Costly state verification with ex post participation constraint

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Abstract: This paper studies a Principal-Agent model with the Agent's private information as his type. Transfers are not feasible. The Principal can engage in costly inspection to learn the type. The paper shows if the Principal can commit to the mechanism, the optimal (deterministic) inspection mechanism inspects only intermediate types, sets a cap on efficient types' actions and mandates the first best action to intermediate types with no reward. A stochastic mechanism is helpful when the inspection cost is intermediate. When the Principal cannot commit, types pool to (maximum) two messages and the optimal structure (for the Principal) is similar to the deterministic mechanism, with different thresholds. Finally, if the inspection cost is not high, and if the Principal commits (only) to the inspection policy she can achieve the full commitment payoff.

Keywords: Costly state verification, mechanism design, cheap talk, inspection, regulation.

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1 Introduction

A Principal (she) wants to mandate an action to an Agent (he) who has a private information about his type. The principal prefers lower actions and the Agent prefers higher actions. The Principal can learn the Agent's private information at a cost, and can mandate an action with or without knowing the agent's private information. The Principal cannot use transfers. The Agent is protected by Ex-post participation constraint; therefore, the Principal faces a trade-off between low actions and the risk that the Agent rejects the action, and chooses his outside option.

How does the Principal maximize the expected gain from mandating actions with and without inspection net the learning cost? In the absence of commitment power, what are the best and worst equilibria for the Principal? What is the most effective tool for her to commit to? By committing on which tool can she reach the highest payoff?

The paper develops a framework to analyze the complementarity between information acquisition and the ability to mandate optimal actions. This framework allows studying optimal mechanisms and the effect of the commitment ability of the Principal on the implementable policies.

Several economic environments correspond to our setting. Grants: An agency (Principal) awards grants for research proposals, and a researcher (Agent) requests a budget (action) for a project. The World Bank wants to give a loan to a country. A governmental agency gives loans to small businesses or entrepreneurs. A unit of a firm asks for a budget to run a project. Environment: A potential polluter (Agent) causes pollution, and the environmental protector (Principal) tries to reduce the pollution levels. In regulation: A regulator (Principal) wants to regulate the prices of a monopolist (Agent). In all of this examples, the Agents is likely to have more information than the Principal. The researcher knows how much is needed to run the project. The polluter firm knows its technology and the minimal amount of the pollution that can produce. The monopolist knows the marginal cost of production, or knows the outside option by leaving the country better than the regulator.[1]

In all of these applications, under mild assumptions, the Principal prefers a lower actions and the Agent likes higher actions. However, the Principal cannot impose a very low action since the Agent refuses to take the action, by opting out. A researcher or an entrepreneur likes a higher budget or loan, the agency prefers to allocate a low budget or loan. However, the agency cannot force the researcher or the entrepreneur to run their projects with a very low budget or loan. The environmental protection agency wants to decrease the amount of pollution, but cannot impose a very low level since the firm does not have the required technology for the targeted pollution level. The regulator cannot mandate a very low price, since the marginal production cost of the monopolist is higher

[1] We implicitly assume the existence of the agent while undertaking the optimal action is valuable for the Principal. The Principal can set an upper bound on actions and exclude all agents that cannot accept actions less than the upper bound.

than that price.

In most applications, the Principal can investigate and collect information to learn more about the private information of the Agent. The grant-making agencies can engage in costly investigations and find the minimal budget that is required for running the project. The environmental protection agency can hire some inspectors and engineers to determine the technology level of the firm (polluter), and specify a reasonable level of harm that they allow to produce. The regulator can check the accounting documents of the monopolist as well as study the production cost of similar companies to learn the marginal cost of the monopolist.

In practice, we do not see that grant-making agencies use transfers while negotiating with entrepreneurs or researchers. In other settings like controlling a polluter sometimes the shadow cost of public funds is high, so the government cannot design an efficient mechanism with transfers. In some cases, it is not possible for environmental protection agencies to provide subsidies to polluters to reduce their pollution levels.[2]

In some U.S. industries, such as telecommunications, the regulator does not subsidize (or tax) firms.[3] If the regulator or the monopolist faces a hard budget constraint, then a mechanism with the transfer is not implementable. Other cases are when contingent transfers between the regulator and regulated firm are limited or banned by rules. In some circumstances, transfers are not desirable due to social and moral considerations.

Section 2 considers the problem of a Principal who can commit to a mechanism. In section 3, we show that the optimal mechanism with deterministic inspection is a cutoffs policy that splits types into three regions. Low types (efficient types) are never inspected and and face a cap on their actions. The intermediate types are always inspected and are mandated to a first best action. Finally, high types (inefficient types) are excluded. The Principal offers a low action to high types, and they refuse to undertake this action. This structure highlights the importance of inspecting intermediate types which limits the low types' rents while obtaining a low action for efficient types. Later in this section we study the optimal mechanism with stochastic inspection.

In section 3, after finding the optimal policy for deterministic inspection, we find the optimal policy when the principal can commit to a stochastic inspection policy. We introduce a novel idea by transforming the problem to a log space. Writing the truth-telling condition in the log space we can replace the global truth-telling conditions to local truth-telling conditions. We write a payoff equivalent lemma and find the optimal inspection as

[2]The Environmental Protection Agency (EPA) regulates polluter companies in a variety of industries, including refineries, manufacturing, and energy production. The EPA sets strict standards for emissions, water quality, and waste management and has the authority to enforce these standards. For example see: https://www.epa.gov/laws-regulations/summary-clean-air-act.

[3]The Federal Communications Commission (FCC) is the primary regulator of communications in the United States, and it has implemented a number of price cap policies in the telecommunications industry. These include caps on the prices for services such as internet access and long-distance calling services. See: https://docs.fcc.gov/public/attachments/FCC-18-46A1.doc. For more information see also Laffont and Tirole [1990], and Laffont and Tirole [1993].

a function of mandated action. Finally we solve the problem using Portraying maximum principal, and find the ex-ante payoff of the principal.

In practice the Principal may not be able to commit to a policy. When the Principal faces electoral or political pressure, or when she observes a new opportunity that has a short-term benefit, she may deviate from the committed policy. Commitment requires a stable government and constitutional guarantee. In section 4 relax the commitment assumption. It can be shown that the stochastic inspection is not helpful for the Principal, and the deterministic inspection strategies can be used to find all equilibria payoffs (for the Principal).

In the equilibrium with the lowest expected payoff (babbling equilibrium) for the Principal, all types of the Agent pool, by choosing a same message. In this case, if the inspection cost is low, the Principal inspects the Agent and mandates the first best action. For high inspection costs, the Principal does not inspect and mandates a low action.

Continuing the analyze of the no-commitment case, in section 4, we show that to find all equilibrium payoffs, we can simply focus on the equilibrium structures with at most two groups of types. The types of each group pool at one message and they separate from each other by setting different actions. Thus the Principal observes at most two actions on the equilibrium path. The structure of the maximum payoff equilibrium for the Principal is very similar to the structure of the commitment policy (with deterministic inspection). The intermediate types pool together at one message. The Principal inspects them and mandates an action equal to the first best. Low types and high types pool together by setting a different action. The Principal does not inspect them. She excludes inefficient types by mandating a low action.

In section 5, we compare the optimal policy with commitment and the highest equilibrium payoff for the Principal without commitment. If the Principal cannot commit to any of the instruments, then she may not achieve the commitment payoff.[4] The loss comes from two sources. First by excluding some intermediate types (which were not be excluded at the optimal policy under the commitment case), and second by inspecting some low types (which again were not be inspected at the optimal policy under the commitment assumption).

Now a question is that by committing on which tool the Principal can reach the highest payoff? In section 5, we examine a partial commitment environment. If the inspection cost is not high, and if the Principal grantees to inspect once the Agent requests an inspection, then the Principal can achieve the commitment payoff. In other words, there is no loss if the Principal cannot commit to the mandating actions (with or without inspection). In section 6, we study three different applications.

Relationship to the literature. The paper contributes to two literature: (i) on mechanism design with costly state verification (CSV), and (ii) CSV without commitment.

[4]Depending on the parameters of the setting, the Principal may or may not achieve the commitment payoff. For this purpose, we provide some examples in section 5.

The literature on mechanism design with commitment and CSV starts with the well known paper Becker [1968]. The paper states that high punishments and low probability of monitoring is the best policy for the Principal. However, the analysis relies on the assumptions that very high punishments are enforceable.

The literature continuous with an application on financial markets by Townsend [1979]. Townsend studied the optimal insurance contract between a lender and a borrower. The borrower is a risk-averse agent with private information about the project's income. The investor is a risk-neutral agent who can audit the borrower's report about the income by incurring a cost. A contract specifies for each report of income, a probability of audit, and non negative (so limited punishments) rewards in the absence and presence of audit. The optimal contract is auditing incomes that are below a threshold. Gale and Hellwig [1985] assume a risk-neutral borrower and find a similar result. Both papers assume deterministic audit mechanisms.

Border and Sobel [1987] consider a more general mechanism with stochastic audit and bounded pre-audit and post-audit transfers. Moreover, they assume the Principal may never make a net payment (reward) to the Agent. They show that the probability of audit should decrease in the agent's wealth. They pointed out with an example if the principal wants to maximize expected revenue net of audit cost, then the optimal contract pays a large rewards and audit with small probabilities.

Mookherjee and Png [1989] assume the borrower is risk-averse and show that the optimal contract should be stochastic. The current paper differs from this literature for three reasons. First, it does not consider transfer as a tool for the designer. Second, most of the works in the CSV literature on financial markets assume the market for borrowing money is competitive. Thus it maximizes the borrower's utility subject to the outside option of the lender, and truth telling condition for the borrower. Third, unlike here, the state of nature is not yet known by any party at the contract date.

Another application of CSV is for optimal allocation and collective choice problems. Ben-Porath et al. [2014] considers a principal allocates an object to one of I Agents. The principal cannot use transfers but can check the private information of each agent at a cost. The private information is the value of the object for each agent. Mylovanov and Zapechelnyuk [2017], study a similar problem with a different verification technology, and limited punishments. They assume the principal can verify information after allocating the object, and contingent on this observation, can destroy a fraction of the agent's payoff. Li [2020] studies the connection between costly verification and limited punishment. Patel and Urgun [2022] assume money burning as a new instrument for the Principal and study the optimal allocation problem with CSV, and Erlanson and Kleiner [2020] investigates the optimal allocation and collective choice problems. Our paper is different from this literature. First the current paper is not an allocation problem. Second our paper is more general in which the principal can design a mechanism depending on the information that is acquired after inspection.

Another branch of CSV is in Monopoly regulation. Baron and Myerson [1982] consider a price regulation and transfer mechanism. Amador and Bagwell [2013], and Amador and Bagwell [2022] consider mechanisms only with price regulation which the regulator's problem would take the form of a delegation problem. However, they do not consider the audit mechanism as a regulatory instrument. Baron and Besanko [1984] extend Baron and Myerson's model to allow random audits of cost. In their setting, monopolist pricing is a two-part tariff consisting of a fixed charge and unit price. By assuming that fixed charge does not affect quantity demand, they could be able to relax IC conditions. They show the optimal mechanism audits high reported costs and imposes a punishment if the observed cost is low. Our paper is different from Baron and Besanko [1984] since it does not have monetary transfer, and monopolist price is not a two-part tariff. Unlike their setting we cannot relax IC conditions since price always affects the demand.

Palonen and Pekkarinen [2022] consider a CVS regulation principal-agent problem with a different approach. They assume the Agent can reduce the probability of being verified, by engaging in costly avoidance action. The paper assumes a linear and exogenous punishment function if the the agent caught being untruthful, and no reward if the agent is truthful. The principal maximizes the expected weighed sum of the agent's payoff and transfers net of monitoring costs subject to the incentive compatibility and the participation constraints. Comparing without avoidance case to with commitment case of our paper, it departs from ours for two reasons. First, we have ex-post participation constraint, so punishments are endogenous and restricted to the utility of the agent. Second, unlike their setting in our paper Principal has a general utility function net inspection cost (not only weighed sum of the Agent's payoff and transfers).

Perhaps the closest paper to our paper is Halac and Yared [2020]. They study a CSV principal-agent delegation problem where the agent is biased toward higher actions. If the agent becomes extremely biased, their model under full commitment is similar to ours. However, unlike our paper, we do not restrict the inspection to deterministic mechanisms and the agent is not protected by ex-post participation constraint. As a result, a threshold with an escape clause (TEC) policy is not optimal. Their analysis for the lack commitment case differs from ours since they assume limited commitment power for the regulator (the regulator can commit to the inspection policy). In contrast, we analyze both limited and without commitment cases.[5]

Section 4 (no commitment) links with the literature on cheap talk models that follows Crawford and Sobel [1982].[6] To the best of our knowledge, there is no paper that links cheap talk models with regulation and CVS. Khalil [1997] considers a regulatory problem (procurement) in which the regulator cannot commit to the audit mechanism.[7] The paper considers two types for the monopolist. Transfers and exogenous punishments are available for the regulator. The paper finds the probability of the audit is higher when the

[5] Thus they analyze a delegation problem, and we analysis a cheap talk problem.

[6] Our model is not completely the same as a cheap talk model, since the Agent (sender) can accept or reject the proposed action by the Principal (receiver).

[7] However, the regulator can commit to the other regulatory instruments, and the model is not a cheap talk model.

Principal cannot commit compared to when he can. Our paper is different from Khalil [1997] since there is no transfer in our model, we assume (in section 4) the Principal cannot commit to all instruments, and we allow the Principal to design a policy after inspection.

2 Model

Players, and information structure. There are two players, an Agent (he) and a Principal (she). The Agent has type $\theta \in [\underline{\theta}, \overline{\theta}]$ drawn from a commonly known cumulative distribution function $F(.)$. The type is the Agent's private information.

Mechanism. The Principal chooses and commits to a mechanism.[8] The mechanism $\mathbb{M} = \big(M, x(m), a^I(m,\theta), a^{NI}(m)\big)$ has four components: the message space M, the probability of inspection and two mandated actions conditional on whether she inspects or not. The probability of inspection as a function of message $m \in M$ is $x(m) \in [0,1]$, and inspection allows learning the true type of the Agent. Action $a^I(m,\theta) \in \mathbb{R}_+$ is mandated in case of inspection, and $a^{NI}(m) \in \mathbb{R}_+$ is in case of no inspection which determine the Agents' actions as a function of the message m and the true type θ if observed through inspection.

Inspection costs $\phi > 0$ for the Principal. We assume **ex-post participation constraint** for the Agent: the Agent can accept or reject the final action. He rejects when the mandated action generates a negative payoff (the outside option is zero) for him.[9]

Payoffs. The payoff of the Agent with type θ and action a is $\max\{U(\theta,a),0\}$. The Principal's payoff is

$$V(\theta,a)\mathbb{1}_{U(\theta,a)\geq 0} - \phi\mathbb{1}_{\text{inspection}}.$$

Timing. The Principal commits to the mechanism. The nature draws a type θ, and the Agent learns it privately. The Agent sends a message $m \in M$. The Principal inspects with probability $x(m)$. If she inspects mandates an action $a^I(m,\theta)$, if she does not inspect mandates an action $a^{NI}(m)$. The Agent accepts or rejects the action. Figure 1 shows the timing of the model.

We maintain the following assumptions throughout the paper.

Assumption 1 *Utility functions $U(\theta,a)$, and $V(\theta,a)$ are C^2 for all $(\theta,a) \in [\underline{\theta},\overline{\theta}]^2$.*

Assumption 2 *A type $\theta \in [\underline{\theta},\overline{\theta}]$ of the Agent gets a **zero** payoff when $a = \theta$, and prefers **higher actions**, i.e.*

$$U(\theta,\theta) = 0, \text{ and } U_a(\theta,a) > 0 \text{ for all } (\theta,a) \in [\underline{\theta},\overline{\theta}]^2.$$

[8]In section 4, we relax the commitment assumption.

[9]Another interpretation of this model is that the Agent has a private outside option θ.

