Does Public Information Help Social Learning: An Anti-Transparency Result

(Preliminary Draft)

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Abstract

This paper studies the effect of releasing exogenous public information in rational social-learning models that predicts informational cascades and incomplete learning. Despite the fact that informational cascades can be triggered by incorrect early actions, we show that, to improve social learning, it is better to postpone the disclosure of public information in a canonical setting with binary states and actions. More importantly, it is suboptimal to ever release public information less precise than people's private information even through contingent disclosure strategies, since noisy public signals crowd out more informative private signals and thus harm information aggregation. In other words, anti-transparency turns out to be the correct public information policy for social learning when public information is relatively coarse.

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1 Introduction

As important as the rationalization of herd behavior, one contribution of the theoretical literature on social learning is the prediction of incomplete learning, *i.e.*, inefficient information

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aggregation among the population, when people have boundedly accurate private information. In the canonical model due to Bikhchandani *et al.* (1992), agents make binary choices between A and B sequentially over an infinite time horizon, and before taking her action, each agent receives a private binary signal indicating which option is better with identical precision and observes all the past actions. Eventually rational agents herd without fully learning the truth and informational cascade arises.[1]

Like every other economic model that predicts inefficient outcomes, we naturally ask ourselves of potential ways to improve efficiency in such environments. In fact, as Bikhchandani *et al.* (1992) pointed out, informational cascades are fragile: since information stops to aggregate, the cascades and hence the herds are vulnerable to new pieces of information. Hence it is of interest to investigate whether and how disclosure of exogenous public information can improve social learning.

For such purpose, we introduce a social planner to the canonical binary model, who has access to an exogenous piece of information and decides whether and when to release it to the public to maximize the asymptotic efficiency of the population. It can be interpreted that the social planner is actively looking for the optimal timing (if any) of running a public experimentation that provides informative evidence about the underlying state with given precision, which maximizes the expected payoff of the long-run agents.[2]

We first provide an *anti-transparency* result: the social planner should never release a noisy public signal that is less precise than people's private signals. An informational cascade arises when one action, say A, has been chosen at least twice more than B, and agents start to herd on A.[3] Releasing a noisy public signal then cannot break down the informational cascade as agents will continue to herd on A even when the public signal suggests B: combining the history and the public signal, future agents still find action A sufficiently attractive. Hence a noisy public signal has no effect on the limiting expected payoff if it is released after an informational cascade has arisen. On the other hand, when a cascade has not yet formed, releasing a noisy public signal may induce a wrong cascade in the future more likely than people's own private signals

[1] Strictly speaking, incomplete learning does not necessarily imply informational cascades when private signals are continuous rather than discrete. See Herrera and Hörner (2012) for a discussion about a necessary and sufficient condition on the distribution of private signals for informational cascades.

[2] Due to the presence of herd behavior, the expected payoff of the long-run agents is equivalent to the average expected payoff of the population, hence it has a welfare interpretation as well. This is in fact a common objective of interest in the literature of social learning. In Section 3 we will discuss an alternative objective function of the social planner by introducing time preference.

[3] We assume each agent follows her private signal when indifferent.

due to its lower precision, hence lowers the limiting expected payoff. Therefore overall to release a noisy public signal is a bad idea for the social planner. Moreover, this result is robust when sophisticated releasing strategies are allowed, *i.e.*, a noisy public signal should never be released even if the social planner can make the timing of disclosure contingent on the history of past actions.[4]

The other finding in company with the anti-transparency result is a monotonicity result: the expected payoff of long-run agents is (weakly) increasing in the period at which the public signal is released, regardless of its precision. In other words, the social planner should always postpone the disclosure of any public information, despite the fact that informational cascades can be triggered by incorrect early actions.[5] The intuition behind this result is that the benefit of releasing a public signal is greater when an informational cascade has arisen than when it has not. Before a cascade starts the information aggregation of private signals is still going on, so a public signal released then may crowd out the next private signal(s) in terms of updating people's belief. Hence the "net" informational contribution of the public signal is lower than when it is released after a cascade has started, in which case the information content of the public signal is fully absorbed into the public belief. Since the probability of entering an informational cascade is weakly increasing over time, the benefit of releasing a public signal is, as a result, also weakly increasing over time.

Nevertheless the monotonicity result seems not compelling especially for extremely precise public signals: if the social planner holds a public signal that perfectly reveals the truth, then she should naturally release it as early as possible so that everyone can learn the truth from it and choose the right action. This thought experiment casts doubt on whether the asymptotic efficiency is a proper objective for the social planner, and we reconsider the whole problem assuming that the social planner has time preference and thus wants to maximize the discounted sum of people's expected payoffs instead. Although the optimal timing of disclosure is not yet clear to us (but definitely finite), we can expect that the monotonicity result no longer holds. In particular, if the social planner is indifferent between two periods to release the public signal before, now she strictly prefers the earlier period of the two due to her impatience.

[4]In general numerous Bayesian Nash equilibria exist with contingent releasing strategies, so we focus on a selection of equilibria to make meaningful prediction. See Subsection 2.3 for details.

[5]Note that the monotonicity result is true even for noisy public signals, but does not contradict with the anti-transparency result: releasing a noisy public signal is bad, but if the social planner were forced to release one, she should postpone as much as possible.

We also present an alternative setting with three states and three actions, and show that the monotonicity result does not hold either, however through a totally different channel. In this setting, at some point in the history an action could be excluded by all the agents afterwards: *e.g.*, an agent observing history (A,B,A,B,A,B) would not choose C regardless of her private signal and neither would all her successors. We call this situation a *trap* away from action C, and unlike a herd, the informational "depth" of a trap can increase over time; hence a public signal could fail to break down a wrong trap when released too late and therefore the social planner would not always prefer a postponed disclosure even if she is infinitely patient as in the benchmark setting.

Related literature. This paper is clearly related to the social learning literature initiated by Banerjee (1992), Bikhchandani *et al.* (1992), and Smith and Sørensen (2000). Nevertheless few papers talked about disclosure of public information in social-learning models. Bikhchandani *et al.* (1992) pointed out the fragility of informational cascades and only briefly discussed the effect of releasing extra information, while this work further looks into this issue and investigates the optimal timing of release. Gill and Sgroi (2008) also augmented the standard model to allow a principal to provide public information to the agents by subjecting herself to a test of certain toughness at the beginning.[6] On the other hand, as discussed before, the monotonicity result in this paper might question the plausibility of limiting efficiency, which is the common objective of interest in most of the literature, as a good measure of social welfare in social-learning models.

It is also related to a stream of papers on anti-transparency of information disclosure. Morris and Shin (2002) presented a model where every agent wants to minimize a loss function made up of two components: loss in the distance between her action and the underlying state, and loss in the distance between her action and the average action in the population, *i.e.*, a "beauty-contest" term.[7] With later comments by Svensson (2006) and Morris *et al.* (2006), it can be shown that in such a model the welfare with noisy public information could indeed be worse than the welfare without.[8] Demertzis and Hoeberichts (2007) further explored this anti-transparency result by introducing costly information acquisition to the model, where people

[6] Essentially the outcome of the test is like a public signal with certain precision (based on the toughness) that is released at the beginning. Note that the principal in Gill and Sgroi (2008) does not have the same objective of the social planner in this paper though.

[7] See Keynes (1936).

[8] In that model, public information serves as a coordination device for the second loss term, and people could overlook their private signals when they put a sufficiently high weight on the second loss term.

might free-ride on public information and abandon private information acquisitions. In this paper we get an anti-transparency result as well, though without payoff interdependence or costs of obtaining information.[9]

The remainder of the paper is structured as follows. Section 2 sets up the canonical binary model and provides the main result. Section 3 discusses alternative settings with impatient social planner and with ternary states/actions. Section 4 concludes.

2 The Binary Model

2.1 Setup and Preliminaries

There is a population of countably infinite agents who are exogenously ordered to make a binary choice sequentially. Each agent is labelled by the period of her turn, $t \in T = \{1, 2, 3, ...\}$.

The state of the world θ is realized out of a binary state space $\Theta \equiv \{1, -1\}$ before anyone makes the choice, with $\Pr\{\theta = 1\} = 1/2$. After the realization of θ, every agent t receives a private signal $s_t \in \{1, -1\}$ and the private signals are conditionally i.i.d. with

$$\Pr\{s_t = 1|\theta = 1\} = \Pr\{s_t = -1|\theta = -1\} = q \in (\frac{1}{2}, 1),$$

where the precision q is common knowledge to the whole population.

Before exerting her action $a_t \in A = \{1, -1\}$, agent t is allowed to observe the history of all her predecessors' choices, $\boldsymbol{h}_t \in H_t \equiv \{\emptyset\} \cup A^{t-1}$, where $\boldsymbol{h}_1 \equiv \emptyset$ denotes the empty history at period 1. Agents have identical utility function

$$u(a_t; \theta) = 1_{\{\theta = a_t\}}$$

and are assumed to follow their own signal when indifferent.

We call $w_t \equiv \log_{q/(1-q)}[\frac{\Pr(\boldsymbol{h}_t|\theta=1)}{\Pr(\boldsymbol{h}_t|\theta=-1)}]$ the *public belief* after history $\boldsymbol{h}_t$.[10]

This is essentially the canonical model in Bikhchandani *et al.* (1992) and it is well known that the Bayesian Nash equilibrium exhibits herd behavior eventually. For the purpose of future

[9] Compared to Morris and Shin (2002), for example, noisy public information distorts social welfare in this model through informational externality rather than payoff interdependence.

[10] It is convenient to use this particular log likelihood ratio w_t here as it only takes integer values in equilibrium (prior to the disclosure of public information) and in fact represents the *net* number of private signals revealed by the history.

analysis though, let us restate the existing results as lemmata.

Lemma 2.1 *The Bayesian Nash equilibrium strategy of each agent t is given by*

$$a_t^*(\boldsymbol{h}_t, s_t) = a^*(w_t, s_t) \equiv \begin{cases} s_t & \text{if } |w_t| \leq 1 \\ sgn(w_t) & \text{otherwise} \end{cases},$$

where $sgn(x) \equiv \begin{cases} x/|x| & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$.

Hence, along the equilibrium path $\boldsymbol{h}_t^* = (a_1^*, a_2^*, ..., a_{t-1}^*)$ with $\boldsymbol{h}_1^* \equiv \emptyset$, the dynamic of public beliefs is given by

$$w_1^* = 0; \; w_{t+1}^* = \begin{cases} w_t^* + a_t^* & \text{if } |w_t^*| \leq 1 \\ w_t^* & \text{otherwise} \end{cases}.$$

Proof. See Appendix. ■

Lemma 2.1 shows that public belief w_t serves as a sufficient statistic for agent t's decision problem and in equilibrium w_t^* stops to update once it leaves interval $[-1,1]$, which is exactly when an informational cascade, or a herd, starts.

