On the Relationship between Borrower and Bank risk*

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February 26, 2023

Abstract

We study bank default risk in a model with a competitive banking sector where banks lend to risky borrowers and adjust their interest rates accordingly. Our main result is the following unconventional equilibrium relationship: banks can be more likely to default with less risky borrowers than with more risky borrowers. The reason is that competition may force the bank to lend at excessively low interest rates to less risky borrowers, eroding its capital buffer in case of borrower default and thus increasing its own failure risk. We also show that such an outcome is surprisingly ruled out by workhorse credit risk models such as the single risk factor model of Vasicek (2002), while it naturally arises if one extends the baseline setup to multiple risk factors. Our analysis explains why this happens and indicates that canonical single-risk factor models used extensively by regulators are too restrictive and may be misleading as regards the relationship between borrower and bank risk - especially if loan terms (i.e., interest rate, LTV, LTI, collateral) are endogenous to borrower credit quality.

Keywords: Credit Risk Models, Bank Failure Risk, Banking Regulation

JEL Codes: G21, G28, E43

*We thank Frederic Malherbe, David Martinez-Miera, Christian Opp, Galo Nuno, Nicola Gennaioli, Todd Keister, Andrea Modena, Ansgar Walther, Victoria Vanasco, Javier Suarez, and seminar participants at the University of Bonn, Rutgers University, the CRC conference in Mainz, and the Credit and Financial Frictions Workshop in Alghero. Yuliyan Mitkov acknowledges funding by the German Research Foundation (DFG) through CRC TR 224 (Project C03) and EXC 2126/1-390838866 under Germany's Excellence Strategy.

1 Introduction

An improved understanding of bank failure risk is crucial for efficient capital and liquidity regulation, valuing explicit and implicit government guarantees, and identifying forward-looking indicators of bank failure risk (Acharya et al. (2014), Goldstein and Leitner (2018)). A growing literature investigates how bank failure risk is shaped by factors such as capital and liquidity regulation, government guarantees, interest rate risk, and competition (Merton (1977), Froot and Stein (1998), Repullo and Suarez (2013), Hugonnier and Morellec (2017)). Further, there is a growing consensus in the literature that the endogenous reaction of banks to changes in the credit risk of their borrowers can lead to unexpected outcomes.[1]

Our main question is the following. How is bank failure risk related to the credit risk of its borrowers if loan terms, such as interest rate, are determined as part of equilibrium? The answer seems obvious at first glance and summarized by the *conventional view* depicted by the solid line in Figure 1. This view holds that any given bank with a riskier loan portfolio will be more likely to default unless it simultaneously engages in some form of risk mitigation, such as more equity capital, reduced interest rate risk, sales of loans, credit derivatives, or smaller loan portfolios (Berger et al. (2017)).

In this paper, we challenge the conventional wisdom: the equilibrium relationship between borrower and bank risk might be more complex, including positive and inverse relationships, as shown by the dashed line in Figure 1. Specifically, we show that a bank with safer borrowers might, in fact, be more likely to fail than an otherwise identical bank with riskier borrowers.[2]

The intuition is the following. Suppose an aggregate state determines the fraction of the bank's loan in default, m. The probability of bank failure is equal to the probability that m exceeds a given cutoff $\bar{m}$, namely $1 - F(\bar{m})$, where $F(m)$ denotes the cumulative distribution function of m and the cutoff $\bar{m}$ depends on the interest rate on loans. We show that an improvement in the distribution of m (i.e., a uniform shift to the left of $F(m)$ can sometimes lead to an increase in the probability of bank failure. It is not hard to explain how this might happen: if the probability distribution over m shifts to the left, then the

[1] Benoit et al. (2017) write: "It is also likely that many of the new tools introduced to deal with systemic risk will suffer from the Lucas critique. For instance, while loans with a low loan-to-value ratio are, on average safer, it is not clear how banks will endogenously react to caps imposed on this measure. As many tools are new, their modeling and the empirical evaluation of their impact will be an important research topic going forward."

[2] We *do not* obtain the unconventional outcome through increasing loan correlations, as in the following example. Situation A: each loan has a 10 % probability of default, but the defaults are uncorrelated, so the bank never fails. Situation B: each loan has a 1 % probability of default, but defaults are perfectly correlated, so the bank fails in this event. Throughout this paper, a lower probability of loan default is always associated with a lower correlation in loan defaults, so the above example does not apply.

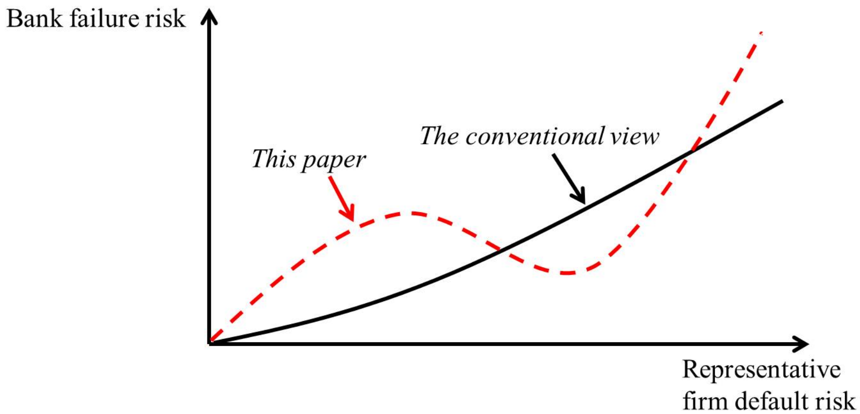


Figure 1: The figure illustrates schematically the probability of bank failure as a function of the default risk of a representative firm that obtains a loan from the bank.

corresponding equilibrium interest rate will be lower. As a result, the bank's revenue from performing loans is also lower, implying it can withstand fewer loan defaults before becoming insolvent. The question is whether this second (indirect) effect can be significant enough to overturn the direct effect. If so, what changes in the distribution of m would imply such an unconventional outcome?

Section 2 provides a simple illustration of the unconventional outcome and the mechanism behind it. Next, we discipline the analysis in Sections 3 - 4 using a canonical approach to generate the distribution of m. Specifically, we follow the literature on credit risk by building the distribution of m through systematic risk factors.[3] First, we fix a set of systematic risk factors Y_1, ..., Y_k and the conditional probability of default for each borrower, given the realization of these factors. Second, we fix a joint distribution of the risk factors to obtain $F(m)$. A uniform shift to the left in $F(m)$ is obtained by assuming that each borrower is less likely to default for each realization of the systematic risk factors due to, for example, higher or less volatile income, lower LTV, or LTI, shorter loan maturity or some other reason. We then show that the unconventional outcome (i.e., the indirect dominating the direct effect) can indeed happen: each borrower can be less likely to default for each realization of the systemic risk factors, but, at the same time, the bank's probability of failure can be higher.

While explaining how the unconventional outcome can happen within a simple model is relatively straightforward, obtaining such an outcome within canonical credit risk models is significantly more complicated. As it turns out, it requires extending the baseline setup to multiple risk factors. Specifically, we show in Section 4 that using the canonical single risk factor model in Vasicek (2002) to generate $F(m)$ implies that direct always dominates

[3]Factors models are essential for credit risk modeling because they generate realistic correlated loan default patterns while remaining analytically tractable.

the indirect effect. Intuitively, the Vasicek model imposes a tight structure on firm default risk, which does not allow $F(m)$ to vary in ways consistent with the unconventional outcome depicted in Figure 1. The implication that single-risk factor models are restrictive and might not capture all sources of systematic risk is consistent with empirical and theoretical findings.[4] However, to the best of our knowledge, we are the first to point out that a widely used single risk factor model rules out an important class of changes, namely, those that imply the indirect effect of an "improvement" in the distribution of m dominates the direct effect.[5]

We contribute to the literature on the determinants of bank failure risk. Most models of bank default risk either assume a log-normal distribution of bank returns following Merton (1974) or postulate that banks invest directly in productive assets instead of issuing loans (see Van den Heuvel (2008), and Gertler and Kiyotaki (2010) among others). However, bank assets consist of loans to firms and households with limited upside and significant downside risk. They are thus not well-described by the log-normal distribution, which assumes unlimited upside potential. Several recent papers have shown that credit risk models that do not take into account this special feature of bank assets will tend to understate bank failure risk (Nagel and Purnanandam (2020), Mendicino et al. (2019)). We complement this literature by showing that even unambiguous gains or losses in firm credit quality can have surprising effects on bank risk.

The most popular approach to modeling the special nature of bank assets is the Vasicek single-factor model (Vasicek (2002)), which is a simplified multi-borrower version of Merton (1974).[6] Regulators have adopted this model as an essential building block in computing bank capital requirements. In addition, extensive literature uses the Vasicek model to study capital regulation, banking competition, loan pricing, and leverage choices (Repullo and Suarez (2004), Martinez-Miera and Repullo (2010), Gornall and Strebulaev (2018)). We show that the Vasicek model imposes substantial restrictions on the firm risk profile that rule out the unconventional effect depicted in Figure 1. At the same time, the two-risk factor model we propose to illustrate the unconventional effect is a simple extension of the canonical Vasicek model that is not more complicated and still allows for closed-form expressions.

[4] For example, single-risk factor models exhibit properties, such as portfolio invariance, that do not necessarily extend to multiple-factor models (Gordy (2003)).

[5] To be clear, an inverse equilibrium relationship between borrower and bank risk *can* arise in the Standard Vasicek model if one varies other factors such as the level of competition (Martinez-Miera and Repullo (2010)) or bank capital requirements (Repullo and Suarez (2004)). We apply the term unconventional outcome *only* if the bank is not simultaneously earning greater profits due to lower competition or holding more capital due to stricter capital regulation.

[6] The setup in Vasicek (2002) is part of a framework known as Gaussian asymptotic single-factor model of portfolio losses developed by Vasicek (1987), Finger (1999), Schönbucher (2001), Gordy (2003), and Frey and McNeil (2003)) among others.

2 An example

Before going to the baseline model in Section 3, we provide a simple example of the unconventional outcome.

The environment. There are two dates (0 and 1) and a continuum of firms on date 0. Each firm demands a loan of \$1 on date 0 that matures on date 1. On date 1, the firm either repays $1 + r$, where r is the loan interest rate, or defaults. In case of default, the bank gets $1 - \triangle$ where $\triangle \in (0, 1]$ is the loss given default. There are two possible states on date 1: a *bad* state occurring with probability $q \in [0, 1]$ and a *good* state occurring with probability $1 - q$. The rate of firm default in the *bad* state is $m(B) \in [0, 1]$ and the rate of firm default in the *good* state is $m(G) \in [0, 1]$ where $m(B) \geq m(G)$. The unconditional probability of firm default is

$$p = (1 - q)m(G) + qm(B).$$

Loans are issued by a perfectly competitive banking sector subject to free entry. Bank deposits are fully insured by the government, in perfectly elastic supply, and with a net interest rate r_D normalized to zero. Each bank is subject to a mandatory minimum capital ratio $k_{min} \in (0, 1)$ implying a maximum leverage ratio of $1/k_{min}$. Consider a bank with a portfolio of size \$1 on date 0 financed by \$$k$ units of capital and \$$1 - k$ units of deposits (k is thus also the bank's capital ratio). The net expected payoff for the bank's shareholders is

$$\begin{aligned}\pi(r, k) = \quad -k \quad &+ (1 - q) \max\{(1 - m(G))(1 + r) + m(G)(1 - \triangle) - (1 - k), 0\} \\ &+ q \max\{(1 - m(B))(1 + r) + m(B)(1 - \triangle) - (1 - k), 0\}\end{aligned}$$

We assume the bank's equity holders require an expected net return of at least $\delta > 0$.[7] Each bank chooses to operate with the minimum possible capital ratio $k^* = k_{min}$ since deposits are insured, and bank equity is privately costly. The equilibrium interest rate on loans r^* ensures that the bank's shareholders break even in expectation $\pi(r^*, k_{min}) = \delta k_{min}$. Finally, the bank defaults on date 1 in state $S \in \{G, B\}$ if the income from performing loans in that state $(1 - m(S))(1 + r)$ plus the income from non-performing loans $m(S)(1 - \triangle)$ is insufficient to repay the amount promised to the depositors $1 - k$.

