \int_{v(2)}^{1^-} \left[x\tilde{q}_i(x) - \int_{v(2)}^{x} \tilde{q}_i(t)dt\right] d\bar{\bar{F}}(x) + a\left(1 \cdot \tilde{q}_i(1) - \int_{v(2)}^{1} \tilde{q}_i(x)dx\right) \\
&= \int_{v(2)}^{1^-} \tilde{q}_i(x) \left[x\bar{\bar{F}}'(x)) - (\bar{\bar{F}}(1) - \bar{\bar{F}}(x))\right] dx + a\tilde{q}_i(1) \\
&\leq a,
\end{aligned}$$

where the equality uses integration by parts, and the inequality holds because $x\bar{\bar{F}}'(x) - (\bar{\bar{F}}(1) - \bar{\bar{F}}(x)) = 0$ when $x_1 < x < 1$ and $\tilde{q}_i(1) \leq 1$.

Case 3: $v(2) = 1$. Bidder i is possible to pay only if his value is 1, and pays at most 1 if he wins the good. The same holds for other highest bidders. Therefore, the total expected revenue in this case is weakly lower than 1.

Now the total expected revenue

$$E[\sum_{i \in I} t_i(v)] \leq n \cdot a \cdot (1-a)^{n-1} + 1 \cdot \left[1 - (1-a)^n - n(1-a)^{n-1}a\right] = 1 - (1-a)^n,$$

where the inequality holds because the probability that Case 1 or Case 2 happens is $(1-a)^{n-1}$, and the probability that at least two bidders have the value of 1 is $1-(1-a)^n - n(1-a)^{n-1}a$. The details about the construction of the value distribution $\bar{\bar{F}}$ is provided below the proof.

If (LM) fails, we discuss two cases.

Case 1': $v(2) = x_0$. The conditional expected revenue from bidder i

$$\begin{aligned}
E[\tilde{t}_i(v_i)] &\leq \left(1 - \frac{1}{(n-1)^2}\right) x_0\tilde{q}_i(x_0) + \frac{1}{(n-1)^2}\left(1 \cdot \tilde{q}_i(1) - \int_{x_0}^{1} \tilde{q}_i(x)dx\right) \\
&= \left(1 - \frac{1}{(n-1)^2}\right) x_0\tilde{q}_i(x_0) - \frac{1}{(n-1)^2}\int_{x_0}^{1} \tilde{q}_i(x)dx + \frac{1}{(n-1)^2}\tilde{q}_i(1) \\
&\leq \left(1 - \frac{1}{(n-1)^2}\right) \cdot \frac{x_0}{n} - \frac{1-x_0}{(n-1)^2} + \frac{1}{(n-1)^2}\tilde{q}_i(1) \\
&\leq \left(1 - \frac{1}{(n-1)^2}\right) \cdot \frac{x_0}{n} + \frac{x_0}{(n-1)^2},
\end{aligned}$$

where the second inequality holds because $\tilde{q}_i(x) \geq n\tilde{q}_i(x_0)$ when $x > x_0$ and $\left(1 - \frac{1}{(n-1)^2}\right) x_0 -$

$\frac{1}{(n-1)^2} \cdot n \cdot (1-x_0) \geq 0$, implied by Equation (8') and that (LM) fails, and the last inequality holds because $\tilde{q}_i(1) \leq 1$.

Case 3': $v(2) = 1$. This is exactly the same as Case 3.

Now the total expected revenue

$$\begin{aligned} E[\sum_{i \in I} t_i(v)] &\leq n \cdot \left(1 - \frac{1}{(n-1)^2}\right)^{n-1} \left[\left(1 - \frac{1}{(n-1)^2}\right) \cdot \frac{x_0}{n} + \frac{x_0}{(n-1)^2}\right] \\ &\quad + 1 \cdot \left[\left(1 - \frac{1}{(n-1)^2}\right)^n - \frac{n}{(n-1)^2}\left(1 - \frac{1}{(n-1)^2}\right)^{n-1}\right] \\ &= 1 - (1-x_0)\left[\left(1 - \frac{1}{(n-1)^2}\right)^n - \frac{n}{(n-1)^2}\left(1 - \frac{1}{(n-1)^2}\right)^{n-1}\right], \end{aligned}$$

where the inequality holds because the probability that Case 1' happens is $\left(1 - \frac{1}{(n-1)^2}\right)^{n-1}$, and the probability that at least two bidders have the value of 1 is $\left(1 - \frac{1}{(n-1)^2}\right)^n - \frac{n}{(n-1)^2}\left(1 - \frac{1}{(n-1)^2}\right)^{n-1}$. □