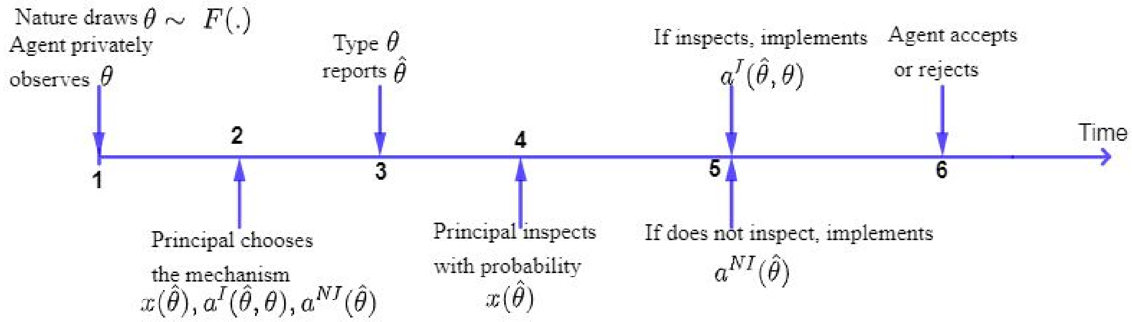

Figure 1: Timing

Assumption 3 *__Lower__ types are more valuable for the Principal than __higher__ types, i.e.*

$$V_\theta(\theta, a) \leq 0 \text{ for all } (\theta, a) \in [\underline{\theta}, \overline{\theta}]^2.$$

Assumption 4 *All types of the Agent are __valuable__ for the Principal, i.e.* $V(\theta, \theta) > 0$.

Assumption 5 *The Principal prefers lower actions for all types of the Agent, i.e.*

$$V_a(\theta, a) < 0 \text{ for all } (\theta, a) \in [\underline{\theta}, \overline{\theta}]^2.$$

Assumption 2 simply says the Agent's utility starts from zero at action equal to his type ($a = \theta$), and it is increasing in action. More precisely, the Agent prefers higher actions by only considering the support of CDF $F(.)$, i.e. $a \in [\underline{\theta}, \overline{\theta}]$. The assumption is **silent** for actions above $\overline{\theta}$.

Assumption 3 is just a sorting assumption on types of the Agent. Lower types are more efficient than higher types. Assumption 4 assures that the existence of all types of the Agent is valuable for the Principal. This assumption is without loss of generality. If we assume $V(\overline{\theta}, \overline{\theta}) < 0$, then the Principal can exclude this type by mandating actions to be less that $\overline{\theta}$.

Assumption 5 explains that the Principal prefers lower actions. More precisely, the principal prefers lower actions down to θ. If the principal chooses an action less than θ, the agent will reject the action (due to the ex-post participation constraint).

By Assumption 2 we can conclude, the payoff of the Agent at final action a given type θ is $u(\theta, a)\mathbb{1}_{a \geq \theta}$, and the Principal's payoff is $V(\theta, a)\mathbb{1}_{a \geq \theta} - \phi\mathbb{1}_{\text{inspection}}$.

3 Results

We analyze the model. By the revelation principle we can restrict the messages space M to the types space $[\underline{\theta}, \overline{\theta}]$, and mechanisms to direct mechanisms. The Principal chooses a direct mechanism $\left(x(\hat{\theta}), a^I(\hat{\theta}, \theta), a^{NI}(\hat{\theta})\right)$, which $\hat{\theta}$ is the reported type and θ is the realized type. The Agent's expected payoff given its type θ and the report $\hat{\theta}$ is

$$\pi(\hat{\theta},\theta) \equiv (1-x(\hat{\theta}))\Big(U(\theta,a^{NI}(\hat{\theta}))\Big)\mathbb{1}_{a^{NI}(\hat{\theta})\geq\theta} + x(\hat{\theta})\Big(U(\theta,a^{I}(\hat{\theta},\theta))\Big)\mathbb{1}_{a^{I}(\hat{\theta},\theta)\geq\theta}.$$

The Principal's expected payoff if the Agent with type θ reports $\hat{\theta}$ is

$$(1-x(\hat{\theta}))\Big(V(\theta,a^{NI}(\hat{\theta}))\Big)\mathbb{1}_{a^{NI}(\hat{\theta})\geq\theta} + x(\hat{\theta})\Big(V(\theta,a^{I}(\hat{\theta},\theta))\mathbb{1}_{a^{I}(\hat{\theta},\theta)\geq\theta} - \phi\Big).$$

The Principal's problem:

$$\max_{x(.),a^{NI}(.),a^{I}(.,.)} \mathbb{E}\Big[(1-x(\theta))\Big(V(\theta,a^{NI}(\theta))\Big)\mathbb{1}_{a^{NI}(\theta)\geq\theta} + x(\theta)\Big(-\phi + V(\theta,a^{I}(\theta,\theta))\mathbb{1}_{a^{I}(\theta,\theta)\geq\theta}\Big)\Big],$$

subject to the ex-post participation constraint and the truth telling conditions for the Agent:

$$\theta \in \underset{\hat{\theta}}{\operatorname{argmax}}\Big[(1-x(\hat{\theta}))\Big(U(\theta,a^{NI}(\hat{\theta}))\Big)\mathbb{1}_{a^{NI}(\hat{\theta})\geq\theta} + x(\hat{\theta})\Big(U(\theta,a^{I}(\hat{\theta},\theta))\Big)\mathbb{1}_{a^{I}(\hat{\theta},\theta)\geq\theta}\Big],$$

for all $\theta \in [\underline{\theta},\overline{\theta}]$. Note that the mandated action without inspection $a^{NI}(\hat{\theta})$, can be less than the true type of the Agent, which implies that the Agent rejects the offered action, or in other words, the Principal can exclude some types. Mandated action with inspection $a^{I}(\hat{\theta},\theta)$ optimally has to be weakly higher than the true type. The reason is when the Principal pays the inspection cost, and has full information, it is not efficient to exclude the Agent by mandating an action less than the true type, so $a^{I}(\hat{\theta},\theta) \geq \theta$. If the Agent lies and the Principal realizes, she implements the maximum punishment; i.e. $a^{I}(\hat{\theta},\theta) = \theta$ if $\hat{\theta} \neq \theta$. However, it is not clear that the optimal policy gives a reward, when the Agent tells the truth. Formally it is not clear whether $a^{I}(\theta,\theta)$ is equal to or strictly higher than θ. First, let us solve the problem when there are only two types of the Agent.

3.1 Illustrative Example:

To describe the main intuitions of the results, we begin with a simple example. Assume the Agent has only two types θ_L, and θ_H, where $\theta_L < \theta_H$. The efficient type is θ_L, and θ_H corresponds to the inefficient type. The prior information of the Principal is

$$f(\theta) = \begin{cases} f_H & \theta = \theta_H, \\ f_L & \theta = \theta_L, \end{cases}$$

and $f_L + f_H = 1$. Under the optimal policy, the expected payoff of θ_H should be zero (no rent for the most inefficient type), therefore $a^{I}(\theta_H,\theta_H) = \theta_H$, and $a^{NI}(\theta_H) \leq \theta_H$. If $a^{NI}(\theta_H) < \theta_H$, then $a^{NI}(\theta_H) = \theta_L$. If the Principal excludes θ_H when does not inspect, the is no reason to leave a rent for θ_L. Later we show this argument is true (even) for a continuum of types. Now we consider two cases, when $a^{NI}(\theta_H)$ is equal to θ_L, or θ_H.

Case 1: $a^{NI}(\theta_H) = \theta_L$. The objective of the Principal is

$$\max_{x(.),a^{NI}(.),a^I(.,.)} f_L(1-x(\theta_L))\Big(V(\theta_L,a^{NI}(\theta_L))\Big) + f_L x(\theta_L)\Big(-\phi + V(\theta_L,a^I(\theta_L,\theta_L))\Big)$$
$$+ f_H(1-x(\theta_H))\Big(V(\theta_H,\theta_L)\Big) + f_H x(\theta_H)\Big(-\phi + V(\theta_H,\theta_H)\Big),$$

Subject to the truth-telling condition for θ_L

$$x(\theta_L)U(\theta_L,a^I(\theta_L,\theta_L)) + (1-x(\theta_L))U(\theta_L,a^{NI}(\theta_L)) \geq (1-x(\theta_H))U(\theta_L,a^{NI}(\theta_H))$$
$$= (1-x(\theta_H))U(\theta_L,\theta_L) = 0.$$

Since the objective is decreasing in $a^I(\theta_L,\theta_L)$, and $a^{NI}(\theta_L)$, we conclude $a^I(\theta_L,\theta_L) = a^{NI}(\theta_L) = \theta_L$. Thus an immediate observation is that $x(\theta_L) = 0$. In addition, $x(\theta_H) = 0$ if $V(\theta_H,\theta_H) < \phi$, and $x(\theta_H) = 1$ if $V(\theta_H,\theta_H) \geq \phi$. Finally, the ex-ante payoff of the Principal is

$$f_L V(\theta_L,\theta_L) + f_H \max\{0, -\phi + V(\theta_H,\theta_H)\}.$$

In summary the inspection policy, in this case, is deterministic. The Principal does not inspect the efficient type (θ_L), and may inspect with probability one or may exclude the inefficient type (θ_H). The Principal mandates an action equal to the type for both types which leaves no rent of types.

Case 2: $a^{NI}(\theta_H) = \theta_H$. The objective of the Principal is

$$\max_{x(.),a^{NI}(.),a^I(.,.)} f_L(1-x(\theta_L))\Big(V(\theta_L,a^{NI}(\theta_L))\Big) + f_L x(\theta_L)\Big(-\phi + V(\theta_L,a^I(\theta_L,\theta_L))\Big)$$
$$+ f_H(1-x(\theta_H))\Big(V(\theta_H,\theta_H)\Big) + f_H x(\theta_H)\Big(-\phi + V(\theta_H,\theta_H)\Big).$$

Subject to the truth-telling condition for θ_L

$$x(\theta_L)U(\theta_L,a^I(\theta_L,\theta_L)) + (1-x(\theta_L))U(\theta_L,a^{NI}(\theta_L)) \geq (1-x(\theta_H))U(\theta_L,a^{NI}(\theta_H))$$
$$= (1-x(\theta_H))U(\theta_L,\theta_H).$$

Writing the optimization problem with a Lagrangian multiplier $\lambda \geq 0$, for the constraint, the optimal $a^{NI}(\theta_L)$ maximizes

$$(1-x(\theta_L))\big(f_L V(\theta_L,a^{NI}(\theta_L)) + \lambda U(\theta_L,a^{NI}(\theta_L))\big).$$

Similarly the optimal $a^I(\theta_L,\theta_L)$ maximizes

$$x(\theta_L)\big(f_L V(\theta_L,a^I(\theta_L,\theta_L)) + \lambda U(\theta_L,a^I(\theta_L,\theta_L))\big).$$

Therefore $a^I(\theta_L,\theta_L) = a^{NI}(\theta_L)$. Since $V(\theta_L,a^I(\theta_L,\theta_L)) - \phi < V(\theta_L,a^I(\theta_L))$ so $x(\theta_L) = 0$. If $x(\theta_H) = 1$, we are back to case 1. Now we consider two sub-cases.

Case 2.1: $x(\theta_H) = 0$. Using the truth-telling condition we get $a^{NI}(\theta_L) = \theta_H$. The ex-ante payoff of the Principal is

$$f_L V(\theta_L, \theta_H) + f_H V(\theta_H, \theta_H).$$

The policy, in this case, is no inspection at all. This policy leaves a high rent for the efficient type θ_L, since θ_L can mimic θ_H. Later we show this policy is generally optimal when the inspection cost ϕ is very high.

Case 2.2: $x(\theta_H) \in (0, 1)$. The optimality condition respect to $x(\theta_H)$ gives us

$$\lambda = \frac{f_H \phi}{U(\theta_L, \theta_H)}.$$

The mandated action $a^{NI}(\theta_L)$ maximizes

$$f_L V(\theta_L, a^{NI}(\theta_L)) + \frac{f_H \phi}{U(\theta_L, \theta_H)} U(\theta_L, a^{NI}(\theta_L)).$$

The truth-telling condition binds and $x(\theta_H)$ solves

$$U(\theta_L, a^{NI}(\theta_L)) = (1 - x(\theta_H)) U(\theta_L, \theta_H).$$

Finally, the ex-ante payoff of the Principal in this case is

$$f_L V(\theta_L, a^{NI}(\theta_L)) + f_H V(\theta_H, \theta_H) - f_H x(\theta_H) \phi.$$

The policy in case 2.2 suggests a stochastic inspection for the inefficient type θ_H. By inspecting θ_H with a positive probability, θ_L has less incentive to mimic θ_H, so the Principal can mandate a lower action $a^{NI}(\theta_L)$. The mandated action $a^{NI}(\theta_L)$ maximizes a weighted sum of the utilities of the Agent and the Principal

$$V(\theta_L, a^{NI}(\theta_L)) + \frac{\lambda}{f_L} U(\theta_L, a^{NI}(\theta_L)).$$

When the inspection cost is small, the multiplier λ is small and the coefficient of the Agent's utility is small. Therefore Principal is "biased" toward herself. When the inspection cost is big, the Principal is "biased" toward the Agent.

Using simple algebra, one can rewrite the ex-ante payoff of the Principal in case 2.2 as

$$f_L \big(V(\theta_L, a^{NI}(\theta_L)) - \frac{U(\theta_L, a^{NI}(\theta_L))}{U_a(\theta_L, a^{NI}(\theta_L))} V_a(\theta_L, a^{NI}(\theta_L)) \big) + f_H \big(V(\theta_H, \theta_H) - \phi \big).$$

In the above expression, there is no probability of inspection $x(\theta_H)$. It says the ex-ante payoff of the Principal is equivalent to a deterministic inspection policy; inspecting θ_H with probability one and not inspecting θ_L. However, there is a difference. The mandated action $a^{NI}(\theta_L)$ is higher than θ_L which is a loss for the Principal but in exchange, there is a gain which is

$$-f_L \frac{U(\theta_L, a^{NI}(\theta_L))}{U_a(\theta_L, a^{NI}(\theta_L))} V_a(\theta_L, a^{NI}(\theta_L)) > 0.$$

If the gain does not overcome the loss, the optimal policy should be deterministic. Later we show a similar expression for the continuum types of the Agent.

The optimal policy:

To find the optimal policy we should compare all cases. One can explain all cases with the following equation

$$\max\{f_L V(\theta_L, a^{NI}(\theta_L)) + f_H(V(\theta_H, \theta_H) - x(\theta_H)\phi), f_L V(\theta_L, \theta_L)\}.$$

When the inspection cost is very high, the optimal policy does not inspect at all. Therefore $x(\theta_H) = 0$, and $a^{NI}(\theta_L) = \theta_H$. The Principal excludes the inefficient type (θ_H) if and only if

$$f_L V(\theta_L, \theta_H) + f_H V(\theta_H, \theta_H) < f_L V(\theta_L, \theta_L).$$

Then the policy in case 1, is the optimal policy. Otherwise (the reversed inequality), case 2.1 represents the optimal policy.

If the inspection cost is low, the Principal inspects θ_H with probability one. There is no reason to inspect θ_L, since the θ_L cannot mimic θ_H. The optimal policy corresponds to case 1, and the Principal does not exclude θ_H.

When the Principal faces an intermediate inspection cost, case 2.2 represents the optimal policy. The Principal inspects θ_H with a positive probability; i.e. $x(\theta_H) > 0$. The Principal pays the ex-ante inspection cost $f_H x(\theta_H)\phi$ to decrease the mandated action $a^{NI}(\theta_L)$.

Inspection policies in cases 1, and 2.1 are deterministic. This leads us to study the optimal policy (as a benchmark) when only the deterministic inspection is available for the Principal.

3.2 Benchmark - Deterministic Inspection

In order to state the optimal policy, we need to define two thresholds. Define the problem $\mathbb{P}$ as follows

$$\mathbb{P} : \max_{\theta^* \in [\underline{\theta}, \overline{\theta}]} \left\{ \int_{\underline{\theta}}^{\theta^*} \Big(V(\theta, \theta^*) \Big) \, dF(\theta) + \int_{\theta^*}^{\overline{\theta}} \Big(V(\theta, \theta) - \phi \Big) \mathbb{1}_{V(\theta,\theta) \geq \phi} \, dF(\theta) \right\}.$$

Let Θ^* be the set of of the solutions of $\mathbb{P}$. For $\theta^* \in \Theta^*$, define θ^{**} such that

$$\theta^{**} = \begin{cases} \theta^* & \text{if } \; V(\theta^*, \theta^*) \leq \phi \\ \overline{\theta} & \text{if } \; V(\overline{\theta}, \overline{\theta}) > \phi \end{cases},$$

otherwise define θ^{**} as a solution of $v(\theta^{**}, \theta^{**}) = \phi$. Using θ^*, and θ^{**}, the below proposition expresses the optimal policy.

