IAMs and CO_2 Emissions – An Analytic Discussion

Abstract: The paper analyzes the core drivers of CO_2 emissions in a generic integrated assessment model of climate change (IAM). I use the general framework to compare emissions in DICE, Golosov et al. (2014), and Traeger (2021). Conditional on the SCC, DICE's emissions have a closed-form solution that is crucially driven by the backstop price of a fossil-substitute. Golosov et al. (2014) have a more detailed representation of the energy sector, which exhibits non-satiation in fossil fuels as a result of a CES aggregator. Traeger (2021) combines features of both models and permits a more detailed analysis of technological progress and changing energy substitutabilities.

By the time of the SURED conference, I expect the paper to additionally contain a detailed quantitative section.

JEL Codes: Q54, H23, E13, D62

Keywords: climate change, integrated assessment, emissions, social cost of carbon, carbon tax, Hotelling rent, technological progress, energy, substitutability, CES

1 Introduction

Integrated assessment models (IAMs) of climate change analyze the long-term interactions of economic production, greenhouse gas (GHG) emissions, and global warming. Given their complexity, IAMs are often considered "black boxes" by outsiders. The present paper analyzes the drivers of CO_2 emissions, their response to carbon taxes, and their dependence on technological progress and substitutabilities across energy supplies in a general analytic framework.

Analytic approaches to the integrated assessment of climate change date back to at least Heal's (1984) insightful non-quantitative contribution. Several papers have used the linear quadratic model for a quantitative analytic discussion of climate policy (Hoel & Karp 2002, Newell & Pizer 2003, Karp & Zhang 2006, Karp & Zhang 2012, Valentini & Vitale 2019, Karydas & Xepapadeas 2019, Karp & Traeger 2021). A disadvantage of these linear quadratic approaches is their highly stylized representation of the economy and the climate system. In particular, these models have no production or energy sector. Golosov et al. (2014) broke new ground by amending the log-utility and full-depreciation version of Brock & Mirman's (1972) stochastic growth model with an energy sector and an impulse response of production to emissions. Golosov et al.'s (2014) framework has sparked a growing literature on Analytic Integrated Assessment Models (AIAMs), including applications to a multi-regional setting (Hassler & Krusell 2012, Hassler et al. 2018, Hambel et al. 2018), non-constant discounting (Gerlagh & Liski 2018*b*, Iverson & Karp 2020), intergenerational games (Karp 2017), and regime shifts (Gerlagh & Liski 2018*a*). Traeger (2021) merges analytic IAMs with a full complexity climate system and generalizes the representation of economic production and Traeger (2018) integrates uncertainty into the framework.[1]

The analytic discussion in these papers focuses almost entirely on the optimal price of CO_2 emissions. The actual emission trajectories are either numerically simulated or follow very simple rules. In contrast, the present paper

[1]Anderson et al. (2014), Brock & Xepapadeas (2017), Dietz & Venmans (2018), and van der Ploeg (2018) spearhead the use of a simplified and yet descriptively convincing climate model (known as the TCRE model).

discusses the resulting emissions analytically for the complex production and energy sectors of Nordhaus's (2017) DICE model, Golosov et al. (2014), and a more general production system based on Traeger's (2018) ACE model. It derives closed-form expressions for DICE's abatement rate and the energy-specific fossil-based CO_2 emissions in Golosov et al. (2014) and ACE. The paper compares the structural response of these models to a carbon tax analyzing their respective production elasticities to fossil inputs, both within a period and over time. Finally, it incorporates the (potential) scarcity of fossil fuels and the endogenous response of Hotelling's intertemporal scarcity rents to climate policy. The background for the present analysis is a generic IAM, able to generate somewhat arbitrary trajectories for the optimal carbon tax. Alongside, I specify a particular specification based on Traeger (2021), which also generates the SCC in closed-form.

The present paper's focus is on analytic insight and tractability. This focus complements a large number of indispensable complex IAMs with detailed energy models studying the transition of energy supply and fossil fuel use numerically. WITCH (Bosetti et al. 2006, Emmerling et al. 2016) uses a large variety of energy inputs including renewables such as biomass and biofuels and backstop technologies including carbon capture and sequestration. REMIND (Luderer et al. 2021) first splits the economy into sectors (including detailed transportation modes) before specifying a variety of sector-specific energy inputs (with elasticities of substitution larger than unity). Related structures are shared by MERGE (Manne et al. 1995), EPPA (Paltsev et al. 2005), and MIND (Edenhofer et al. 2005), among others. The paper also complements contributions on technical change including the papers by Acemoglu et al. (2019) on directed technical change, Bretschger et al. (2019) with a focus on endogenous growth, and Hassler et al. (2019) who integrate a form of directed technical change in analytic IAMs.