Definition 1 *We say a* ***herd*** *on action $1(-1)$ starts at period T if*

$$\forall t \geq T, \; a_t(\boldsymbol{h}_t, s_t) = a_t(\boldsymbol{h}_t, \cdot) = 1(-1).$$

Lemma 2.2 *Along the equilibrium path described by Lemma 2.1, a herd starts eventually with probability 1.*

Proof. See Appendix. ■

Note that the eventual herd could be incorrect, as Smith and Sørensen (2000) argued, if agents have *bounded private beliefs*, which is exactly the case here. The probability of a correct herd eventually is nevertheless important for our analysis later on social welfare, as it determines expected payoffs for future agents in the long run. Hence we would like to calculate this probability here.

For convenience, let us assume the realization of θ is 1 without loss of generality for the remainder of this section.[11] Define

$$p(x) \equiv \Pr(\text{plim}_{t\to\infty} a_t^*(\boldsymbol{h}_t^*, s_t) = 1 | w_1 = x, \theta = 1), \; \forall x \in \mathbb{R},$$

the probability of a correct herd eventually conditional on some initial public belief $w_1 = x$, which can be explicitly calculated according to the following useful lemma.

Lemma 2.3 *$p(x)$ can only take the following 7 discrete values:*

$$\begin{aligned}
&p(x) = 0 \equiv \alpha_1, \; \forall x < -1; \\
&p(-1) = \frac{q^3}{1-2q(1-q)} \equiv \alpha_2; \\
&p(x) = \frac{q^2}{1-q(1-q)} \equiv \alpha_3, \; \forall x \in (-1,0); \\
&p(0) = \frac{q^2}{1-2q(1-q)} \equiv \alpha_4; \\
&p(x) = \frac{q}{1-q(1-q)} \equiv \alpha_5, \; \forall x \in (0,1); \\
&p(1) = q + \frac{(1-q)q^2}{1-2q(1-q)} \equiv \alpha_6; \\
&p(x) = 1 \equiv \alpha_7, \; \forall x > 1.
\end{aligned}$$

In fact, $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7)^\top$ satisfies $Q\boldsymbol{\alpha} = \boldsymbol{\alpha}$ where

$$Q \equiv \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-q & 0 & 0 & q & 0 & 0 & 0 \\
1-q & 0 & 0 & 0 & q & 0 & 0 \\
0 & 1-q & 0 & 0 & 0 & q & 0 \\
0 & 0 & 1-q & 0 & 0 & 0 & q \\
0 & 0 & 0 & 1-q & 0 & 0 & q \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

Proof. See Appendix. ■

[11] It is without loss of generality from an *ex-ante* perspective due to the symmetric setting.

Note that the subgame starting from a period T is identical to the original game (and the equilibrium strategy is stationary according to Lemma 2.1), hence Lemma 2.3 actually tells us how to calculate the probability of a correct herd eventually if the public belief at period T is w_T. On the other hand, the matrix Q introduced in Lemma 2.3 also helps us to characterize the equilibrium public beliefs as a *monotone* Markov chain.

Definition 2 *A transition matrix $C = (c_{ij})_{n\times n}$ is **monotone** if*

$$\forall 1 \le i < j \le n, \forall k \le n, \sum_{m=1}^{k} c_{mi} \ge \sum_{m=1}^{k} c_{mj}.$$

*A Markov chain is **monotone** if it has a monotone transition matrix.*[12]

Lemma 2.4 *Let $\mathscr{P} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$ be a finite partition of $\mathbb{R}$ with:*

$$P_1 = (-\infty, -1),\ P_2 = \{-1\},\ P_3 = (-1, 0),\ P_4 = \{0\},$$
$$P_5 = (0, 1),\ P_6 = \{1\},\ P_7 = (1, +\infty)\}.$$

Define $\pi_i^t \equiv \Pr(w_t^* \in P_i)$ and $\boldsymbol{\pi}^t = (\pi_1^t, \pi_2^t, \pi_3^t, \pi_4^t, \pi_5^t, \pi_6^t, \pi_7^t)$ is hence the probability vector of w_t^* over partition $\mathscr{P}$. We have

$$\boldsymbol{\pi}^{t+1} = \boldsymbol{\pi}^t Q \text{ with } \boldsymbol{\pi}^1 = (0,0,0,1,0,0,0),$$

where the transition matrix Q is given in Lemma 2.3 and it is monotone.

Proof. See Appendix. ■

2.2 Timed Release of Public Information

As we have seen in the previous subsection, a herd starts eventually but it is possibly on the wrong action. Bikhchandani *et al.* (1992) referred to the eventual herd as an informational cascade and pointed out that it is vulnerable to public information disclosure. Here we look into this issue more specifically by introducing public information release into the model.

In addition to the population of agents as before, there is a social planner who also receives

[12] These definitions come from Keilson and Kester (1977).

a signal $\widetilde{s} \in \{1, -1\}$ after the realization of θ and

$$\Pr\{\widetilde{s} = 1 | \theta = 1\} = \Pr\{\widetilde{s} = -1 | \theta = -1\} = \widetilde{q} \in (\frac{1}{2}, 1).$$

The precision $\widetilde{q}$ is common knowledge to the whole population and $\widetilde{s}$ is conditionally independent of any s_t.

The social planner can decide whether and when to release the signal $\widetilde{s}$ to the public. Once $\widetilde{s}$ is released at period $\tau \geq 1$ it becomes public information and every agent afterwards, $t \geq \tau$, can take it into account before she makes her decision. The social planner wants to maximize the expected average payoff of the whole population,

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} Eu(a_t^*; \theta).$$

Note that due to the existence of herd behavior, the social planner's objective is essentially to maximize the probability of a correct herd eventually, or say, the probability of learning the truth eventually.

In this subsection we particularly consider the situation where the releasing strategy is *timed*, namely she has to decide a period $\tau \in \{1, 2, 3...\}$ to release or not to release at all before anything happens and commits to that. Notice that the realization of θ is still assumed to be 1 without loss of generality.

Note that private signals are equally precise, hence the public belief w_t can also be interpreted as the *net* number of correct private signals revealed by history $\boldsymbol{h}_t$. We would like to first have a similar interpretation of the public signal by measuring its precision with respect to private signals:

Definition 3 *The public signal $\widetilde{s}$ has **relative precision** $\lambda \in \mathbb{R}^+$ if*

$$\log_{q/(1-q)}[\frac{\widetilde{q}}{1-\widetilde{q}}] = \lambda, \textit{ or equivalently, } \widetilde{q} = \frac{q^\lambda}{q^\lambda + (1-q)^\lambda}.$$

When the public signal has **relative precision** λ, we have

$$\frac{\Pr\{\widetilde{s}|\theta=1\}}{\Pr\{\widetilde{s}|\theta=-1\}} = (\frac{\widetilde{q}}{1-\widetilde{q}})^{\widetilde{s}} = [(\frac{q}{1-q})^{\lambda}]^{\widetilde{s}}$$
$$= [(\frac{q}{1-q})^{s_t}]^{\lambda} = [\frac{\Pr(s_t|\theta=1)}{\Pr(s_t|\theta=-1)}]^{\lambda} \text{ whenever } s_t = \widetilde{s}.$$

That is, learning a public signal in favor of one state with relative precision λ is equivalent to learning λ *net* private signals in favor of that state.

Now suppose the social planner releases the public signal at period τ. Then the subgame after the release is equivalent to the original game without public information, which we discussed in the previous subsection, but with an initial public belief inferred from bother the history before period τ and the public signal. Hence, a herd still starts eventually and the expected average payoff of the population is just the probability of a correct herd eventually, which depends only on the initial public belief according to Lemma 2.3. Meanwhile, using the relative precision, we can linearly describe the effect of the public signal on the public belief. These observations are summarized in the following lemma.

Lemma 2.5 *Suppose the social planner releases $\widetilde{s}$ with relative precision λ at period $\tau \geq 1$ and the history before that has generated a public belief w_τ. Then the new public belief after release will be*

$$\widetilde{w}_\tau = w_\tau + \lambda\widetilde{s},$$

and (under the assumption that the realization of θ is 1) the expected average payoff of the population conditional on $\widetilde{w}_\tau$ is simply given by

$$\text{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}u(a_t^*;\theta=1|\widetilde{w}_\tau) = p(\widetilde{w}_\tau) = p(w_\tau+\lambda\widetilde{s}).$$

Furthermore, let us denote $v(\tau;\lambda)$ as the (unconditional) expected average payoff of the population when the social planner is to release the public signal with relative precision λ at period τ, then

$$v(\tau;\lambda) = E_{w_\tau,\widetilde{s}}[p(w_\tau+\lambda\widetilde{s})].$$

Proof. See Appendix. ■

Keep in mind that without no release at all, the expected average payoff of the population, which is just the probability of a correct herd eventually, is equal to α_4 given in Lemma 2.3. Now we are in a position to provide the main result of this section.

Proposition 2.1 *1. It is never optimal to release a public signal less precise than the private signals. That is,*

$$\forall \lambda \in (0,1),\ \forall \tau \geq 1,\ v(\tau;\lambda) < \alpha_4.$$

2. It is strictly better to release a public signal no less precise than the private signals than not to release at all. That is,

$$\forall \lambda \in [1,+\infty),\ \exists \tau < \infty \textit{ such that } v(\tau;\lambda) > \alpha_4.$$

3. It is always (weakly) better to release a public signal later than sooner regardless of its precision. That is,

$$\forall \lambda \in \mathbb{R}^+,\ \forall \tau \geq 1,\ v(\tau+1;\lambda) \geq v(\tau;\lambda).$$

Proof. See Appendix. ∎

Here we would like to talk about the third statement of Proposition 2.1 in particular. The weak monotonicity of $v(\tau;\lambda)$ in τ mathematically comes from the fact that equilibrium public beliefs evolve according to a *monotone* transition matrix until the public signal is released, regardless of the value of λ. However, the intuition for this weak monotonicity is not as universal as the property itself. For illustrative purposes, let us focus on two cases, $\lambda = 1$ and $\lambda < 1$.

We can think of releasing a public signal as an "additional" agent joining in the sequence who always follows his own private signal $\tilde{s}$. When $\lambda = 1$, the release has no ex-ante effect if a herd has not started yet because every agent t just follows her own private signal s_t, which has the same precision as $\tilde{s}$, before a herd starts. In this case the ex-ante benefit of releasing $\tilde{s}$ arises after a herd starts, where $\tilde{s}$ is more likely to break down a wrong herd than to break down a correct herd as $\tilde{q} > 1/2$. So the benefit of releasing $\tilde{s}$ is increasing in the probability of herding at the time of release. It is easy to verify that the probability of herding is weakly increasing in t, which explains the weak monotonicity of $v(\tau;\lambda)$ in τ.

When $\lambda < 1$, however, releasing $\tilde{s}$ has no effect once a herd starts: $|\tilde{w}_\tau| = |w^*_\tau + \lambda \tilde{s}| > 1$ and $sgn(\tilde{w}_\tau) = sgn(w^*_\tau)$ when $w^*_\tau = \pm 2$ and $\lambda < 1$.[13] But it brings ex-ante disadvantage before a herd starts since it is more likely to induce a wrong herd than what a normal agent does, due to the lower precision $\tilde{q} < q$. Therefore the harm of release is decreasing in the probability of herding, which in turn is weakly decreasing over time and hence explains the weak monotonicity of $v(\tau;\lambda)$ in τ.