Proposition 1 *The optimal policy for all* $(\hat{\theta}, \theta) \in [\underline{\theta}, \overline{\theta}]^2$ *is*

$$a^I(\hat{\theta}, \theta) = \theta,$$

$$x(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} \leq \theta^* \\ 1 & \theta^{**} \geq \hat{\theta} > \theta^* \\ 0 & \hat{\theta} > \theta^{**}, \end{cases}$$

$$a^{NI}(\hat{\theta}) = \begin{cases} \theta^* & \hat{\theta} \leq \theta^* \\ \hat{\theta} & \theta^{**} \geq \hat{\theta} > \theta^* \\ \underline{\theta} & \hat{\theta} > \theta^{**}, \end{cases}$$

The proof of Proposition 1 is in Appendix.

Proposition 1 states that the optimal policy **does not give a reward** for telling the truth in case of inspection. The reason is that the Principal is free to choose types to inspect and verify the information. By excluding inefficient types ($\theta > \theta^{**}$) and choosing to inspect intermediate types ($\theta \in [\theta^*, \theta^{**}]$), the Principal can decrease the rent of efficient types and leave a zero rent for intermediate types. Thus intermediate types cannot mimic higher types types, and the Principal optimally can set $a^I(\theta, \theta) = \theta$.

The Principal does not waste resources (cost of inspection) for inefficient types ($\theta > \theta^{**}$), so inspection is zero. Instead she mandates an action without inspection ($a^{NI}(.)$) and excludes the inefficient types. By inspecting intermediate types, the optimal policy hits two goals. First does not allow efficient types ($\theta < \theta^*$) to mimic intermediate types. Second by having full information on the intermediate types, she can implement full information mechanisms for these types.

Proposition 1 argues that the optimal mandated action without inspection ($a^{NI}(.)$) sets a fixed action equal to θ^* for efficient types. By Assumption 5 all efficient types like to have the highest possible action, so we do not delve into a delegation problem. Figure 2 illustrates the optimal policy. The Principal sets a **cap** on actions equal to θ^*. She inspects types between θ^* and θ^{**}, and mandates an action equal to the type $a^I(\hat{\theta}, \theta) = \theta$. She excludes types above θ^{**} by mandating a very low action $a^{NI}(\hat{\theta}) = \underline{\theta}$.

The following Corollary demonstrates the implication of the mechanism through an indirect mechanism in the message space.

Corollary 1 ***Implementation using indirect mechanisms****. The optimal policy is implementable through the following indirect mechanism: The Principal inspects if and only if observes a message $m \in (\theta^*, \theta^{**}]$. After inspecting message m, the Principal mandates action $a^I(m, \theta) = \theta$. If the principal does not inspect m, then mandates action $a^{NI}(m) = \theta^*$.*

3.3 Stochastic Inspection

Now we consider the environment that the stochastic inspection mechanism is available for the Principal. In addition, we assume $a^I(\theta, \theta) = \theta$.[10] In support of this assumption, we

[10] Note that $a^I(\theta, \theta) = \theta$ is a result when the Principal can only use a deterministic mechanism.

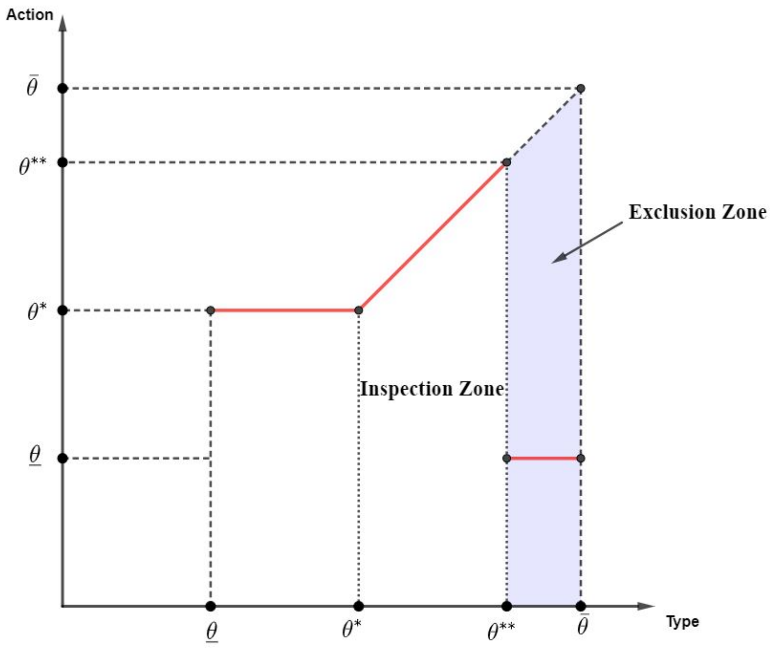

Figure 2: The optimal policy

present two arguments. First, this is the case when the Agent has only two types. Second, without this assumption and under the optimal policy, the Principal might choose a very high action $a^I(\theta,\theta)$ along with a small probability of inspection $x(\theta)$.[11] This is not a policy that normally we see in practice. For example, consider the regulation of a polluter with an unknown cost (or equivalently with an unknown technology). If we allow $a^I(\theta,\theta)$ to be higher than θ, then the regulator inspects efficient types of the polluter with a small probability and guarantees a very high pollution rate to the polluter. This is not a policy that we can implement in practice. Therefore we see $a^I(\theta,\theta)=\theta$ as a reasonable assumption.

For simplicity (and only in this section) we drop the superscript of $a^{NI}(\theta)$, and we write $a(\theta)$. We know $a^I(\hat{\theta},\theta)=\theta$, then the truth-telling condition is

$$\pi(\theta) \equiv (1-x(\theta))\Big(U(\theta,a(\theta))\Big)\mathbb{1}_{a(\theta)\geq\theta} \geq \ (1-x(\hat{\theta}))\Big(U(\theta,a(\hat{\theta}))\Big)\mathbb{1}_{a(\hat{\theta})\geq\theta},$$

for all for all $(\theta,\hat{\theta})\in[\underline{\theta},\overline{\theta}]^2$.

[11] The intuition of this fact is as follows: For simplicity assume $V(\theta,a)=V-a$, and $U(\theta,a)=a-\theta$. Let $V>\overline{\theta}$. It is easy to see Assumptions 1 to 5 hold in this environment. Assume θ is a type in which the Principal inspect with a small probability (this is the case for the efficient types). The Principal can decrease the information rent by decreasing $a^{NI}(\theta)$. At the same time can increase $a^I(\theta,\theta)$ to keep the expected payoff of type θ, and herself (for type θ) constant. If the Principal decreases $a^{NI}(\theta)$ by d, she should increase $a^I(\theta,\theta)$ by $\frac{(1-x(\theta))d}{x(\theta)}$. When $x(\theta)$ goes to zero, $\frac{(1-x(\theta))d}{x(\theta)}$ goes to infinity. Therefore the principal inspects θ with a small probability and promises a very high action $a^I(\theta,\theta)$.

The optimal policy for types with zero payoffs:

Let us begin the analysis with types that have zero payoffs. Define

$$\tilde{\theta} = \{\inf \theta | \pi(\theta) = 0\}.$$

An immediate observation is that if $\pi(\theta) = 0$, then either $a(\theta) \leq \theta$ or $x(\theta) = 1$. If $x(\theta) = 1$, then $a(\theta)$ is irrelevant, and for simplicity we assume $a(\theta) = \underline{\theta}$. The expected payoff function $\pi(\theta)$ is a weakly decreasing function, so $\pi(\theta) = 0$ for all $\theta \geq \tilde{\theta}$. The following lemma studies the structure of the optimal policy for $\theta > \tilde{\theta}$.

Lemma 1 *(i) For type θ, if $a(\theta) < \theta$, then it is without loss of generality if we assume $a(\theta) = \underline{\theta}$. (ii) For type θ, if $a(\theta) = \underline{\theta}$, then for all $\theta' > \theta$, $a(\theta') = \underline{\theta}$. (iii) For type θ, if $a(\theta) \leq \theta$, then for all $\theta' > \theta$, either $a(\theta') = \underline{\theta}$ or $x(\theta') = 1$.*

The proof of Lemma 1 is in the Appendix.

An instant result of Lemma 1 (ii), and (iii) is that there exists $\tilde{\tilde{\theta}} \in [\tilde{\theta}, \overline{\theta}]$, such $x(\theta) = 1$ for $\theta \in [\tilde{\theta}, \tilde{\tilde{\theta}}]$, and $a(\theta) = \underline{\theta}$ for $\theta \in [\tilde{\tilde{\theta}}, \overline{\theta}]$. This means that the inspection and exclusion areas are completely separated. Using the structure of Lemma 1, one can conclude that types $\theta \leq \tilde{\theta}$ do not have the incentive to mimic types higher than $\tilde{\theta}$. In addition, a necessary and sufficient condition for incentives of types higher than $\tilde{\theta}$, is that $a(\theta) \leq \tilde{\theta}$ for $\theta \leq \tilde{\theta}$. This implies $a(\tilde{\theta}) = \tilde{\theta}$.

Global truth-telling conditions to local truth-telling conditions:

Now we study the optimal mechanism for types lower than $\tilde{\theta}$. For simplicity restrict attention to C^2 policy functions $x(\theta)$, and $a(\theta)$ for $\theta \leq \tilde{\theta}$.[12] The policy functions are C^2, and $a(\theta) > \theta$ for $\theta < \tilde{\theta}$, so for a type $\hat{\theta}$ close enough to θ, $a(\hat{\theta}) > \theta$. Now we can write the local truth telling condition

$$(1 - x(\theta))\Big(U(\theta, a(\theta))\Big) \geq \ (1 - x(\hat{\theta}))\Big(U(\theta, a(\hat{\theta}))\Big).$$

The following assumption (only for this section) on the Agent's utility function allows us to transform the global truth telling conditions to the local truth telling conditions. The assumption is similar to a log transformation of Spence-Mirrlees condition.

Assumption 6 *The Log Spence-Mirrlees condition*

$$\frac{\partial^2 lnU(\theta, a)}{\partial\theta\partial a} \geq 0,$$

for all $(\theta, a) \in [\underline{\theta}, \overline{\theta}]^2$.

[12]Later we relax this assumption.

Define $y(\hat{\theta}) = ln(1 - x(\hat{\theta}))$, and $u(\theta, a(\hat{\theta})) = ln(U(\theta, a(\hat{\theta})))$. A logarithm transformation of the local truth telling condition, and the first order condition respect to $\hat{\theta}$ at point $\hat{\theta} = \theta$ gives us

$$\dot{y}(\theta) + \dot{a}(\theta) u_a(\theta, a(\theta)) = 0.$$

The second order condition gives us

$$\ddot{y}(\theta) + \ddot{a}(\theta) u_a(\theta, a(\theta)) + (\dot{a}(\theta))^2 u_{aa}(\theta, a(\theta)) \leq 0.$$

Differentiating the first order condition respect to θ, we can simplify the second order condition and simply

$$-\dot{a}(\theta) u_{a\theta}(\theta, a(\theta)) \leq 0.$$

By using the Log Spence-Mirrlees condition, we can conclude $\dot{a}(\theta) \geq 0$, and the local truth telling condition is a sufficient condition for the global truth telling condition. By employing the Envelope Theorem, the local truth telling condition reduces to

$$\frac{\partial ln(\pi(\theta))}{\partial \theta} = \frac{\partial u(\theta, a(\theta))}{\partial \theta},$$

or equivalently for $\pi(\theta) > 0$

$$\dot{\pi}(\theta) = \pi(\theta) \frac{U_\theta(\theta, a(\theta))}{U(\theta, a(\theta)}. \tag{1}$$

Lemma 2 *(Payoff Equivalence) Let $a^*(.)$ be the optimal mandated action in case of no inspection. Then the expected payoff for the Agent, and the optimal inspection policy for $\theta \leq \tilde{\theta}$, are*

$$\pi(\theta) = U(\underline{\theta}, a^*(\underline{\theta}))\ exp\Big(\int_{\underline{\theta}}^{\theta} \frac{U_\theta(t, a^*(t))}{U(t, a^*(t))}\ dt \Big),$$

and

$$x^*(\theta) = 1 - \frac{U(\underline{\theta}, a^*(\underline{\theta}))\ exp\big(\int_{\underline{\theta}}^{\theta} \frac{U_\theta(t, a^*(t))}{U(t, a^*(t))}\ dt \big)}{U(\theta, a^*(\theta))}.$$

Proof. From local truth telling condition 1, and knowing that the inspection for the lowest type $(\underline{\theta})$ is zero, we can write

$$ln(\pi(\theta)) - ln(\pi(\underline{\theta})) = \int_{\underline{\theta}}^{\theta} \frac{d\ ln(\pi(t))}{d\ t}\ dt = \int_{\underline{\theta}}^{\theta} \frac{U_\theta(t, a^*(t))}{U(t, a^*(t))}\ dt.$$

Therefore

$$\pi(\theta) = U(\underline{\theta}, a^*(\underline{\theta}))\ exp\Big(\int_{\underline{\theta}}^{\theta} \frac{U_\theta(t, a^*(t))}{U(t, a^*(t))}\ dt \Big).$$

We know $\pi(\theta) = (1 - x^*(\theta)) U(\theta, a^*(\theta))$, so $x^*(\theta) = 1 - \frac{\pi(\theta)}{U(\theta, a^*(\theta))}$. ■

Principal's optimization program:

Now we use the local truth-telling condition 1 to solve the Principal's program. We know $a(\theta) \geq \theta$ for $\theta \leq \tilde{\tilde{\theta}}$, we can rewriting the Principal's objective as

$$\max_{x(.),a(.),\tilde{\tilde{\theta}}} \int_{\underline{\theta}}^{\tilde{\tilde{\theta}}} \Big[(1-x(\theta))\Big(V(\theta,a(\theta)) - V(\theta,\theta) + \phi\Big) + \Big(V(\theta,\theta) - \phi\Big) \Big] f(\theta)d\theta,$$

Replacing $1-x(\theta)$ by $\frac{\pi(\theta)}{U(\theta,a(\theta))}$, finally the optimization program of the Principal becomes

$$\max_{\pi(.),a(.),\tilde{\theta},\tilde{\tilde{\theta}}} \int_{\underline{\theta}}^{\tilde{\theta}} \Big[\frac{\pi(\theta)}{U(\theta,a(\theta))}\Big(V(\theta,a(\theta)) - V(\theta,\theta) + \phi\Big)\Big] f(\theta)d\theta + \int_{\underline{\theta}}^{\tilde{\tilde{\theta}}} \Big(V(\theta,\theta) - \phi\Big) f(\theta)d\theta,$$

subject to (for $\theta < \tilde{\theta}$)

$$\dot{\pi}(\theta) = \pi(\theta)\frac{U_\theta(\theta,a(\theta))}{U(\theta,a(\theta))},$$

$$\dot{a}^{NI}(\theta) \geq 0,$$

$$\pi(\theta) > 0,$$

$$\pi(\theta) \leq U(\theta,a(\theta)),$$

$$a(\theta) > \theta,$$

and for $\theta \geq \tilde{\theta}$, we have $a(\tilde{\theta}) = \tilde{\theta}$, and $\pi(\theta) = 0$. Finally $\tilde{\theta}$, and $\tilde{\tilde{\theta}}$ should be in a right order; i.e. $\tilde{\theta} \leq \tilde{\tilde{\theta}}$. Consider two cases $\tilde{\theta} < \tilde{\tilde{\theta}}$ and $\tilde{\theta} = \tilde{\tilde{\theta}}$.