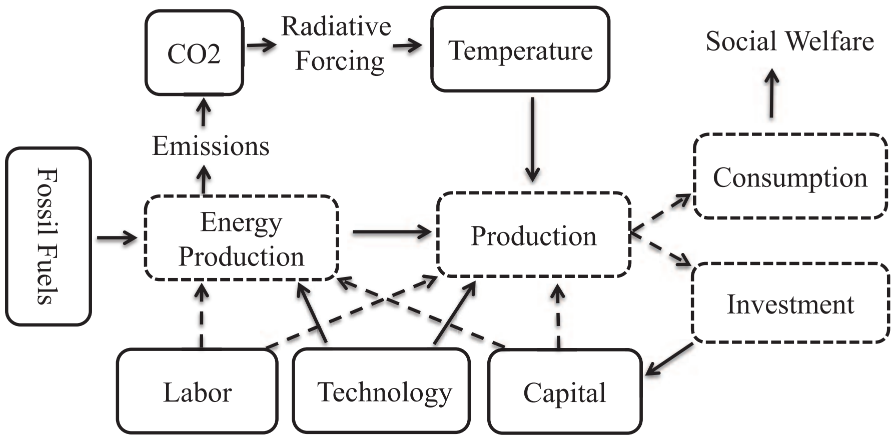

Figure 1: The structure of ACE and most IAMs. Solid boxes characterize the model's state variables, dashed boxes are flows, and dashed arrows mark choice variables.

2 The Generic IAM

The IAM's structure follows (and generalizes) that of most IAMs, see Figure 1. Labor, capital, technology, and energy produce output that is either consumed or invested. "Dirty" energy sectors consume fossil fuels and cause emissions, which accumulate in the atmosphere, cause radiative forcing (greenhouse effect), and increase global temperature(s), thus reducing output. This section introduces the basic model of the economy and the climate system. I simultaneously introduce a "fully general" model, and a version with constraints that permits an analytic solution of the optimal carbon tax. The constraints of this "analytic model" are summarized in the starred versions of the equations below.

Production and energy sectors. Final output Y_t is a function of (vectors specifying) exogenous technologies $\boldsymbol{A}_t$, the optimally allocated labor and capital distributions $\boldsymbol{N}_t$ and $\boldsymbol{K}_t$, a flow of potentially scarce resource inputs $\boldsymbol{E}_t$, and it is reduced by damages resulting from the current atmospheric temperature increase $T_{1,t}$ over 1900 levels[2]

$$Y_t^{net} = \underbrace{F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t)}_{\equiv Y_t}[1 - D(T_{1,T})] \tag{1}$$

If the production function is homogenous in capital

$$\text{Equation (1) \& } F(\boldsymbol{A}_t, \boldsymbol{N}_t, \gamma\boldsymbol{K}_t, \boldsymbol{E}_t) = \gamma^\kappa F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t)\ \forall \gamma \in \mathbb{R}_+. \tag{1*}$$

the model permits an analytic solution of the SCC. The subsequent section discusses several concrete realizations of this generic economic structure. I assume a generic capital stock that accumulates and depreciates as

$$K_{t+1} = (1-\delta)K_t + Y_t^{net} - C_t. \tag{2}$$

At the beginning of each period, the capital stock is optimally distributed across sectors.[3] The model permits a closed-form solution for the optimal carbon tax if we approximate the growth rate of capital, $g_{k,t}$, exogenously and use the multiplicative formulation

$$K_{t+1} = (Y_t - C_t)\left[\frac{1+g_{k,t}}{\delta + g_{k,t}}\right]. \tag{2*}$$

Equation (2*) is equivalent to equation (2) if the capital growth rate, which is usually calibrated to exogenous observation (also when using equation **??**), is correct.