It is worth pointing out that when $\lambda > 3$, the weak monotonicity is actually uniformity. In that case, the public signal is so strong that people start to herd on the action same as the realization of $\tilde{s}$ immediately after it is released, so releasing at different periods makes no difference.

2.3 Contingent Release of Noisy Public Information

In addition to the *monotonicity* result, Proposition 2.1 also makes another observation: it is better not to release the public signal at all when it is less precise than private signals. This can be interpreted as an *anti-transparency* result: more (but noisy) public information can be bad for social welfare.[14] However, the social planner has so far been restricted to use exogenous releasing strategies, hence a natural question would be whether this suboptimality of release when $\lambda < 1$ still holds if *contingent* releasing strategies are allowed, namely the social planner can decide whether to release or not at period t based on the realization of $\tilde{s}$ and w_t.

Let $g(\tilde{s}, w_t) \in \{0, 1\}$ be the strategy of the social planner: $g(\tilde{s}, w_t) = 1$ means the social planner releases the public signal at period t after seeing $\tilde{s}$ and w_t; $g(\tilde{s}, w_t) = 0$ means not. And $g_t \in \{0, 1\}$ denotes the corresponding action. We restriction attention on pure strategies by the social planner.

Note that g_t is now relevant information for agents $\tau \geq t$, because, given a releasing strategy by the social planner, agents can possibly infer the realization of $\tilde{s}$ from g_t and w_t. A natural issue arises here, like in lots of games with incomplete information, that there could potentially exist undesired equilibria due to lack of restriction on *off-equilibrium* beliefs. So we want to impose the following refinement on certain off-equilibrium path.

Definition 4 *Given a releasing strategy $g(\tilde{s}, w_t)$ by the social planner, let $\mu(w_t, g_t)$ denote agents'*

[13] Bikhchandani *et al.* (1992) argued that releasing a public signal less informative than the private signal can still be beneficial when there is an information cascade. This is true under their assumption that agents play mixed strategies when indifferent, but not under the tie-breaking rule here.

[14] The seminal paper on anti-transparency, Morris and Shin (2002), also used the average payoff of the population to refer to social welfare.

belief at period t about the realization of $\tilde{s}$ after observing w_t and g_t. That is,

$$\mu(w_t, g_t) \equiv \Pr(\tilde{s} = 1 | w_t, g_t, g(\cdot,\cdot)).$$

*We say $\mu(w_t, g_t)$ is **non-excessive** if*

$$\mu(w_t, 0) = \frac{1}{2}, \ \forall w_t \text{ s.t. } g(\tilde{s}, w_t) = 1 \text{ for any } \tilde{s} \in \{-1, 1\}.$$

Non-excessive belief requires that, on an off-equilibrium path where the social planner does not release $\tilde{s}$ while she should have released it regardless of its realization, agents should not make excessive inference about the realization of $\tilde{s}$ in this symmetric world. We think this is a reasonable refinement and it indeed helps us get rid of meaningless equilibria which do not serve for the purpose of our analysis here.[15]

Note that the agents' behavior still follows what has been described in Lemma 2.1 but under a public belief generated from both the previous actions and their inference about $\tilde{s}$. Hence we will give the main result here that focuses on the social planner's releasing strategy.

Proposition 2.2 *When $\lambda < 1$ and agents' belief about $\tilde{s}$ is non-excessive, there are 3 Bayesian Nash equilibria where the social planner's contingent releasing strategies are respectively:*

$$g^1(\tilde{s}, w_t) = 0;$$
$$g^2(\tilde{s}, w_t) = 1_{\{w_t = -\tilde{s}\}};$$
$$g^3(\tilde{s}, w_t) = 1_{\{w_t = \pm 2\tilde{s}\}}.$$

Proof. See Appendix. ■

The interesting addition compared to the case with exogenous releasing strategy is g^2. With g^2, the social planner will release the public signal once he saw an history that is not a herd yet but against the realization of $\tilde{s}$, which is reasonable because he wants to prevent the agents from starting an herd against the public signal too early. Unfortunately, from an ex-ante perspective, social welfare is *not* improved under contingent releasing strategies.

[15] Without restriction on non-excessive beliefs, one can show that to release the public signal after any history could be an equilibrium. But these equilibria do not help improve the social welfare.

Corollary 2.1 *g^1 generates the same ex-ante average payoff of the population in equilibrium as g^3 does, which is better than what g^2 does. And none of them can improve social welfare compared to exogenous releasing strategies.*

Proof. g^1 means no release at all, which is also the best the social planner can do under exogenous releasing strategies. g^3 means to disclose the public signal when a herd has already started but in that case disclosure makes no difference as the noisy public signal can never break down a herd, hence social welfare is the same as with no release at all. On the other hand, the "separating" strategy g^2 implies that the agents can perfectly infer the realization of $\tilde{s}$ after period 1, hence social welfare is the same as with exogenous release at period 2, which is worse than with no release at all when $\lambda < 1$ as we saw in Proposition 2.1. ■

3 Alternative Settings

Recall that the benefit of public information disclosure is weakly increasing over time in the binary model. In this section, however, we are about to introduce two alternative settings under which postponing disclosure of public information is not necessarily a good decision for the social planner.

3.1 Impatient Social Planner

So far we have assumed that the social planner cares about the expected average payoff of the population *without discounting*, hence she essentially cares only about whether people eventually herd on the correct action or not, *i.e.*, limiting efficiency. Although limiting efficiency is the common objective of interest in the literature of social learning, it might not be a plausible measure of social welfare for a social planner.

For example, Proposition 2.1 says that the social planner is indifferent among all periods to release a public signal that is sufficiently precise. Imagine that the social planner has a public signal with perfect precision. Then naturally she should release the public signal as early as possible, because any delay would hurt some earlier agents. However this natural observation is not captured by the non-discounted average payoff as the social planner only cares about people in the limit. Hence in this subsection we introduce a discount factor δ in the social planner's objective and reconsider the timing of information disclosure. In particular, we show that it is not always better to postpone the release of a public signal.

Formally, with all the other configurations identical to the benchmark model, we assume the social planner now wants to maximize the discounted sum of people's expected payoff,

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}Eu(a_t^*;\theta).$$

For simplicity we restrict attention on exogenous releasing strategies and assume the public signal has the same precision q as the private signals.

Claim 3.1 *Let $V(\tau)$ be the discounted sum of people's expected payoff when the public signal is released at period τ. Then $\forall q \in (\frac{1}{2},1)$, $V(3) > V(4)$.*

Proof. See Appendix. ■

In the benchmark without discounting, the social planner is indifferent between releasing at period 3 and at period 4.[16] Hence with discounting it is not surprising to see that the social planner now strictly prefers to release the public signal at periods 3 than period 4. The optimal timing of disclosure with discounting is yet to be explicitly characterized, nevertheless we present this example mainly to bring up the concern for the plausibility of treating limiting efficiency as the main objective in social-learning models.

3.2 Ternary Setting

In this subsection we expand the binary setting in the benchmark model and allow the state/action/signal space to have three elements.[17] Under this new setting, even for a patient social planner who only cares about limiting efficiency as in the benchmark model, it is not always better to postpone the disclosure of public information.

Formally, the state of the world ψ is realized out of $\{L,M,R\}$ with

$$\Pr(\psi=L)=\Pr(\psi=M)=\Pr(\psi=R)=1/3.$$

[16] See the proof of Proposition 2.1 for details.

[17] Ternary spaces are sufficient to capture the intuition we want to describe, yet not too complicated for analysis.

After the realization of ψ, every agent t receives a private signal $s_t \in \{L,M,R\}$ and the private signals are conditionally i.i.d. with

$$\Pr\{s_t = L|\psi = L\} = \Pr\{s_t = M|\psi = M\} = \Pr\{s_t = R|\psi = R\} = q \in (\frac{1}{3},1);$$
$$\Pr\{s_t = L|\psi = M\} = \Pr\{s_t = R|\psi = M\} = \Pr\{s_t = M|\psi = R\} =$$
$$\Pr\{s_t = L|\psi = R\} = \Pr\{s_t = R|\psi = L\} = \Pr\{s_t = M|\psi = L\} = \frac{1-q}{2}.$$

Each agent chooses a_t from $\{L,M,R\}$ and observes the history of past actions $\boldsymbol{h}_t \equiv (a_1,a_2,\dots,a_{t-1})$. They have identical utility function $u(a_t;\psi) = 1_{\{\psi=a_t\}}$, and note that this degenerate utility function implies that the three states cannot be linearly ordered, unlike many other economic models with multiple states.[18]

With ternary spaces, we want to specify a tie-breaking rule:

$$a_t^*(\boldsymbol{h}_t,s_t) = s_t \text{ if } s_t \in \underset{a\in\{L,M,R\}}{\arg\max}\; E_\psi[u(a;\psi)|\boldsymbol{h}_t,s_t],$$
$$a_t^*(\boldsymbol{h}_t,s_t) = a_{t-1} \text{ if } \underset{a\in\{L,M,R\}}{\arg\max}\; E_\psi[u(a;\psi)|\boldsymbol{h}_t,s_t] = \Psi\backslash\{s_t\}.$$

Namely, agent t follows s_t if it is one of the maximizers and chooses to follow her immediate predecessor if the two actions different from s_t are both maximizers.[19]

Again there is a social planner who receives a signal $\tilde{s} \in \{L,M,R\}$ and decide whether/when to release it to the public. Here for simplicity $\tilde{s}$ is assumed to be *equally* precise as the private signals. The (patient) social planner's objective is still to maximize the ex-ante average payoff of the population and we restrict attention on *exogenous* releasing strategies only in this subsection.

Definition 5 $\forall a \in \{L,M,R\}$, *a* ***trap*** *away from action a starts at period T if $a_t(\boldsymbol{h}_t,\cdot) \neq a$ for all $t \geq T$.*

It is easy to see that a trap away from one action is equivalent to a herd on the other action in the binary model. However, with three possible actions, a herd on action $a' \neq a$ is a trap away from action a, but *not* vice versa. And the difference between a herd and a trap is exactly what

[18] Specifically, the degenerate utility function rules out the scenario where an agent believes one state, say M, is more likely after observing an action L and an action R. This setting, though complicates the analysis, is crucial for the result we will present in this subsection.

[19] The specification itself is not very important; we just want a tie-breaking rule to guarantee deterministic outcomes and hence tractability.

drives the following result that the benefit of releasing the public signal is no longer weakly monotone over time.

Claim 3.2 *Let* $\overline{G}$ *be the ex-ante average payoff of the population with* no *release at all, and let* $G(\tau)$ *be the ex-ante average payoff of the population if the public signal is released at period* $\tau \geq 1$. *Then* $\forall q \in (\frac{1}{3}, 1)$, $G(3) > \overline{G}$ *and* $G(4) > G(5)$.