The unconventional outcome. Next, we illustrate the unconventional outcome by fixing parameters $q = 0.05$, $\triangle = 0.3$, $k_{min} = 0.05$, $\delta = 0.01$ and comparing two economies A and B that only differ in the rate of firm default in each state as depicted in Figure 2. Specifically,

[7] The parameter δ is exogenous and captures in a reduced form the scarcity of bank capital due to agency costs, equity issuance costs, the tax advantage of debt, or liquidity premium of deposits (see Holmstrom and Tirole (1997), Diamond and Rajan (2001), and Gorton and Pennacchi (1990) among others).

each firm in economy B is less likely to default in each state than in economy A. Nevertheless, we show that the bank in economy B is *more likely* to fail than in A.

Economy A: $(m(G), m(B)) = (0.2, 0.3)$. The equilibrium loan interest rate and bank capital ratio is then $r^* = 0.0789$ and $k^* = k_{min} = 0.05$. The bank remains solvent in the *bad* state since its portfolio income in that state $(1 - m(B))(1 + r^*) + m(B)(1 - \triangle) = 0.9653$ exceeds the amount promised to the depositors $1 - k_{min} = 0.95$. The probability of bank failure in this economy is thus zero.[8]

Economy B: $(\tilde{m}(G), \tilde{m}(B)) = (0.05, 0.25)$. The equilibrium loan interest rate is then $\tilde{r}^* = 0.0197$, and the bank's capital ratio is the same as before $k^* = k_{min} = 0.05$. Notice, however, that something interesting has happened: the bank's portfolio income in the *bad* state, namely $(1 - \tilde{m}(B))(1 + \tilde{r}^*) + \tilde{m}(B)(1 - \triangle) = 0.9398$, falls below the amount promised to the depositors $1 - k_{min} = 0.95$ implying that the bank fails in that state. As a result, the probability of bank failure now equals the likelihood of the *bad* state $q = 0.05$.

We refer to this situation as the *unconventional outcome* and derive parameter conditions that lead to it. Specifically, the bank will fail in the *bad* state if and only if the parameters satisfy the following.

$$\frac{1-m(B)}{1-m(G)} < \left(k_{min}\left(\frac{r_K+q}{1-q}\right) + \frac{\triangle}{\triangle - k_{min}}\right)^{-1} \equiv \chi.$$

The unconventional outcome, therefore, obtains if and only if the following conditions jointly hold

$$\tilde{m}(G) < m(G), \quad \tilde{m}(B) < m(B) \quad \text{and} \quad \frac{1-\tilde{m}(B)}{1-\tilde{m}(G)} < \chi < \frac{1-m(B)}{1-m(G)}.$$

The above implies the following necessary condition for the unconventional outcome: $\frac{1-\tilde{m}B)}{1-m(B)} < \frac{1-\tilde{m}(G)}{1-m(G)}$. Thus, if $1 - m(S)$ is interpreted as the credit quality of the firm in state $S \in \{G, B\}$, then the percentage gain in the firm's credit quality in the *good* state must be higher than the percentage gain in the *bad* state. Recall that in our example, the rate of firm default in economy B compared to A is much lower in the *good* state (0.2 vs. 0.05) but only slightly lower in the *bad* state (0.3 vs. 0.25). Nevertheless, competition and the likelihood of the *bad* state being only 5 percent implies that the equilibrium loan interest rate is significantly lower in economy B than in A, leading the bank in economy B to fail in the *bad* state. In contrast, the bank in economy A remains solvent in both states.

Summary and the plan ahead. The example in this section shows that state-contingent changes in firm credit quality are central to generating the unconventional outcome. This situation is illustrated in Figure 2 where all combinations of $(m(G), m(B))$ on the dashed line

[8]The zero failure probability is an artifact of our two-state example and is not critical for the results.

would lead to the same unconditional probability of firm default as in Economy B, namely $(1-q)m(G)+qm(B) = 0.06$ (recall $q = 0.05$). At the same time, only those pairs belonging to this line's red segment would lead to the unconventional outcome. For example, Economies B and C are characterized by the same unconditional probability of firm default even though only the bank in economy B defaults in the *bad* state. In addition, systematic risk, defined as the joint default probability of any two firms, does not explain the unconventional outcome since economy A has greater systematic risk than B in both states. Yet, the bank in this economy is safe, whereas the bank in B is risky. Finally, the unconventional outcome is not due to distortion generated by miss-priced government guarantees since it arises with or without flat-rate deposit insurance, as we show in Section 4.

Next, we turn to the Vasicek (2002) credit risk model, which implies a particular shape of the distribution of the proportion of loan defaults $F(m)$. It also imposes restrictions on how $F(m)$ varies in response to changes in the underlying parameters of the environment. A natural question, then, is whether the unconventional outcome also arises in this canonical credit risk framework. We turn to this question in the following sections, showing that it does, but only if one extends the baseline setup to more than one risk factor.

3 The model

We start by summarizing the main idea of risk factor models (see Gordy (2003), and Frey and McNeil (2003) for more details).

Risk factor models. The firm's unconditional probability of default (denoted PD) is its probability of default before some fixed horizon. The firm's conditional PD is its probability of default given the realization of K systematic risk factors $Y = (Y_1, ..., Y_K)$. The vector of risk factors Y is drawn from a known probability distribution H and captures macroeconomic, regional, and industry-specific variables. The conditional PD of a given firm i is $p_i(y)$ where $y = (y_1, ..., y_K)$ is a particular realization of the risk factors. The firm's unconditional PD is $p_i = \int_{\Omega_Y} p_i(y)dH(y)$ where Ω_Y is the domain of Y. The central assumption is that any correlation in default is entirely due to the firm's common exposure to systematic risk factors. The remaining credit risk is idiosyncratic and thus can be eliminated in a well-diversified loan portfolio. Now, suppose all firms are identically exposed to the systematic risk factors as we will assume to be the case: $p_i(y) = p(y)$ for each i. In that case, the fraction of loan defaults in a large loan portfolio equals $p(y)$, where y is a given realization of the risk factors.

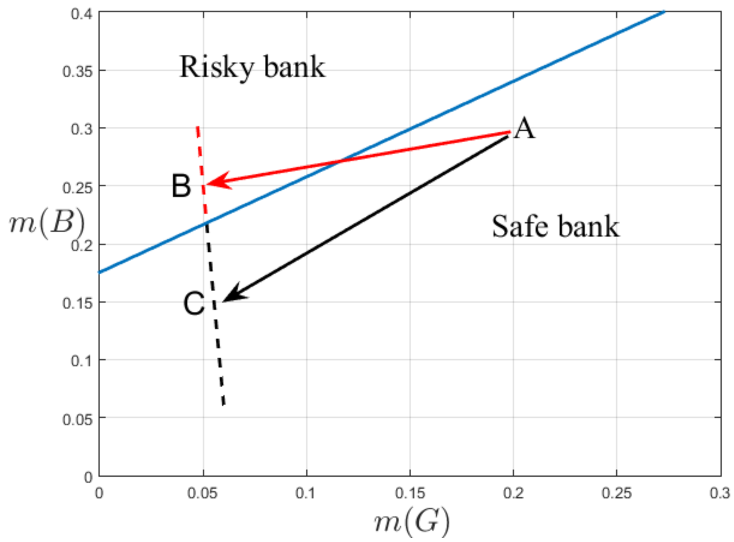


Figure 2: A two-state example. We set $q = 0.05$, $\triangle = 0.3$, $k_{min} = 0.05$, $\delta = 0.01$ and vary $m(B)$ and $m(B)$. Those values of $(m(G), m(B))$ below the solid blue line imply the bank remains solvent in both states (a *safe bank*). Those values of $(m(G), m(B))$ above the solid blue line imply the bank fails in the *bad* state (a *risky bank*). The unconventional outcome: moving from the economy at point A to the one at point B implies that each firm is less likely to default for each realization of the state. Still, the bank in economy B will be more likely to fail than in economy A. Finally, all combinations of $(m(G), m(B))$ on the dashed line lead to the same unconditional probability of firm default as in Economy B, namely $(1 - q)m(G) + qm(B) = 0.06$, but only those pairs belonging to this line's red segment lead to the unconventional outcome.

3.1 Environment

Our model generalizes the setup in Repullo and Suarez (2004) to multiple risk factors. There is a unit continuum of firms indexed by $i \in [0, 1]$ and many banks. The economy has two dates (0 and 1), two systematic risk factors $s \sim G$ and $z \sim N(0, 1)$, and a firm-specific shock $\epsilon_i \sim N(0, 1)$. The two risk factors and the firm-specific shock are all realized on date 1. The distribution of the risk factor s is G. The risk factor z is independent of s and follows the standard normal distribution $s \sim N(0, 1)$ as in the baseline Vasicek model. Finally, the firm-specific shocks $\{\epsilon_i\}_{i=0}^{1}$ are independent of s and z, independent and identically distributed across all firms, and follow the standard normal distribution: $\epsilon_i \sim N(0, 1)$ for each $i \in [0, 1]$. The sequence of events is depicted in Figure 3. Table 1 in Appendix A provides a summary of the model notation.[9]

[9] In terms of the general risk factor setup at the start of this section: $K = 2$, $y_1 = s$, $y_2 = z$, $\Omega_Y = S \times Z$, and $H(t) = \Pr[s \leq t, z \leq t] = G(t)\Phi(t)$ where Φ is the c.d.f. of the standard normal.

3.1.1 Firms

Each firm i demands a loan of \$1 on date 0. All loans come due on date 1 after the realization of the two risk factors s and z, and the firm-specific shocks $\{\epsilon_i\}_{i=0}^{1}$. The required repayment on date 1 is $1+r$, where r is the loan interest rate which will be determined in equilibrium.