Emissions and resources. The first I^d resources $E_1, ..., E_{I^d}$ are fossil fuels and emit CO_2; I collect them in the subvector $\boldsymbol{E}_t^d$ ("dirty"). I measure these fossil fuels in terms of their carbon content and total emissions from production amount to $\sum_{i=1}^{I^d} E_{i,t}$. In addition, land conversion, forestry, and agriculture emit smaller quantities of CO_2. Following DICE and ACE, I treat

[2] I use a multiplicative damage formulation merely because it is assumed in DICE, Golosov et al. (2014), and ACE. The proof of Proposition 1 explicitly shows that the assumption is not crucial.

[3] This putty-putty-structure can be extended to putty-clay structure, including a variety of capital goods and a distinction between investment and consumption sectors. Yet, for the main results of the paper such extensions are secondary.

these additional anthropogenic emissions as exogenous and denote them by E_t^{exo}.

Renewable energy production relies on the inputs indexed by I^{d+1} to I_E such as water, wind, or sunlight, which I assume to be abundant. By contrast, fossil fuel use reduces the resource stock in the ground $\boldsymbol{R_t} \in \mathbb{R}_+^{I^d}$:

$$\boldsymbol{R}_{t+1} = \boldsymbol{R}_t - \boldsymbol{E}_t^d, \tag{4}$$

with initial stock levels $\boldsymbol{R}_0 \in \mathbb{R}_+^{I^d}$ given. Possible extraction costs are part of the general production function. I take the following assumption to avoid boundary value complications; if a resource is scarce along the optimal path, its use is stretched over the infinite time horizon.[4]

Damages. The next section explains how the carbon emissions increase the *global atmospheric temperature* $T_{1,t}$ measured as the increase over the preindustrial temperature level. This temperature increase causes damages, which destroy a fraction $D_t(T_{1,t})$ of output. *Damages at the preindustrial temperature level are* $D_t(0) = 0$. In general, this damage function can take an almost arbitrary form, say

$D(T_{1,t})$ is weakly increasing and convex.

The optimal carbon tax solved in closed form if this damages function is of the form

$$D(T_{1,t}) = 1 - \exp(-\xi_0 \exp[\xi_1 T_{1,t}] + \xi_0) \tag{5*}$$

with $\xi_0 \in \mathbb{R}$ and $\xi_1 = \frac{\log 2}{s} \approx \frac{1}{4}$, where $s \approx 3$ is the climate sensitivity parameter (Traeger 2018). The cited paper calibrates ξ_0 for different estimates in the literature including DICE, Howard & Sterner (2017), and Pindyck (2020), and extends it to a stochastic process.

Climate. Carbon dioxide emissions accumulate in the atmosphere. Let $M_{1,t}$ denote the atmospheric carbon content and let $M_{2,t}, ..., M_{m,t}$, $m \in \mathbb{N}$,

[4] A sufficient but not necessary condition is that the scarce resources are essential in production, i.e., that production is not possible without the input of the scarce resource.

denote the carbon content of a finite number of non-atmospheric carbon reservoirs. These carbon stocks evolve as

$$\boldsymbol{M}_{t+1} = Z_M\big(\boldsymbol{M}_t, \textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big) \tag{5}$$

Most IAMs implement the function Z_M by a linear carbon cycle equation,

$$\boldsymbol{M}_{t+1} = \boldsymbol{\Phi}\boldsymbol{M}_t + \boldsymbol{e}_1(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}), \tag{6*}$$

where $\boldsymbol{\Phi}$ captures the transfer coefficients of carbon across different reservoirs and $\boldsymbol{e}_1$ is the first unit vector channeling new emissions into the atmosphere, or as a linear impulse response model for atmospheric carbon. Both of these particular specifications permit a closed-form solution of the optimal carbon tax (Traeger 2018).

Atmospheric carbon causes a greenhouse effect, usually characterized by radiative forcing. Radiative forcing essentially characterizes the warming and is logarithmic in atmospheric carbon. Temperatures respond with some delay to this warming. Let the vector $\boldsymbol{T}_t$ characterize atmospheric temperature ($T_{t,1}$) and the temperature of a finite number of ocean layers, which keep cooling our planet for some time. In general, I assume

$$\boldsymbol{T}_t = Z_T(\boldsymbol{T}_{i,t}, M_{1,t}, G_t). \tag{6}$$

where G_t denotes the radiative forcing (greenhouse effect) from other greenhouse gas that I treat as exogenous. Many IAMs implement the function Z_T as logarithmic in $M_{1,t}$ and linear in temperatures and G_t. Traeger (2021) shows that the following specification permits a closed-form solution of the optimal carbon tax