Construction of $\bar{\bar{F}}$. If the second highest bidder's value is not x_0, then the highest bidder's conditional virtual value is 0 for any value except for 1, and therefore any highest-bidder lottery is optimal. If the second highest bidder's value is x_0, in order for any highest-bidder lottery to be optimal, we require that

$$\left(1 - \frac{a}{x_1}\right) x_0 - \frac{a}{x_1}(x_1 - x_0) \cdot n = 0.$$

By Suzdaltsev (2020), both 0 and $1 - \frac{1}{(n-1)^2}$ are minimizers of the integrand for some deterministic reserve price, we thus conjecture that

$$1 - \frac{a}{x_1} = 1 - \frac{1}{(n-1)^2}.$$

It is straightforward to show that the above two equations are equivalent to (9) and (11).

Theorem 2 follows immediately from Proposition 3 and Proposition 4.

Remark 2 (Not a strong maxmin solution)**.** For $n \geq 3$, a second-price auction with a random reserve price is not a maxmin mechanism across all participation-securing mechanisms. We will use an example below to show that C2 fails.

Let $n = 3$ and $\mu = \frac{2}{3}$. By Theorem 2, the second-price auction with a reserve price of zero is a maxmin mechanism among highest-bidder lotteries, and the minmax value distribution is a two-point discrete distribution as follows:

$$\overline{\overline{\overline{F}}}(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{5}{9}), \\ \frac{3}{4} & \text{if } x \in [\frac{5}{9}, 1). \\ 1 & \text{if } x = 1. \end{cases}$$

The expected revenue using a second-price auction is $\left(\frac{3}{4}\right)^3 \cdot \frac{5}{9} + \left(1 - \left(\frac{3}{4}\right)^3 - 3 \cdot \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}\right) \cdot 1 = \frac{5}{8}$. However, the seller can do better against $\overline{\overline{\overline{F}}}(x)$ by using the following mechanism.

A bidder never wins or pays, if his bid is less than $\frac{5}{9}$; a bidder wins with a probability of $\frac{1}{3}$ and pays $\frac{5}{9}$ conditional on winning, if his bid is in $[\frac{5}{9}, 1)$; a bidder wins with probability one and pays $\frac{23}{27}$, if his bid is 1 and the other two bidders' bids are both less than 1; a bidder wins with a probability of $\frac{1}{3}$ (resp, $\frac{1}{2}$) and pays 1, if the other two bidders' bids are both 1 (resp, only one of the other bidder's bids is 1).

Truth-telling is a dominant strategy (and thus an undominated strategy) under this mechanism. To see this, consider the case in which one bidder (assume bidder 1) has a value of 1, and the other two have values of $\frac{5}{9}$. The ex-post payoff for bidder 1 from truth-telling in this case is $1 - \frac{23}{27} = \frac{4}{27}$. By reporting less than 1 (weakly higher than $\frac{5}{9}$), the bidder would win with probability of $\frac{1}{3}$, and would pay $\frac{5}{9}$ conditional on winning. This results in an ex-post payoff of $\frac{1}{3} \cdot (1 - \frac{5}{9}) = \frac{4}{27}$ as well. It is straightforward to show that truth-telling is a best strategy for the other cases. Therefore, the mechanism is strategy-proof. The expected revenue in the truth-telling equilibrium is $\frac{5}{9} \cdot \left(\frac{3}{4}\right)^3 + \frac{23}{27} \cdot 3 \cdot \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + 1 \cdot \left(\frac{1}{4^3} + 3 \cdot \frac{1}{4^2} + 3 \cdot \frac{1}{4^2} \cdot \frac{3}{4}\right) = \frac{3}{4} > \frac{5}{8}$.

5 A Reversed Bulow-Klemperer Result

The maxmin mechanism relies on knowledge of the expectation of a generic bidder's value, while the Vickrey auction does not. What is the relationship between the revenue guarantee of the Vickrey auction, as compared to the maxmin mechanism? We give below a reversed Bulow-Klemperer result.