Credit quality. We model the credit quality of firm i, denoted x_i, as follows

$$x_i = \underbrace{\mu(s)}_{\text{exposure to risk factor } s} + \underbrace{\sqrt{\rho}z}_{\text{exposure to risk factor } z} + \underbrace{\sqrt{1-\rho}\epsilon_i}_{\text{firm-specific shock}} . \tag{1}$$

where $\mu : S \to \mathbb{R}$, the parameter $\rho \in [0,1]$ governs the firm's exposure to the risk factor z, and ϵ_i is an i.i.d. firm-specific shock independent across all firms, and independent of the two risk factors. Firm i defaults on its loan on date 1 if (and only if) its credit quality, denoted by x_i, falls below a cutoff normalized to equal zero

$$\text{Firm } i \text{ defaults iff} \quad x_i < 0. \tag{2}$$

If $\rho = 1$, the effect of the firm-specific shocks in (1) disappears, and all firms default on date 1, or they all repay their loan. On the other hand, if $\rho \in [0,1)$, loan defaults on date 1 are only partially correlated. The function $\mu(s) = E[x_i|s]$ gives the average credit quality across all firms for a given value of the risk factor s. We impose the normalization $\frac{d\mu(s)}{ds} \geq 0$. The Standard Vasicek model is the special case $\mu(s) = \mu$ for all s, i.e., the risk factor s is irrelevant. In this case, the model reduces to the one in Repullo and Suarez (2004).[10]

Conditional PD. Conditional on the realization of s and z firm credit quality is normally distributed with mean $\mu(s) + \sqrt{\rho}z$ and variance $1-\rho$. Thus, the probability of firm default conditional on s and z, denoted by $p(s,z)$, is the following

$$\begin{aligned} p(s,z) &\equiv \Pr[x_i < 0 \mid s, z] \\ &= \Pr\left[\mu(s) + \sqrt{\rho}z + \sqrt{1-\rho}\epsilon_i < 0 \mid s, z\right] \\ &= \Phi\left(\frac{-\mu(s)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right) \end{aligned} \tag{3}$$

where the last line uses $\epsilon_i \sim N(0,1)$ and Φ denotes the cumulative distribution function (c.d.f.) of the standard normal. Given that there is a continuum of firms, $p(s,z)$ is also the fraction of loan defaults on date 1 as a function of s and z. Note that the rate of firm default

[10] Our baseline model assumes for simplicity that the probability of firm default does not depend on the loan interest rate r (since the default condition in (2) is independent of r). Appendix B extends the baseline setup to a model where the probability of firm default depends on r and shows that the key results continue to apply (see Figures (7) - (8)).

on date 1, namely $p(s,z)$, is decreasing in s and z. Figure 4(a) displays $p(s,z)$ as a function of z for two different realizations of $s \in \{s_L, s_H\}$ such that $s_L < s_H$. Next, the probability of firm default conditional on s, denoted $p(s)$, is given by

$$\begin{aligned} p(s) &\equiv \Pr[x_i < 0 \mid s] \\ &= \Pr\left[\mu(s) + \sqrt{\rho}z + \sqrt{1-\rho}\epsilon_i < 0 \mid s\right] \\ &= \Phi(-\mu(s)) \end{aligned} \tag{4}$$

The last line uses that the random variable $\sqrt{\rho}z + \sqrt{1-\rho}\epsilon_i \sim N(0,1)$. Notice from (4) that $p(s)$ and $\mu(s)$ are inversely related: higher average credit quality for a given s implies a lower probability of firm default for that s and vice versa.

Unconditional PD. The unconditional distribution of firm credit quality is governed by a mixture distribution with s acting as the mixing variable. That is,

$$\Pr[x_i \leq t] = \int_S \Phi(t - \mu(s))dG(s),$$

where S is the domain of the risk factor $s \sim G$. The unconditional probability of default for any firm i, denoted p, is then

$$p \equiv \Pr[x_i < 0] = \int_S \Phi(-\mu(s))dG(s). \tag{5}$$

The above is interpreted as follows: nature draws s from the distribution G giving a rate of firm default $p(s) \equiv \Phi(-\mu(s))$. The unconditional probability of default is then obtained by integrating over all values of s weighted by their probability $p = \int_S p(s)dG(s)$.

Standard Vasicek model. The *Standard Vasicek* model is a special case of our setup where $\mu(s)$ in (1) does not vary with the risk factor s (making this risk factor irrelevant), implying $p(s,z) = p(z) = \Phi\left(\frac{-\mu - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)$ and $p(s) = p = \Phi(-\mu)$ for each $s \in S$. We will refer to the general case where $\mu(s)$ varies with s as the *Mixture Vasicek* model since s behaves as a mixture variable (see (5)).

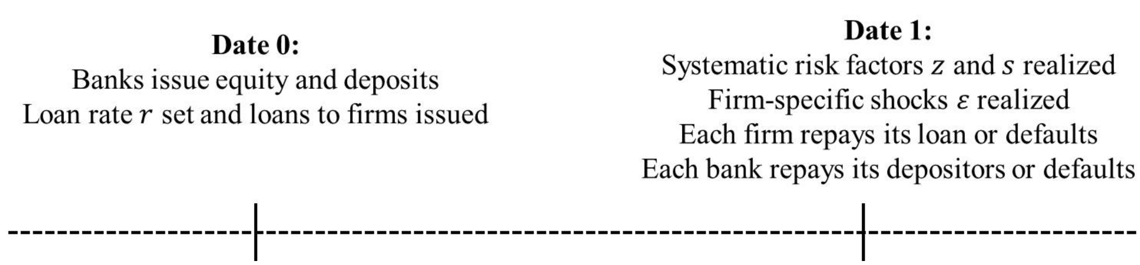
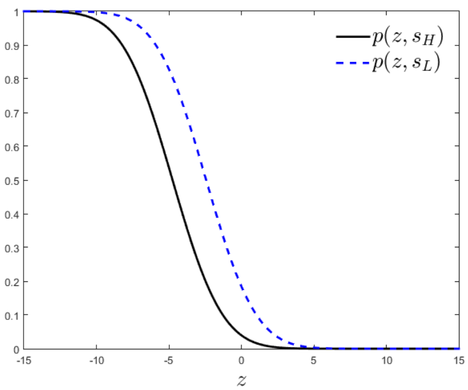
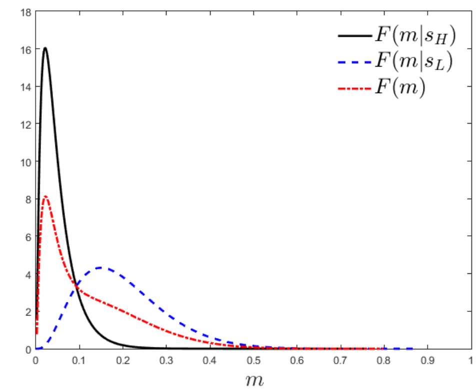


Figure 3: Timeline.



(a) Conditional PD of firm default $p(s, z)$ in the mixture model. Note that $p(s, z)$ is not a function of s in the standard model, that is, $p(s, z) = p(z)$ for each z.



(b) CDF of the proportion of loan defaults m. The Standard model can generate $F(m|s_H)$ and $F(m|s_L)$ but not the mixture distribution $F(m) = (1 - q)F(m|s_H) + qF(m|s_L)$.

3.1.2 Banks

The banking sector is standard: banks issue loans on date 0 financed by a mix of capital (bank equity) and deposits. The banking sector is perfectly competitive, with no legacy assets and no intermediation costs. Deposits are fully insured by the government, in perfectly elastic supply, with a net interest rate normalized to zero. Each firm gets a loan of \$1 on date 0 that must be rapid on date 1 after the realization of both risk factors s and z and the firm-specific shocks $\{\epsilon_i\}_{i=0}^{1}$ (see the timeline in Figure 3). The interest rate r on each loan is determined on date 0. If the loan is repaid on date 1, the bank gets $1 + r$. On the other hand, if the firm defaults on date 1, the bank gets $1 - \triangle$ where the parameter $\triangle$ measures the bank's loss-given default (LGD).

Net worth. Consider a bank with a portfolio of size \$1 on date 0 financed by \$$k$ units of capital and \$$1 - k$ units of deposits. Since the net interest rate on deposits is normalized to zero, the bank must repay $1 - k$ to the depositors on date 1. The bank's net worth on date 1, denoted y, is then

$$y = \underbrace{(1-m)(1+r)}_{\text{income from performing loans loans}} + \underbrace{m(1-\triangle)}_{\text{income from non-performing loans}} - \underbrace{1-k}_{\text{amount promised to the depositors}} \quad (6)$$

where $m \in [0, 1]$ denotes the proportion of loan defaults on date 1. The proportion of loan defaults on date 1 is determined by the realization of the two risk factors, namely $m = p(s, z)$, where $p(s, z)$ is the conditional probability of firm default in (3).

Default condition. The bank defaults on date 1 if its portfolio income $(1-m)(1+r)+m(1-\triangle)$ is insufficient to repay the amount owed to depositors $1-k$ (i.e., the bank's net worth in (6) is negative). In particular, the bank defaults if the proportion of loan defaults on date 1 exceeds a cutoff given by

$$\text{Bank defaults on date 1 if: } m > \bar{m} \equiv \tfrac{r+k}{r+\triangle}, \tag{7}$$

The cutoff $\bar{m}$ is determined in equilibrium since it depends on two endogenous variables: the interest rate on loans r and the bank's capital ratio k. If $k < \triangle$ then $\bar{m} \in (0,1)$ and $\frac{\partial \bar{m}}{\partial r} > 0$, $\frac{\partial \bar{m}}{\partial k} > 0$, and $\frac{\partial \bar{m}}{\partial \triangle} < 0$.

Failure risk. We denote by $F(m)$ the cumulative distribution function (c.d.f.) of the proportion of loan defaults on date 1, which will be derived in Section 4. Figure 4(b) depicts one example of $F(m)$. The probability of bank default on date 1 is denoted by q, and it equals the probability that the proportion of loan defaults m exceeds the cutoff in (7), namely

$$\text{Probability of bank default:} \quad q \equiv 1 - F(\bar{m}) = 1 - F\left(\tfrac{r+k}{r+\triangle}\right), \tag{8}$$

where the last line uses the definition of $\bar{m}$ in (7). Note that q is inversely related to $\bar{m}$ and $\frac{\partial q}{\partial r} < 0$, $\frac{\partial q}{\partial k} < 0$, and $\frac{\partial q}{\partial \triangle} > 0$. Irrespective of F, the probability of bank default is decreasing in the interest rate on loans r and the bank's capital ratio k and increasing in the bank's loss given default $\triangle$.

Bank's objective. The bank is operated in the best interest of its shareholders, who are risk-neutral and protected by limited liability. The payoff for the bank's shareholders on date 1 is $\max\{y, 0\}$ where y is the bank's net worth in (6). The net expected payoff of the bank's shareholders is

$$\pi(r,k) \equiv \tfrac{1}{1+\delta} \int_0^1 \max\{(1-m)(1+r) + m(1-\triangle) - (1-k), 0\}\, dF(m) - k, \tag{9}$$

where F is the c.d.f. of the fraction of loan defaults on date 1 and $\delta > 0$ is the minimum expected net rate of return required by the bank's shareholders. We impose a minimum mandatory capital ratio for the bank $k_{min} \in (0,1)$ set by regulation and, to make the model interesting, assume that $k_{min} < \triangle$ (if $k_{min} \geq \triangle$ the bank cannot fail, see (7)).

3.2 Definitions

The exogenous variables are $(\mu, G, \rho, k_{min}, \triangle, \delta)$ and the endogenous variables are the interest rate on loans r^*, the bank's capital ratio k^*, and the bank's default probability q^*.

Definition 1. *An economy* Θ *is defined by* $(\mu, G, \rho, k_{min}, \triangle, \delta)$ *with* $\mu : S \rightarrow \mathbb{R}$*,* $s \sim G$*,* $\rho \in [0,1]$*,* $k_{min} \in (0,1)$*,* $\triangle \in [0,1)$*, and* $\delta > 0$ *where*

(i) $\mu(s) = E[x_i|s]$ *is the average credit quality across all firms given* s*,*

(ii) G *is the c.d.f. of the risk factor* s*,*

(iii) ρ *is firm's exposure to the risk factor* $z \sim N(0,1)$ *(*s *and* z *are independent),*

(iv) k_{min} *is the mandatory minimum capital ratio,*

(v) $\triangle$ *is the bank's loss given loan default (LGD),*

(vi) δ *is the unit cost of bank equity (the expected net return for the bank's shareholders must be at least* δ*).*

Definition 2.

(i) ***Universal decrease in firm risk****. Economy* Θ *is characterized by a universal decrease in firm risk relative to economy* Θ' *(denoted* $\Theta \succ_U \Theta'$*) whenever* $p(s,z|\Theta) < p(s,z|\Theta')$ *almost surely or equivalently* $\Pr\left[(s,z) \in S \times Z \quad s.t. \quad p(s,z|\Theta) \geq p(s,z|\Theta')\right] = 0$*.*

(ii) ***Unconventional outcome****. The unconventional outcome happens whenever a universal decrease in firm risk is associated with a greater probability of bank failure. That is,* $\Theta \succ_U \Theta'$ *and* $q(\Theta) > q(\Theta')$*.*

4 Results

Recall that the Standard Vasicek model is a special case of the baseline setup where $\mu(s)$ in (1) does not vary with s, making this risk factor irrelevant. We refer to the more general case where $\mu(s)$ does very with s as the Mixture Vasicek model. The main result in this section is the following.