$$\begin{aligned} &T_{i,t+1} = \mathfrak{M}_i^{\boldsymbol{\sigma}}(T_{i,t}, T_{i-1,t}, T_{i+1,t}) \text{ for } i \in \{1, ..., l\}, \text{ where } T_{l+1,t} = 0, \\ &T_{0,t} = \log\left(\frac{M_{1,t}}{M_{pre}}\right) + G_t \end{aligned} \tag{7*}$$

specifies radiative forcing and $\mathfrak{M}_i^{\boldsymbol{\sigma}}$ is a quasi-arithmetic mean with weighting function $f(\cdot) = \exp(\xi_1 \cdot)$ and weight matrix $\boldsymbol{\sigma}$.[5] M_{pre} denotes the preindustrial

[5] A quasi-arithmetic mean takes the form $\mathfrak{M}_i^{\boldsymbol{\sigma}}(T_{i-1,t}\ ,\ T_{i,t}\ ,\ T_{i+1,t}) =$

atmospheric carbon concentration and, in this specification, any doubling of atmospheric carbon results in a medium to long-term temperature increase equaling the climate sensitivity s.

Objective and Bellman Equation. A social planner optimizes consumption level and energy inputs and she distributes labor and capital optimally across sectors. The planner has a per-period welfare objective $u(C_t)$, an infinite time horizon and discounts next period welfare with factor $\beta < 1$. Formally, she solves the dynamic programming problem

$$V_t(K_t, \boldsymbol{T}_t, \boldsymbol{M}_t, \boldsymbol{R}_t) = \max_{\boldsymbol{C}_t, \boldsymbol{N_t}, \boldsymbol{K}_t, \boldsymbol{E_t}} u(\boldsymbol{C}_t) + \beta V_{t+1}(K_{t+1}, \boldsymbol{T}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}) \quad (7)$$

subject to equations (1-6).

Social Cost of Carbon (SCC). The SCC is the intertemporally aggregated cost from emitting an additional unit of carbon dioxide. The optimal carbon tax is the SCC along the optimal trajectory, which is

$$SCC_t = -\frac{\beta \frac{\partial V_{t+1}(K_{t+1}, \boldsymbol{T}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1})}{\partial M_{t+1}}}{u'(C_t)}$$

Traeger (2021) shows that, using the starred versions of equations (1-6) and $u(\boldsymbol{C}_t) = \log(C_t)$, the optimal carbon tax has the closed-form solution[6]

$$SCC_t = \frac{\beta^2 Y_t^{net}}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1} \quad (8)$$

where σ^{forc} is the temperature equation's weight on radiative forcing and $[\cdot]_{1,1}$ denote the first ("atmospheric") element of the inverted matrices. The optimal

---

$f^{-1}[\sigma_{i,i-1} f(T_{i-1,t}) + \sigma_{i,i} f(T_{i,t}) + \sigma_{i,i+1} f(T_{i+1,t})]$. The weights have to sum to unity so that $\sigma^{forc} \equiv \sigma_{1,0} = 1 - \sigma_{1,1} - \sigma_{1,2}$ and $\sigma_{i,i} = 1 - \sigma_{i,i-1} - \sigma_{i,i+1}$ for $i > 1$, where $\sigma_{l,l+1} = 0$. We obtain the standard arithmetic mean for linear f and for $f(\cdot) = \exp(\xi_1 \cdot)$ the equation of motion is $T_{i+1,t} = \frac{1}{\xi_1} \log\Big((1 - \sigma_{i,i-1} - \sigma_{i,i+1}) \exp[\xi_1 T_{i,t}] + \sigma_{i,i-1} \exp[\xi_1 T_{i-1,t}] + \sigma_{i,i+1} \exp[\xi_1 T_{i+1,t}]\Big)$.

[6] Traeger (2021) defines the SCC as the cost of an instantaneous increase of the atmospheric carbon stock, i.e., $SCC_t^* = \frac{\partial V_t(K_t, \boldsymbol{T}_t, \boldsymbol{M}_t, \boldsymbol{R}_t)}{\partial M_t} / u'(C_t)$, which results in "$\beta$" rather than "$\beta^2$" in equation (8). As a result, Traeger (2021) has an additional β in the first order conditions, which is absent in the present paper. The definition given here is motivated by the model's timing that current emissions only enter the atmosphere in the next period, and it is more convenient in the general framework; both definition coincide in continuous time.

carbon tax increases in damages, the forcing response of temperature, and the persistence of temperature change and atmospheric carbon (and falls in warming delay), see Traeger (2021) for a detailed discussion. It turns out helfpul to define as well the social cost per unit of output

$$\widetilde{SCC}_t = \frac{SCC_t}{Y_t^{net}}.$$

In the analytic version of the model, equation (8) shows that $\widetilde{SCC}_t$ is independent of the endogenous output.