First assume $\tilde{\theta} < \tilde{\tilde{\theta}}$. It is easy to see $\tilde{\tilde{\theta}}$ solve $v(\tilde{\tilde{\theta}},\tilde{\tilde{\theta}}) = \phi$ if $v(\overline{\theta},\overline{\theta}) < \phi$, otherwise $\tilde{\tilde{\theta}} = \overline{\theta}$. We solve the problem by using the Pontryagin's maximum principle (π is the state and a is the control variable). The Hamiltonian for $\theta < \tilde{\theta}$ is

$$H(a,\pi,\mu,w,\theta) = \frac{\pi}{U(\theta,a)}\Big(V(\theta,a) - V(\theta,\theta) + \phi\Big) f(\theta) + \mu\, \pi \frac{U_\theta(\theta,a)}{U(\theta,a)} + w(U(\theta,a) - \pi),$$

where the Lagrangian multiplier for $\pi(\theta) \leq U(\theta,a(\theta))$ is $w(\theta)$. From the Portraying principle for the co-state variable $\mu(\theta)$ we have

$$\dot{\mu}(\theta) = -\frac{\partial H}{\partial \pi} = -\frac{1}{U(\theta,a^*(\theta))}\Big(V(\theta,a^*(\theta)) - V(\theta,\theta) + \phi\Big) f(\theta) - \mu(\theta)\, \frac{U_\theta(\theta,a^*(\theta))}{U(\theta,a^*(\theta))} + w(\theta).$$

Since $U(\theta,a^*(\theta)) > 0$, then

$$\dot{\mu}(\theta)U(\theta,a^*(\theta)) + \mu(\theta)\, U_\theta(\theta,a^*(\theta)) - w(\theta)U(\theta,a^*(\theta)) = -\Big(V(\theta,a^*(\theta)) - V(\theta,\theta) + \phi\Big) f(\theta). \tag{2}$$

From the Pontryagin's maximum principle, we know for all $\theta < \tilde{\theta}$, a maximizes the following

$$\frac{\pi(\theta)}{U(\theta,a)}\Big(V(\theta,a) - V(\theta,\theta) + \phi\Big) f(\theta) + \mu(\theta)\ \pi(\theta)\frac{U_\theta(\theta,a)}{U(\theta,a)} + w(\theta)(U(\theta,a) - \pi(\theta)). \quad (3)$$

Optimizing equation 3 respect to a gives us

$$\begin{aligned} H_a = \pi(\theta)\Bigg(&\frac{U(\theta,a^*(\theta))\big(V_a(\theta,a^*(\theta))f(\theta) + \mu(\theta)U_{\theta a}(\theta,a^*(\theta))\big)}{U^2(\theta,a^*(\theta))} \\ &- \frac{U_a(\theta,a^*(\theta))\Big((V(\theta,a^*(\theta)) - V(\theta,\theta) + \phi)f(\theta) + \mu(\theta)\ U_\theta(\theta,a^*(\theta))\Big)}{U^2(\theta,a^*(\theta))}\Bigg) \\ &+ w(\theta)U_a(\theta,a^*(\theta)) = 0. \end{aligned}$$

Replacing 2, we get

$$\begin{aligned} &H_a = \\ &\pi(\theta)U(\theta,a^*(\theta))\Bigg(\frac{V_a(\theta,a^*(\theta))f(\theta) + \mu(\theta)U_{\theta a}(\theta,a^*(\theta)) + U_a(\theta,a^*(\theta))\big(\dot{\mu}(\theta) - w(\theta)\big)}{U^2(\theta,a^*(\theta))}\Bigg) \\ &\qquad + w(\theta)U_a(\theta,a^*(\theta)) = 0. \end{aligned}$$

Simple algebra gives us

$$\begin{aligned} &\pi(\theta)U(\theta,a^*(\theta))\Bigg(\frac{V_a(\theta,a^*(\theta))f(\theta) + \mu(\theta)U_{\theta a}(\theta,a^*(\theta)) + U_a(\theta,a^*(\theta))\dot{\mu}(\theta)}{U^2(\theta,a^*(\theta))}\Bigg) \\ &\qquad = U_a(\theta,a^*(\theta))w(\theta)\Big(\frac{\pi(\theta)}{U(\theta,a^*(\theta))} - 1\Big) = 0. \end{aligned}$$

The last equality is due to the fact that $w(\theta)(U(\theta,a^*(\theta)) - \pi(\theta)) = 0$. Finally since $\pi(\theta)$, and $U(\theta,a^*(\theta))$ are strictly positive we can write the optimality condition as

$$\mu(\theta)U_{\theta a}(\theta,a^*(\theta)) + U_a(\theta,a^*(\theta))\dot{\mu}(\theta) = -V_a(\theta,a^*(\theta))f(\theta). \quad (4)$$

In order to study $w(\theta)$, we need to show the optimal inspection policy is a weakly increasing function.

Lemma 3 *The optimal inspection function $x^*(\theta)$ is weakly increasing in θ.*

Proof. By contradiction assume there exist types $\theta < \theta'$ such that $x(\theta) > x(\theta')$. Since $a^*(.)$ is a weakly increasing function, then type θ prefers to mimic type θ'; i.e.

$$(1 - x(\theta))U(\theta,a(\theta)) < (1 - x(\theta'))U(\theta,a(\theta')).$$

A contradiction. ■

Since $x(\theta)$ is a weakly increasing function we can define θ^* the supremum of θ such that $\pi(\theta) = U(\theta, a(\theta))$, so for $\theta \geq \theta^*$, $w(\theta) = 0$. We separate the analysis for types below and above θ^*. First for types below θ^*. From the truth telling condition $a^*(\theta) = a^*(\underline{\theta})$ for $\theta \leq \theta^*$. So $a^*(.)$ is constant. Using equation 4 we get

$$\int_{\underline{\theta}}^{\theta} \frac{d\, \mu(t) U_a(t, a^*(t))}{d\, t}\, dt = \int_{\underline{\theta}}^{\theta} -V_a(t, a^*(t)) f(t)\, dt.$$

The transversely condition at $\underline{\theta}$ is $\mu(\underline{\theta}) = 0$. Hence

$$\mu(\theta) U_a(\theta, a^*(\underline{\theta})) = \int_{\underline{\theta}}^{\theta} -V_a(t, a^*(\underline{\theta})) f(t)\, dt. \tag{5}$$

For types above θ^*, using equations 2, and 4, and the fact that $w(\theta) = 0$, we get

$$\mu(\theta) = \frac{\big(U(\theta, a^*(\theta)) V_a(\theta, a^*(\theta)) - U_a(\theta, a^*(\theta))(V(\theta, a^*(\theta)) - V(\theta, \theta) + \phi)\big) f(\theta)}{U_\theta(\theta, a^*(\theta)) U_a(\theta, a^*(\theta)) - U(\theta, a^*(\theta)) U_{\theta a}(\theta, a^*(\theta))} \equiv \Psi(\theta, a^*(\theta)).$$

$$\dot{\mu}(\theta) = \frac{\big(U_\theta(\theta, a^*(\theta)) V_a(\theta, a^*(\theta)) - U_{a\theta}(\theta, a^*(\theta))(V(\theta, a^*(\theta)) - V(\theta, \theta) + \phi)\big) f(\theta)}{-U_a(\theta, a^*(\theta)) U_\theta(\theta, a^*(\theta)) + U(\theta, a^*(\theta)) U_{\theta a}(\theta, a^*(\theta))} \equiv \psi(\theta, a^*(\theta)).$$

The above two equations generate a first order differential equation for $a^*(\theta)$. Therefore using $a^*(\theta^*) = a^*(\underline{\theta})$ as an initial condition for this first order differential equation we can find $a^*(\theta)$. Now given all explanations we can write the following Proposition for the optimal $a^*(\theta)$.

Proposition 2 *(The optimal mandated action) The optimal mandated action for types* $\theta \leq \theta^*$ *is* $a^*(\theta) = a^*(\underline{\theta})$. *For types* $\theta \in [\theta^*, \tilde{\theta}]$, $a^*(\theta)$ *solves*

$$\frac{d\, \Psi}{d\theta} = \psi.$$

In addition $\dot{a}^*(\theta^*) = 0$. [13]

Solving the above first-order differential equation with an initial condition $\dot{a}^*(\theta^*) = 0$. gives us the optimal mandated action $a^*(.)$. Then by using Lemma 2, we can find the optimal inspection policy $x^*(.)$.

[13]Later we check under what conditions $a^*(.)$ is a weakly increasing function.

Principal's ex-ante payoff:

Surprisingly, for computing the ex-ante payoff of the Principal we only need to know $a^*(\theta^*)$. In order to find the expected payoff of the Principal for types $\theta \in [\theta^*, \tilde{\theta}]$ from equation 2 we get:

$$\frac{d\ \mu(\theta)\pi(\theta)}{d\ \theta} = -\frac{\pi(\theta)}{U(\theta, a^*(\theta))}\Big(V(\theta, a^*(\theta)) - V(\theta, \theta) + \phi\Big) f(\theta)$$

Therefore

$$\int_{\theta^*}^{\tilde{\theta}} \frac{d\ \mu(\theta)\pi(\theta)}{d\ \theta}\ d\theta = \int_{\theta^*}^{\tilde{\theta}} -\frac{\pi(\theta)}{U(\theta, a^*(\theta))}\Big(V(\theta, a^*(\theta)) - V(\theta, \theta) + \phi\Big) f(\theta)\ d\theta$$

So since $\pi(\tilde{\theta}) = 0$

$$\mu(\theta^*)\pi(\theta^*) = \int_{\theta^*}^{\tilde{\theta}} \frac{\pi(\theta)}{U(\theta, a^*(\theta))}\Big(V(\theta, a^*(\theta)) - V(\theta, \theta) + \phi\Big) f(\theta)\ d\theta, \tag{6}$$

Now, the Principal's payoff becomes

$$\int_{\underline{\theta}}^{\theta^*} \Big(V(\theta, a^*(\underline{\theta})) - V(\theta, \theta) + \phi\Big) f(\theta)\ d\theta + \mu(\theta^*)\pi(\theta^*).$$

Using equation 5, and the fact that $\pi(\theta^*) = U(\theta^*, a^*(\underline{\theta}))$

$$\int_{\underline{\theta}}^{\theta^*} \Big(V(\theta, a^*(\underline{\theta})) - V(\theta, \theta) + \phi\Big) f(\theta)\ d\theta + \frac{U(\theta^*, a^*(\underline{\theta}))}{U_a(\theta^*, a^*(\underline{\theta}))} \int_{\underline{\theta}}^{\theta^*} -V_a(\theta, a^*(\underline{\theta})) f(\theta)\ d\theta.$$

Equivalently

$$\int_{\underline{\theta}}^{\theta^*} \Big(V(\theta, a^*(\underline{\theta})) - V(\theta, \theta) + \phi - \frac{U(\theta^*, a^*(\underline{\theta}))}{U_a(\theta^*, a^*(\underline{\theta}))} V_a(\theta, a^*(\underline{\theta}))\Big) f(\theta)\ d\theta.$$

Proposition 3 *The Principal's expected payoff given $a^*(\underline{\theta})$ is*

$$\max_{\tilde{\tilde{\theta}}\ and\ \theta^* \leq a^*(\underline{\theta})} \int_{\underline{\theta}}^{\theta^*} \Big(V(\theta, a^*(\underline{\theta})) - \frac{U(\theta^*, a^*(\underline{\theta}))}{U_a(\theta^*, a^*(\underline{\theta}))} V_a(\theta, a^*(\underline{\theta}))\Big) f(\theta)\ d\theta + \int_{\theta^*}^{\tilde{\tilde{\theta}}} \Big(V(\theta, \theta) - \phi\Big) f(\theta)\ d\theta.$$

The proof of Proposition 3 is in the Appendix.

Now let $\tilde{\theta} = \tilde{\tilde{\theta}}$. This is the case when the optimal $\tilde{\theta}$, and $\tilde{\tilde{\theta}}$, from the previous part are in the wrong order; i.e. $\tilde{\theta} > \tilde{\tilde{\theta}}$, otherwise we do not consider this case. Note that $a^*(\theta) = \underline{\theta}$, and $x^*(\theta) = 0$ for all $\theta > \tilde{\theta}$. The analysis is almost the same with a different Hamiltonian. The Hamiltonian for $\theta < \tilde{\theta}$ is

$$H(a, \pi, \mu, w, \theta) = \Big(\frac{\pi}{U(\theta, a)}\big(V(\theta, a) - V(\theta, \theta) + \phi\big) + V(\theta, \theta) - \phi\Big) f(\theta) + \mu\ \pi \frac{U_\theta(\theta, a)}{U(\theta, a)} + w(U(\theta, a) - \pi).$$

One can see that equations 2, and 4 are the same. Moreover, given θ^*, and $a^*(\underline{\theta})$, Proposition 2 holds. Equations 5, and 6 are the same, but the expected payoff of the Principal is different. The expected payoff is

$$\int_{\underline{\theta}}^{\theta^*} \Big(V(\theta, a^*(\underline{\theta})) - \frac{U(\theta^*, a^*(\underline{\theta}))}{U_a(\theta^*, a^*(\underline{\theta}))} V_a(\theta, a^*(\underline{\theta}))\Big) f(\theta)\, d\theta + \int_{\theta^*}^{\tilde{\theta}} \Big(V(\theta, \theta) - \phi\Big) f(\theta)\, d\theta.$$

Therefore an equivalent result for the Principal's expected payoff given $a^*(\underline{\theta})$ is

$$\max_{\theta^* \leq a^*(\underline{\theta})} \int_{\underline{\theta}}^{\theta^*} \Big(V(\theta, a^*(\underline{\theta})) - \frac{U(\theta^*, a^*(\underline{\theta}))}{U_a(\theta^*, a^*(\underline{\theta}))} V_a(\theta, a^*(\underline{\theta}))\Big) f(\theta)\, d\theta + \int_{\theta^*}^{\tilde{\theta}} \Big(V(\theta, \theta) - \phi\Big) f(\theta)\, d\theta.$$

Similar to the case that the Agent has only two types, there is no probability of inspection $x(.)$ in the above expressions. The ex-ante payoff of the Principal is equivalent to a deterministic inspection policy; inspecting $\theta > \theta^*$ with probability one and not inspecting $\theta \leq \theta^*$. The difference is that the mandated action $a^*(\underline{\theta})$ is higher than θ^* which is a loss for the Principal but instead there is a gain

$$-\int_{\underline{\theta}}^{\theta^*} \Big(\frac{U(\theta^*, a^*(\underline{\theta}))}{U_a(\theta^*, a^*(\underline{\theta}))} V_a(\theta, a^*(\underline{\theta}))\Big) f(\theta)\, d\theta > 0.$$

4 Without commitment

Assume the Principal does not have commitment power. Timing is as follows: the Agent privately observes the type θ and sends a message $m \in \mathbb{R}$. The Principal observes the message and inspects with probability $x(m) \in [0,1]$, and mandates an action $a^I(m,\theta) \in \mathbb{R}_+$ in case of inspection. If it does not inspect, the Principal can mandate an action $a^{NI}(m) \in \mathbb{R}_+$ (maybe different from $a^I(m,\theta)$). The Agent decides to accept or reject the mandated action. Figure 3 shows the timing of the model.

Strategies, and beliefs. The strategy of the Agent with type θ in pure strategies is to send a message $m(\theta) \in \mathbb{R}_+$.[14] The Principal strategy is $\big(x(m), a^I(m,\theta), a^{NI}(m)\big)$. The belief $\beta(\theta|m)$, of the Principal after observing message m, is a probability distribution over types.

Payoffs. The ex-post payoff of the Agent which sends message m, and has type θ if the Principal inspects is

$$U\left(\theta, a^I(m,\theta)\right) \mathbb{1}_{a^I(m,\theta) \geq \theta},$$

and if does not inspect is

$$U\left(\theta, a^{NI}(m)\right) \mathbb{1}_{a^{NI}(m) \geq \theta}.$$

The Principal's ex-post payoff if inspects is

$$V\left(\theta, a^I(m,\theta)\right) \mathbb{1}_{a^I(m,\theta) \geq \theta} - \phi,$$

[14]Here message (signal) space is $\mathbb{R}_+$.