Theorem 1. *The unconventional outcome:*

(i) cannot happen within the Standard Vasicek model,

(ii) but it can happen within the Mixture Vasicek model.

All proofs can be found in Appendix C (see Definition 2 for the unconventional outcome). For the rest of this section, we proceed as follows. *First*, we characterize the equilibrium probability of bank failure after going through the numerical example below. *Second*, we explain why the unconventional outcome cannot happen within the Standard Vasicek model. *Third*, we investigate when the unconventional outcome happens within the Mixture Vasicek model.

Numerical example. Figure 4 displays the equilibrium probability of bank default q^* as a function of the probability of firm default p implied by the Mixture Vasicek model (panels (1) - (3)) and the Standard Vasicek model (panels (4) - (6)). The unconventional outcome happens only in panels (2) and (3), with both panels generated by the Mixture model. We stress the following.

- The baseline model assumes for simplicity that firm default probability is independent of the required loan repayment. Appendix B extends the baseline setup to a model where the probability of firm default increases in the required repayment and finds that the same pattern: the unconventional outcome happens only within the Mixture Vasicek model (see Figures 7 - 8).

- The baseline model also assumes that banks can issue insured deposits leading to the question of whether the unconventional outcome in this setup is attributed solely to frictions due to miss-priced government guarantees. To test whether this is the case, we recomputed our baseline examples assuming no deposit insurance. Figure 4 (and Figure 7) show that distortions generated by government guarantees do not drive our results since we obtain similar qualitative patterns with or without deposit insurance.

4.1 Capital ratio, interest rate, and bank failure risk

We now characterize the equilibrium capital ratio k^*, interest rate r^*, and finally, the probability of bank failure q^*.

Capital ratio. Deposits are fully insured, and bank equity capital is (privately) costly for the bank's shareholders $\delta > 0$. As a result, the bank's shareholders prefer to operate with the minimum mandatory capital ratio k_{min}. Specifically, differentiating the shareholder's net expected payoff in (9) with respect to k and using the definition of the default cutoff $\bar{m}$ in (7) yields

$$\frac{\partial \pi}{\partial k} = \tfrac{1}{1+\delta} F\left(\tfrac{r+k}{r+\triangle}\right) - 1 < 0. \tag{10}$$

The strict inequality above follows from $\delta > 0$. It implies that the optimal choice of capital ratio from the perspective of the bank's shareholders is k_{min}.[11]

[11]Even if bank equity is not privately costly ($\delta = 0$), the bank's shareholders still prefer to operate with the minimum possible capital if the bank fails with positive probability since this would maximize the value of the deposit insurance subsidy.

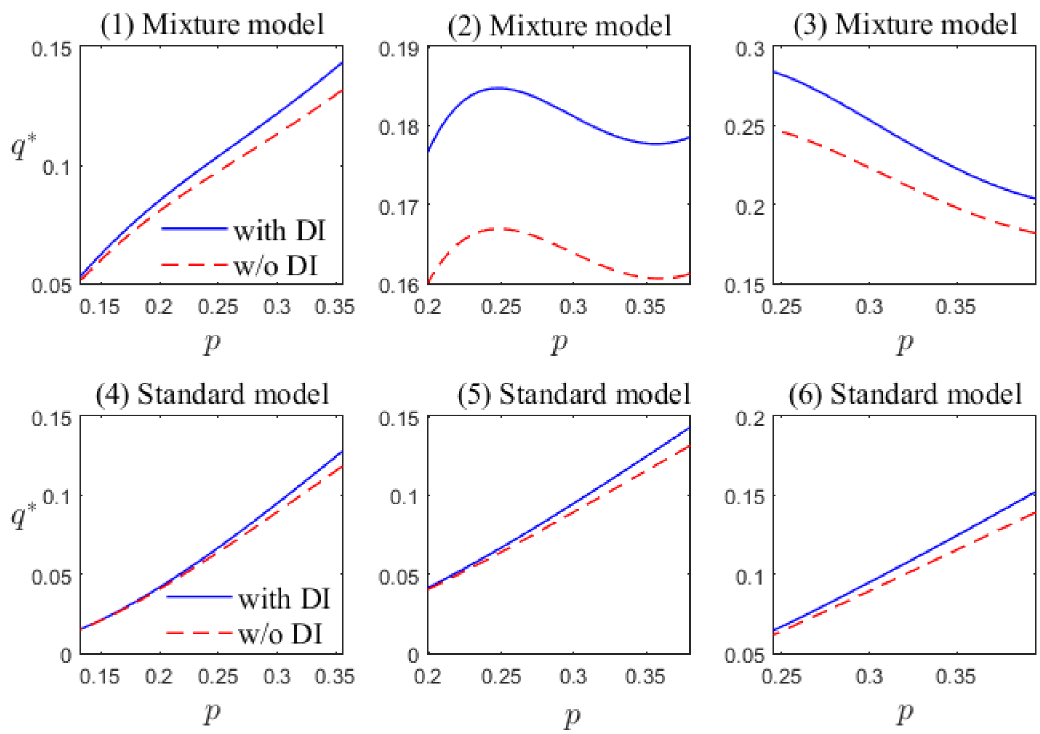


Figure 4: Probability of bank failure q^* as a function of the firm's probability of default p. The top three panels are generated by the Mixture Vasicek model, and the bottom three by the Standard Vasicek model. The solid lines correspond to deposit insurance and the dashed lines to no deposit insurance. The parameter values are $k = 0.05$, $\triangle = 0.2$, $\delta = 0.01$, and $\rho = 0.12$. The distribution of s is binary as in (14), with each value of s equally likely. The unconventional outcome happens in panels (2) – (3), where a higher probability of firm default can be associated with a lower probability of bank failure.

Interest rate. Since the banking sector is perfectly competitive, the equilibrium value of the interest rate on loans, denoted r^*, is such that the bank's shareholders break even in expectation $\pi(r^*, k_{min}) = 0$ where $\pi(r, k)$ is the net expected payoff of the bank's shareholders (see (9)). The equilibrium interest rate can then be shown to equal

$$r^* = \underbrace{\tfrac{p\triangle + k_{min}\delta - (1+\delta)T^*_{DI}}{1-p}}_{\text{actuarially fair rate}} - \underbrace{(1+\delta)\tfrac{T^*_{DI}}{1-p}}_{\text{effect of deposit insurance}}, \tag{11}$$

where $T^*_{DI} \geq 0$ denotes the value of the implicit deposit insurance subsidy received by the bank's shareholders (the Merton put Merton (1977)). Notice that the interest rate r^* is defined implicitly since T^*_{DI} is function of r^*. That is,

$$T_{DI}^{*} \equiv \int_{\bar{m}^{*}}^{1} \underbrace{(m(r^{*}+\triangle)-r^{*}-k_{min})}_{\geq 0} dF(m) = q^{*} \underbrace{E[-y \,|\, m > \bar{m}^{*}]}_{\geq 0}, \tag{12}$$

where y is the bank's net worth (see (6)), $\bar{m}^{*} \equiv \frac{r^{*}+k_{min}}{r^{*}+\triangle}$ is the equilibrium default cutoff, and q^{*} is the bank's equilibrium probability of default (see Lemma 1). The expected shortfall of the depositors conditional on their bank defaulting (which happens with probability q^{*}) is thus $E[-y \,|\, m > \bar{m}^{*}]$, which equals the expected transfer from the deposit insurance fund since deposits are insured. As long as $q^{*} > 0$, the equilibrium interest rate r^{*} falls below the actuarially fair rate in (11) since banks do not internalize the losses they impose on the deposit insurance fund.[12]

Bank failure risk. Lemma 1 derives the equilibrium probability of bank failure q^{*}, which is a function of the equilibrium interest rate on loans r^{*}, the minimum capital ratio k_{min}, the bank's loss given default $\triangle$, and the credit risk profile of the firms (μ, G, ρ).

Lemma 1. *The equilibrium probability of bank default is given by*

$$q^{*} = \int_{S} \left[1 - \Phi\left(\tfrac{1}{\sqrt{\rho}}\left[\sqrt{1-\rho}\,\Phi^{-1}\left(\tfrac{r^{*}+k_{min}}{r^{*}+\triangle}\right) + \mu(s)\right]\right)\right] dG(s). \tag{13}$$

As long as the two risk factors are independent as we maintain throughout the analysis, Lemma 1 applies to the Mixture Vasicek model where $\mu(s)$ varies with s and to the Standard Vasicek model where $\mu(s)$ is independent of s.[13] Note that q^{*} is decreasing in r^{*} and k_{min} and increasing in $\triangle$. One way to interpret (13) is that nature first draws the risk factor s from the distribution G and then draws the fraction of loan defaults from the conditional distribution of loan defaults for a given realization of s given by

$$F(m|s) \equiv \Phi\left(\tfrac{1}{\sqrt{\rho}}\left[\sqrt{1-\rho}\,\Phi^{-1}(m) + \mu(s)\right]\right).$$

Then $q^{*} = \int_{S} q^{*}(s) dG(s)$ where $q^{*}(s) \equiv 1 - F(\bar{m}^{*}|s)$ denotes the probability of bank failure conditional on s. Figure 4(b) displays the conditional distribution of the fraction of loan defaults $F(m|s)$ for two different values of the risk factor $s \in \{s_L, s_H\}$ such that $s_L < s_H$. Notice that a higher value of s shifts the distribution to the left, thus making lower values of m more likely. The figure also displays the unconditional distribution of loan defaults,

[12] If deposits are not insured then $T_{DI}^{*} = 0$ and r^{*} equals the actuarially fair rate in (11). It is well-known that deposit insurance and other forms of miss-priced government guarantees distort loan pricing and can lead to the under-pricing of risky loans (see Rochet (1992), Repullo and Suarez (2004), Bahaj and Malherbe (2020), and Harris et al. (2020).

[13] One should bear in mind that the value of the equilibrium interest rate r^{*} is different across the two models since the probability of firm default is affected by s in the mixture model but not in the standard model.

namely $F(m)$.

Two additional lemmas. As a final ingredient for the analysis, we need two lemmas. The first describes two equivalent ways to induce a universal change in firm risk, and the second describes how such a change would affect the equilibrium interest rate on loans. Recall that for a fixed value of the risk factor s, the average credit quality across firms is given by $\mu(s) = E[x_i|s]$ and the probability of firm default is given by $p(s) = \Phi(-\mu(s))$ (see (1) and (4)).

Lemma 2. *Consider two economies Θ and Θ' (see Definition 1) where firms have the same exposure to the risk factor z, that is, $\rho(\Theta) = \rho(\Theta')$. The following statements are equivalent:*

(i) Θ exhibits a universal decrease in firm risk relative to Θ' (i.e., $\Theta \succ_U \Theta'$).

(ii) $\mu(s|\Theta) > \mu(s|\Theta')$ for each $s \in S$.

(ii) $p(s|\Theta) < p(s|\Theta')$ for each $s \in S$.

In other words, one can induce a universal decrease in firm risk in two equivalent ways: by taking $\mu(s|\Theta) > \mu(s|\Theta')$ for each s or by taking $p(s|\Theta) < p(s|\Theta')$ for each s. Moreover, we can go from $\mu(s)$ to $p(s)$ and vice versa through the relationship $p(s) = \Phi(-\mu(s))$.