Remark 1: Generalizations of the analytic model. Traeger (2021) generalizes the analytic solutions (starred equations) to incorporate inter- as well as (some forms of) intragenerational population-weighting. The paper also shows how to extend the present model to the case where consumers have preferences over a variety of goods (log-CES) and the investment composite is not a perfect substitute to the consumption bundle. These extensions imply slight modifications to equation (8) but do not otherwise affect the results discussed in the present paper.

Remark 2: Uncertainty. The results presented here extend to settings with climate and damage uncertainty. Such scenarios require additional informational variables $\boldsymbol{I}_t$ to capture persistent shocks or learning and the Bellman equation generalizes to

$$V(K_t, \boldsymbol{T}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, \boldsymbol{I}_t, t) =$$
$$\max_{\boldsymbol{c}_t, \boldsymbol{N_t}, \boldsymbol{K}_t, \boldsymbol{E_t}} u(\boldsymbol{c}_t) + \beta f_t^{-1} \left(\mathbb{E}\, f_t \left[V(K_{t+1}, \boldsymbol{T}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, \boldsymbol{I}_t, t{+}1) \right] \right) ,$$

where the nonlinear uncertainty aggregation (quasi-arithmetic mean with weighting function f_t) permits for preferences that disentangle risk attitude from the desire to smooth consumption, e.g., Epstein-Zin-Weil preferences (Epstein & Zin 1989, Weil 1990, Traeger 2019). Traeger (2018) presents such a model with a closed-form solution for the optimal carbon tax extending equation (8).

3 Specifications of Production

3.1 Prevailing Specifications: Simple, DICE, Golosov et al. (2014)

Cobb-Douglas. A possible starting point for an energy enriched production system uses a Cobb-Douglas production function adding an energy component

$$Y_t = F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t) = A_t K_t^{\kappa} N_t^{\eta} \; G\left(\boldsymbol{A}_t^E, \boldsymbol{N}_t^E, \boldsymbol{E}_t\right). \tag{9}$$

I present three such examples below, "simple", DICE, and Golosov et al. (2014) before generalizing the production function to a CES-form. In each approach, the energy sector(s) $G\left(\boldsymbol{A}_t^E, \boldsymbol{N}_t^E, \boldsymbol{E}_t\right)$ take different functional forms and my notation drops the potential arguments of $G(\cdot)$ not used in the particular approach.

Simple. A simple example specifies energy use in the form

$$G\left(E_t\right) = E_t^{\nu} \tag{10}$$

with $\kappa + \eta + \nu = 1$. Here E_t denotes the aggregate fossil-based energy input into production. Specifying production fully in Cobb-Douglas form prevents the full elimination of fossil fuels from production. However, we can reduce fossil-based energy increasing the production share of capital and labor.

DICE. The arguably most famous integrated assessment model DICE does not model energy use explicitly. Nordhaus formulates DICE in terms of abatement costs that are a function of the abatement rate. Let E_t^{BAU} denote business as usual (BAU) emissions, i.e., emissions in the absence of climate policy. The DICE model's control variable is the abatement rate

$$\mu_t = \frac{E_t^{BAU} - E_t}{E_t^{BAU}} = 1 - \frac{E_t}{E_t^{BAU}},$$

which characterizes avoided emissions as a fraction of BAU emissions. The abatement rate is zero if no mitigation takes place and unity if all industrial CO_2 emissions are abated. DICE characterizes production, abatement costs, and BAU emissions with half a dozen equations including inductively defined exogenous trends. Under a simplifying assumption that BAU emissions are fully rather than partly exogenous (see Appendix B.1 for details), these equations fit the functional form (9) with an energy sector

$$G\left(A_t^E, E_t^{BAU}, E_t\right) = 1 - \tfrac{1}{A_t^E} f\left(\tfrac{E_t}{E_t^{BAU}}\right), \tag{11}$$

where I continue to measure fossil-based energy use in CO_2 equivalents. The function f decreases (convexly) in emissions, implying that current output increases (concavely) in emissions. As green technological progress A_t^E increases exogenously over time, the corresponding factor reduces the economy's fossil fuel dependence.