Proof. See Appendix. ■

$G(3) > \overline{G}$ is not a surprising: the public signal is equally precise as a private signal, so releasing it would not bring any harm but could possibly break down a wrong herd starting at period 3, if the first two actions are the same but wrong.[20] Meanwhile we lose weak monotonicity as $G(4) > G(5)$ for the following reason: releasing $\tilde{s}$ at period 4 is possible to break down a trap away from true state ψ if $\tilde{s} = \psi$, no matter what $\boldsymbol{h}_3$ is; however, if $\psi = R$ but $\boldsymbol{h}_4 = (L, M, L, L)$, the trap away from R could not be broken down even if $\tilde{s} = R$ as long as it is released at period 5. In general, weak monotonicity fails here because the existence of traps rather than herds: a trap is not necessarily an informational cascade and information can still aggregate over time for the two "surviving" actions before a herd finally starts, hence the social planner could face the danger of not being able to break down a wrong trap if the public signal is released too late. On the other hand, the optimal timing of release is unclear to us and in principle it shall depends on the value of q. See Figure 1 for some examples.

4 Conclusions

We treat this work as a contribution to the literature on social learning, with a focus on exogenous information intervention. In particular, we look into the effect of public information disclosure on the asymptotic efficiency of social learning. In the canonical binary model, if a social planner were to choose a certain period to release a public signal, she should release it as late as possible regardless of the precision of the public signal: a *monotonicity* result. Meanwhile, when the public signal is less precise than people's private signals, releasing it would do no good on social welfare even if the timing of release can be contingent on the history of actions: an *anti-transparency* result.

[20] It is not difficult to see that a herd will arise when, in the history, the number of one action is larger than the number of the other two actions by at least 2.

We present two alternative settings where the monotonicity result may fail. Postponing information disclosure could be bad for a social planner, if her objective is the discounted sum of people's expected payoffs, or if the state/action spaces are richer. Characterizing the optimal timing of disclosure in these two settings is a challenging but interesting follow-up to this work.

As to the anti-transparency result, a relevant and interesting question is: what is the lower bound of the (relative) precision of a public signal that could improve social welfare once released in a more general setting, *e.g.*, agents have private signals of heterogeneous precisions? Some preliminary work suggests that this lower bound is lower and could be substantially lower than the "average "precision of people's private signals.[21]

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[21] Based on the insight of Herrera and Hörner (2012), we further conjecture that exogenous public information could, with a careful timing of release, be always welcome regardless of its precision, under certain private information structures.

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Appendix

Proof of Lemma 2.2.1**.** By standard Bayesian Nash equilibrium definition and the tie-breaking rule,

$$
\begin{aligned}
& a_t^*(\boldsymbol{h}_t, s_t) \\
&= \underset{a\in\{1,-1\}}{\arg\max}\, E_\theta(1_{\{\theta=a\}}|\boldsymbol{h}_t, s_t) = \underset{a\in\{1,-1\}}{\arg\max}\, \Pr(\theta = a|\boldsymbol{h}_t, s_t) \\
&= \begin{cases} s_t & \text{if } \Pr(\theta=1|\boldsymbol{h}_t,s_t) = \Pr(\theta=-1|\boldsymbol{h}_t,s_t) \\ sgn(\Pr(\theta=1|\boldsymbol{h}_t,s_t) - \Pr(\theta=-1|\boldsymbol{h}_t,s_t)) & \text{otherwise} \end{cases}.
\end{aligned}
$$

By Bayes' Rule and uniform prior,

$$
\begin{aligned}
& \Pr(\theta=1|\boldsymbol{h}_t,s_t) = \frac{\Pr(\boldsymbol{h}_t,s_t|\theta=1)}{\Pr(\boldsymbol{h}_t,s_t|\theta=1)+\Pr(\boldsymbol{h}_t,s_t|\theta=-1)} = 1-\Pr(\theta=-1|\boldsymbol{h}_t,s_t) \\
& \Rightarrow sgn(\Pr(\theta=1|\boldsymbol{h}_t,s_t) - \Pr(\theta=-1|\boldsymbol{h}_t,s_t)) = sgn(\Pr(\boldsymbol{h}_t,s_t|\theta=1) - \Pr(\boldsymbol{h}_t,s_t|\theta=-1)) \\
& \Rightarrow a_t^*(\boldsymbol{h}_t,s_t) \\
&= \begin{cases} s_t & \text{if } \Pr(\boldsymbol{h}_t,s_t|\theta=1) = \Pr(\boldsymbol{h}_t,s_t|\theta=-1) \\ sgn(\Pr(\boldsymbol{h}_t,s_t|\theta=1) - \Pr(\boldsymbol{h}_t,s_t|\theta=-1)) & \text{otherwise} \end{cases}.
\end{aligned}
$$

By definition of w_t and independence between s_t and $\boldsymbol{h}_t$,

$$\frac{\Pr(\boldsymbol{h}_t, s_t|\theta=1)}{\Pr(\boldsymbol{h}_t, s_t|\theta=-1)} = \frac{\Pr(\boldsymbol{h}_t|\theta=1)}{\Pr(\boldsymbol{h}_t|\theta=-1)} \frac{\Pr(s_t|\theta=1)}{\Pr(s_t|\theta=-1)}$$
$$= (\frac{q}{1-q})^{w_t} (\frac{q}{1-q})^{s_t} = (\frac{q}{1-q})^{w_t+s_t}, \text{ where } \frac{q}{1-q} > 1$$
$$\Rightarrow sgn(\Pr(\boldsymbol{h}_t, s_t|\theta=1) - \Pr(\boldsymbol{h}_t, s_t|\theta=-1)) = sgn(w_t+s_t) = \begin{cases} s_t \text{ or } 0 & \text{if } |w_t| \leq 1 \\ sgn(w_t) & \text{otherwise} \end{cases},$$

hence we get $a_t^*(\boldsymbol{h}_t, s_t)$ characterized in the Lemma.

On the other hand, apparently $w_1^* = 0$ and by definition of w_t^*,

$$(\frac{q}{1-q})^{w_{t+1}^*} = \frac{\Pr(\boldsymbol{h}_{t+1}^*|\theta=1)}{\Pr(\boldsymbol{h}_{t+1}^*|\theta=-1)} = \frac{\Pr(\boldsymbol{h}_t^*, a_t|\theta=1)}{\Pr(\boldsymbol{h}_t^*, a_t|\theta=-1)}$$
$$= \frac{\Pr(\boldsymbol{h}_t^*|\theta=1)}{\Pr(\boldsymbol{h}_t^*|\theta=-1)} \frac{\Pr(a_t^*|\boldsymbol{h}_t^*, \theta=1)}{\Pr(a_t^*|\boldsymbol{h}_t^*, \theta=-1)}$$
$$= (\frac{q}{1-q})^{w_t^*} \frac{\Pr(a_t^*|\boldsymbol{h}_t^*, \theta=1)}{\Pr(a_t^*|\boldsymbol{h}_t^*, \theta=-1)}.$$

Meanwhile, if $|w_t^*| \leq 1$,

$$a_t^* = s_t \Rightarrow \frac{\Pr(a_t^*|\boldsymbol{h}_t^*, \theta=1)}{\Pr(a_t^*|\boldsymbol{h}_t^*, \theta=-1)} = \frac{\Pr(s_t|\boldsymbol{h}_t^*, \theta=1)}{\Pr(s_t|\boldsymbol{h}_t^*, \theta=-1)} = \frac{\Pr(s_t|\theta=1)}{\Pr(s_t|\theta=-1)} = (\frac{q}{1-q})^{s_t}$$
$$\Rightarrow w_{t+1}^* = w_t^* + s_t = w_t^* + a_t^*;$$

otherwise,

$$a_t^* = sgn(w_t^*) \Rightarrow \frac{\Pr(a_t^*|\boldsymbol{h}_t^*, \theta=1)}{\Pr(a_t^*|\boldsymbol{h}_t^*, \theta=-1)} = \frac{\Pr(sgn(w_t^*)|\boldsymbol{h}_t^*, \theta=1)}{\Pr(sgn(w_t^*)|\boldsymbol{h}_t^*, \theta=-1)} = 1$$
$$\Rightarrow w_{t+1}^* = w_t^*.$$

■

Proof of Lemma 2.2.2. According to Lemma 2.1 and Definition 2.1, a herd on action $sgn(w_t^*)$

starts at period t if and only if $|w_t^*| > 1$. Note that

$$\forall t \in \mathbb{N}^*,\ |w_t^*| \leq 1 \Rightarrow \begin{cases} \forall k \in \mathbb{N}^*,\ a_{2k-1}^* + a_{2k}^* = 0 \\ \forall t \in \mathbb{N}^*,\ a_t^* = s_t \end{cases}$$

$$\Rightarrow \forall k \in \mathbb{N}^*,\ s_{2k-1} + s_{2k} = 0.$$

Hence a herd starts eventually unless $s_{2k-1} + s_{2k} = 0$, $\forall k \in \mathbb{N}^*$. However,

$$\Pr(\forall k \in \mathbb{N}^*,\ s_{2k-1} = -s_{2k}) \leq 1 - \Pr(\exists k' \in \mathbb{N}^* \text{ s.t. } s_{k'} = s_{k'+1} = s_{k'+2})$$

$$= 1 - E_\theta[\Pr(\exists k' \in \mathbb{N}^* \text{ s.t. } s_{k'} = s_{k'+1} = s_{k'+2}|\theta)] = 1 - 1 = 0,$$

where $\forall \theta \in \Theta$, $\Pr(\exists k' \in \mathbb{N}^* \text{ s.t. } s_{k'} = s_{k'+1} = s_{k'+2}|\theta) = 1$ due to $\{s_t\}_{t=1}^{\infty}$ being conditional i.i.d. and Law of Large Numbers. ■

Proof of Lemma 2.2.3**.** By Lemma 2.1, a herd starts in equilibrium when $|w_t^*| > 1$ and the herd is correct(wrong) if $w_t^* > 1 (< -1)$. Then we can immediately see that $\alpha_1 = 0$ and $\alpha_7 = 1$. For the remaining cases, let us look into the transition of w_t^θ. (Recall that we have assumed that the realization of θ is 1 without loss of generality)

If $w_1 = 0$,

$$a_1^* = s_1 \text{ and } w_2^* = s_1$$

$$\Rightarrow \Pr(w_2^* = 1|w_1 = 0) = \Pr(s_1 = 1) = q,$$

$$\Pr(w_2^* = -1|w_1 = 0) = \Pr(s_1 = -1) = 1 - q.$$

Note that $\forall T < \infty$ (especially $T = 2$ here),

$$p(x) \equiv \Pr(\text{plim}_{t\to\infty} a_t^*(\boldsymbol{h}_t^*, s_t) = \theta | w_1 = x)$$

$$= \Pr(\text{plim}_{t\to\infty} a_t^*(\boldsymbol{h}_t^*, s_t) = \theta | w_T = x)$$

since the subgame starting from agent T is identical to the original game, thus we have $\alpha_4 =$

$(1-q)\alpha_2+q\alpha_6$. Through similar arguments,

$$\Pr(w_2^*=2|w_1=1)=q \text{ and } \Pr(w_2^*=0|w_1=1)=1-q$$
$$\Rightarrow \alpha_6=q\alpha_7+(1-q)\alpha_4=q+(1-q)\alpha_4;$$
$$\Pr(w_2^*=0|w_1=-1)=q \text{ and } \Pr(w_2^*=-2|w_1=-1)=1-q$$
$$\Rightarrow \alpha_2=q\alpha_4+(1-q)\alpha_1=q\alpha_4.$$

Solve the three linear equations together to get α_2, α_4 and α_6 as stated in the Lemma.