Lemma 3. *If $\Theta \succ_U \Theta'$ then $r^*(\Theta) < r^*(\Theta')$. A universal decrease in firm risk leads to a lower equilibrium interest rate on loans.*

A universal decrease in firm risk implies the rate of firm default for each realization of the risk factors will be lower and bank profits higher. The equilibrium interest rate on loans must then fall to ensure that the bank free-entry condition is satisfied, namely that the shareholders of each bank break even in expectation.

4.2 Standard Vasicek model

We now show that the unconventional outcome will not arise in the Standard Vasicek model: a universal decrease in firm risk would always lead to a lower probability of bank failure. We proceed by assuming (to derive a contradiction) that the conventional outcome can happen in equilibrium, even in this case. We then show that if this were true, then it would lead to a situation such as the one depicted in Figure 5 where the bank's net worth in the high credit risk economy crosses the bank's net worth in the low credit risk economy at least twice - once from above and once from below. But then we establish the following lemma.

Lemma 4. *The situation depicted in Figure 5 cannot emerge in equilibrium.*

Lemma 4 shows that if the outcome depicted in 5 were true, the risk factor z, which recall follows the standard normal distribution, must have a non-monotone hazard function. However, this is a contradiction since the hazard function of the standard normal is monotone-increasing.[14] Specifically, suppose the unconventional outcome is true. In that case, the net worth of the bank in the high-risk economy $\tilde{y}^*(z)$ must become positive for lower values of z than the net worth of the bank in the low-risk economy $y^*(z)$, as depicted in Figure 5. At the same time, $\tilde{y}^*(z)$ cannot stay above $y^*(z)$ since, due to free entry, the expected payoff of the bank's shareholders in each economy must be the same and equal to $(1+\delta)k_{min}$. Hence, $\tilde{y}^*(z)$ must also cross $y^*(z)$ from above at least once (for example, at point z_1 on the figure), but Lemma 1 shows that this cannot be true. Intuitively, the Standard model is not sufficiently flexible to allow for the changes in firm risk that would generate the unconventional outcome and satisfy the free-entry condition of the bank's shareholders.

Discussion. To be clear, an inverse equilibrium relationship between borrower and bank risk *can* arise in the Standard Vasicek model if one varies other factors such as the level of competition (Martinez-Miera and Repullo (2010)) or bank capital requirements (Repullo and Suarez (2004)). We apply the term unconventional outcome *only* if the bank is not simultaneously earning greater profits due to lower competition or holding more capital due to stricter capital regulation. Specifically, regulation might force the bank to hold more capital as it lends to firms with higher default risk. If this risk-weighted capital channel is strong enough, the bank will become less likely to fail. Further, the banks in the high-risk economy can become less likely to fail even within the Standard Vasicek model if they compete less aggressively and thus earn higher profits than the banks in the low-risk economy.[15]

4.3 Mixture Vasicek model

Finally, we investigate why the unconventional outcome can happen in the Mixture Vasicek setup. For simplicity, we assume that the risk factor s is binary: $s = B$ with probability $\eta \in (0, 1)$ and $s = G$ with the complement probability

$$\mu(s) = \left\{ \begin{array}{c} \mu(G) \\ \mu(B) \end{array} \right\} \qquad \text{as} \qquad \left\{ \begin{array}{c} s = G \\ s = B \end{array} \right\} \tag{14}$$

[14]The hazard function of a random variable $x \sim F$ with a p.d.f. f is defined as $h(x) \equiv \frac{f(x)}{1-F(x)}$.

[15]The effect of this competition channel was investigated by Martinez-Miera and Repullo (2010). They show the relationship between competition and bank failure risk is generally non-monotone. At the same time, Figure 2 in Martinez-Miera and Repullo (2010) shows that higher firm default risk leads to a weakly greater bank failure risk for a fixed level of competition, which is consistent with our Theorem 1.

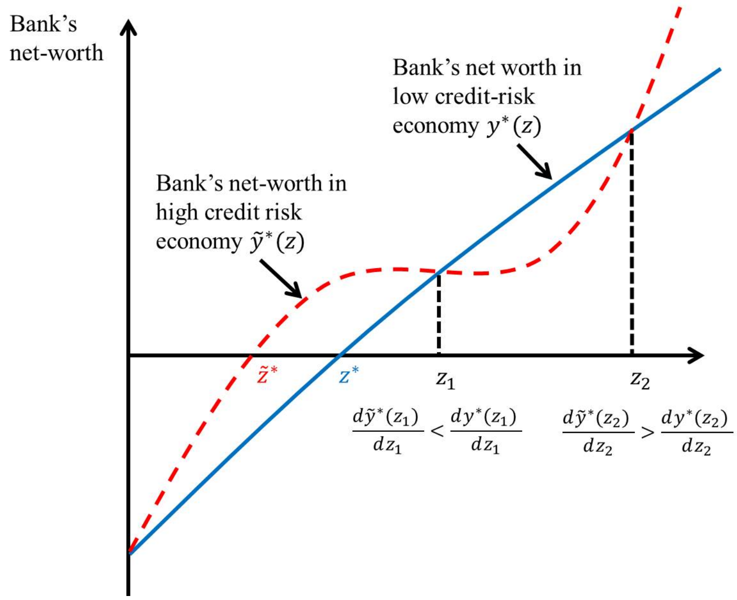


Figure 5: The solid line depicts the bank's net worth in the low-risk economy where each firm's unconditional probability of default is p. The dashed line depicts the bank's net worth in the high-risk economy where each firm's unconditional probability of default is $p+\sigma$ with $\sigma > 0$. Credit risk in each economy is governed by the Standard Vasicek model, and the figure is constructed under the assumption that the bank in the high-risk economy is less likely to default than the bank in the low-risk economy. However, this situation cannot arise since it implies that the hazard rate function of the standard normal random variable z is non-monotone when in fact, it is a monotone-increasing function.

with $\mu(G) \geq \mu(B)$. Note $p(G) \leq p(B)$ with $s = B$ interpret as the bad realization of s. The unconditional firm default probability is $p = (1-\eta)p(G) + \eta p(B)$ where $p(s) = \Phi(-\mu(s))$ is the probability of firm default in state s.

Proposition 1. *Starting from $p(G) = p(B)$ with a corresponding bank failure probability q^*, suppose firm credit risk improves slightly for $s = G$ while remaining unchanged for $s = B$. That is, $\tilde{p}(G) = p(G) - \epsilon$ and $\tilde{p}(B) = p(B)$ where $\epsilon > 0$. The bank in the lower credit risk economy is then more likely to fail than the bank in the higher credit risk economy (i.e., $\tilde{q}^* > q^*$) if and only if the following parameter condition is satisfied $\frac{p}{1-p}\left(1 - \frac{k_{min}}{\triangle}\right) < (1-\rho)^{1/4}$.*

One way to think of the setup in Proposition 1 is as follows. The risk factor $z \sim N(0,1)$ is the state of the economy, whereas the binary risk factor s is a policy outcome or some other feature of the environment that used not to matter for firm risk ($p(G) = p(B)$) but now becomes relevant (i.e., $\tilde{p}(G) < \tilde{p}(B)$). Note that such changes in firm risk are ruled out by the Standard Vasicek model where $\mu(s)$ is not allowed to vary with s, implying that s does not affect credit risk.

The parameter condition in Proposition 1 holds for a wide range of parameter values, particularly those most relevant for a typical commercial bank, but it does not hold if firm defaults are perfectly correlated, i.e., $\rho = 1$. If loan defaults are perfectly correlated, then all firms repay or all default, implying for each s and each z, the firms' default rate is either zero or one. The bank thus remains solvent if the default rate is zero and fails if the default rate is one, but the probability that all firms default is now lower (since we assumed that firm credit risk has declined), implying that the bank failure risk can only fall.

Numerical example cont. Proposition 1 examined the effect of small changes starting from $p(G) = p(B)$ and derived necessary and sufficient conditions for the unconventional outcome. Figure 6 extends the analysis to situations where the starting point is $p(G) < p(B)$, showing that the unconventional outcome still emerges. For example, it happens as we move from the economy located at point A to the economy at point B in Figure 6. First, moving from economy A to B implies that each firm is less likely to default for each realization of the risk factors s and z (a universal decrease in firm risk), but the percentage reduction in firm default risk is larger for $s = G$. Second, the equilibrium interest rate on loans is lower in economy B than in A (see Lemma 3). However, the reduction in the rate of firm default is only marginal for $s = B$, which, combined with the lower interest rate, implies the bank is likely to fail when $s = B$. Finally, the bank's failure risk in economy B compared to A goes down for $s = G$ and up for $s = B$. However, the elevated failure risk for $s = B$ outweighs the reduced failure risk for $s = G$, implying that the overall bank failure risk is higher in economy B than in A.

To summarize, the situation depicted in Figure 6 is similar to the example in Section 2 (see Figure 2). In both cases, the percentage decrease in the firm default rate happened for all risk factors values - a universal decline in firm risk. At the same time, the gain in credit quality mainly occurred for those values of the risk factors characterized by lower rates of firm default and, thus, higher bank profitability.

5 Conclusion

The conventional view is that other things being equal, a bank with safer borrowers will be automatically safer than a bank with riskier borrowers. As a result, the bank with the safer borrowers should not be as heavily scrutinized by regulators and allowed to hold less capital or, more generally, reduce the risk mitigation it undertakes compared to the bank with the riskier borrowers.

We presented a model building on the canonical credit risk framework that challenges this conventional view. Our model implies that once we allow the interest rate on loans to adjust to reflect borrower risk, the bank with the safer borrowers might be more likely to

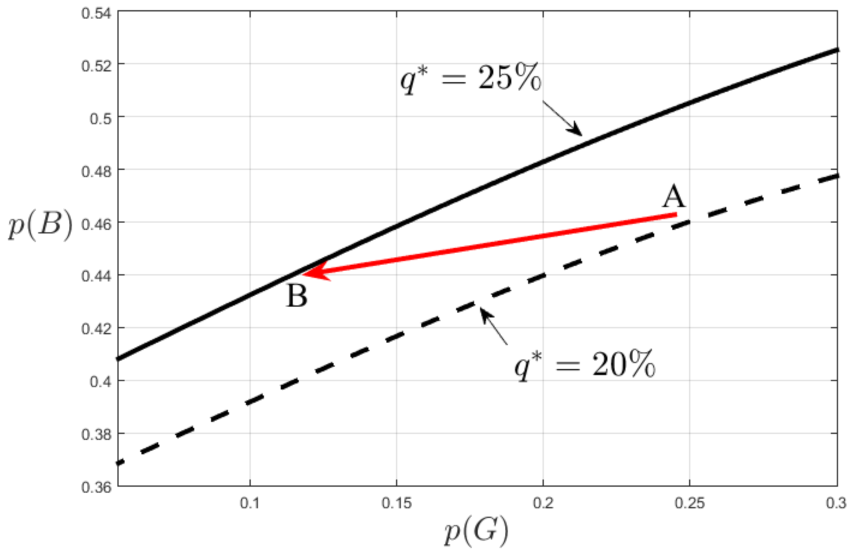


Figure 6: The Mixture Vasicek model generated this figure. It displays the equilibrium probability of bank failure q^* as a function of $p(G)$ and $p(B)$ where $p(s)$ is the probability of firm default conditional on $s \in \{G, B\}$. The parameter values (i.e., k_{min}, $\triangle$, δ, ρ) are the same as in Figure 4. Pairs of $p(G)$ and $p(B)$ located on the dashed (solid) line correspond to $q^* = 0.2$ ($q^* = 0.25$). The unconventional outcome: moving from the economy located at point A to the one located at the point B implies that each firm will be less likely to default for each realization of s, but the bank in economy B will be more likely to fail than the bank in economy A.

fail than the bank with the riskier borrowers. This unconventional outcome is not solely resulting from frictions due to miss-priced deposit insurance since it occurs with or without government guarantees. Instead, the underlying reason is that credit market competition pushes the bank with the safer borrowers to set a lower interest rate on loans which erodes its profitability and can make it more likely to fail than the bank with the riskier borrowers. Specifically, such an outcome tends to happen when the safer borrowers are only marginally less likely to default than the riskier borrowers in those states of the economy where both types of banks have relatively low profits and are thus close to failing (i.e., in a recession).