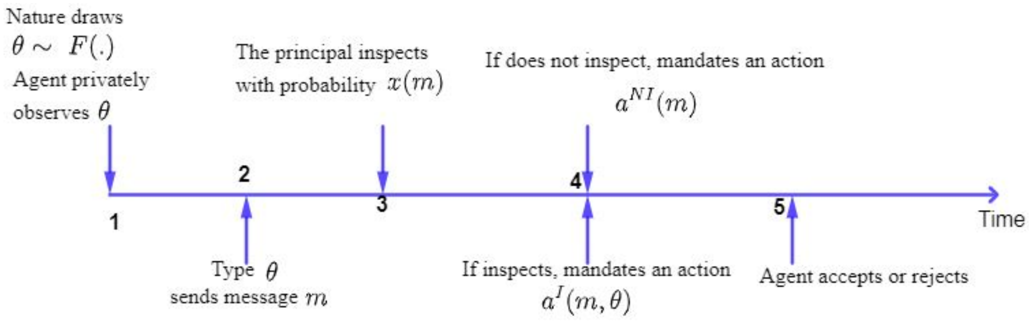

Figure 3: Timing without commitment

and if does not inspect is

$$V\left(\theta, a^{NI}(m)\right) \mathbb{1}_{a^{NI}(m)\geq\theta}.$$

Equilibrium. We focus on the Perfect Bayesian Equilibrium. The payoff of the Principal if inspects is decreasing in a^I, so the Principal optimally chooses $a^I(m,\theta) = \theta$ in a Perfect Bayesian Equilibrium. Define

$$E(m,\tilde{a}) \equiv \int_{\underline{\theta}}^{\overline{\theta}} \left[V\left(\theta,\tilde{a}\right) \mathbb{1}_{\tilde{a}\geq\theta}\right] \beta(\theta|m)\, d\theta.$$

By Assumption 2.

$$E(m,\tilde{a}) = \int_{\underline{\theta}}^{\tilde{a}} \left[V\left(\theta,\tilde{a}\right)\right] \beta(\theta|m)\, d\theta.$$

If the Principal does not inspect then she optimally chooses $a^{NI}(m)$, such that

$$a^{NI}(m) = \underset{\tilde{a}}{\operatorname{argmax}}\, E(m,\tilde{a}).$$

Let $V(m) = E(m, a^{NI}(m))$. The Principal's best reply given the strategy of the Agent is:

$$x(m) = \begin{cases} 1 & \mathbb{E}\big[V(\theta,\theta)|m\big] - \phi > V(m) \\ 0 & \mathbb{E}\big[V(\theta,\theta)|m\big] - \phi < V(m) \\ [0,1] & \mathbb{E}\big[V(\theta,\theta)|m\big] - \phi = V(m) \end{cases}$$

Finally the Agent with type θ chooses message $m(\theta)$ such that

$$m(\theta) \in \underset{\tilde{m}}{\operatorname{argmax}}\Big[\big(1-x(\tilde{m})\big)u(\theta, a^{NI}(\tilde{m}))\mathbb{1}_{a^{NI}(\tilde{m})\geq\theta} + x(\tilde{m})u(\theta, a^{I}(\tilde{m},\theta))\mathbb{1}_{a^{I}(\tilde{m},\theta)\geq\theta}\Big].$$

Let

$$a^* = \underset{\tilde{a}}{\operatorname{argmax}} \int_{\underline{\theta}}^{\tilde{a}} \left[V\left(\theta,\tilde{a}\right)\right] f(\theta)\, d\theta.$$

and

$$V^* = \int_{\underline{\theta}}^{a^*} \left[V\left(\theta,a^*\right)\right] f(\theta)\, d\theta,$$

In the absence of inspection, the Principal can reduce actions at least to $\bar{\theta}$, without incurring any cost. Therefore the final action of all types cannot be higher $\bar{\theta}$. Another noteworthy point is that when the inspection cost is strictly positive, a fully separated equilibrium does not exist. The reason is if the Agent chooses a strategy $m(\theta)$ such that fully separate the types (an injective function), then the Principal after observing the message, completely learns the type of the Agent. If the Principal knows the type of the Agent, then she mandates an action without inspection equal to the type, i.e. $a^{NI}(m(\theta)) = \theta$. This means that the profits of all types are zero. However, this cannot happen in equilibrium due to the inefficiency which is created by the cost of verifying information. In other words, the message should not contain full information.

The model is an extended cheap talk game. The Agent sends a signal, then the Principal chooses a final action (with inspection or without inspection), and finally the Agent accepts or rejects the action. In a cheap talk setting normally the fully pooling equilibrium exists (the babbling equilibrium). This is the case in our paper. Let $m(\theta) = s$, a constant function. By observing signal s the Principal cannot learn anything more than her prior $F(.)$. Now if the gain from inspecting is higher than the gain from not inspecting, the Principal inspects and mandates an action equal to the type of the Agent. Formally if

$$\mathbb{E}[V(\theta,\theta)] - \phi \geq V^* \;\Rightarrow\; x(s) = 1, \text{and } a^I(s,\theta) = \theta.$$

if the gain from inspecting is less than gain from not inspecting, the Principal does not inspect and mandates an action with inspection equal to the a^*.

$$\mathbb{E}[V(\theta,\theta)] - \phi < V^* \;\Rightarrow\; x(s) = 0, \text{and } a^{NI}(s) = a^*.$$

Now if we assume the belief of the Principal, off the equilibrium path is equal to the lowest type, then the Agent does not have any incentive to deviate from the pooling action. Formally if

$$\beta(\theta|m \neq s) = \begin{cases} 1 & \theta = \underline{\theta} \\ 0 & \theta \neq \underline{\theta}. \end{cases}$$

Then the best reply of all types of the Agent is to choose s. The pooling equilibrium is the worst equilibrium for the Principal since it generates the lowest payoff among all Perfect Bayesian Equilibria for the Principal. The reason is that the Principal does not learn any information on equilibrium. Later we examine the ex-ante payoff of the Principal more carefully.

The following Proposition characterizes the pooling equilibrium that we described.

Proposition 4 *(The Pooling equilibrium) Assume $m(\theta) = s$, for all $\theta \in [\underline{\theta}, \overline{\theta}]$. The following belief and strategies establish a pooling equilibrium*

$$\beta(\theta|m = s) = f(\theta), \quad \beta(\theta|m \neq s) = \begin{cases} 1 & \theta = \underline{\theta} \\ 0 & \theta \neq \underline{\theta}. \end{cases}$$

$$a^{NI}(m) = \begin{cases} a^* & m = s \\ \underline{\theta} & m \neq s \end{cases},$$

$$a^I(m, \theta) = \theta,$$

$$x(m) = \begin{cases} 1 & m = s, \ \textit{and}\ \phi \leq \mathbb{E}[V(\theta, \theta)] - V^* \\ 0 & \textit{otherwise} \end{cases}$$

The proof of Proposition 4 is in the Appendix.

Proposition 4 argues that if the inspection cost is not high ($\phi \leq \mathbb{E}[V(\theta, \theta)] - V^*$), then the Principal inspects the on-path equilibrium message s. This means that she inspects all types, so the payoffs of all types will be zero after inspection. If the inspection cost is high compare to the alternative policy ($\phi > \mathbb{E}[V(\theta, \theta)] - V^*$), then the Principal prefers to not inspect and mandates a^*.

In the next step we study all equilibria. We start with the following lemma

Lemma 4 *In all equilibria $a^{NI}(m)$ is a constant function for all m such that $x(m) < 1$.*

Proof. Given $m(\theta)$, let m_1, and m_2 be two messages on the equilibrium path, in which the Principal inspects with probability less than one, i.e. $x(m_1) < 1$, and $x(m_2) < 1$. Let $\Theta_i = \{\theta \in [\underline{\theta}, \overline{\theta}] : m(\theta) = m_i\}$ for $i \in \{1, 2\}$. By Assumption 4, for $i \in \{1, 2\}$ we have

$$\max_{\tilde{a}} \int_{\underline{\theta}}^{\tilde{a}} \Big[V(\theta, \tilde{a})\Big] \beta(\theta | m = m_i)\, d\theta > 0,$$

since inside of the integral for $\tilde{a} \leq \overline{\theta}$ is strictly positive. This means that the argument of the maximum of the above equation which by definition is $a^{NI}(m)$ is strictly bigger than the lowest possible type, i.e. $a^{NI}(m_i) > \inf\{\Theta_i\}$. Moreover for all ϵ we can find a type $\theta_i \in \Theta_i$ such that $a^{NI}(m_i) - \theta_i < \epsilon$. Otherwise $a^{NI}(m_i)$ is not optimal and the Principal can decrease it. Using this fact we want to show $a^{NI}(m_1) = a^{NI}(m_2)$. By contradiction assume $a^{NI}(m_1) > a^{NI}(m_2)$, then types in Θ_2 that are very close but less that $a^{NI}(m_2)$ want to deviate and send message m_1. Formally

$$(1 - x(m_1))U(\theta, a^{NI}(m_1))\mathbb{1}_{a^{NI}(m_1) \geq \theta} < (1 - x(m_2))U(\theta, a^{NI}(m_2))\mathbb{1}_{a^{NI}(m_2) \geq \theta},$$

for $\theta \in \Theta_2$, and $a^{NI}(m_2) - \theta < \epsilon$. For small enough ϵ the left side goes to zero, and the right side is always higher than a positive amount. This is a contradiction. Hence $a^{NI}(m_1) = a^{NI}(m_2)$.

∎

Lemma 4 states that the mandated action without inspection should be unique. The intuition is as follows: if there are two different mandated actions, types that are very close to the lower action want to mimic types with the higher action. This fact is independent of the probability of inspection (if it is strictly less than one). The reason is that always there is a type very close to an action, otherwise the Principal can reduce that action. As a consequence of Lemma 4, the next Lemma shows, in a equilibrium, there are maximum two probability inspections.

Lemma 5 *In an equilibrium if $x(m_1) < 1$ and $x(m_2) < 1$, then $x(m_1) = x(m_2)$.*

Proof. By contradiction assume $x(m_1) > x(m_2)$. By lemma 4 $a^{NI}(m_1) = a^{NI}(m_2)$. Let $\Theta_i = \{\theta \in [\underline{\theta}, \overline{\theta}] : m(\theta) = m_i\}$ for $i \in \{1, 2\}$. Then types in Θ_1 that are less than $a^{NI}(m_1)$ want to deviate to m_2. A contradiction.

■

Now we study the strategy of the Agent. We want to show that given an equilibrium we can find another equilibrium in which the set of equilibrium messages of the Agent has a maximum of two elements, i.e. $|\{m(\theta) : \theta \in [\underline{\theta}, \overline{\theta}]\}| \leq 2$. First, let us introduce some useful definitions and notations.

Definitions and Notation: Given $m(\theta)$, let M_{NI} be the set of on the equilibrium path messages, in which the Principal inspects with probability less than one, i.e. $x(m) < 1$, for all $m \in M_{NI}$. Similarly define M_I be the set of on the equilibrium path messages, in which the Principal inspects with probability one, i.e. $x(m) = 1$, for all $m \in M_I$. Let

$$\Theta_I = \{\theta \in [\underline{\theta}, \overline{\theta}] : \ x(m(\theta)) = 1\},$$

$$\Theta_{NI} = \{\theta \in [\underline{\theta}, \overline{\theta}] : \ x(m(\theta)) < 1\},$$

and

$$\Theta_m = \{\theta \in [\underline{\theta}, \overline{\theta}] : m(\theta) = m\},$$

for all $m \in \mathbb{R}_+$.

Lemma 6 *Fix an equilibrium with a strategy $\big(x(m), a^I(m, \theta) = \theta, a^{NI}(m)\big)$ and belief $\beta(\theta|m)$ for the Principal, and a strategy $m(\theta)$ for the Agent. Given two messages $m_I \in M_I$ and $m_{NI} \in M_{NI}$ define the strategy $\big(\check{x}(m), \check{a}^I(m, \theta) = \theta, \check{a}^{NI}(m)\big)$ and the belief $\check{\beta}(\theta|m)$ for the Principal and the strategy $\check{m}(\theta)$ for the Agent as follows*

- *Strategies:*

$$\check{m}(\theta) = \begin{cases} m_I & \theta \in \Theta_I, \\ m_{NI} & \theta \in \Theta_{NI}, \end{cases}$$

$$\check{x}(m) = \begin{cases} 1 & m = m_I, \\ 0 & otherwise \end{cases}$$

$$\check{a}^{NI}(m) = \begin{cases} a^{NI}(m) & m \in \{m_I, m_{NI}\}, \\ \underline{\theta} & otherwise \end{cases}$$

- *Beliefs: On the equilibrium path, beliefs are consistent with strategies and the off-path belief puts probability 1 on $\underline{\theta}$, i.e.*

$$\check{\beta}(\underline{\theta}|m \neq m_I, m_{NI}) = 1.$$

1. *The new strategies and beliefs form an equilibrium.*

2. *The ex-ante payoff of the Principal under the new strategies and beliefs remains the same.*

The proof of Lemma 6 is in the Appendix.

Lemma 6 states that in order to characterize all equilibrium payoffs for Principal, we do not need think about complex functions of $m(\theta)$, the strategy of the Agent. If we restrict our attention to all functions with two values, we can find all equilibria payoffs. Equivalently we can restrict the space of messages two values instead of $\mathbb{R}_+$. We can think about the strategy of the Agent as follows: A type θ of the Agent sends a message m_I high enough in which the Principal decides to inspect, simply type θ says "inspect me", and the Principal inspects. Another type θ of the Agent sends a message θ_{NI} low enough in which the Principal decides to not inspect, or inspect with probability less than one. Simply type θ says "do not inspect me", and the Principal does not inspect, or inspect with probability less than one.

In other words, the strategy of the Agent is separating and pooling at the same time. Type $\theta \in \Theta_I$ pool together in one message. Type $\theta \in \Theta_{NI}$ pool together in another message. But they separate from each other by choosing different messages. The reader may wonder this is a truthfully report, but types of the Agent do not reveal their full information which is their types. They reveal in which group of the types are they, either in a group that the Principal inspect with probability one or in a group that the Principal inspects with probability less than one.

Lemma 6 does not describe the exact form of the Agent's strategy. It does not explain what is the set of Θ_I, the types that send m_I, and what is the set Θ_{NI}, the types that sends m_{NI}. The following Proposition characterizes the set Θ_I, and Θ_{NI} for the maximum ex-ante payoff of the Principal.

Proposition 5 *Let a strategy* $\big(x(m), a^I(m,\theta) = \theta, a^{NI}(m)\big)$ *and belief* $\beta(\theta|m)$ *for the Principal and a strategy* $m(\theta)$ *for the Agent with a form of Lemma 6 generate the maximum ex-ante payoff for the Principal. There exists another equilibrium with a strategy* $\big(\check{x}(m), \check{a}^I(m,\theta) = \theta, \check{a}^{NI}(m)\big)$ *and belief* $\check{\beta}(\theta|m)$ *for the Principal and a strategy* $\check{m}(\theta)$ *for the Agent such that:*

1. *Strategies of the Principal remain the same, i.e.* $\check{x}(m) = x(m)$, $\check{a}^{NI}(m) = a^{NI}(m)$, *and* $\check{a}^I(m,\theta) = a^I(m,\theta)$.

2. *There exists* $s^*, s^{**} \in [\underline{\theta}, \overline{\theta}]$ *with,* $s^* \leq s^{**}$ *such that*

$$\check{m}(\theta) = \begin{cases} m_{NI} & \theta \in [\underline{\theta}, s^*] \cup [s^{**}, \overline{\theta}], \\ m_I & \theta \in (s^*, s^{**}) \end{cases}$$

3. *On the equilibrium path, beliefs are consistent with strategies and the off-path belief puts probability* 1 *on* $\underline{\theta}$, *i.e.*

$$\check{\beta}(\underline{\theta}|m \neq m_I, m_{NI}) = 1.$$

4. *The ex-ante payoff of the Principal and the ex-post payoff of the Agent remain the same.*[15]

The proof of Proposition 5 is in Appendix.[16]

Assume $\check{x}(m_{NI}) > 0$. Since $\check{a}^{NI}(m_{NI}) \leq s^*$, if the Principal reduces $\check{x}(m_{NI})$ to zero, the incentive of types do not change. Furthermore, $\check{x}(m_{NI}) > 0$ means that the Principal is indifferent to inspect or not the message m_{NI}. Therefore reducing $\check{x}(m_{NI})$ to zero does not affect Principal's payoff. Thus for the maximum ex-ante payoff we can focus on deterministic inspection strategies.

Following Lemma 6, Proposition 5 states that for the equilibrium with the maximum ex-ante payoff, the Principal can restrict the strategy of the Agent to $\check{m}(.)$. There are three regions. Efficient types ($\theta \leq s^*$) chooses m_{NI}, and the Principal mandates an action without inspection. Intermediate types ($s^* < \theta < s^{**}$) set send m_I, and the Principal mandates an action with inspection. Inefficient types ($\theta \geq s^{**}$) sends the same message as the efficient types (pool in message m_{NI}), and the Principal mandates an action without inspection. The mandated action is always less than their types, so these types rejects the action, which means that the Principal excludes them. [17]

The structure of the maximum payoff equilibrium without commitment (Proposition 5), and the above explanation is very close to the structure of the optimal policy with commitment, and when only the deterministic inspection is available (Proposition 1). However, thresholds s^*, and s^{**} are not necessarily the same as thresholds in Proposition 1. In the next section, We study the effect of the commitment ability of the Principal on the implementable policies.

5 Commitment versus partial commitment

In this section, we study under what conditions the Principal can reach the commitment payoff with the deterministic inspection (Proposition 1).