If $w_1=x\in(-1,0)$,

$$a_1^*=s_1 \text{ and } w_2^*=x+s_1$$
$$\Rightarrow \Pr(w_2^*=x+1\in(0,1)|w_1=x)=q,$$
$$\Pr(w_2^*=x-1<-1|w_1=x)=1-q$$
$$\Rightarrow \alpha_3=q\alpha_5+(1-q)\alpha_1=q\alpha_5;$$

through similar argument,

$$\Pr(w_2^*=x'+1>1|w_1=x'\in(0,1))=q,$$
$$\Pr(w_2^*=x'-1\in(-1,0)|w_1=x'\in(0,1))=1-q$$
$$\Rightarrow \alpha_5=q\alpha_7+(1-q)\alpha_3=q+(1-q)\alpha_3.$$

Solve the two linear equations together to get α_3 and α_5 as stated in the Lemma.

Combing all these linear equations together, we have exactly $Q\boldsymbol{\alpha}=\boldsymbol{\alpha}$. In other words, $\boldsymbol{\alpha}$ is an eigenvector of P associated with eigenvalue 1, with restriction that $\alpha_7=1$ and $\alpha_1=0$. ■

Proof of Lemma 2.2.4**.** $\boldsymbol{\pi}^1=(0,0,0,1,0,0,0)$ simply because $w_1^*=0$. As to the transition between w_t^* to w_{t+1}^*, it is identical to the transition between w_1^* and w_2^* illustrated in the proof of Lemma

2.3:

$$\Pr(w^*_{t+1} > 1 | w_t > 1) = 1,\ \Pr(w^*_{t+1} < -1 | w^*_t < -1) = 1;$$

$$\Pr(w^*_{t+1} = 1 | w^*_t = 0) = q,\ \Pr(w^*_{t+1} = -1 | w^*_t = 0) = 1 - q;$$

$$\Pr(w^*_{t+1} = 2 | w^*_t = 1) = q,\ \Pr(w^*_{t+1} = 0 | w^*_t = 1) = 1 - q;$$

$$\Pr(w^*_{t+1} = 0 | w^*_t = -1) = q,\ \Pr(w^*_{t+1} = -2 | w^*_t = 0) = 1 - q;$$

$$\Pr(w^*_{t+1} \in (0,1) | w^*_t \in (-1,0)) = q,\ \Pr(w^*_{t+1} < -1 | w^*_t \in (-1,0)) = 1 - q;$$

$$\Pr(w^*_{t+1} > 1 | w^*_t \in (0,1)) = q,\ \Pr(w^*_{t+1} \in (-1,0) | w^*_t \in (0,1)) = 1 - q.$$

Therefore the transition matrix is exactly matrix Q, which is indeed monotone according to Definition 2.2.

Note that by Lemma 2.1, w^*_t can only take values ± 2, ± 1 and 0, hence $\pi^t_3 = \pi^t_5 = 0$ for any t and $w^*_t > 1 (< -1)$ indicates $w^*_t = 2(-2)$. ■

Proof of Lemma 2.2.5. By definition of public beliefs,

$$\left(\frac{q}{1-q}\right)^{w_\tau} = \frac{\Pr(\boldsymbol{h}_\tau | \theta = 1)}{\Pr(\boldsymbol{h}_\tau | \theta = -1)} \text{ and } \left(\frac{q}{1-q}\right)^{\widetilde{w}_\tau} = \frac{\Pr(\boldsymbol{h}_\tau, \widetilde{s} | \theta = 1)}{\Pr(\boldsymbol{h}_\tau, \widetilde{s} | \theta = -1)}.$$

Since $\boldsymbol{h}_t$ and $\widetilde{s}$ are independent,

$$\begin{aligned}
\left(\frac{q}{1-q}\right)^{\widetilde{w}_\tau} &= \frac{\Pr(\boldsymbol{h}_\tau, \widetilde{s} | \theta = 1)}{\Pr(\boldsymbol{h}_\tau, \widetilde{s} | \theta = -1)} \\
&= \frac{\Pr(\boldsymbol{h}_\tau | \theta = 1)}{\Pr(\boldsymbol{h}_\tau | \theta = -1)} \cdot \frac{\Pr(\widetilde{s} | \theta = 1)}{\Pr(\widetilde{s} | \theta = -1)} \\
&= \left(\frac{q}{1-q}\right)^{w_\tau} \left(\frac{\widetilde{q}}{1-\widetilde{q}}\right)^{\widetilde{s}} = \left(\frac{q}{1-q}\right)^{w_\tau} \left(\frac{q}{1-q}\right)^{\lambda \widetilde{s}}.
\end{aligned}$$

Hence we have $\widetilde{w}_\tau = w_\tau + \lambda \widetilde{s}$.

On the other hand,

$$\text{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}u(a_t^*;\theta=1|\widetilde{w}_\tau)$$
$$=\text{plim}_{T'\to\infty}\frac{1}{T'}\sum_{t=\tau}^{\tau+T'}u(a_t^*;\theta=1|\widetilde{w}_\tau)$$
$$=\Pr(\text{plim}_{t\to\infty}a_t^*=1|w_1=\widetilde{w}_\tau)=p(\widetilde{w}_\tau),$$

where the last equation comes from Lemma 2.3. Finally, the (unconditional) expected average payoff is just

$$v(\tau;\lambda)=E_{\widetilde{w}_\tau}p(\widetilde{w}_\tau)=E_{w_\tau,\widetilde{s}}[p(w_\tau+\lambda\widetilde{s})].$$

■

Proof of Proposition 2.2.1**.** For $\tau\geq 1$, let $\widetilde{\boldsymbol{\pi}}^\tau=(\widetilde{\pi}_1^\tau,\widetilde{\pi}_2^\tau,\widetilde{\pi}_3^\tau,\widetilde{\pi}_4^\tau,\widetilde{\pi}_5^\tau,\widetilde{\pi}_6^\tau,\widetilde{\pi}_7^\tau)$ be the probability vector of $\widetilde{w}_\tau$ on the partition $\mathscr{P}$ introduced in Lemma 2.3. Then by Lemma 2.3 we have

$$E_{\widetilde{w}_\tau}(p(\widetilde{w}_\tau))=\widetilde{\boldsymbol{\pi}}^\tau\cdot\boldsymbol{\alpha}.$$

Note that the public information is irrelevant for agents before period τ so in equilibrium w_t^* for $t\leq\tau$ still evolves according to Lemma 2.4. Bearing in mind as well that $\widetilde{s}$ is independent of w_t^* and distributed according to

$$\Pr\{\widetilde{s}=1|\theta=1\}=\Pr\{\widetilde{s}=-1|\theta=-1\}=\widetilde{q},$$

we can derive $\widetilde{\boldsymbol{\pi}}^\tau$ explicitly and prove the proposition case by case on λ as follows: (we then omit the argument λ in $v(\cdot;\cdot)$ in each case)

Case I ($0 < \lambda < 1 \Leftrightarrow \frac{1}{2} < \widetilde{q} < q$)

$\widetilde{w}_1 = w_1^* + \lambda \widetilde{s}$ and $\boldsymbol{\pi}^1 = (0,0,0,1,0,0,0)$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^1 = (0,0,1-\widetilde{q},0,\widetilde{q},0,0)$

$\Rightarrow v(1) = \widetilde{q}\alpha_5 + (1-\widetilde{q})\alpha_3 = \dfrac{\widetilde{q}q + q^2(1-\widetilde{q})}{1-q(1-q)} < \dfrac{2q^2-q^3}{1-q(1-q)} < \alpha_4$;

$\widetilde{w}_2 = w_2^* + \lambda \widetilde{s}$ and $\pi^2 = (0,1-q,0,0,0,0,q,0)$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^1 = ((1-q)(1-\widetilde{q}),0,(1-q)\widetilde{q},0,q(1-\widetilde{q}),0,q\widetilde{q})$

$\Rightarrow v(2) = (1-q)\widetilde{q}\alpha_3 + q(1-\widetilde{q})\alpha_5 + q\widetilde{q} = \dfrac{q\widetilde{q} + q^2(1-\widetilde{q})}{1-q(1-q)} = v(1) < \alpha_4$;

$\widetilde{w}_\tau = w_\tau^* + \lambda \widetilde{s}$ and $\boldsymbol{\pi}^\tau = 2q(1-q)\boldsymbol{\pi}^{\tau-2} + ((1-q)^2,0,0,0,0,0,q^2)$ for $\tau \geq 3$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^\tau = 2q(1-q)\widetilde{\boldsymbol{\pi}}^{\tau-2} + ((1-q)^2,0,0,0,0,0,q^2)$;

$\Rightarrow v(\tau) = 2q(1-q)v(\tau-2) + q^2$

$\Rightarrow sgn(v(\tau+1) - v(\tau)) = sgn(v(\tau-1) - v(\tau-2))$.

If $v(\tau-2) < \alpha_4 = \dfrac{q^2}{1-2q(1-q)}$

$\Rightarrow v(\tau-2) < 2q(1-q)v(\tau-2) + q^2 = v(\tau) < 2q(1-q)\alpha_4 + q^2 = \alpha_4$

$\Rightarrow v(1) = v(2) < v(3) = v(4) < \alpha_4$.

Recursively we have $v(1) = v(2) < v(3) = v(4) < v(5) = v(6) < \cdots$ and $v(\tau) < \alpha_4$ for any $\tau \geq 1$.

Case II ($\lambda = 1 \Leftrightarrow \widetilde{q} = q$)

$\widetilde{w}_1 = w_1^* + \lambda\widetilde{s}$ and $\boldsymbol{\pi}^1 = (0,0,0,1,0,0,0)$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^1 = (0,1-\widetilde{q},0,0,0,\widetilde{q},0)$

$\Rightarrow v(1) = \widetilde{q}\alpha_6 + (1-\widetilde{q})\alpha_2 = q(q + \frac{(1-q)q^2}{1-2q(1-q)}) + \frac{(1-q)q^3}{1-2q(1-q)} = \alpha_4$;

$\widetilde{w}_2 = w_2^* + \lambda\widetilde{s}$ and $\pi^2 = (0,1-q,0,0,0,0,q,0)$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^2 = ((1-q)(1-\widetilde{q}),0,0,(1-q)\widetilde{q} + q(1-\widetilde{q}),0,,0,q\widetilde{q})$

$\Rightarrow v(2) = [(1-q)\widetilde{q} + q(1-\widetilde{q})]\alpha_4 + q\widetilde{q} = 2q(1-q)\alpha_4 + q^2 = \alpha_4$;

$\widetilde{w}_\tau = w_\tau^* + \lambda\widetilde{s}$ and $\boldsymbol{\pi}^\tau = 2q(1-q)\boldsymbol{\pi}^{\tau-2} + ((1-q)^2,0,0,0,0,0,q^2)$ for $\tau \geq 3$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^\tau = 2q(1-q)\widetilde{\boldsymbol{\pi}}^{\tau-2} + ((1-\widetilde{q})(1-q)^2,\widetilde{q}(1-q)^2,0,0,0,(1-\widetilde{q})q^2,\widetilde{q}q^2)$

$\Rightarrow v(\tau) = 2q(1-q)v(\tau-2) + \widetilde{q}(1-q)^2\alpha_2 + (1-\widetilde{q})q^2\alpha_6 + \widetilde{q}q^2 =$

$2q(1-q)v(\tau-2) + (1-q)q\alpha_4 + q^3 > 2q(1-q)v(\tau-2) + q^2$

$\Rightarrow \; sgn(v(\tau+1) - v(\tau)) = \; sgn(v(\tau-1) - v(\tau-2))$.