Theorem 3. *Let us denote by $Reg(Maxmin_n)$ the revenue guarantee of the maxmin mechanism with n bidders, and by $Reg(VA_{n+1})$ the revenue guarantee of the Vickrey auction with $n+1$ bidders. Then there exists $\mu_n > 0$ such that when $\mu < \mu_n$,*

$$Reg(Maxmin_n) > Reg(VA_{n+1}).$$

Furthermore, for any $m \geq 2$, there exists μ_{n_m} such that when $\mu < \mu_n$,

$$Reg(Maxmin_n) > Reg(VA_{n+m}).$$

Proof. The proof is in Appendix A.3. □

That is, when the expectation of a generic bidder's value is low, running a Vickrey auction with an additional bidder would not surpass the revenue guarantee of the maxmin mechanism. Furthermore, when the expectation of a generic bidder's value is sufficiently low, running a Vickrey auction with any number of additional bidders would not surpass the revenue guarantee of the maxmin mechanism.

6 Discussion and Extension

6.1 Value of randomization

Recall that Suzdaltsev (2020) characterizes the optimal deterministic reserve price for the second-price auction in the same framework, which is zero in our setting. For the two-bidder case, our proposed mechanism, which involves randomization, achieves a strictly higher revenue guarantee. For a parametric example, if $\mu = 0.5$, then the revenue guarantee under our proposed mechanism is about 0.3385, whereas the one under his mechanism is 0.25. Intuitively, randomization hedges against uncertainty towards value distributions, rendering a higher revenue guarantee. To our knowledge, it is an open question what the optimal deterministic mechanism is in this setting. Therefore, the difference between the revenue guarantee of our mechanism and the one of his can be interpreted as an upper bound on the "value of randomization".

6.2 Cost of correlation

Recall that Che (2019) finds that a second-price auction with a random reserve price maximizes the revenue guarantee across a wide range of mechanisms in the private value environment in which the seller only knows the expectation of the value distribution. That is, values across bidders can be correlated. As expected, the revenue guarantee of our proposed mechanism is strictly higher in our setting for the two-bidder case than the one in his setting. This is because the seller in our setting knows more: he knows that the correlation structure is the independent one. For a parametric example, if $\mu = 0.5$, then the revenue guarantee of his mechanism in his setting is about 0.317, which is strictly smaller than 0.3385. Intuitively, independently and identically distributed value distributions make our setting more competitive. Indeed, in his worst-case value distribution, the competitor of the high-valuation bidder always has the lowest possible valuation. To our knowledge, it is not known what the maxmin mechanism is across all participation-securing mechanisms (even across all dominant-strategy mechanisms) in his setting. Therefore, the difference between the revenue guarantee in our paper and the one in his paper can be interpreted as an upper bound on the "cost of correlation".

6.3 Other strong maxmin solutions

Corollary 1. *Let $\bar{\mathcal{M}}^*$ be a second price auction with a random reserve price whose cumulative distribution is H^*. Then for the two-bidder case, $(\bar{\mathcal{M}}^*, \bar{F}, \bar{\sigma})$ is a strong maxmin solution with a revenue guarantee of $\bar{R}$ if H^* satisfies the following properties:*
P1. $H^*(x) = \bar{H}(x)$ for $x \geq a$.
P2. $H^*(x) - x \cdot (H^*)'(x) \geq 0$ for $x < a$.

Proof. By the proof of Proposition 1, the property $P1$ implies that for any $a < x < 1$, $F(x) = 1 - \frac{a}{x}$ is the unique minimizer of $\mathcal{I}(F, H^*, \lambda)$. Moreover, the properties $P1$ and $P2$ together imply that for any $0 < x < a$, $F(x) = 0$ is the unique minimizer of $\mathcal{I}(F, H^*, \lambda)$, as $\mathcal{I}(F, H^*, \lambda)$ is a (weakly) convex function of F. □

In a strong maxmin solution, there is some flexibility for the distribution of a random reserve price when the reserve is below a. One example that satisfies $P2$ is that $H^*(x) = \bar{H}(a)$ for $x < a$, i.e., there is a probability mass of size $\bar{H}(a)$ on zero.

6.4 Knowing the second moment

Suppose that the seller knows only the second moment of the value distribution as well as that the two values are independently and identically distributed. Let δ denote the known second moment, i.e., $\delta = \int_0^1 x^2 dF(x)$. Then we will show that the second-price auction with the uniformly distributed random reserve price $(\hat{\mathcal{M}})$, the equal-revenue distribution with $a = 1 - \sqrt{1-\delta}$ $(\hat{F})$, and the truth-telling strategy profile $(\hat{\sigma})$ constitute a strong maxmin solution in this case.