[15]Thresholds s^*, and s^{**} are different from thresholds in Proposition 1.

[16]Conjecture: the Proposition works for all equilibrium payoffs.

[17]Consider an equilibrium in which there are two types $\theta_1 > \theta_2$ such that $m(\theta_1) = m_I$, and $m(\theta_2) = m_{NI}$ (by Lemma 6 we restrict the strategy of the Agent to these two messages). This equilibrium is not stable since by assuming a small benefit (positive, but we can consider the limit when it goes to zero) from not being excluded (a positive profit for running the firm for instance) then θ_2 prefers to send message m_I. By this restriction we can think about **all** equilibria payoffs with only one threshold. All types below a threshold send the message m_{NI}, and types above send m_I. Now when the inspection cost is low $\phi < V(\overline{\theta}, \overline{\theta})$, the exclusion region does not exist and the maximum payoff equilibrium is stable.

First, if the Principal cannot commit to the mandated action with inspection ($a^I(.,.)$), the payoff and the optimal inspection policy do not change. The reason is that with or without commitment the Principal chooses the same policy, always $a^I(\theta,\theta) = \theta$.

Second, if the Principal cannot commit to any policies, from Proposition 5, we know if thresholds s^*, and s^{**} become the same as thresholds in Proposition 1, then the equilibrium reaches the maximum ex-ante payoff for the Principal which is the commitment payoff. This is not the case always; thresholds s^*, and s^{**} are not necessarily the same as thresholds in Proposition 1. First, threshold s^* can be different from θ^* which means that the Principal needs to inspect more or less than optimal. Second, threshold s^{**} can be different from θ^{**}, so the Principal may exclude more types or keep (instead of excluding) some inefficient types.

What is the main policy that the Principal gains the most by committing to it? To answer this question we restrict our attention to the situation that the inspection cost is not very high i.e. $\phi < V(\overline{\theta}, \overline{\theta})$. [18] Based on Proposition 1, in this case the Principal does not exclude types, i.e. $\theta^{**} = \overline{\theta}$.

Let us assume the Principal can **only** commit to the inspection when the Agent requests an inspection. In other word the principal cannot deny to inspect while the Agent requests for an inspection. This is weaker than assuming the Principal can commit to the inspection policy (for all messages). Formally assume there exists a "safe" message m_I, such that if the Agent sends this message, the Principal commits to inspect. Therefore the Principal cannot exclude or cannot refuses to inspect the Agent. Note that the Principal does not need to commit to inspect or not inspect other messages.

Proposition 6 *(Partial commitment) If the inspection cost is not high, i.e. $\phi < V(\tilde{\theta}, \tilde{\theta})$, and if the Principal can commit to inspect message m_I, then there exists an equilibrium such that the ex-ante payoff of the Principal is the same as the full commitment payoff with deterministic inspection.*

The proof of Proposition 6 is in the Appendix. In the next section, we examine the applications of the model.

6 Applications

6.1 The funding agency and the researcher

Consider an agency (Principal) awards grants for research proposals. A researcher (Agent) requests a budget (action) for a project. Let us assume the following utilities for the

[18]If the support of distribution is infinity, then this situation correspondence to the case that $1 - F(\tilde{\theta})$ is small, where $\phi = V(\tilde{\theta}, \tilde{\theta})$.

Principal and the Agent

$$V(\underbrace{e}_{type}, \overbrace{b}^{action}) = \big(v - b + \alpha(b-e)\big)\mathbb{1}_{b\geq e},$$

$$U(e,b) = \big(b-e\big)\mathbb{1}_{b\geq e}.$$

The type of the researcher is e, which is the amount of the effort that he has to exert to run the project. The requested budget is b, and the value of the project for the Principal is a known parameter $v > \bar{e}$. Similar to the main model assume $e \sim F(.)$, and assume the distribution function $F(.)$ satisfies the monotone hazard rate i.e. $(F(e)/f(e))' > 0$, for all $e \in [\underline{e}, \bar{e}]$. We show that the stochastic inspection is not helpful with these preferences.

It is easy to see that assumptions 1 to 5 hold in this environment. First we derive the optimal policy when only the deterministic inspection is available. The solution of problem $\mathbb{P}$ (in Proposition 1 is e^*, and it solves $F(e^*)/f(e^*) = \phi/(1-\alpha)$. Using Proposition 1, the optimal policy is as follows: $b^I(\hat{e}, e) = e$, and
(i) If $\phi \leq V - \bar{e}$, then:

$$x(e) = \begin{cases} 0 & e \leq e^* \\ 1 & e > e^*, \end{cases} \text{ and } b^{NI}(e) = \begin{cases} e^* & e \leq e^* \\ e & e > e^*. \end{cases}$$

(ii) If $V - e^* \geq \phi > V - \bar{e}$, then:

$$x(e) = \begin{cases} 0 & e \leq e^* \\ 1 & e^{**} \geq e > e^* \\ 0 & e > e^{**}, \end{cases} \; b^{NI}(e) = \begin{cases} e^* & e \leq e^* \\ e & e^{**} \geq e > e^* \\ \underline{e} & e > e^{**}, \end{cases}$$

where $e^{**} = V - \phi$.
(iii) If $V - e^* \leq \phi$ then

$$x(e) = 0, \text{ and } b^{NI}(e) = \begin{cases} e^{**} & c \leq e^{**} \\ \underline{e} & e > e^{**}, \end{cases}$$

where e^{**} solves $(1-\alpha)F(e^{**}) = (V - e^{**})f(e^{**})$. The cutoff e^* is the point at which the marginal benefit of the Principal cancels the marginal loss. To see that, at point e^*, the gain by inspecting is $F(e^*)$. The reason is that all types below e^* should decrease their requested budget which increases the ex-ante payoff of the Principal by $F(e^*)$. The ex-ante cost of inspection at point e^* is the probability of inspection $f(e^*)$, times the normalized cost $\phi/(1-\alpha)$.

Now assume the Principal can commit to a stochastic inspection. From Proposition 2, we have

$$\frac{d\,\Psi}{de} = \psi,$$

where $\Psi(e, b^*(e)) = \phi f(e)$, and $\psi(e, b^*(e)) = (1-\alpha)f(e)$. Solving the above differential equation we get

$$\phi f(e) = (1-\alpha)F(e).$$

The distribution function $F(.)$ satisfies the monotone hazard rate, therefore the answer of the equation is only one point e^*. In other words, the policy is deterministic.

6.2 The environmental protector and the polluter firm (regulation with unknown control cost)

An environmental protector (Principal), EPA for example, aims to control the amount of the harm (action) of a potential polluter (Agent). EPA is uncertain about the characteristic of the polluter. EPA does not know what is the technology of the polluter, therefore does not know the minimal level of the harm that the agent can produce. The following utilities are for the Principal and the Agent respectively

$$V(\underbrace{t}_{type}, \overbrace{h}^{action}) = \big(v - h\big)\mathbb{1}_{h \geq t}$$

$$U(t, h) = \big(R + (h-t)^2\big)\mathbb{1}_{h \geq t}$$

v is the value of the production for the Principal. h is the amount of harm that the polluter generates. The production revenue of the polluter is R. The type t of the polluter is the technology. For example the technology of reducing the amount of producing harmful chemical material (carbon, methane, ...), or using harmful material in production (palm oil, paraffin, alcohol, ...). Another example is a refinery that is using river's water for cooling the constructions and increases the temperature of a river. Of course using harmful material is easier and less costly for the firm. So the profit of the firm is **increasing** in the amount of the harm $h \leq \overline{h}$. And is increasing function of the difference of the harm and technology $h - t$, and it goes to zero when $h = t$. The environmental protector does not know t, so does not know how much she can force the firm to reduce the harm. Using Proposition 2, one can show the stochastic inspection is helpful, and generates a strictly higher payoff for the Principal compared to the optimal deterministic inspection mechanism.

6.3 Monopoly price regulation

The regulator (Principal) wants to regulate the price p of a monopolist (Agent). The monopolist faces a weakly decreasing and differentiable demand $D(.)$ of consumers, and has cost $c \in [\underline{c}, \bar{c}]$. The cost is the monopolist's private information. The payoffs of the regulator and the monopolist are as follows:

$$V(\underbrace{c}_{type}, \overbrace{p}^{action}) = \big(CS(p) + \alpha D(p)(p-c)\big)\mathbb{1}_{p \geq c}.$$

$$U(c,p) = D(p)(p-c)\mathbb{1}_{p\geq c},$$

where $0 \leq \alpha < 1$, and $CS(p)$ is the consumer surplus is at price p. Therefore regulator's payoff is the social surplus which is a weighted sum of consumer surplus and profit of the monopolist.

Assumption 5 guarantees that the profit of the monopolist is increasing for prices less than $\bar{c}$. Normally the profit of the monopolist is increasing before the monopoly price. Moreover, this assumption says the support of this distribution (the prior information of the regulator) is less than the monopoly prices. In other words, the prior information of the monopolist is not "too bad", and she can distinguish between the cost and the monopoly prices.

Another application is the regulation of the monopolist with unknown outside option. Assume the regulator knows the cost c of the the monopolist, and for simplicity normalize this cost to zero. However, the outside option is the monopolist's private information. We can write the payoff as follows:

$$V(\underbrace{c}_{type}, \overbrace{p}^{action}) = \big(CS(p) + \alpha p D(p)\big)\mathbb{1}_{U\geq u}.$$

$$U(c,p) = pD(p)\mathbb{1}_{U\geq u}.$$

7 Conclusion

In this paper we consider a Principal-Agent model in which the Agent has a private information. This private information is the type of the Agent which can be the outside option, technology or production cost. The Principal can gather information about this type using a costly information verification instrument. The Principal can mandate an action with or without inspection. However, transfers are ruled out, and the Agent is opportunistic; he has the opportunity to refuse to undertake the action.

The paper finds the optimal policy when the inspection policy is restricted to deterministic mechanism is simply a cap on actions, and inspection of an interval of types (intermediate types) above the cap. When the stochastic inspection is available for the Principal, the Principal inspects more types (with less probability of inspection), and at the same time mandates higher actions. The optimal inspection policy inspects inefficient types with a higher probability than efficient types.

If the Principal cannot commit to the policy, the model becomes an extended cheap talk game. We show the stochastic inspection (mixed strategies for inspection) is not useful for the Principal, and for finding all equilibria payoffs we can restrict the strategy of the Agent to at most two messages. Each group pools at one message, and they separate from each other by setting different messages. The best equilibrium for the Principal is the one with a similar structure to the commitment policy (with deterministic inspection), but with different thresholds. If the inspection cost is not high, and if the Principal grantees to inspect once the Agent requests an inspection, then the Principal can achieve the commitment payoff. This fact highlighters the importance of the commitment ability on the inspection instrument.

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8 Appendix

Proof of Proposition 1.
Due to the maximum punishment rule, and Assumption 2, $a^I(\hat{\theta},\theta) \leq \theta$ if $\hat{\theta} \neq \theta$. Observe that if $a^{NI}(\theta) < \theta$, then the mechanism can choose $a^{NI}(\theta) = \underline{\theta}$. Now given a mechanism $(x(\hat{\theta}), a^I(\hat{\theta},\theta), a^{NI}(\hat{\theta}))$, define $\theta^* = \sup\{a^{NI}(\theta) | x(\theta) = 0\}$. Assumption 2, and the global IC imply that

$$\begin{aligned} x(\theta)\Big(u(\theta, a^I(\theta,\theta))\Big)\mathbb{1}_{a^I(\theta,\theta)\geq\theta} + (1-x(\theta))\Big(u(\theta, a^{NI}(\theta))\Big)\mathbb{1}_{a^{NI}(\theta)\geq\theta} \\ \geq x(\hat{\theta})\Big(u(\theta, a^I(\hat{\theta},\theta))\Big)\mathbb{1}_{a^I(\hat{\theta},\theta)\geq\theta} + (1-x(\hat{\theta}))\Big(u(\theta, a^{NI}(\hat{\theta}))\Big)\mathbb{1}_{a^{NI}(\hat{\theta})\geq\theta} \\ = (1-x(\hat{\theta}))\Big(u(\theta, a^{NI}(\hat{\theta}))\Big)\mathbb{1}_{a^{NI}(\hat{\theta})\geq\theta}. \end{aligned}$$

for all $\hat{\theta} \neq \theta$, and $(\theta,\hat{\theta}) \in [\underline{\theta},\overline{\theta}]^2$. The last inequality comes from the fact that either $a^I(\hat{\theta},\theta) = \theta$, then $u(\theta,\theta) = 0$ (by Assumption 2), or $a^I(\hat{\theta},\theta) < \theta$, then $u(\theta, a^I(\hat{\theta},\theta)\mathbb{1}_{a^I(\hat{\theta},\theta)\geq\theta} = 0$. Thus we have

$$x(\theta)\Big(u(\theta, a^I(\theta,\theta))\Big)\mathbb{1}_{a^I(\theta,\theta)\geq\theta} + (1-x(\theta))\Big(u(\theta, a^{NI}(\theta))\Big)\mathbb{1}_{a^{NI}(\theta)\geq\theta} \geq \Big(u(\theta,\theta^*)\Big)\mathbb{1}_{\theta^*\geq\theta}.$$

First it means that if $x(\theta) = 0$, and $a^{NI}(\theta) = \underline{\theta}$, then $\theta^* \leq \theta$. Second if $x(\theta) = 0$, and $a^{NI}(\theta) \neq \underline{\theta}$ means that $a^{NI}(\theta) \geq \theta$, and then $a^{NI}(\theta) \geq \theta^*$. However, by the definition of θ^*, we conclude $a^{NI}(\theta) = \theta^*$. Therefore if $x(\theta) = 0$, then

$$a^{NI}(\theta) = \begin{cases} \theta^* & \theta > \theta^* \\ \{\theta^*, \underline{\theta}\} & \theta = \theta^* \\ \underline{\theta} & \theta < \theta^*. \end{cases} \tag{7}$$

Third if $x(\theta) = 1$, and $\theta \leq \theta^*$, then $a^I(\theta,\theta) \geq \theta^*$. On the other hand, we know if $x(\theta) = 1$, then $a^I(\theta,\theta) \geq \theta$. To see this by contradiction assume $a^I(\theta,\theta) < \theta$, then if the Principal chooses $x(\theta) = 0$, and $a^{NI}(\theta) = \underline{\theta}$, will have higher payoff with no effect on the global IC. The higher payoff comes from the fact that inspection has a positive cost $\phi > 0$. Thus from IC two necessary conditions are $a^I(\theta,\theta) \geq \max\{\theta^*,\theta\}$, and condition 7.

Rewrite the Principal's problem

$$\max_{x(.), a^I(.,.), a^N(.)} \int_{\underline{\theta}}^{\overline{\theta}} \Big[x(\theta)V(\theta, a^I(\theta,\theta)) + (1-x(\theta))V(\theta, a^{NI}(\theta))\mathbb{1}_{a^{NI}(\theta)\geq\theta} - \phi x(\theta)\Big]\, dF(\theta),$$

subject to the IC constraints. For the moment we consider weaker conditions, that are $a^I(\theta,\theta) \geq \max\{\theta^*,\theta\}$, and condition 7. Later we check the global IC condition. By Assumption 5, the objective function is decreasing in $a^I(\theta,\theta)$. Therefore $a^I(\theta,\theta) = \max\{\theta^*,\theta\}$. Rewriting the objective function

$$\max_{x(.),\theta^*} \left\{ \int_{\underline{\theta}}^{\overline{\theta}} \Big[x(\theta)\Big(V(\theta, \max\{\theta^*,\theta\}) - \phi\Big) + (1-x(\theta))\Big(V(\theta,\theta^*)\Big)\mathbb{1}_{\theta^*\geq\theta}\Big]\, df(\theta) \right\}.$$

Now if $\theta \leq \theta^*$, then

$$V(\theta, \max\{\theta^*, \theta\}) - \phi < V(\theta, \theta^*),$$

therefore the optimal policy chooses $x(\theta) = 0$ for all $\theta \leq \theta^*$. For $\theta > \theta^*$, the optimal policy chooses $x(\theta) = 1$, iff $V(\theta, \theta) - \phi \geq 0$. Therefore the objective becomes to solve problem $\mathbb{P}$

$$\max_{\theta^* \in [\underline{\theta}, \overline{\theta}]} \left\{ \int_{\theta^*}^{\overline{\theta}} \Big(V(\theta, \theta) - \phi \Big) \mathbb{1}_{V(\theta,\theta) \geq \phi} \, dF(\theta) + \int_{\underline{\theta}}^{\theta^*} \Big(V(\theta, \theta^*) \Big) \, dF(\theta) \right\}.$$

The above explanation, and the optimal θ^* for problem $\mathbb{P}$, together suggests that the following policy is optimal for the weaker IC conditions.

$$a^I(\hat{\theta}, \theta) = \theta,$$

$$x(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} \leq \theta^* \\ 1 & \theta^{**} \geq \hat{\theta} > \theta^* \\ 0 & \hat{\theta} > \theta^{**}, \end{cases}$$

$$a^{NI}(\theta) = \begin{cases} \theta^* & \hat{\theta} \leq \theta^* \\ \theta & \theta^{**} \geq \hat{\theta} > \theta^* \\ \underline{\theta} & \hat{\theta} > \theta^{**}, \end{cases}$$

Now we have to show the above policy is globally IC. The argument is as follows. Types $\theta \leq \theta^*$, cannot mimic types $\hat{\theta} \geq \theta^*$, since they are either inspected or excluded. Types $\theta \leq \theta^*$, are indifferent to mimic types $\hat{\theta} < \theta^*$, since $a^{NI}(\hat{\theta}) = \theta^*$. Types $\theta > \theta^*$, cannot mimic types $\hat{\theta} \geq \theta^*$, since they are either inspected or excluded. Types $\theta > \theta^*$, cannot mimic types $\hat{\theta} < \theta^*$, since $a^{NI}(\hat{\theta}) = \theta^* < \theta$.