As it turns out, however, the unconventional outcome is ruled out by Vasicek's canonical model of credit risk since it is not flexible enough to capture those changes in credit risk that are most likely to generate the unconventional outcome (due to the single risk factor and the assumption of normality). At the same time, the unconventional outcome naturally arises if one considers a multiple risk factor extension of the baseline Vasicek setup. We presented one such extension that involves only a marginal complication relative to the canonical setup and still allows for closed-form expressions.

The main policy takeaway of our analysis is thus straightforward: models relying on a single risk factor to evaluate the failure risk of banks can be misleading and predict that bank failure risk goes down when it goes up or vice versa. Moreover, the likelihood of such a misleading conclusion is especially high when loan terms (i.e., interest rate, loan-to-value, loan-to-income, and so on) adjust endogenously in response to credit market competition and borrower credit risk.

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Appendix

A. Summary of notation

Variable:	Explanation:
$x_i = \mu(s) + \sqrt{\rho}z + \sqrt{1-\rho}\epsilon_i$	Firm $i \in [0,1]$ credit quality on date 1 (default $x_i < 0$)
$s \sim G$, $z \sim N(0,1)$, s and z independent	Systematic risk factors realized on date 1
$\epsilon_i \sim N(0,1)$ (i.i.d. and independent of s, z)	Firm-specific shock realized on date 1
$\mu(s) = E[x_i\|s]$	Mean credit quality given risk factor s
$p(s,z) = \Pr[x_i < 0 \mid s, z]$	Prob. of firm default on date 1 given s and z
$p(s) = \Pr[x_i < 0\|s]$	Prob. of firm default on date 1 given s
$p = \Pr[x_i < 0]$	Unconditional prob. of firm default on date 1
$y = (1-m)(1+r) + m(1-\triangle) - (1-k)$	Bank net-worth on date 1
k; $k_{min} \in (0,1)$	Bank capital ratio; min capital requirement
$\triangle \in (0,1]$	Bank's loss given loan default
$\delta > 0$	Unit cost of capital for the bank's shareholders
r	Interest rate on loans
$m \in [0,1]$	Fraction of loan defaults on date 1
$F(m\|s)$	CDF of the fraction of loan defaults on date 1 given s
$F(m)$	CDF of the fraction of loan defaults on date 1
$q(s) = \Pr[y < 0\|s]$	Prob. bank defaults on date 1 given s
$q = \Pr[y < 0]$	Unconditional prob. bank defaults on date 1

Table 1: Summary of model notation in Section 3.

B. Structural model of firm default

Here we extend the baseline model in Section 3 to include a structural model of firm default and show that the main results continue to apply (see Figures 7 - 8). Specifically, each firm i gets a loan of L on date 0 and must repay $(1+r)L$ on date 1, where the loan amount L is fixed. Firm i defaults if the value of its assets at loan maturity, denoted A_i, falls below the required repayment $A_i < (1+r)L$. Firm i's asset value on date 1 is given by

$$\ln(A_i) = \mu(s) + \sigma(s)X_i \qquad \text{and} \qquad X_i = \sqrt{\rho}z + \sqrt{1-\rho}\varepsilon_i, \tag{15}$$

where $s \sim G$, $z \sim N(0,1)$, $\epsilon_i \sim N(0,1)$, and s, z, $\{\epsilon_i\}_{i=0}^{1}$ are independent. Since $X_i \sim N(0,1)$ the distribution of firm asset value condition on s is log-normal $A_i(s) \sim LN(\mu(s), \sigma^2(s))$. The firm's probability of default conditional on s and z can be shown to equal

$$p(s,z|r) = \Phi\left(\frac{c(s|r) - z\sqrt{\rho}}{\sqrt{1-\rho}}\right) \qquad \text{where} \qquad c(s|r) \equiv \frac{\ln(1+r)L - \mu(s)}{\sigma(s)}.$$

The firm's probability of default conditional on s is given by

$$p(s|r) = \int_{-\infty}^{\infty} p(s, z|r) d\Phi(z),$$

and the firm's unconditional probability of default is

$$p(r) = \int_S p(s|r) dG(s).$$

where S is the domain of the risk factor $s \sim G$. Note that $p(s, z|r)$, $p(s|r)$, and $p(r)$ are all increasing in the required loan repayment $(1 + r)L$. Next, the distribution of the fraction of loan defaults m for given s can be shown to equal

$$F(m, s|r) = \Phi\left(\frac{1}{\sqrt{\rho}}\left[c(s|r) - \sqrt{1-\rho}\Phi^{-1}(m)\right]\right),$$

and the unconditional distribution of the fraction of loan defaults is given by

$$F(m|r) = \int_S F(m, s|r) dG(s).$$

Note that $F(m|r)$ depends on r. Finally, the probability of bank default on date 1 is given by

$$q^* = 1 - F\left(\frac{r^*+k}{r^*+\triangle}|r^*\right),$$

where r^* is the equilibrium loan interest rate determined through perfect competition as in the baseline model. Figure 7 shows the equilibrium relationship between the bank's default probability q^* and the unconditional default probability of a representative firm $p^* \equiv p(r^*)$. We take $\sigma(s) = 0.5$ for all s and $\mu(s)$ as in (14) with μ_G going from 0 to 1 and μ_B going from 0.05 to -0.1. For each panel: $L = 0.8$, $\triangle = 0.2$, $\delta = 0.02$, and we vary k_{min} and η across the four panels. Figure 8 is constructed under the same parameter values as Figure 7, but we switch off the effect s by setting $\mu(s) = (1 - \eta)\mu(G) + \eta\mu(B)$ for each s.

C. Proofs

Theorem 1.

Case (i): We will establish that the unconventional outcome cannot happen in the Standard Vasicek model by using Figure 5 and Lemma 4, but to get there, we need to establish some preliminary results first. To begin, since $\mu(s) = \mu$ for each $s \in S$ we have

$$p(s, z) = p(z) = \Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right) \qquad \text{and} \qquad p(s) = p = \Phi^{-1}(\mu) \qquad \text{for each } s \in S. \quad (16)$$

The rate of firm default on date 1 for given z also equals $p(z)$. The bank's net worth as a function of z is the following.

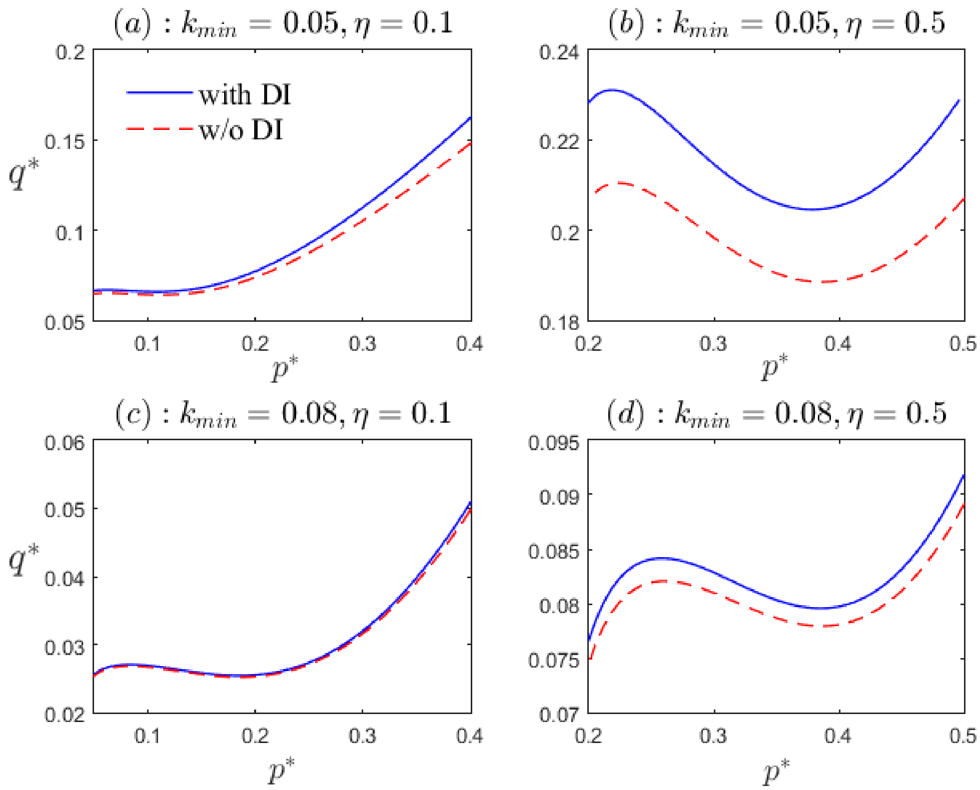

Figure 7: Probability of bank failure q^* as a function of the unconditional default probability of each firm p^* using a structural model of firm default and two risk factors (Mixture Vasicek model). The solid line in each panel is with deposit insurance, and the dashed line is without deposit insurance.

$$y(z) = (1 - p(z))(1 + r) + p(z)(1 - \triangle) - (1 - k). \tag{17}$$

In equilibrium, the bank operates with the minimum mandatory capital ratio $k^* = k_{min}$ (see Section 4), and the interest rate on loans r^* is such that the bank's shareholders break even in expectation. That is,

$$\pi(r^*, k_{min}|\text{low risk}) = \tfrac{1}{1+\delta} \int_{-\infty}^{\infty} \max\{y^*(z), 0\}\, d\Phi(z) - k_{min} = 0, \tag{18}$$

where $y^*(z)$ denotes the bank's net worth as a function of z when $r = r^*$ and $k = k_{min}$. Denote by z^* the value of z such that $y^*(z^*) = 0$. Using (16) to solve for z^*

$$z^* \equiv \tfrac{1}{\sqrt{\rho}} \left[\Phi^{-1}(p) - \sqrt{1-\rho}\,\Phi^{-1}\left(\tfrac{r^*+k_{min}}{r^*+\triangle} \right) \right]. \tag{19}$$

Since $p(z)$ is decreasing in z (see (16)) $y^*(z)$ is increasing in z. As a result, the bank is insolvent if $z < z^*$ and is solvent if $z > z^*$. Since $z \sim N(0, 1)$ the probability of bank default is $\Pr[z < z^*]$ becomes