Golosov et al. (2014). The authors introduce green energy explicitly and split fossil-based energy into its two main sources, oil and coal. They assume that oil E_1 is extracted and converted into energy for free, but oil supply is scarce and follows a one-dimensional version of equation (3). In contrast, coal E_2 is not scarce and its use and emissions are proportional to the corresponding labor input and the exogenous technology level in the coal industry. Like coal, green energy E_3 is proportional to the corresponding labor input and the exogenous technology level in the renewable energy sector. Their model combines equation (9) with the energy sector

$$G\left(E_1, A_2N_2, A_3N_3\right) = \left(a_1 E_1^s + a_2(A_2N_2)^s + a_3(A_3N_3)^s\right)^{\frac{\nu}{s}} \tag{12}$$

$$\text{and } E_2 = A_2N_2 \quad \text{(required for emissions)}.$$

The parameters $a_1, a_2, a_3 \in \mathbb{R}_+$ balance energy and carbon content of the different energy sources. This energy formulation has been criticized for the lack of capital in energy production, a production factor of major empirical importance.

3.2 ACE-based Generalizations

Capital in Energy Production and CES Aggregator of Final Consumption (CapCES). Traeger (2021) suggests a more general production example, deviating from the Cobb-Douglas specification in equation (9). Here,

$I_N \in \mathbb{N}$ energy sectors depend on capital, labor, and explicit fossil fuel or renewable input $E_{i,t}$,

$$e_{i,t} = g_{i,t}(A_{i,t}, K_{i,t}, N_{i,t}, E_{i,t}) \tag{13}$$
$$\text{satisfying } \; g_{i,t}(A_{i,t}, \gamma K_{i,t}, N_{i,t}, E_{i,t}) = \gamma^{\tilde{\alpha}} g_{i,t}(A_{i,t}, K_{i,t}, N_{i,t}, E_{i,t}),$$

where some fossil fuels will be subject to the intertemporal scarcity constraint (3). The general function g accomodates Cobb-Douglas as well as formulations with a "bliss point", i.e., a maximal level of energy input beyond which more coal or more oil no longer increase production – a feature satisfied in DICE's implicit energy sector formulation. The different energy sources are either used directly ($d_{l,t} = e_{i,t}$) in a set of $I_c \in \mathbb{N}$ different final good production processes

$$Y_t = \Big(\sum_l a_{l,t} c_{l,t}^{s_t} \Big)^{\frac{\bar{s}}{s_t}} \text{ with } c_{l,t} = A_{l,t} K_{l,t}^{\alpha} N_{l,t}^{1-\alpha-\nu} d_{l,t}^{\nu} \tag{14}$$

where $\sum_l a_{l,t} = 1$, or they are first combined defining intermediate goods

$$d_{l,t} = \Big(\sum_i e_{i,t}^{\tilde{s}_{l,t}} \Big)^{\frac{1}{\tilde{s}_{l,t}}}, \tag{15}$$

which are limited substitutability combination of the different energy sources. I left out CES-weights in the energy intermediates to compactify notation.[7]

Example. The two main sources of global (and US) CO_2 emissions are electricity & heating and transport. The present example connects these two sectors to the use of oil versus coal. Let the final goods be transport $c_{trans,t}$ and other consumption $c_{other,t}$, i.e., $l \in \{trans, other\}$. Transport uses the intermediate $d_{trans,t}$ ("propulsion"), which relies on energy $e_{oil,t}$ from oil and electricity from renewable energy $e_{renew,t}$. Other consumption relies on the intermediate $d_{other,t}$ (electricity & heat) that relies on energy $e_{coal,t}$ from coal and renewable energy $e_{renew,t}$. Given electricity is highly substitutable across

[7]Possible CES-weights can be absorbed into the general production functions $g_{i,t}$. For a time-changing degree of substitutability, constant CES-weights will imply time changing adjustments of the functions $g_{i,t}$. Appendix B.3 solves for the relevant expressions including CES-weights.

sources, the substitutability index $\tilde{s}_{other,t}$ between coal-based and renewable energy is high (close to unity). In contrast, oil-based and renewable energy in the transport sector exhibit a much lower substitutability $\tilde{s}_{trans,t}$. This substitutability $\tilde{s}_{trans,t}$ is expected to increase over time reflecting (exogenous) technological progress that improves the substitutability of oil-based propulsion by renewable electricity.[8]