If $v(\tau-2) \geq \alpha_4 \Rightarrow v(\tau) > 2q(1-q)v(\tau-2) + q^2 \geq \alpha_4$

$\Rightarrow v(4) = v(3) > v(2) = v(1) = \alpha_4$.

Recursively we have $\alpha_4 = v(1) = v(2) < v(3) = v(4) < v(5) = v(6) < \cdots$.

Case III ($1 < \lambda < 2 \Leftrightarrow q < \widetilde{q} < \frac{q^2}{q^2+(1-q)^2} = \alpha_4$)

$$\widetilde{w}_1 = w_1^* + \lambda \widetilde{s} \text{ and } \boldsymbol{\pi}^1 = (0,0,0,1,0,0,0)$$

$$\Rightarrow \widetilde{\boldsymbol{\pi}}^1 = (1-\widetilde{q},0,0,0,0,0,\widetilde{q})$$

$$\Rightarrow v(1) = \widetilde{q} < \alpha_4;$$

$$\widetilde{w}_2 = w_2^* + \lambda \widetilde{s} \text{ and } \pi^2 = (0,1-q,0,0,0,0,q,0)$$

$$\Rightarrow \widetilde{\boldsymbol{\pi}}^2 = ((1-q)(1-\widetilde{q}),0,q(1-\widetilde{q}),0,(1-q)\widetilde{q},0,q\widetilde{q})$$

$$\Rightarrow v(2) = (1-q)\widetilde{q}\alpha_5 + q(1-\widetilde{q})\alpha_3 + q\widetilde{q} = \frac{2q(1-q)\widetilde{q}+q^3}{1-q(1-q)}$$

$$\Rightarrow v(2)-v(1) = \frac{q^3(1-\widetilde{q})-(1-q)^3\widetilde{q}}{1-q(1-q)} > 0 \text{ as } \widetilde{q} < \alpha_4 < \frac{q^3}{q^3+(1-q)^3};$$

$$\widetilde{w}_\tau = w_\tau^* + \lambda \widetilde{s} \text{ and } \boldsymbol{\pi}^\tau = 2q(1-q)\boldsymbol{\pi}^{\tau-2} + ((1-q)^2,0,0,0,0,0,q^2) \text{ for } \tau \geq 3$$

$$\Rightarrow \widetilde{\boldsymbol{\pi}}^\tau = 2q(1-q)\widetilde{\boldsymbol{\pi}}^{\tau-2} + ((1-\widetilde{q})(1-q)^2,0,\widetilde{q}(1-q)^2,0,(1-\widetilde{q})q^2,0,\widetilde{q}q^2)$$

$$\Rightarrow v(\tau) = 2q(1-q)v(\tau-2) + \widetilde{q}(1-q)^2\alpha_3 + (1-\widetilde{q})q^2\alpha_5 + \widetilde{q}q^2$$

$$\Rightarrow sgn(v(\tau+1)-v(\tau)) = sgn(v(\tau-1)-v(\tau-2)).$$

$$v(3) = 2q(1-q)v(1) + \widetilde{q}(1-q)^2\alpha_3 + (1-\widetilde{q})q^2\alpha_5 + \widetilde{q}q^2 \text{ with } v(1) = \widetilde{q}$$

$$\Rightarrow v(3)-v(2) = q(1-q)\widetilde{q} - \widetilde{q}(1-q)q = 0 \Rightarrow v(3) = v(2) > v(1).$$

Recursively we have $v(1) < v(2) = v(3) < v(4) = v(5) < v(6) < \cdots$. As the sequence $\{v(\tau)\}_{\tau=1}^\infty$ is weakly monotonic and bounded between 0 and 1, $\lim_{\tau\to\infty} v(\tau)$ exists and it satisfies

$$\lim_{\tau\to\infty} v(\tau) = 2q(1-q)\lim_{\tau\to\infty} v(\tau) + \widetilde{q}(1-q)^2\alpha_3 + (1-\widetilde{q})q^2\alpha_5 + \widetilde{q}q^2$$
$$\Rightarrow \lim_{\tau\to\infty} v(\tau) = \frac{\widetilde{q}(1-q)^2\alpha_3 + (1-\widetilde{q})q^2\alpha_5 + \widetilde{q}q^2}{1-2q(1-q)} >$$
$$\frac{q(1-q)^2\alpha_3 + (1-q)q^2\alpha_5 + q^3}{1-2q(1-q)} > \frac{q^2}{1-2q(1-q)} = \alpha_4.$$

Thus $\exists T < \infty$ s.t. $v(T) > \alpha_4$.

Case IV ($\lambda = 2 \Leftrightarrow \widetilde{q} = \frac{q^2}{q^2+(1-q)^2} = \alpha_4$)

Similar to Case III, we have $v(1) = \widetilde{q} = \alpha_4$;

$\widetilde{w}_2 = w_2^* + \lambda\widetilde{s}$ and $\pi^2 = (0, 1-q, 0, 0, 0, 0, q, 0)$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^2 = ((1-q)(1-\widetilde{q}), q(1-\widetilde{q}), 0, 0, 0, (1-q)\widetilde{q}, q\widetilde{q})$

$\Rightarrow v(2) = (1-q)\widetilde{q}\alpha_6 + q(1-\widetilde{q})\alpha_2 + q\widetilde{q} = (1-q)\alpha_4\alpha_6 + q(1-\alpha_4)\alpha_2 + q\alpha_4 =$

$\alpha_4[\alpha_4 + 2q(1-\alpha_4)] > \alpha_4$;

$\widetilde{w}_\tau = w_\tau^* + \lambda\widetilde{s}$ and $\boldsymbol{\pi}^\tau = 2q(1-q)\boldsymbol{\pi}^{\tau-2} + ((1-q)^2, 0, 0, 0, 0, 0, 0, q^2)$ for $\tau \geq 3$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^\tau = 2q(1-q)\widetilde{\boldsymbol{\pi}}^{\tau-2} + ((1-\widetilde{q})(1-q)^2, 0, 0, \widetilde{q}(1-q)^2 + (1-\widetilde{q})q^2, 0, 0, \widetilde{q}q^2)$

$\Rightarrow v(\tau) = 2q(1-q)v(\tau-2) + [\widetilde{q}(1-q)^2 + (1-\widetilde{q})q^2]\alpha_4 + \widetilde{q}q^2$

$\Rightarrow$ $sgn(v(\tau+1) - v(\tau)) =$ $sgn(v(\tau-1) - v(\tau-2))$.

$v(3) = 2q(1-q)v(1) + [\widetilde{q}(1-q)^2 + (1-\widetilde{q})q^2]\alpha_4 + \widetilde{q}q^2$ with $v(1) = \widetilde{q} = \alpha_4$

$\Rightarrow v(3) = [2q(1-q) + \alpha_4(1-q)^2 + (1-\alpha_4)q^2 + q^2]\alpha_4 = v(2) > v(1)$.

Recursively we have $\alpha_4 = v(1) < v(2) = v(3) < v(4) = v(5) < v(6) = \cdots$.

Case V ($2 < \lambda < 3 \Leftrightarrow \alpha_4 = \frac{q^2}{q^2+(1-q)^2} < \widetilde{q} < \frac{q^3}{q^3+(1-q)^3}$)

Similar to Case III, we have $v(1) = \widetilde{q} > \alpha_4$;

$\widetilde{w}_2 = w_2^* + \lambda\widetilde{s}$ and $\pi^2 = (0, 1-q, 0, 0, 0, 0, q, 0)$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^2 = ((1-\widetilde{q}), 0, 0, 0, 0, 0, \widetilde{q})$

$\Rightarrow v(2) = \widetilde{q} = v(1)$;

$\widetilde{w}_\tau = w_\tau^* + \lambda\widetilde{s}$ and $\boldsymbol{\pi}^\tau = 2q(1-q)\boldsymbol{\pi}^{\tau-2} + ((1-q)^2, 0, 0, 0, 0, 0, 0, q^2)$ for $\tau \geq 3$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^\tau = 2q(1-q)\widetilde{\boldsymbol{\pi}}^{\tau-2} + ((1-\widetilde{q})(1-q)^2, 0, (1-\widetilde{q})q^2, 0, \widetilde{q}(1-q)^2, 0, \widetilde{q}q^2)$

$\Rightarrow v(\tau) = 2q(1-q)U(\tau-2) + \widetilde{q}(1-q)^2\alpha_5 + (1-\widetilde{q})q^2\alpha_3 + \widetilde{q}q^2$

$\Rightarrow$ $sgn(v(\tau+1) - v(\tau)) =$ $sgn(v(\tau-1) - v(\tau-2))$.

$v(3) = 2q(1-q)v(1) + \widetilde{q}(1-q)^2\alpha_5 + (1-\widetilde{q})q^2\alpha_3 + \widetilde{q}q^2$ with $v(1) = \widetilde{q}$

$$\Rightarrow v(3) - \widetilde{q} = -(1-q)^2\widetilde{q} + \alpha_5(1-q)^2\widetilde{q} + (1-\widetilde{q})\alpha_3 q^2 = \frac{q^4(1-\widetilde{q}) - (1-q)^4\widetilde{q}}{1-q(1-q)} > 0$$

$\Rightarrow v(3) > \widetilde{q} = v(2) = v(1)$.

Recursively we have $\alpha_4 < v(1) = v(2) < v(3) = v(4) < v(5) = v(6) < \cdots$.

Case VI ($\lambda = 3 \Leftrightarrow \widetilde{q} = \frac{q^3}{q^3+(1-q)^3}$)

Similar to Case V, we have $v(1) = v(2) = \widetilde{q} > \alpha_4$;

$\widetilde{w}_\tau = w^*_\tau + \lambda\widetilde{s}$ and $\boldsymbol{\pi}^\tau = 2q(1-q)\boldsymbol{\pi}^{\tau-2} + ((1-q)^2, 0,0,0,0,0,q^2)$ for $\tau \geq 3$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^\tau = 2q(1-q)\widetilde{\boldsymbol{\pi}}^{\tau-2} + ((1-\widetilde{q})(1-q)^2, (1-\widetilde{q})q^2, 0,0,0, \widetilde{q}(1-q)^2, \widetilde{q}q^2)$

$\Rightarrow v(\tau) = 2q(1-q)v(\tau-2) + \widetilde{q}(1-q)^2\alpha_6 + (1-\widetilde{q})q^2\alpha_2 + \widetilde{q}q^2$

$\Rightarrow \ sgn(v(\tau+1) - v(\tau)) = \ sgn(v(\tau-1) - v(\tau-2))$.