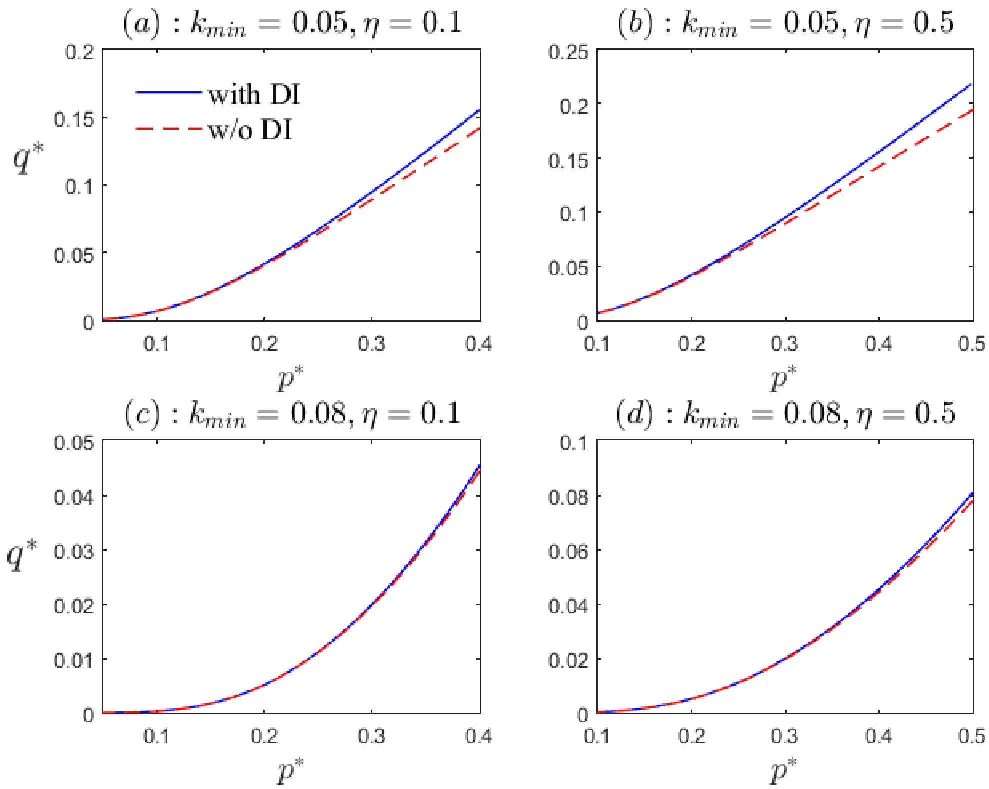

Figure 8: Probability of bank failure q^* as a function of the unconditional default probability of each firm p^* using a structural model of firm default and a single risk factor (Standard Vasicek model). The solid line in each panel is with deposit insurance, and the dashed line is without deposit insurance.

$$q^* = \Phi(z^*). \tag{20}$$

Next, consider another high-risk economy such that each firm's unconditional default probability is $\tilde{p} \equiv p + \sigma$ for $\sigma > 0$. The rate of firm default as a function of z in this economy is

$$\tilde{p}(z) = \Phi\left(\frac{\Phi^{-1}(\tilde{p}) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right).$$

The bank's net worth as a function of z in the high-risk economy is

$$\tilde{y}(z) = (1 - \tilde{p}(z))(1 + r) + \tilde{p}(z)(1 - \triangle) - (1 - k). \tag{21}$$

The bank in the high-risk economy also operates with the minimum possible capital ratio k_{min} and the equilibrium loan interest rate is such that the bank's shareholders break even in expectation

$$\pi(\tilde{r}^*, k_{min}|\text{high risk}) = \tfrac{1}{1+\delta} \int_{-\infty}^{\infty} \max\{\tilde{y}^*(z), 0\}\, d\Phi(z) - k_{min} = 0, \tag{22}$$

where $\tilde{y}^*(z)$ denotes the bank's net worth in (21) with $k = k_{min}$ and $r = \tilde{r}^*$. Observe the following

$$p(z) = \Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right) < \Phi\left(\frac{\Phi^{-1}(\tilde{p})-\sqrt{\rho}z}{\sqrt{1-\rho}}\right) = \tilde{p}(z).$$

The high-risk economy features a universal decrease in firm risk (see Definition 2). Then Lemma 3 implies $\tilde{r}^* > r^*$. Finally, the probability of bank failure in the high-risk economy is $\tilde{q}^* = \Phi(\tilde{z}^*)$ where

$$\tilde{z}^* \equiv \frac{1}{\sqrt{\rho}}\left[\Phi^{-1}(\tilde{p}) - \sqrt{1-\rho}\,\Phi^{-1}\left(\frac{\tilde{r}^*+k_{min}}{\tilde{r}^*+\triangle}\right)\right]. \quad (23)$$

Recall that we must show that the bank in the high-risk economy is more likely to default than the bank in the low-risk economy $\tilde{q}^* > q^*$. We proceed by contradiction: suppose $\tilde{q}^* \le q^*$. The solid line in Figure 5 shows the bank's net worth $y^*(z)$ as a function of z in the low-risk economy. The dashed line on the same figure shows the bank's net worth $\tilde{y}^*(z)$ as a function of z in the high-risk economy. Note that

$$\tilde{q}^* \le q^* \qquad \Rightarrow \qquad \tilde{z}^* \le z^*,$$

which combined with $\tilde{r}^* > r^*$ and the break-even condition for the bank's shareholders implies that $\tilde{y}^*(z)$ must cross $y^*(z)$ at least twice once from below and once from above as show in Figure 5. To see why note the following.

(a) If $\tilde{y}^*(z)$ crosses $y^*(z)$ only from below then this implies $\tilde{z}^* > z^*$ which is inconsistent with $\tilde{q}^* \le q^*$.

(b) If $\tilde{y}^*(z)$ crosses $y^*(z)$ only from above, then $\tilde{y}^*(z) < y^*(z)$ for z large enough which is inconsistent with $\tilde{r}^* > r^*$ since the rate of default in both economies goes to zero as z goes to ∞ (see (16)).

(c) If $\tilde{y}^*(z)$ does not cross $y^*(z)$ either from above or below then $\tilde{z}^* \le z^*$ and $\tilde{r}^* > r^*$ implies

$$\pi(r^*, k_{min}|\text{low risk}) < \pi(\tilde{r}^*, k_{min}|\text{high risk}).$$

But then the free entry condition will be violated either for the low risk economy or for the high-risk economy (or both) since the left and the right-hand side of the above cannot both equal zero.

We are now ready to invoke Lemma 4, which shows that the situation depicted in Figure 5 cannot emerge since it implies that the hazard ratio function of the standard normal is non-monotone, which is a contradiction. Therefore, we must have $\tilde{q}^* > q^*$ as desired.

Case (ii): The statement that the unconventional outcome can emerge in the Mixture Vasicek model follows from Proposition 1 and also from the numerical examples in Figures 4 - 6 (see also Figure 7).

Lemma 1.

The bank defaults on date 1 if $m < \bar{m}^*$ and remains solvent if $m > \bar{m}^*$ where the cutoff $\bar{m}^*$ equals the value of m such that

$$(1-\bar{m}^*)(1+r^*) + \bar{m}^*(1-\triangle) = 1 - k_{min} \qquad \Rightarrow \qquad \bar{m}^* = \frac{r^*+k_{min}}{r^*+\triangle}.$$

The parameter restriction $0 < k_{min} < \triangle$ implies that the cutoff $\bar{m}^*$ is interior $\bar{m}^* \in (0,1)$, increasing in r^* and k_{min} and decreasing in $\triangle$. Next, the c.d.f. of the proportion of loan defaults m conditional on s is denoted $F(m;s)$ and given by

$$\begin{aligned} F(m|s) &= \Pr\left[p(s,z) \le m \mid s\right] \\ & \Pr\left[\Phi\left(\tfrac{-\mu(s)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right) \le m \mid s\right] \\ &= \Phi\left(\tfrac{1}{\sqrt{\rho}}\left[\sqrt{1-\rho}\Phi^{-1}(m) + \mu(s)\right]\right) \end{aligned} \tag{24}$$

The second line in (24) uses the definition of $p(s,z)$ in (3). The bank's probability of default for a given s is then

$$\begin{aligned} \Pr[m > \bar{m}^*|s] &= 1 - F(\bar{m}^*; s) \\ &= 1 - \Phi\left(\tfrac{1}{\sqrt{\rho}}\left[\sqrt{1-\rho}\Phi^{-1}(\bar{m}^*) + \mu(s)\right]\right) \end{aligned}$$

Finally, the unconditional probability of bank failure is

$$q^* = \int_S \Pr[m > \bar{m}^*|s]dG(s),$$

which gives the expression in (13).

Lemma 2.

First, $\rho(\Theta) = \rho(\Theta')$ implies

$$\mu(s|\Theta) > \mu(s|\Theta') \qquad \Leftrightarrow \qquad \Phi\left(\frac{-\mu(s|\Theta)-\sqrt{\rho(\Theta)}z}{\sqrt{1-\rho(\Theta)}}\right) < \Phi\left(\frac{-\mu(s|\Theta')-\sqrt{\rho(\Theta')}z}{\sqrt{1-\rho(\Theta')}}\right) \tag{25}$$

Second, $p(s) = \Phi(-\mu(s))$ implies

$$\mu(s|\Theta) > \mu(s|\Theta') \qquad \Leftrightarrow \qquad p(s|\Theta) < p(s|\Theta') \tag{26}$$

Lemma 2 follows by combining (26) and (25) and Definition 2.

Lemma 3.

If economy Θ features universally lower firm risk than economy Θ' then

$$p(s,z|\Theta) < p(s,z|\Theta') \qquad \text{for each } s \in S.$$

Fix some $r \ge 0$ and note the following.

$$\begin{aligned} y(s,z|\Theta) &= (1-p(s,z|\Theta))(1+r) + p(s,z|\Theta)(1-\triangle) - (1-k) \\ &> (1-p(s,z|\Theta'))(1+r) + p(s,z|\Theta')(1-\triangle) - (1-k) \\ &= y(s,z|\Theta') \end{aligned}$$

The above then implies

$$\begin{array}{rcl} \pi(r,k|\Theta) & = & \frac{1}{1+\delta}\int_Z\int_S \max\{y(s,z|\Theta),\,0\}\,d\Phi(z)dG(s)-k \\ & > & \frac{1}{1+\delta}\int_Z\int_S \max\{y(s,z|\Theta'),\,0\}\,d\Phi(z)dG(s)-k \\ & = & \pi(r,k|\Theta') \end{array}$$

Finally, applying the free entry condition and using $k^*(\Theta)=k^*(\Theta')=k_{min}$ we get

$$\pi(r^*(\Theta),k_{min}|\Theta)=\pi(r^*(\Theta'),k_{min}|\Theta')=0.$$

which implies $r^*(\Theta)<r^*(\Theta')$ as desired since $\frac{\partial\pi}{\partial r}>0$.

Lemma 4.

Recall that the net worth of the low-risk economy bank is

$$y^*(z)=(1-p(z))(1+r^*)+p(z)(1-\triangle)-(1-k_{min}),$$

where $p(z)$ is the conditional probability of firm default in that economy

$$p(z)=\Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right).$$

The net worth of the high-risk economy bank is

$$\tilde{y}^*(z)=(1-\tilde{p}(z))(1+\tilde{r}^*)+\tilde{p}(z)(1-\triangle)-(1-k_{min}),$$

where $\tilde{p}(z)$ is the conditional probability of firm default in that economy

$$\tilde{p}(z)=\Phi\left(\frac{\Phi^{-1}(\tilde{p})-\sqrt{\rho}z}{\sqrt{1-\rho}}\right).$$

Note that $p<\tilde{p}$ implies

$$p(z)<\tilde{p}(z)\qquad\text{for each } z$$

Thus, the high-risk economy features a universal decrease in firm risk relative to the low-risk economy (see Definition 2). Next, assume that $\tilde{y}^*(z)$ crosses $y^*(z)$ first from above at some point z_1 and then from below at another point z_2 such that $z_1<z_2$ as shown in Figure 5. The following relations between $y^*(z)$ and $\tilde{y}^*(z)$ must then be jointly satisfied

$$y^*(z_1)=\tilde{y}^*(z_1)\qquad\text{and}\qquad y^*(z_2)=\tilde{y}^*(z_2) \tag{27}$$

$$\frac{dy^*(z_1)}{dz_1}>\frac{d\tilde{y}^*(z_1)}{dz_1}\qquad\text{and}\qquad\frac{dy^*(z_2)}{dz_2}<\frac{d\tilde{y}^*(z_2)}{dz_2} \tag{28}$$