■

Proof of Lemma 1.
(i) When $a(\theta) < \theta$, it means that the Principal excludes type θ, if she does not inspect. In this case, type θ's expected payoff does not change if the principal decreases $a(\theta)$ to $\underline{\theta}$.
(ii) Let $a(\theta) = \underline{\theta}$. Since the expected payoff is a weakly decreasing function, then for all types $\theta' > \theta$, we have $a(\theta') \leq \theta'$. If $a(\theta') < \theta'$, then we can assume $a(\theta') = \underline{\theta}$. If $a(\theta') = \theta'$, then $x(\theta') = 1$, otherwise type θ, can mimic θ'. If $x(\theta') = 1$, the Principal can strictly gain by switching the policies for type θ, and θ'. Excluding type θ, while keeping θ' is not efficient; i.e $V(\theta, \theta) - \phi > V(\theta', \theta') - \phi$.
(iii) If $a(\theta) \leq \theta$, then either $a(\theta') \leq \theta'$ or $x(\theta') = 1$ for all $\theta' > \theta$. If $a(\theta') < \theta'$, we can assume $a(\theta') = \underline{\theta}$. If $a(\theta') = \theta'$, then $x(\theta') = 1$, otherwise θ can mimic θ'.

■

Proof of Proposition 3.

The derivative respect to θ^* gives us

$$\frac{-U_a(\theta^*, a^*(\underline{\theta}))U_\theta(\theta^*, a^*(\underline{\theta})) + U(\theta^*, a^*(\underline{\theta}))U_{\theta a}(\theta^*, a^*(\underline{\theta}))}{U_a^2(\theta^*, a^*(\underline{\theta}))} \int_{\underline{\theta}}^{\theta^*} V_a(\theta, a^*(\underline{\theta}))f(\theta)\, d\theta$$
$$- \Big(V(\theta^*, a^*(\underline{\theta})) - V(\theta^*, \theta^*) + \phi - \frac{U(\theta^*, a^*(\underline{\theta}))}{U_a(\theta^*, a^*(\underline{\theta}))} V_a(\theta^*, a^*(\underline{\theta}))\Big) f(\theta^*).$$

Using equations 5, and 2 we get

$$\frac{\big(U_a(\theta^*, a^*(\underline{\theta}))U_\theta(\theta^*, a^*(\underline{\theta})) - U(\theta^*, a^*(\underline{\theta}))U_{\theta a}(\theta^*, a^*(\underline{\theta}))\big)\mu(\theta^*)}{U_a(\theta^*, a^*(\underline{\theta}))}$$
$$\dot{\mu}(\theta)U(\theta, a^*(\theta)) + \mu(\theta)\, U_\theta(\theta, a^*(\theta)) + \frac{U(\theta^*, a^*(\underline{\theta}))}{U_a(\theta^*, a^*(\underline{\theta}))} V_a(\theta^*, a^*(\underline{\theta}))f(\theta^*).$$

Employing equation 4, we find that the above equation is zero.

■

Proof of Proposition 4.[19]

The Agent does not have any incentive to deviate. If the Agent with type c deviates from $m(\theta) = s$, then the Principal does not inspect and mandates a price equal to $\underline{\theta}$, i.e. $a^{NI}(p \neq s) = \underline{\theta}$. The mandated price generates zero profit for the Agent.

The Principal does want to deviate. If the Principal observe a price not equal to s (off-path signal), then based on the belief she should not inspect and mandates a price equal to $\underline{\theta}$, i.e. $a^{NI}(p \neq s) = \underline{\theta}$. Now consider two cases. First $V^* \leq \mathbb{E}[V(c) - cD(c)] - \phi$. After observing s, the Principal optimally should inspect, since the value of inspection is higher. Second $V^* > \mathbb{E}[V(c) - cD(c)] - \phi$. After observing s, the Principal optimally should not inspect, since the value of inspection is higher. The optimal price regulation without inspection is by definition p^*, i.e. $a^{NI}(p = s) = p^*$.

■

Proof of Lemma 6

Call the first equilibrium (strategy $\big(x(m), a^I(m, \theta) = \theta, a^{NI}(p)\big)$ and belief $\beta(\theta|m)$ for the Principal, and a strategy $m(\theta)$ for the Agent) S, and the suggested one (strategy $\big(\check{x}(m), \check{p}^I(m, \theta) = \theta, \check{a}^{NI}(m)\big)$ and belief $\check{\beta}(\theta|m)$ for the Principal and a strategy $\check{m}(\theta)$ for the Agent) $\check{S}$.

1) The Agent does not have any incentive to deviate under $\check{S}$. Type $\theta \in \Theta_I$ has a zero payoff under S, when this type does not choose m_{NI} it means that sending this message generates a zero payoff for this type, or in other words $a^{NI}(m_{NI}) \leq \theta$. Since $\check{a}^{NI}(m_{NI}) = a^{NI}(m_{NI})$, type θ should not have any incentive to send message m_{NI} under $\check{S}$. A type $\theta \in \Theta_{NI}$ does not have any incentive to send messages other than m_{NI} since the Principal inspects all of other messages, and the profit after inspection is zero.

[19](To be updated for the general utility functions)

The Principal does not have any incentive to deviate under $\check{S}$. Assume the Principal observes message m_{NI}, we know S is an equilibrium so

$$
\begin{aligned}
M(\Theta_m) &\int_{\underline{\theta}}^{a^{NI}(m)} \Big[V\left(\theta, a^{NI}(m)\right)\Big] \beta(\theta|m)\, d\theta \\
&\geq M(\Theta_m)\Big(\int_{\underline{\theta}}^{\overline{\theta}} \Big[V\left(\theta, a^{I}(m,\theta)\right) \mathbb{1}_{a^I(m,\theta)\geq\theta} - \phi\Big] \beta(\theta|m)\, d\theta\Big) \\
&= M(\Theta_m)\Big(\int_{\underline{\theta}}^{\overline{\theta}} \Big[V\left(\theta, \theta\right) - \phi\Big] \beta(\theta|m)\, d\theta\Big)
\end{aligned}
$$

for all $m \in M_{NI}$. Where $M(.)$ is a measure of a set. Therefore under $\check{S}$ does not have incentive to deviate since for all $m \in M_{NI}$, $\check{a}^{NI}(m_{NI}) = a^{NI}(m)$, and

$$
\begin{aligned}
&\int_{\underline{\theta}}^{\check{a}^{NI}(m_{NI})} \Big[V\left(\theta, a^{NI}(m)\right)\Big] \left(\sum_{m\in M_{NI}} M(\Theta_m)\beta(\theta|m)\right) d\theta \\
&\qquad \geq \int_{\underline{\theta}}^{\overline{\theta}} \Big[V\left(\theta, \theta\right) - \phi\Big] \left(\sum_{m\in M_{NI}} M(\Theta_m)\beta(\theta|m)\right) d\theta.
\end{aligned}
$$

Assume the Principal observes message m_I, S is an equilibrium so

$$
\begin{aligned}
&M(\Theta_m) \int_{\underline{\theta}}^{\overline{\theta}} \Big[V\left(\theta, \theta\right) - \phi\Big] \beta(\theta|m) d\theta \\
&\geq \max_{\tilde{a}} \int_{\underline{\theta}}^{\tilde{a}} \Big[V\left(\theta, \tilde{a}\right)\Big] \left(M(\Theta_m)\beta(\theta|m)\right) d\theta,
\end{aligned}
$$

for all $m \in M_I$. Therefore we have

$$
\begin{aligned}
\int_{\underline{\theta}}^{\overline{\theta}} \Big[V(\theta,\theta)-\phi\Big] \left(\sum_{m\in M_I} M(\Theta_m)\beta(\theta|m)\right) d\theta &\geq \sum_{m\in M_I} \max_{\tilde{a}} \int_{\underline{\theta}}^{\tilde{a}} \Big[V\left(\theta, \tilde{a}\right)\Big] \left(M(\Theta_m)\beta(\theta|m)\right) d\theta \\
&\geq \max_{\tilde{a}} \int_{\underline{\theta}}^{\tilde{a}} \Big[V\left(\theta, \tilde{a}\right)\Big] \left(\sum_{m\in M_I} M(\Theta_m)\beta(\theta|m)\right) d\theta.
\end{aligned}
$$

The second inequality is due to the fact that the sum of the maximum of each term is weakly higher than the maximum of sum of the terms. This means under $\check{S}$ the Principal inspects if observes message m_I. Other messages are off-path equilibrium and given the beliefs the Principal does not have incentive to deviate.

2) The ex-ante payoff of the Principal remains the same. The ex-ante payoff of the Principal if observes messages M_{NI} under S is

$$
\sum_{m\in M_{NI}} \frac{M(\Theta_m)}{\sum_{m\in M_{NI}} M(\Theta_m)} \int_{\underline{\theta}}^{a^{NI}(m)} \Big[V(\theta, a^{NI}(m))\Big] \beta(\theta|m) d\theta,
$$

where $M(\Theta_m)$ is the mass of types that send m. The Principal's ex-ante payoff by observing m_{NI} under $\check{S}$ is

$$\max_{\tilde{a}} \int_{\underline{\theta}}^{\tilde{a}} \Big[V(\theta, \tilde{a})\Big] \left(\sum_{m \in M_{NI}} \frac{M(\Theta_m)\beta(\theta|m)}{\sum_{m \in M_{NI}} M(\Theta_m)} \right) d\theta.$$

The sum of the maximum of each term (the first equation) is weakly higher than the maximum of sum of the terms (the second equation). By Lemma 4 we know $a^{NI}(m) = a^{NI}(m^{'})$ for all $m, m^{'} \in M_{NI}$. If we choose $\tilde{a} = a^{NI}(m)$ for the second equation, the Principal will have the same payoff as the first equation. This means that the argument of maximum of the second equation is $a^{NI}(m)$ for $m \in M_{NI}$. Therefore The Principal's ex-ante payoff by observing m_{NI} under $\check{S}$ is equal to the ex-ante payoff of the Principal if observes a message from M_{NI} under S.

The ex-ante payoff of the Principal if observes a price from M_I under S is

$$\sum_{m \in M_I} \frac{M(\Theta_m)}{\sum_{m \in M_I} M(\Theta_m)} \int_{\underline{\theta}}^{\overline{\theta}} \Big[V(\theta, \theta) - \phi\Big] \beta(\theta|m) d\theta.$$

The Principal's ex-ante payoff by observing messages from M_I under $\check{S}$ is

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V(\theta, \theta) - \phi\Big] \left(\sum_{m \in M_I} \frac{M(\Theta_m)\beta(\theta|m)}{\sum_{m \in M_I} M(\Theta_m)} \right) d\theta.$$

Therefore the Principal's ex-ante payoff by observing m_I under $\check{S}$ is equal to the ex-ante payoff of the Principal if observes a message from M_I under S.

■

Proof of Proposition 5.[20]

First we show the strategy of the monopolist under the equilibrium that generates the maximum payoff is $\check{p}(c)$ for some c^*, and c^{**}. By lemma 6 we can focus on $\check{p}(.)$ with maximum two values. The problem to find the maximum payoff is as follows

$$\max_{\check{p}(c) \in \{p_{NI}, p_I\}} \Bigg\{ \int_{\underline{c}}^{\check{P}^{NI}(p_{NI})} \Big[V(\check{P}^{NI}(p_{NI})) - \check{P}^{NI}(p_{NI}) D(\check{P}^{NI}(p_{NI})) + \alpha \left\{ D\left(\check{P}^{NI}(p_{NI})\right) \left(\check{P}^{NI}(p_{NI}) - c\right) \right\}^+ \Big] \check{\beta}(c|p_{NI}) dc + \int_{\underline{c}}^{\overline{c}} \Big[V(c) - cD(c) - \phi \Big] \check{\beta}(c|p_I) dc \Bigg\},$$

[20](To be updated for the general utility functions)

subject to the incentive of the regulator for not inspecting p_{NI}

$$\int_{\underline{c}}^{\check{P}^{NI}(p_{NI})} \Big[V(\check{P}^{NI}(p_{NI})) - \check{P}^{NI}(p_{NI})D(\check{P}^{NI}(p_{NI})) \\ + \alpha \left\{D\left(\check{P}^{NI}(p_{NI})\right)\left(\check{P}^{NI}(p_{NI}) - c\right)\right\}^{+}\Big] \check{\beta}(c|p_{NI})dc \\ \geq \int_{\underline{c}}^{\overline{c}} \Big[V(c) - cD(c) - \phi\Big] \check{\beta}(c|p_{NI})dc, \quad (8)$$

subject to the incentive of the regulator for inspecting p_I

$$\int_{\underline{c}}^{\overline{c}} \Big[V(c) - cD(c) - \phi\Big] \check{\beta}(c|p_I)dc \geq \\ \max_{\tilde{p}} \int_{\underline{c}}^{\tilde{p}} \Big[V(\tilde{p}) - \tilde{p}D(\tilde{p}) + \alpha \left\{D\left(\tilde{p}\right)\left(\tilde{p}\right) - c)\right\}^{+}\Big] \check{\beta}(c|p_I)dc, \quad (9)$$

and finally subject to the incentive of the monopolist

$$\check{P}^{NI}(p_{NI}) \leq \inf\{c: \ \check{p}(c) = p_I\} \quad (10)$$

Constraints **??**, and **??** are by definition, but we need to show why constraint **??** is a necessary and sufficient for the incentive of the monopolist. The reason is that type $c < \check{P}^{NI}(p_{NI})$ has to set a price equal to p_{NI} to receive a positive profit. Otherwise by setting a price equal to p_I the profit is zero. Type $c \geq \check{P}^{NI}(p_{NI})$ is indifferent to set a price equal to p_I or p_{NI} since in any case the profit is zero.

Assume $\check{S}$ (by an abuse of the notation) is the equilibrium with highest payoff for the regulator (I need to show the existence). Define

$$\check{C}_I = \{c \in [\underline{c}, \overline{c}]: \ x(\check{p}(c)) = 1\},$$

$$\check{C}_{NI} = \{c \in [\underline{c}, \overline{c}]: \ x(\check{p}(c)) = 0\}.$$

Without loss of generality we can assume $\check{C}_I$ does not have a set of measure zero. The reason is that we can remove the set of measure zero of $\check{C}_I$, and add it to $\check{C}_{NI}$ without affecting the incentive any players. Now if we show $\check{C}_I \cap \check{C}_{NI}$ is a measure zero set, then $\check{C}_I$ is almost an interval. Finally we can remove $\check{C}_I \cap \check{C}_{NI}$ from $\check{C}_{NI}$, add it to $\check{C}_I$ to get a complete interval.