Non-constant Elasticity of Fuel Substitution (N-CES). Empirically, most estimates find that coal and electricity are far less substitutable than oil and electricity.[9] Yet, technically, it is easier to substitute most of our coal usage by renewable electricity than to substitute our oil use in the transport sector by electricity. The main reason for the low substitutability in the first case is the current cost differential, whereas the second is partly technical infeasibility. We cannot suddenly create the amount of battery power to guarantee mobility and transport around the world. And we currently do not have the technology to electrify marine and air traffic.[10] Thus, at the margin, oil and electricity seem to be more substitutable, but only at the margin. If we impose sufficiently high carbon taxes, the substitutability ranking of coal versus oil is likely to flip. Given that we commonly employ IAMs to perdict emissions and fuel substitutions for major changes in relative prices, I suggest the following generalization of the ubiquitously applied CES-aggregator. In the case of interfuel substitutability, the extension replaces the constants $\tilde{s}_{l,t}$ by functions

[8]At the expense of brevity, one could easily introduce electricity as another layer between the basic energy inputs and the intermediates, entering both intermediates.

[9]Stern (2012) provide a meta-analysis of 47 studies of interfuel substitutability. The study finds that coal-electricity substitution is limited (elasticity below unity) whereas the elasticity of substitution between oil and electricity is above unity. These findings summarize the authors conclusions, who uses different approaches, some of which give rise to different results. The magnitudes of the elasticity estimates vary widely. Ma & Stern's (2016) overview and own estimations for China find suggest limited substitutability with electricity for both oil and coal, but again a lower substitutability between coal and electricity than between oil and electricity, the latter being somewhat closer to (but below) unity.

[10]Luderer et al. (2012) confirm this intuiting in a careful study of the energy transition using complex IAMs. They find that the transport sector remains oil-intensive in the close future also in scenarios that substantially reduce global emissions.

of the form

$$\tilde{s}_{l,t} \to \tilde{s}_{l,t}(\cdot) = h_l\left(\frac{e_{1,t}}{\sum_i e_{i,t}}, ..., \frac{e_{L,t}}{\sum_i e_{i,t}}\right). \tag{16}$$

The elasticity of substitution across fuel types can depend on the share of technologies already employed. Restricting $h_1 = ... = h_L \equiv h$ makes this non-constant elasticity of substitution form (N-CES) directly comparable to Arrow et al.'s (1961) CES function. If $h(\cdot)$ is constant, then (16) coincides with the common CES aggregator. If, e.g., the function $h(\cdot)$ is single peaked, then the elasticity of substitution falls into either direction when moving away from a point of maximal substitutability as more and more of a fuel source has already been substituted.[11] Two nice features of the N-CES form in equation (16) are its direct comparability and easy nesting of the common CES formulation and that it does not break with ACE's analytic tractability, still leading to the closed-form solution presented in equation (8) and its generalizations in Traeger (2021) and Traeger (2018).

The community employs IAMs to predict long-term emissions and temperature change, or to calculate the tax necessary to reach, e.g., the $2°C$ temperature target of the Paris Accord. Such long-term predictions involve major changes in the relative prices of different energy sources and constancy of the elasticity of substitution seems to be an issue. A similar warning has recently been raised by Kaya et al. (2017), who also point out that, historically, technological transitions have often followed and S-shape transition that is not necessarily compatible with the IAM-typical CES aggregators. The same generalization from CES to N-CES can be applied to Golosov et al.'s (2014) substitutability parameter s in equation (13). Yet, given the discussed

[11]It is important to understand that already the usual CES function makes substitution in absolute terms harder as we approach extremes. The CES function assumes that the relative factor inputs follows a power function of the relative factor prices. It is this relation that is independent of the relative input level. Yet, no matter how much more expensive oil becomes relative to electricity, we could not currently fly a major airplane on electricity or even cross the ocean on a ship. On the other hand, at current relative prices, there is little substitution between coal and electricity for minor changes of the relative price. Yet, once the relative price favors renewable electricity sufficiently, most industries are likely to swap out coal quickly.

differences between the substitutability of oil and electricity in the transport sector and the substitutability of coal and electricity in other sectors might make N-CES more suitable in a context where the fuel substitution is specific to sectors and fuel types rather than in the context of a catch-all aggregator. Similarly, the N-CES form can be employed for s_t in equation (14) governing the aggregation of different consumption goods.