$v(3) = 2q(1-q)v(1) + \widetilde{q}(1-q)^2\alpha_6 + (1-\widetilde{q})q^2\alpha_2 + \widetilde{q}q^2$ with $v(1) = \widetilde{q}$

$\Rightarrow v(3) - \widetilde{q} = -(1-q)^2\widetilde{q} + \alpha_6(1-q)^2\widetilde{q} + (1-\widetilde{q})\alpha_2 q^2 = \dfrac{q^5(1-\widetilde{q}) - (1-q)^5\widetilde{q}}{1-2q(1-q)} > 0$

$\Rightarrow v(3) > \widetilde{q} = v(2) = v(1)$.

Recursively we have $\alpha_4 < v(1) = v(2) < v(3) = v(4) < v(5) = v(6) < \cdots$.

Case VII ($\lambda > 3 \Leftrightarrow \widetilde{q} > \frac{q^3}{q^3+(1-q)^3}$)

$\forall \tau \geq 1$, $\widetilde{w}_\tau = w^*_\tau + \lambda\widetilde{s}$ with $|w^*_\tau| \leq 2$ by Lemma 2.1

$\Rightarrow |\widetilde{w}_\tau| > 1$ and $sgn(\widetilde{w}_\tau) = sgn(\widetilde{s})$ as $\lambda > 3$

$\Rightarrow \widetilde{\boldsymbol{\pi}}^\tau = ((1-\widetilde{q}), 0,0,0,0,0,\widetilde{q})$

$\Rightarrow v(\tau) = \widetilde{q} > \alpha_4$.

Thus we have $\alpha_4 < v(1) = v(2) = v(3) = v(4) = \cdots$. ■

Proof of Proposition 2.2.2. Firstly, note that *before* the period when the social planner would release the signal according to his releasing strategy, the equilibrium public belief w^*_t still evolves according to Lemma 2.1 and $w^*_t \in \{-2,-1,0,1,2\}$. Thus for the social planner, whether to release the public signal or not depends on just five scenarios:

$$w_t = 2\widetilde{s},\ w_t = \widetilde{s},\ w_t = 0,\ w_t = -\widetilde{s},\ w_t = -2\widetilde{s}.$$

Note also that once the public signal is released or *fully inferred* by the agents, the subgame after that is just the standard case without public information but with an initial public belief $\widehat{w}_t = w_t^* + \lambda \widetilde{s}$, and social welfare is just $p(\widehat{w}_t)$ according to Lemma 2.3 as a herd starts eventually.

Suppose agents believe $g(\widetilde{s}, w_t) = 0$ is the releasing strategy of the social planner:

If $w_t^* = \pm 2\widetilde{s}$

$\Rightarrow$ releasing $\widetilde{s}$ would not break down the herd since $0 < \lambda < 1$

$\Rightarrow$ makes no difference.

If $w_t^* = 0$

$\Rightarrow$ releasing $\widetilde{s}$ makes $\widehat{w}_t = \lambda \widetilde{s}$

$\Rightarrow g(\lambda \widetilde{s}) = \widetilde{q}\alpha_5 + (1-\widetilde{q})\alpha_3 < \alpha_4$ as $\widetilde{q} < q$;

without release, $\widehat{w}_t = w_t^* = 0$

$\Rightarrow g(0) = \alpha_4 \Rightarrow$ not a profitable deviation.

If $w_t^* = \widetilde{s}$

$\Rightarrow$ releasing $\widetilde{s}$ makes $\widehat{w}_t = (\lambda + 1)\widetilde{s}$

$$\Rightarrow p(\widehat{w}_t) = \frac{\widetilde{q}q}{\widetilde{q}q + (1-q)(1-\widetilde{q})}\alpha_7 + \frac{(1-\widetilde{q})(1-q)}{\widetilde{q}q + (1-q)(1-\widetilde{q})}\alpha_1 = \frac{\widetilde{q}q}{\widetilde{q}q + (1-q)(1-\widetilde{q})};$$

without release, $\widehat{w}_t = w_t^* = \widetilde{s}$

$$\Rightarrow p(\widehat{w}_t) = \frac{\widetilde{q}q}{\widetilde{q}q + (1-q)(1-\widetilde{q})}\alpha_6 + \frac{(1-\widetilde{q})(1-q)}{\widetilde{q}q + (1-q)(1-\widetilde{q})}\alpha_2$$

$$> \frac{\widetilde{q}q}{\widetilde{q}q + (1-q)(1-\widetilde{q})} \text{ as } \widetilde{q} < q$$

$\Rightarrow$ not a profitable deviation.

If $w_t^* = -\widetilde{s}$

$\Rightarrow$ releasing $\widetilde{s}$ makes $\widehat{w}_t = (\lambda - 1)\widetilde{s}$

$$\Rightarrow p(\widehat{w}_t) = \frac{\widetilde{q}(1-q)}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_3 + \frac{q(1-\widetilde{q})}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_5;$$

without release, $\widehat{w}_t = w_t^* = -\widetilde{s}$

$$\Rightarrow p(\widehat{w}_t) = \frac{\widetilde{q}(1-q)}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_2 + \frac{q(1-\widetilde{q})}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_6;$$

$$\frac{\widetilde{q}(1-q)}{\widetilde{q}(1-q)+q(1-\widetilde{q})}(\alpha_2-\alpha_3) - \frac{q(1-\widetilde{q})}{\widetilde{q}(1-q)+q(1-\widetilde{q})}(\alpha_5-\alpha_6)$$

$$= \frac{q}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\frac{q}{1-q(1-q)}\frac{(1-q)^2}{1-2q(1-q)}[q^2(1-\widetilde{q})-\widetilde{q}(1-q)^2]$$

$$> 0 \text{ as } \widetilde{q} < \frac{q^2}{q^2+(1-q)^2}$$

$\Rightarrow$ not a profitable deviation.

Therefore $g^1(\widetilde{s}, w_t) = 0$ is indeed an equilibrium strategy of the social planner.

Suppose agents believe $g(\widetilde{s}, w_t) = 1_{\{w_t = \pm 2\widetilde{s}\}}$ is the releasing strategy of the social planner:

If releasing, $\widehat{w}_t = w_t^* + \lambda\widetilde{s}$ and $|\widehat{w}_t| > 1$

$\Rightarrow p(\widehat{w}_t) = sgn(\widehat{w}_t) = sgn(w_t^*)$;

if no release

$\Rightarrow$ by Bayes Rule, $\widehat{w}_t = w_t^* - \lambda\widetilde{s}$ and $|\widehat{w}_t| > 1$

$\Rightarrow p(\widehat{w}_t) = sgn(\widehat{w}_t) = sgn(w_t^*)$

$\Rightarrow$ makes no difference;

if not to release when $w_t^* = \pm 2\widetilde{s}$ but releasing later at $t' > t$

$\Rightarrow$ by Bayes Rule, $\widehat{w}_k = w_t^* - \lambda\widetilde{s}$ and $|\widehat{w}_k| > 1$ for $k = t, t+1, \ldots, t'-1$,

$\widehat{w}_{t'} = \widehat{w}_{t'-1} + 2\lambda\widetilde{s} = w_t^* + \lambda\widetilde{s}$ and $|\widehat{w}_{t'}| > 1$

$\Rightarrow p(\widehat{w}_{t'}) = sgn(\widehat{w}_{t'}) = sgn(w_t^*)$

$\Rightarrow$ makes no difference.

Note that to release earlier at $t'' < t$ when $|w_{t''}| \leq 1$ is also not profitable because the original strategy at t'' is not to release until $w_t = \pm 2\widetilde{s}$ later, which is equivalent to not to release at all since $\lambda < 1$ and has been shown above to give better outcome. Therefore $g^3(\widetilde{s}, w_t) = 1_{\{w_t = \pm 2\widetilde{s}\}}$ is indeed an equilibrium strategy of the social planner.

Suppose agents believe $g(\widetilde{s}, w_t) \equiv 1_{\{w_t=0\}}$ is the releasing strategy of the social planner:

If releasing, $\widehat{w}_t = \lambda\widetilde{s}$

$\Rightarrow p(\widehat{w}_t) = \widetilde{q}\alpha_5 + (1-\widetilde{q})\alpha_3 < \alpha_4$;

if no release at all

$\Rightarrow$ by non-excessive belief, $\widehat{w}_t = 0$ and $p(0) = \alpha_4$;

$\Rightarrow$ it is a profitable deviation.

Therefore $g(\widetilde{s}, w_t) \equiv 1_{\{w_t=0\}}$ is not an equilibrium strategy of the social planner.

Suppose agents believe $g(\widetilde{s}, w_t) = 1_{\{w_t=\widetilde{s}\}}$ is the releasing strategy of the social planner:

If releasing, $\widehat{w}_t = (1+\lambda)\widetilde{s}$

$$\Rightarrow p(\widehat{w}_t) = \frac{\widetilde{q}q}{\widetilde{q}q + (1-q)(1-\widetilde{q})};$$

if no release

$\Rightarrow$ by Bayes Rule, $\widehat{w}_t = (1-\lambda)\widetilde{s}$

$$\Rightarrow p(\widehat{w}_t) = \frac{\widetilde{q}q}{\widetilde{q}q + (1-q)(1-\widetilde{q})}\alpha_5 + \frac{(1-\widetilde{q})(1-q)}{\widetilde{q}q + (1-q)(1-\widetilde{q})}\alpha_3$$

$$> \frac{\widetilde{q}q}{\widetilde{q}q + (1-q)(1-\widetilde{q})} \text{ as } \widetilde{q} < q$$

$\Rightarrow$ it is a profitable deviation.

Therefore $g(\widetilde{s}, w_t) = 1_{\{w_t=\widetilde{s}\}}$ is not an equilibrium strategy of the social planner.