By contradiction assume $K = [\inf\{\check{G}_I\}, \sup\{\check{G}_I\}] \cap \check{G}_{NI}$ has a positive measure. We want to find another PBE with the higher payoff for the regulator. Before that define a set $H(\epsilon)$ of measure ϵ of distribution of highest types of a set $\check{G}_I$, formally

$$H(\epsilon) \subset \check{G}_I \text{ such that } \int_{\inf\{H(\epsilon)\}}^{\sup\{\check{G}_I\}} f(c)1_{c\in\check{G}_I}dc = \epsilon.$$

Let a set $L(\epsilon)$ of measure ϵ of distribution of lowest types of a set K, formally

$$L(\epsilon) \subset K \text{ such that } \int_{\inf\{K\}}^{\sup\{L(\epsilon)\}} f(c)1_{c\in\check{G}_{NI}}dc = \epsilon.$$

If we choose ϵ small enough then

$$\sup\{L(\epsilon)\} < \inf\{H(\epsilon)\}.$$

Now we are ready to define the new PBE. Roughly speaking we want to switch the strategy of $L(\epsilon)$, and $H(\epsilon)$. Formally let

$$\check{p}(c,\epsilon) = \begin{cases} p_I & c \in L(\epsilon), \\ p_{NI} & c \in H(\epsilon), \\ \check{p}(c) & \text{otherwise.} \end{cases}$$

Inspection strategy and price regulation with inspection remain the same, i.e. $\check{x}(p,\epsilon) = \check{x}(p)$, and $\check{p}^I(p,c,\epsilon) = \check{p}^I(p,c) = c$. On the equilibrium path, beliefs are consistent with strategies and the off-path belief puts probability 1 on $\underline{c}$. Define price regulation without inspection as

$$\check{p}^{NI}(p_{NI},\epsilon) = \underset{\tilde{p}}{\operatorname{argmax}} \int_{\underline{c}}^{\tilde{p}} \Big[V(\tilde{p}) - \tilde{p}D(\tilde{p}) + \alpha \max\{D(\tilde{p})(\tilde{p}-c), 0\}\Big] \check{\beta}(c,\epsilon|p_{NI})\, dc.$$

We call the above strategies and beliefs $\check{S}(\epsilon)$. We want to show $\check{S}(\epsilon)$ for small enough $\epsilon > 0$, is a PBE and the regulator's payoff is strictly higher than $\check{S}$.

First we show the payoff is higher. The no inspection term is higher since

$$\begin{aligned}
&\max_{\tilde{p}} \int_{\underline{c}}^{\tilde{p}} \Big[V(\tilde{p}) - \tilde{p}D(\tilde{p}) + \alpha\{D(\tilde{p})(\tilde{p}-c)\}^+\Big] \check{\beta}(c,\epsilon|p_{NI})\, dc \geq \\
&\int_{\underline{c}}^{\check{p}^{NI}(p_{NI})} \Big[V(\check{p}^{NI}(p_{NI})) - \check{p}^{NI}(p_{NI})D(\check{p}^{NI}(p_{NI})) + \alpha\left\{D\left(\check{p}^{NI}(p_{NI})\right)\left(\check{p}^{NI}(p_{NI}) - c\right)\right\}^+\Big] \check{\beta}(c,\epsilon|p_{NI})\, dc \\
&= \int_{\underline{c}}^{\check{p}^{NI}(p_{NI})} \Big[V(\check{p}^{NI}(p_{NI})) - \check{p}^{NI}(p_{NI})D(\check{p}^{NI}(p_{NI})) + \alpha\left\{D\left(\check{p}^{NI}(p_{NI})\right)\left(\check{p}^{NI}(p_{NI}) - c\right)\right\}^+\Big] \check{\beta}(c|p_{NI})\, dc \\
&= \max_{\tilde{p}} \int_{\underline{c}}^{\tilde{p}} \Big[V(\tilde{p}) - \tilde{p}D(\tilde{p}) + \alpha\{D(\tilde{p})(\tilde{p}-c)\}^+\Big] \check{\beta}(c|p_{NI})\, dc
\end{aligned}$$

The first inequality is by definition. The second equality is by the fact that $\check{\beta}(c|p_{NI}) = \check{\beta}(c,\epsilon|p_{NI})$ for $c \leq p_{NI}$. This is because by constraint 3, $\inf\{L(\epsilon\} \geq \inf\{K\} \geq \check{p}^{NI}(p_{NI})$. The inspection term is strictly higher since

$$\begin{aligned}
&\int_{\underline{c}}^{\overline{c}} \Big[V(c) - cD(c) - \phi\Big] \check{\beta}(c|p_I)dc - \int_{\underline{c}}^{\overline{c}} \Big[V(c) - cD(c) - \phi\Big] \check{\beta}(c,\epsilon|p_I)dc \\
&\quad = \int_{H(\epsilon)} \Big[V(c) - cD(c) - \phi\Big] f(c)dc - \int_{L(\epsilon)} \Big[V(c) - cD(c) - \phi\Big] f(c)dc < 0
\end{aligned}$$

The last inequality is due to the fact that $\sup\{L(\epsilon)\} < \inf\{H(\epsilon)\}$, and $V(c) - cD(c)$ is a decreasing function of c.

Second we show the monopolist does not incentive to deviate, i.e.

$$\check{p}^{NI}(p_{NI},\epsilon) \leq \inf\{c:\ \check{p}(c) = p_I\}.$$

We show a stronger fact: $\check{p}^{NI}(p_{NI},\epsilon) \leq \check{p}^{NI}(p_{NI})$.

$$\max_{\tilde{p}} \int_{\underline{c}}^{\tilde{p}} \left[V(\tilde{p}) - \tilde{p}D(\tilde{p}) + \alpha \left\{D\left(\tilde{p}\right)(\tilde{p}-c)\right\}^+\right] \check{\beta}(c,\epsilon|p_{NI})\ dc \geq$$

$$\int_{\underline{c}}^{\check{p}^{NI}(p_{NI})} \left[V(\check{p}^{NI}(p_{NI})) - \check{p}^{NI}(p_{NI})D(\check{p}^{NI}(p_{NI})) + \alpha\left\{D\left(\check{p}^{NI}(p_{NI})\right)\left(\check{p}^{NI}(p_{NI})-c\right)\right\}^+\right] \check{\beta}(c,\epsilon|p_{NI})\ dc$$

$$= \int_{\underline{c}}^{\check{p}^{NI}(p_{NI})} \left[V(\check{p}^{NI}(p_{NI})) - \check{p}^{NI}(p_{NI})D(\check{p}^{NI}(p_{NI})) + \alpha\left\{D\left(\check{p}^{NI}(p_{NI})\right)\left(\check{p}^{NI}(p_{NI})-c\right)\right\}^+\right] \check{\beta}(c|p_{NI})\ dc$$

$$\geq \int_{\underline{c}}^{p} \left[V(p) - pD(p) + \alpha\left\{D\left(p\right))(p-c)\right\}^+\right] \check{\beta}(c|p_{NI})\ dc$$

$$\geq \int_{\underline{c}}^{p} \left[V(p) - pD(p) + \alpha\left\{D\left(p\right)(p-c)\right\}^+\right] \check{\beta}(c,\epsilon|p_{NI})\ dc.$$

The last two inequalities are for all $p \geq \check{p}^{NI}(p_{NI})$. The first inequality is by definition. The second equality is by the fact that $\check{\beta}(c|p_{NI}) = \check{\beta}(c,\epsilon|p_{NI})$ for $c \leq p_{NI}$. The third inequality is by the definition. If we show the last inequality is true then by comparing the first and the last equation we find $\check{p}^{NI}(p_{NI},\epsilon) \leq \check{p}^{NI}(p_{NI})$. The argument for the last inequality is as follows

$$\int_{\underline{c}}^{p} \left[V(p) - pD(p) + \alpha\left\{D\left(p\right))(p-c)\right\}^+\right] \check{\beta}(c|p_{NI})\ dc$$

$$- \int_{\underline{c}}^{p} \left[V(p) - pD(p) + \alpha\left\{D\left(p\right)(p-c)\right\}^+\right] \check{\beta}(c,\epsilon|p_{NI})\ dc$$

$$= \int_{L(\epsilon)\cap[\underline{c},p]} \left[V(p) - pD(p) + \alpha\left\{D\left(p\right))(p-c)\right\}^+\right] f(c)dc$$

$$- \int_{H(\epsilon)\cap[\underline{c},p]} \left[V(p) - pD(p) + \alpha\left\{D\left(p\right))(p-c)\right\}^+\right] f(c)dc$$

Incentive of the regulator if observes price p_{NI}

$$\max_{\tilde{p}} \int_{\underline{c}}^{\tilde{p}} \left[V(\tilde{p}) - \tilde{p}D(\tilde{p}) + \alpha \left\{D\left(\tilde{p}\right)(\tilde{p}-c)\right\}^+\right] \check{\beta}(c,\epsilon|p_{NI})\ dc \geq$$

$$\int_{\underline{c}}^{\check{p}^{NI}(p_{NI})} \left[V(\check{p}^{NI}(p_{NI})) - \check{p}^{NI}(p_{NI})D(\check{p}^{NI}(p_{NI})) + \alpha\left\{D\left(\check{p}^{NI}(p_{NI})\right)\left(\check{p}^{NI}(p_{NI})-c\right)\right\}^+\right] \check{\beta}(c,\epsilon|p_{NI})\ dc$$

$$= \int_{\underline{c}}^{\check{p}^{NI}(p_{NI})} \left[V(\check{p}^{NI}(p_{NI})) - \check{p}^{NI}(p_{NI})D(\check{p}^{NI}(p_{NI}))\right.$$

$$+\alpha\left\{D\left(\check{p}^{NI}(p_{NI})\right)\left(\check{p}^{NI}(p_{NI})-c\right)\right\}^{+}\Big]\,\check{\beta}(c|p_{NI})\,dc$$

$$\geq \int_{\underline{c}}^{p}\left[V(p)-pD(p)+\alpha\left\{D\left(p\right))\left(p-c\right)\right\}^{+}\right]\check{\beta}(c|p_{NI})\,dc$$

$$\geq \int_{\underline{c}}^{p}\left[V(p)-pD(p)+\alpha\left\{D\left(p\right)\left(p-c\right)\right\}^{+}\right]\check{\beta}(c,\epsilon|p_{NI})\,dc$$

$$\geq \int_{\underline{c}}^{\overline{c}}\left[V(c)-cD(c)-\phi\right]\check{\beta}(c|p_{NI})dc$$

$$\geq \int_{\underline{c}}^{\overline{c}}\left[V(c)-cD(c)-\phi\right]\check{\beta}(c,\epsilon|p_{NI})dc,$$

The last inequality is true since $V(c)-cD(c)$ is a decreasing function.

The incentive of the regulator if observes p_I

$$\int_{\underline{c}}^{\overline{c}}\left[V(c)-cD(c)-\phi\right]\check{\beta}(c,\epsilon|p_I)dc \geq$$

$$\max_{\tilde{p}}\int_{\underline{c}}^{\tilde{p}}\left[V(\tilde{p})-\tilde{p}D(\tilde{p})+\alpha D\left(\tilde{p}\right)\left(\tilde{p}\right)-c\right)\right]\check{\beta}(c,\epsilon|p_I)dc.$$

By contradiction, suppose this doesn't hold for any small $\epsilon>0$, though it holds for $\epsilon=0$. It means that it is equality at $\epsilon=0$. To reach a contradiction, one can show that even in this case, the derivative of the left-hand side with respect to ϵ is greater than the right-hand side. Get $c_1=\inf\{L(\epsilon)\}$ and $c_2=\inf\{H(\epsilon)\}$, which are fixed and independent of ϵ. Therefore, the left-hand side derivative is:

$$L'=V(c_1)+c_2D(c_1)-V(c_2)-c_2D(c_2).$$

Denote the set of $\tilde{p}$ that maximizes the right-hand side at $\epsilon=0$ by P^* and suppose it is non-empty. Non of this maximizers could be between c_1 and c_2. One can use the envelope theorem to find the directional derivative of the right-hand side at 0^+:

$$R'=\max_{\tilde{p}\in P^*}\alpha D(\tilde{p})(c_2-c_1)\mathbb{1}_{c_2<\tilde{p}}.$$

If $\tilde{p}^*\leq c_2$, then $L'\geq 0=R'$. Otherwise, one can easily show:

$$L'=V(c_1)-c_1D(c_1)-V(c_2)+c_2D(c_2)\geq D(c_2)(c_2-c_1)\geq \alpha D(\tilde{p}^*)(c_2-c_1)=R'.$$

■

Proof of Proposition 6.

An ideal equilibrium suggestion for the highest equilibrium payoff for the Principal is as follows

- Strategies:

$$m(\theta)=\begin{cases} m_{NI} & \theta\in[\underline{\theta},\theta^*], \\ m_I & \theta\in(\theta^*,\overline{\theta}] \end{cases}$$

$$x(m) = \begin{cases} 1 & m = m_I, \\ 0 & \text{otherwise} \end{cases}$$

$$a^{NI}(m) = \begin{cases} \theta^* & m = m_{NI}, \\ \underline{\theta} & \text{otherwise} \end{cases}$$

$$a^I(m, \theta) = \theta.$$

- Beliefs: On the equilibrium path, beliefs are consistent with strategies and the off-path belief puts probability 1 on $\underline{\theta}$, i.e.

$$\beta(\underline{\theta} | m \neq m_I, m_{NI}) = 1.$$

The above strategy and beliefs generates the commitment payoff according to the Proposition 1. We need to check incentives of both players. The Agent does not have any incentive to deviate. Types above θ^* do not want to deviate to m_{NI} since the mandated action becomes less than their types. Types above θ^* will not deviate to m_I since they will be inspected and their payoff become zero. The principal can commit to inspect m_I, so we do not check the incentive of the Principal after observing m_I. Now we show $a^{NI}(m_{NI}) = \theta^*$, where

$$a^{NI}(m_{NI}) = \underset{\tilde{a}}{\text{argmax}} \int_{\underline{\theta}}^{\tilde{a}} \Big[V\left(\theta, \tilde{a}\right) \Big] \beta(\theta | m_{NI})\, d\theta.$$

By contradiction, first assume $a^{NI}(m_{NI}) < \theta^*$, then

$$\int_{\underline{\theta}}^{a^{NI}(m_{NI})} \Big[V(\theta, a^{NI}(m_{NI})) \Big] f(\theta) d\theta > \int_{\underline{\theta}}^{\theta^*} \Big[V(\theta, \theta^*) \Big] f(\theta) d\theta.$$

Therefore

$$\int_{\underline{\theta}}^{a^{NI}(m_{NI})} \Big[V(\theta, a^{NI}(m_{NI})) \Big] f(\theta) d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[V(\theta, \theta) - \phi \Big] f(\theta) d\theta$$
$$> \int_{\underline{\theta}}^{\theta^*} \Big[V(\theta, \theta^*) \Big] f(\theta) d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[V(\theta, \theta) - \phi \Big] f(\theta) d\theta.$$

The above strict inequality is a contradiction since by the definition, the threshold θ^* should generate higher value than the threshold $a^{NI}(m_{NI})$. Thus the left side should not be higher than the right side. A contradiction. Second, assume $a^{NI}(m_{NI}) > \theta^*$. We can conclude $a^{NI}(m_{NI}) > \theta^{**}$ since after observing m_{NI} by Principal she puts zero probability in interval (θ^*, θ^{**}). However, $a^{NI}(m_{NI}) > \theta^{**}$ is impossible since $\theta^{**} = \overline{\theta}$. Now we need to show by observing m_{NI} the principal does not inspect, formally

$$\int_{\underline{\theta}}^{a^{NI}(m_{NI})} \Big[V(\theta, a^{NI}(m_{NI})) \Big] \beta(\theta | m_{NI}) d\theta \geq \int_{\underline{\theta}}^{\theta^*} \Big[V(\theta, \theta) - \phi \Big] \beta(\theta | m_{NI}) d\theta,$$

we can replace $a^{NI}(m_{NI})$ by θ^*

$$\int_{\underline{\theta}}^{\theta^*} \Big[V(\theta,\theta^*)\Big]\,\beta(\theta|m_{NI})d\theta \geq \int_{\underline{\theta}}^{\theta^*} \Big[V(\theta,\theta)-\phi\Big]\,\beta(\theta|m_{NI})d\theta.$$

By contradiction assume the reverse, thus we have

$$\int_{\underline{\theta}}^{\theta^*} \Big[V(\theta,\theta)-\phi\Big]\,f(\theta)d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[V(\theta,\theta)-\phi\Big]\,f(\theta)d\theta$$
$$> \int_{\underline{\theta}}^{\theta^*} \Big[V(\theta,\theta^*)\Big]\,f(\theta)d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[V(\theta,\theta)-\phi\Big]\,f(\theta)d\theta.$$

This is a contradiction by the definition θ^*. So the Principal does not any incentive to inspect message m_{NI}.

■