It will be useful to define

$$t(z)\equiv\frac{\Phi^{-1}(p)-\sqrt{\rho}z}{\sqrt{1-\rho}}\qquad\text{and}\qquad\tilde{t}(z)\equiv\frac{\Phi^{-1}(\tilde{p})-\sqrt{\rho}z}{\sqrt{1-\rho}}.$$

and then write

$$y^*(z) = r^* + k_{min} - \Phi(t(z))(r^* + \triangle)$$
$$\tilde{y}^*(z) = \tilde{r}^* + k_{min} - \Phi(\tilde{t}(z))(\tilde{r}^* + \triangle)$$

Note that

$$\tilde{p} > p \qquad \Rightarrow \qquad t(z) < \tilde{t}(z) \qquad \text{for all } z. \tag{29}$$

From (27) we obtain for $i = 1, 2$

$$(1 - \Phi(t(z_i)))(r^* + \triangle) = (1 - \Phi(\tilde{t}(z_i)))(\tilde{r}^* + \triangle) \tag{30}$$

From (28), we obtain after differentiating with respect to z

$$\phi(t(z_1))(r^* + \triangle) > \phi(\tilde{t}(z_1))(\tilde{r}^* + \triangle)$$

$$\phi(t(z_2))(r^* + \triangle) < \phi(\tilde{t}(z_2))(\tilde{r}^* + \triangle)$$

where ϕ denotes the p.d.f. of the standard normal. The above inequality can be written as

$$h(t(z_1))\left[(1 - \Phi(t(z_1)))(r^* + \triangle)\right] > h(\tilde{t}(z_1))\left[(1 - \Phi(\tilde{t}(z_1)))(\tilde{r}^* + \triangle)\right]$$

$$h(t(z_2))\left[(1 - \Phi(t(z_2)))(r^* + \triangle)\right] < h(\tilde{t}(z_2))\left[(1 - \Phi(\tilde{t}(z_2)))(\tilde{r}^* + \triangle)\right]$$

where $h(t) \equiv \frac{\phi(t)}{1-\Phi(t)}$ is the hazard rate function of the standard normal. Using (30) to cancel the terms in square brackets in the above inequalities, we arrive at

$$h(t(z_1)) > h(\tilde{t}(z_1)) \qquad \text{and} \qquad h(t(z_2)) < h(\tilde{t}(z_2))$$

However, the first of these inequalities, $h(t(z_1)) > h(\tilde{t}(z_1))$, cannot hold since (i) $h(t)$ is a monotone increasing function and (ii) $t(z_1) < \tilde{t}(z_1)$ from (29). As a result, the situation depicted in 5 cannot emerge. Instead, the bank's net worth in the high-risk economy $\tilde{y}^*(z)$ must cross the bank's net worth in the low-risk economy $y^*(z)$ once and from below, implying that the former bank is more likely to fail than the latter, i.e., the conventional outcome.

Proposition 1.

Step 1. Recall from (4) that $p(s) = \Phi(-\mu(s))$. Substituting for $\mu(s)$ in (13), we can write the bank's probability of default as

$$q^* = \int_S \Phi\left(\tfrac{1}{\sqrt{\rho}}\left[\Phi^{-1}(p(s)) - \sqrt{1-\rho}\,\Phi^{-1}\left(\tfrac{r^*+k_{min}}{r^*+\triangle}\right)\right]\right) dG(s)$$

The above expression is not easy to work with since it depends on the quintile function of the standard normal $\Phi^{-1}(z)$. We will use a well-known approximation (see Shore, 1982).

$$\Phi^{-1}(z) \approx \left\{ \begin{array}{c} -0.4115\left[\frac{1-z}{z} + \log\left(\frac{1-z}{z}\right) - 1\right] \\ 0.4115\left[\frac{z}{1-z} + \log\left(\frac{z}{1-z}\right) - 1\right] \end{array} \right\} \quad \text{for} \quad \left\{ \begin{array}{c} z \geq \frac{1}{2} \\ z < \frac{1}{2} \end{array} \right\} \tag{31}$$

Something to keep in mind

$$\frac{d\Phi^{-1}(z)}{dz} \approx \left\{ \begin{array}{c} 0.4115z^{-3} \\ -0.4115(1-z)^{-2}z^{-1} \end{array} \right\} \quad \text{for} \quad \left\{ \begin{array}{c} z \geq \frac{1}{2} \\ z < \frac{1}{2} \end{array} \right\} \tag{32}$$

We further restrict attention to parameter values such that

$$p(s) < \tfrac{1}{2} \quad \text{for all } s \in S \qquad \text{and} \qquad \frac{1}{2} < \frac{k_{min}}{\triangle} < 1 \tag{33}$$

$$\Rightarrow \qquad m^* \equiv \frac{r^* + k_{min}}{r^* + \triangle} > \frac{1}{2} \tag{34}$$

Combining (33) - (34) with (31) gives

$$\Phi^{-1}(p(s)) \approx 0.4115\left[\tfrac{p(s)}{1-p(s)} + \log\left(\tfrac{p(s)}{1-p(s)}\right) - 1\right] \quad \text{for all } s \in S \tag{35}$$

where the above uses (33), namely $p(s) < \frac{1}{2}$. Also,

$$\Phi^{-1}(m^*) \approx -0.4115\left[\tfrac{m^*}{1-m^*} + \log\left(\tfrac{m^*}{1-m^*}\right) - 1\right] \tag{36}$$

where the above uses (34), namely $m^* > 1/2$. Inserting (35) and (36) into (20) gives

$$q^* = \int_0^1 \Phi\left(T(s)\right) dG(s)$$

where $T(s)$ is defined as

$$\begin{aligned} T(s) &\equiv \tfrac{1}{\sqrt{\rho}}\left[\Phi^{-1}(p(s)) - \sqrt{1-\rho}\,\Phi^{-1}(m^*)\right] \\ &= \tfrac{1}{\sqrt{\rho}}0.4115\left[\left(\tfrac{p(s)}{1-p(s)} + \log\left(\tfrac{p(s)}{1-p(s)}\right) - 1\right) + \sqrt{1-\rho}\left(\tfrac{m^*}{1-m^*} + \log\left(\tfrac{m^*}{1-m^*}\right) - 1\right)\right] \end{aligned} \tag{37}$$

The second line uses (35) and (36). Taking a first-order Taylor series expansion of $T(s)$ around $p(s)$ and m^* yields

$$dT(s) = \frac{\partial T}{\partial p(s)}dp(s) + \frac{\partial T}{\partial m^*}dm^* \tag{38}$$

where $\partial T/\partial p(s)$ denotes the partial derivative of T with respect to $p(s)$ and $\partial T/\partial m^*$ the partial of T with respect to m^*.

Step 2. Assume the risk factor s is binary as in (14) Namely, $s = G$ w.p. $1-\eta$ and $s = B$ w.p. η. Then

$$q^* = (1-\eta)\Phi\left(T(G)\right) + \eta\Phi\left(T(B)\right)$$

To a first-order approximation, the change in q^* is now given by

$$dq^* = (1-\eta)\phi\left(T(G)\right)dT(G) + \eta\phi\left(T(B)\right)dT(B)$$

where ϕ is the p.d.f. of the standard normal. The definition of T in (37) together with (32) yields

$$dT(G) = \frac{1}{\sqrt{\rho}} 0.4115 \left[\frac{1}{(1-p(G))^2} \frac{dp(G)}{p(G)} - \sqrt{1-\rho} \frac{1}{(m^*)^2} \frac{dm^*}{m^*} \right] \tag{39}$$

$$dT(B) = \frac{1}{\sqrt{\rho}} 0.4115 \left[\frac{1}{(1-p(B))^2} \frac{dp(B)}{p(B)} - \sqrt{1-\rho} \frac{1}{(m^*)^2} \frac{dm^*}{m^*} \right] \tag{40}$$

Step 3. Characterize $\frac{dm^*}{m^*}$. To proceed, it is easier to work with no deposit insurance. The equilibrium interest rate in this case, is

$$r^* = \tfrac{p\triangle + k_{min}\delta}{1-p} \tag{41}$$

where $p = (1-\eta)p(G) + \eta p(B)$ is the unconditional borrower default probability. Notice that the interest rate on deposit r_D^* will now adjust to reflect bank-specific risk

$$(1-q^*)(1+r_D^*) + q^*(1 + r_D^* - \text{expected loss}) = 1$$

and the cutoff m^* is obtained from $1 + r - m^*(r^* + \triangle) = (1-k_{min})(1+r_D^*)$ yielding

$$m^* = \frac{r^* + k_{min}}{r^* + \triangle} - \frac{1-k_{min}}{r^* + \triangle} r_D^*$$

(Note: with deposit insurance $r_D^* = 0$ and the second term vanishes.) Inserting the interest rate in (41) into the above expression

$$m^* = \frac{p(\triangle - k_{min}) + k_{min}\delta}{k_{min}\delta + \triangle} - \frac{1-k_{min}}{r^* + \triangle} r_D^*$$

$$\Rightarrow \quad \frac{dm^*}{dp} = \frac{(\triangle - k_{min})}{k_{min}\delta + \triangle} - \frac{1-k_{min}}{r^* + \triangle} \frac{dr_D^*}{dp}$$

The percentage change in m^* is then

$$\frac{dm^*}{m^*} = \frac{\frac{(\triangle - k_{min})}{k_{min}\delta + \triangle} dp - \frac{1-k_{min}}{r^*+\triangle} dr_D^*}{\frac{p(\triangle - k_{min}) + k_{min}\delta}{k_{min}\delta + \triangle} - \frac{1-k_{min}}{r^*+\triangle} r_D^*}$$

Taking δ arbitrarily close to zero and q^* small so that the deposit rate r_D^* is close to zero implies (Equivalently, we can assume deposits have liquidity yield leading to $r_D^* \approx 0$)

$$\frac{dm^*}{m^*} = \frac{dp}{p} = \frac{(1-\eta)dp_G + \eta dp_B}{(1-\eta)p_G + \eta p_B} \tag{42}$$

Inserting (42) into (39) - (40) gives

$$
\begin{aligned}
dq^* \quad \propto \quad & (1-\eta)\phi\left(T(G)\right)\left[\frac{1}{(1-p(G))^2}\frac{dp(G)}{p(G)} - \sqrt{1-\rho}\frac{1}{(m^*)^2}\frac{dp}{p}\right] \\
& +\eta\phi\left(T(B)\right)\left[\frac{1}{(1-p(B))^2}\frac{dp(B)}{p(B)} - \sqrt{1-\rho}\frac{1}{(m^*)^2}\frac{dp}{p}\right]
\end{aligned} \tag{43}
$$

where we have omitted the constant term $\frac{1}{\sqrt{\rho}}0.4115$ from the above expression.

Step 4. We use the expression in (43) to derive necessary and sufficient conditions for the unconventional outcome. Starting from $p(G) = p(B) = p$ and then marginally reducing $p(G)$ while keeping $p(B) = p$. That is, $dp(G) < 0$ and $dp(B) = 0$. We then have

$$
\frac{dp}{p} = (1-\eta)\frac{dp(G)}{p(G)} \tag{44}
$$

$$
\text{and} \qquad \phi\left(T(G)\right) = \phi\left(T(G)\right) = \phi\left(T\right) \tag{45}
$$

The unconventional outcome happens if $dq^* > 0$ which, using (44) and (45) is equivalent to

$$
\frac{1}{(1-p(G))^2} < \sqrt{1-\rho}\frac{1}{(m^*)^2}
$$

Then since $p(G) = p$ and $m^* = \frac{p(\triangle - k_{min})}{\triangle}$ (since $\delta \approx 0$) the above is equivalent to

$$
\frac{p}{1-p}\left(1 - \frac{k_{min}}{\triangle}\right) < (1-\rho)^{1/4}
$$

which is the condition given in Proposition 1.