Suppose agents believe $g(\widetilde{s}, w_t) = 1_{\{w_t = -\widetilde{s}\}}$ is the releasing strategy of the social planner:

If releasing, $\widehat{w}_t = (\lambda - 1)\widetilde{s}$

$\Rightarrow p(\widehat{w}_t) = \frac{\widetilde{q}(1-q)}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_3 + \frac{q(1-\widetilde{q})}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_5;$

if no release

$\Rightarrow$ by Bayes Rule, $\widehat{w}_t = -(1+\lambda)\widetilde{s}$

$\Rightarrow p(\widehat{w}_t) = \frac{\widetilde{q}(1-q)}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_1 + \frac{q(1-\widetilde{q})}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_7$

$= \frac{q(1-\widetilde{q})}{\widetilde{q}(1-q)+q(1-\widetilde{q})} < \frac{\widetilde{q}(1-q)}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_3 + \frac{q(1-\widetilde{q})}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_5$ as $\widetilde{q} < q$

$\Rightarrow$ not a profitable deviation;

if not to release when $w_t = -\widetilde{s}$ but releasing later at $t' > t$

$\Rightarrow$ by Bayes Rule, $\widehat{w}_k = -(1+\lambda)\widetilde{s}$ for $k = t, t+1, \ldots, t'-1$ and

$\widehat{w}_{t'} = \widehat{w}_{t'-1} + 2\lambda\widetilde{s} = (\lambda - 1)\widetilde{s}$

$\Rightarrow p(\widehat{w}_{t'}) = \frac{\widetilde{q}(1-q)}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_3 + \frac{q(1-\widetilde{q})}{\widetilde{q}(1-q)+q(1-\widetilde{q})}\alpha_5 \Rightarrow$ makes no difference;

if instead releasing earlier at $t'' < t$ when $w_{t''} = 0$

$\Rightarrow \widehat{w}_{t''} = \lambda\widetilde{s}$ and $p(\widehat{w}_{t''}) = \widetilde{q}\alpha_5 + (1-\widetilde{q})\alpha_3;$

at t'' the original strategy is not to release until $w_t = -\widetilde{s}$ later

$\Rightarrow$ agents can perfectly infer $\widetilde{s}$ at $t''+1$ since $w_{t''+1} = \pm 1$

$\Rightarrow \widehat{w}_{t''+1} = w_{t''+1} + \lambda\widetilde{s}$ and

$p(\widehat{w}_{t''+1}) = \widetilde{q}[q\alpha_7 + (1-q)\alpha_3] + (1-\widetilde{q})[q\alpha_5 + (1-q)\alpha_1] = \widetilde{q}\alpha_5 + (1-\widetilde{q})\alpha_3$

$\Rightarrow$ makes no difference;

if instead releasing earlier at $t''' < t$ when $w_{t'''} = \widetilde{s}$

$\Rightarrow \widehat{w}_{t'''} = (1+\lambda)\widetilde{s}$ and $p(\widehat{w}_{t'''}) = \frac{\widetilde{q}q}{\widetilde{q}q + (1-q)(1-\widetilde{q})};$

at t''' the original strategy is not to release but agents can infer $\widetilde{s} = w_{t'''}$

$\Rightarrow$ makes no difference.

Therefore $g(\widetilde{s}, w_t) = 1_{\{w_t = -\widetilde{s}\}}$ is indeed an equilibrium strategy of the social planner. ■

Proof of Claim 2.3.1. We start by calculating the discounted sum of people's expected payoffs without any public information, $\overline{V}$. Without loss of generality, we assume $\theta = 1$ and use $EU_t \equiv Eu(a_t^*; \theta = 1)$ to simplify the notation. Then $\overline{V} = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} EU_t$.

Recursive calculation similar to the proof of Proposition 2.1 yields

$$
\begin{aligned}
EU_1 &= EU_2 = q, \\
EU_3 &= EU_4 = q^2 + 2q(1-q)EU_1, \\
&\dots \\
EU_{2k+1} &= EU_{2k+2} = q^2 + 2q(1-q)EU_{2k-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
EU_{2k+1} &= [2q(1-q)]^k [q - \frac{q^2}{(1-q)^2 + q^2}] + \frac{q^2}{(1-q)^2 + q^2}; \\
\overline{V} &= (1-\delta) \sum_{k=0}^{\infty} (\delta^{2k} + \delta^{2k+1}) EU_{2k+1} = \frac{\delta^2 q^2 + (1-\delta^2) q}{1 - \delta^2 2q(1-q)} > q.
\end{aligned}
$$

If the public signal is released at period 1,

$$
\begin{aligned}
EU_1 &= q, \\
EU_2 &= EU_3 = q^2 + 2q(1-q)EU_1, \\
&\dots \\
EU_{2k} &= EU_{2k+1} = q^2 + 2q(1-q)EU_{2k-1}.
\end{aligned}
$$

Compared to the case without public information, we have

$$
V(1) = \frac{\overline{V} - (1-\delta)q}{\delta} > \overline{V} \text{ as } \overline{V} > q.
$$

Clearly there is no difference between releasing at period 1 and at period 2, so $V(2) = V(1) > \overline{V}$.

If the public signal is released at period 3,

$EU_1 = EU_2 = q$;

with probability q^3, a correct herd starts after the release;

with probability $(1-q)^3$, a wrong herd starts after the release;

with probability $3q(1-q)$, it is as if only the public signal is released for agents $t \geq 3$.

Hence

$$\begin{aligned} V(3) &= (1-\delta)(q+\delta q) + (1-\delta)q^3 \frac{\delta^2}{1-\delta} + 3q(1-q)\delta^2 V(1) \\ &= (1-\delta^2)q + q^3\delta^2 + 3q(1-q)\delta^2 V(1). \end{aligned}$$

If the public signal is released at period 4,

$EU_1 = EU_2 = q$; $EU_3 = q^2 + 2q(1-q)EU_1$;

with prob. q^3, a correct herd starts after the release;

with prob. $(1-q)^3$, a wrong herd starts after the release;

with prob. $q(1-q)$, it is as if only the public signal is released for agents $t \geq 4$.

with prob. $2q(1-q)$, the public signal is as if released after one action for agents $t \geq 4$.

Hence

$$\begin{aligned} V(4) &= (1-\delta)(q+\delta q+\delta^2 q^2) + (1-\delta)q^3 \frac{\delta^3}{1-\delta} + q(1-q)\delta^3 V(1) + 2q(1-q)\delta^2 V(2) \\ &= (1-\delta^2)q + (1-\delta)\delta^2 q^2 + q^3\delta^3 + 2q(1-q)(\delta^2 + \frac{\delta^3}{2})V(1). \end{aligned}$$

Therefore,

$$\begin{aligned} V(3) - V(4) &= q^3\delta^2(1-\delta) - (1-\delta)\delta^2 q^2 + q(1-q)\delta^2(1-\delta)V(1) \\ &= q(1-q)\delta^2(1-\delta)[V(1)-q] > 0 \text{ as } V(1) > \overline{V} > q. \end{aligned}$$

■

Proof of Claim 2.3.2. Note that there would still be a herd eventually due to *bounded* private beliefs, so the ex-ante average payoff of the population is again the probability of a correct herd eventually.

Let us first calculate $\overline{G}$, the probability of a correct herd eventually without any public information. Using similar recursive arguments as in the proof of Proposition 2.1 but with more tedious algebra, we have

$$\overline{G} = 6q(\frac{1-q}{2})^2\overline{G} + q^2 + 4q\frac{1-q}{2}q\frac{1+q}{2} + 4q^2(\frac{1-q}{2})^2(A+B),$$

where A and B are given by

$$\begin{cases} A = q\frac{1+q}{2} + q\frac{1-q}{2}A + (1-q)qB \\ B = (\frac{1+q}{2})^2 + \frac{1+q}{2}\frac{1-q}{2}A + \frac{1-q}{2}qB \end{cases}.$$

A and B are in fact the probabilities of a correct herd conditional on the event that a trap has started corresponding to two different tie-breaking situations.

It is easy to see that $G(1) = G(2) = \overline{G}$, because without the public signal the first two agents always follow their own private signals and releasing $\tilde{s}$ is just "adding" an agent who always follow her signal, which does not affect social welfare from an ex-ante perspective. By exploring all possible situations of the first two actions, the value of $G(3)$ can be calculated as follows:

$$\begin{aligned} G(3) &= q^2[q + 2\frac{1-q}{2}(\frac{1+q}{2} + \frac{1-q}{2}A)] + 2(\frac{1-q}{2})^2q^2B \\ &+ 2(\frac{1-q}{2})^2q\overline{G} + 4q\frac{1-q}{2}[q(\frac{1+q}{2} + \frac{1-q}{2}A) + \frac{1-q}{2}\overline{G} + \frac{1-q}{2}qB] \\ &\Rightarrow G(3) - \overline{G} = q^2(1-q)[\frac{1+q}{2} + \frac{1-q}{2}(A+B) - 1] > 0 \text{ as } A+B > 1. \end{aligned}$$

For $G(4)$ and $G(5)$, if the first three actions cancel each other then it is as if the public signal were released three periods earlier; otherwise either a trap or a herd starts and calculations similar to those above can be applied. Indeed we have

$$\begin{aligned} G(4) = & 6q(\frac{1-q}{2})^2 G(1) + 2q^2(\frac{1-q}{2})^2 B + q^2[q + 2\frac{1-q}{2}(\frac{1+q}{2} + \frac{1-q}{2}A)] \\ & + 4q^3\frac{1-q}{2} + 4q^2(\frac{1-q}{2})^2 A \\ & + 4q^2(\frac{1-q}{2})^2[q + 2\frac{1-q}{2}(\frac{1-q}{2}\overline{G} + \frac{1-q}{2}qB + q\frac{1+q}{2} + q\frac{1-q}{2}A)] \\ & + 4(\frac{1-q}{2})^3 q[\frac{1-q}{2}q\overline{G} + q(\frac{1-q}{2}\overline{G} + \frac{1-q}{2}qB + q\frac{1+q}{2} + q\frac{1-q}{2}A)] \\ & + 4(\frac{1-q}{2})^2 q\{qB + \frac{1-q}{2}[\frac{1-q}{2}q\overline{G} + q(\frac{1-q}{2}\overline{G} + \frac{1-q}{2}qB + q\frac{1+q}{2} + q\frac{1-q}{2}A)]\}; \end{aligned}$$

$$\begin{aligned} G(5) = & 6q(\frac{1-q}{2})^2 G(2) + 2q^2(\frac{1-q}{2})^2 B + q^2[q + 2\frac{1-q}{2}(\frac{1+q}{2} + \frac{1-q}{2}A)] \\ & + 4(\frac{1-q}{2})^3\frac{1-q}{2}q^2\overline{G} \\ & + 4(\frac{1-q}{2})q^2\frac{1-q}{2}(q\frac{1+q}{2} + q\frac{1-q}{2}A + q\frac{1-q}{2}B + \frac{1-q}{2}C) \\ & + 4(\frac{1-q}{2})q^2\frac{1+q}{2}[q + \frac{1-q}{2} + \frac{1-q}{2}(\frac{1-q}{2}A + \frac{1-q}{2})] \\ & + 4(\frac{1-q}{2})^2 q[q(q\frac{1+q}{2} + q\frac{1-q}{2}A + \frac{1-q}{2}qB + \frac{1-q}{2}C) + (1-q)q^2 B], \end{aligned}$$

where $C \equiv \frac{1-q}{2}\overline{G} + q\frac{1+q}{2} + q\frac{1-q}{2}A + \frac{1-q}{2}qB$.

It can be verified that

$$sgn(G(4) - G(5)) = sgn(q^2(A + B - 1) + 3\frac{1-q}{2}(1 - \overline{G}) - \frac{1+q}{2}).$$

When $q = 1$ or $\frac{1}{3}$,

$$\begin{aligned} & A = B = \overline{G} = 1 \text{ or } \frac{1}{3} \\ & \Rightarrow q^2(A + B - 1) + 3\frac{1-q}{2}(1 - \overline{G}) - \frac{1+q}{2} = 0; \end{aligned}$$

Meanwhile, $q^2(A + B - 1) + 3\frac{1-q}{2}(1 - \overline{G}) - \frac{1+q}{2}$ is in fact convex in q on $(\frac{1}{3}, 1)$, therefore

$$q^2(A + B - 1) + 3\frac{1-q}{2}(1 - \overline{G}) - \frac{1+q}{2} > 0 \text{ and } G(4) > G(5).$$

■