Corollary 2. *If the seller knows only the second moment of the value distribution as well as that the two values are independently and identically distributed, then for the two-bidder case, $(\hat{\mathcal{M}}, \hat{F}, \hat{\sigma})$ is a strong maxmin solution with a revenue guarantee of δ.*

Proof. It suffices to show that $\hat{F}$ minimizes the expected revene in the truth-telling equilibrium under the mechanism $\hat{\mathcal{M}}$. First, using integration by parts, the constraint $\delta = \int_0^1 x^2 dF(x)$ can be rewritten as $\delta = 2\int_0^1 x(1-F(x))dx$. Similar to the proof of Proposition 1, we construct a Lagrangian as follows:

$$\hat{\mathcal{L}}(F, \hat{H}, \lambda) \equiv \int_0^1 \left\{(1-F^2(x))[x\hat{H}'(x) + \hat{H}(x)] - \hat{H}(x)[2F(x)(1-F(x))] - 2\lambda x(1-F(x))\right\} dx.$$

Let λ be 1. It is straightforward that the integrand of $\hat{\mathcal{L}}(F, \hat{H}, \lambda)$ is a constant of 0 because $\hat{H}(x) = x$, implying that any value distribution with the known second moment yields the same expected revenue in the truth-telling equilibrium under the mechanism $\hat{\mathcal{M}}$. Alternatively, observe that the ex-post revenue at the value profile (v_1, v_2) is $\frac{v_1^2+v_2^2}{2}$ in the truth-telling equilibrium under the mechanism $\hat{\mathcal{M}}$. This also implies that the expected revenue is δ for any value distribution with the known second moment. □

7 Concluding Remarks

In this paper, we find that a second-price auction with a random reserve price is a maxmin mechanism across all participation-securing mechanisms for the two-bidder case. The key step of the result is the construction of a saddle point, which implies that strong duality holds for two-bidder case in our setting. It remains an open question whether strong duality holds and what a maxmin mechanism across all participation-securing mechanisms looks like for more-than-two-bidder cases. This paper provides the first step towards a broad study of robust auction design problems in the independent private value model.

A Omitted Proofs

A.1 Proof of Lemma 2

Taking derivative on $(x_1, 1)$, we obtain that

$$\bar{\bar{H}}'(x) = -(n-1)(1-a)^{n-1}\left(\frac{x}{x-a}\right)^n \frac{\frac{x_1}{n-1}\ln\frac{x}{x_1} + x_1 - x}{x^2\left[(n-2)n - (n-1)\ln x\right]}.$$

To show this is positive, we only need to show that

$$\frac{x_1}{n-1}\ln\frac{x}{x_1} + x_1 - x < 0.$$

This is equivalent to

$$\frac{\ln \frac{x}{x_1}}{n-1} + 1 - \frac{x}{x_1} < 0,$$

which is true for $x \in (x_1, 1)$, for any $n \geq 3$.

A.2 Proof of Lemma 3

We derive the second derivative:

$$\frac{\partial^2 \mathcal{I}(F, \bar{\bar{H}}, \lambda)}{\partial F^2} = n(n-1)\left[((n-1)\bar{\bar{H}}(x) - x\bar{\bar{H}}'(x))F(x) - (n-2)\bar{\bar{H}}(x)\right] F^3(x)$$

Since we care about the sign, we only need to analyze

$$((n-1)\bar{\bar{H}}(x) - x\bar{\bar{H}}'(x))F(x) - (n-2)\bar{\bar{H}}(x). \tag{A.2.1}$$

When $F(x) = 1 - \frac{a}{x}$, we get a positive value for any $x > x_1$.

This proves that we found a local minimum. Since $\frac{\partial^2 \mathcal{I}(F,\bar{\bar{H}},\lambda)}{\partial F^2}$ has only a single root in $(0, \infty)$, this is the single local minimum in this interval. When $F(x) = 0$, we get that (A.2.1) is negative, since $\bar{\bar{H}}(x) > 0$. That proves that the second derivative's single positive root is in $\left(0, 1 - \frac{a}{x}\right)$. Therefore, $\mathcal{I}(F, \bar{\bar{H}}, \lambda)$ is increasing on$\left(1 - \frac{a}{x}, 1\right)$, which means that$\mathcal{I}\left(1 - \frac{a}{x}, \bar{\bar{H}}, \lambda\right) < \mathcal{I}(1, \bar{\bar{H}}, \lambda)$. We now only need to prove that $\mathcal{I}\left(1 - \frac{a}{x}, \bar{\bar{H}}, \lambda\right) \leq \mathcal{I}(0, \bar{\bar{H}}, \lambda)$. To shorten the analysis ,we define $\tilde{\mathcal{I}}(F, \bar{\bar{H}}, \lambda) := \frac{n}{\lambda}\mathcal{I}(F, \bar{\bar{H}}, \lambda)$. Then it is enough to show that $\tilde{\mathcal{I}}(0, \bar{\bar{H}}, \lambda) - \tilde{\mathcal{I}}\left(1 - \frac{a}{x}, \bar{\bar{H}}, \lambda\right) \geq 0$, and indeed,

$$\tilde{\mathcal{I}}(0, \bar{\bar{H}}, \lambda) - \tilde{\mathcal{I}}\left(1 - \frac{a}{x}, \bar{\bar{H}}, \lambda\right) = -\ln \frac{x_1}{x} - \frac{1 - \frac{x_1}{x}}{n-1} \geq 0.$$

A.3 Proof of Theorem 3

By Suzdaltsev (2020), the worst-case value distribution for the Vickrey auction (a second-price auction without a reserve price) is a two-point discrete distribution. When $\mu \leq \frac{1}{(n-1)^2}$, the two points are 0 and 1 with masses of $(1-\mu)$ and μ, respectively. When $\mu > \frac{1}{(n-1)^2}$, the two points are $\frac{(n-1)^2\mu - 1}{(n-1)^2 - 1}$ and 1 with masses of $\left(1 - \frac{1}{(n-1)^2}\right)$ and $\frac{1}{(n-1)^2}$, respectively. Therefore, the revenue guarantee for the Vickrey auction is $1 - (1-\mu)^n - n(1-\mu)^{n-1}\mu$ when $\mu \leq \frac{1}{(n-1)^2}$. We will focus on $\mu \leq \frac{1}{(n-1)^2}$ for the proof.

Recall for $n = 2$, the revenue guarantee of a maxmin mechanism is $R_2(a) := 2a - a^2$ where $a(1 - \ln a) = \mu$. Note that $R_2'(0) = 2 > 0$. Plugging $a(1 - \ln a) = \mu$ to $1 - (1-\mu)^n - n(1-\mu)^{n-1}\mu$, the revenue guarantee of the Vickrey auction is $V_n(a) :=$

$1-(1-a(1-\ln a))^n - n(1-a(1-\ln a))^{n-1}a(1-\ln a)$. Taking the first order derivative to V_n, we obtain that $V_n'(a) = -n(n-1)(1-a(1-\ln a))^{n-2}a\ln a(1-\ln a)$. Using L'Hôpital's rule, we have that $V_n'(0) = 0$ for any $n \geq 3$. This proves the statements in the theorem for $n = 2$.

For $n \geq 3$, a lower bound of the revenue guarantee of a maxmin mechanism is $\tilde{R}_n(a) := 1-(1-a)^n$ where a satisfies Equation (8). Note that $\tilde{R}_n'(0) = n > 0$. Plugging Equation (8) to $1-(1-\mu)^n - n(1-\mu)^{n-1}\mu$, the revenue guarantee of the Vickrey auction is $\tilde{V}_n(a) := 1 - \left[1 - a\left(n - \frac{1}{n-1} - \ln\left((n-1)^2 a\right)\right)\right]^n - n\left[1 - a\left(n - \frac{1}{n-1} - \ln\left((n-1)^2 a\right)\right)\right]^{n-1} a\left(n - \frac{1}{n-1} - \ln\left((n-1)^2 a\right)\right)$. Taking first order derivative to $\tilde{V}_n$, we obtain that
$\tilde{V}_n'(a) = -n(n-1)\left[1 - a\left(n - \frac{1}{n-1} - \ln\left((n-1)^2 a\right)\right)\right]^{n-2} a\left(n - 1 - \frac{1}{n-1} - \ln\left((n-1)^2 a\right)\right)$
$\left(n - \frac{1}{n-1} - \ln\left((n-1)^2 a\right)\right)$. Using L'Hôpital's rule, we have that $\tilde{V}_n'(0) = 0$ for any $n \geq 3$. This proves the statements in the theorem for $n \geq 3$.

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