4 Optimal Emission Response

The optimal carbon tax (SCC) goes along with an optimal emission response. The emission response depends on the particular model specification including the potential scarcity of fossil fuels. This section discusses general insights using the four model structures introduced in section 3 as examples of the general production system. I start with a generic result based on the production elasticity of fossil use. It translates straight into a solution for optimal emission levels if the production elasticity is constant, as in the simple Cobb-Douglas economy. DICE features a more realistic single-peaked production elasticity in a single fossil fuel ("emissions") and I present a closed-form expression for DICE's optimal abatement rate. Subsequently, I derive some analytic insights governing Golosov et al.'s (2014) findings on the role of (abundant) coal versus (scarce) oil use. I expand these insights discussing how the richer CapCES model affects the relative use of oil versus coal over time. Finally, I discuss the endogenous model response to changes in the SCC, relating to changes in economic structure and Hotelling rents.

4.1 Hotelling rent and general result

The so-called Hotelling rent captures the scarcity value of a resource, which is the opportunity cost of foregoing future use. It results from equation (3) together with the non-negativity constraint (if the resource is scarce). In general model, the Hotelling rent is

$$HOT_{i,t} = \frac{\beta \frac{\partial V_{t+1}(\cdot)}{\partial R_{i,t+1}}}{u'(c_t)}.$$

It is the intertemporally aggregated welfare loss from extracting a unit of the resource today (along an optimal consumption trajectory), transformed into consumption equivalents using marginal utility. Extracting a unit of the resource today, reduces the next period stock whose change in value is captured by the partial derivative of the next period value function.[12] In the analytic model, i.e. the starred versions of equations (1-6), the Hotelling rent grows at the rate $\widehat{HOT}_{i,t} = \hat{Y}_t^{net} + prtp$ (pure rate of time preference). The total social cost of using a (CO_2-content-measured) unit of fossil fuel i in period t is

$$\Gamma_{i,t} = HOT_{i,t} + SCC_t.$$

As for the SCC, we define the fundamental (part of the) Hotelling rent and total social cost as

$$\tilde{\Gamma}_{i,t} = \frac{\Gamma_{i,t}}{Y_t^{net}} \quad \text{and} \quad \widetilde{HOT}_{i,t} = \frac{HOT_{i,t}}{Y_t^{net}}.$$

It is the social cost per unit of output.

Proposition 1 *Under the assumptions of section 2, the optimal emissions from a dirty resource $i \in \{1, ..., I^d\}$ satisfy*

$$E_{i,t}^* = \frac{\sigma_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t^*, \boldsymbol{K}_t^*, \boldsymbol{E}_t^*)Y_t^{net}}{HOT_{i,t} + \beta SCC_t} = \frac{\sigma_{Y,E_i}(\cdot)}{\tilde{\Gamma}_{i,t}} \tag{17}$$

where $\sigma_{Y,E_{i,t}}(\cdot) = \frac{\partial F(\cdot)}{\partial E_{i,t}} \frac{E_{i,t}}{Y_t}$ is the production elasticity of the resource and stars denote the optimal allocation.

Equation (17) states that emissions from the dirty energy source E_i are higher if its production elasticity is high and if output is high. The emissions decrease

[12]Here, the definition relying on extracting the resource over the course of the period is equivalent to an instantaneous extraction at the beginning of the period, i.e., to $\frac{\frac{\partial V_t(\cdot)}{\partial R_{i,t}}}{u'(c_t)}$.

in the total social cost of using the resource, i.e., the sum of its scarcity-driven opportunity cost and the SCC. In general, changes in the SCC can change the composition of the economy. Unless σ_{Y,E_i} is constant, equation (17) is an implicit equation. At this point, equation (17) is a generic statement; the insights derive from discussing the particular production examples introduced in section 3. The corresponding calculations are gathered in Appendix B. The subsequent sections revisit the cases of a simple Cobb-Douglas economy, DICE, Golosov et al. (2014), the richer CapCES economy, and the case of N-CES substitutability.

4.2 The simple Cobb-Douglas economy

In the case of the simple Cobb-Douglas production, characterized by equations (9) and (10), emissions are

$$E_t^* = \frac{\nu Y_t^{net}}{HOT_{R,t} + SCC_t} = \frac{\nu}{\tilde{\Gamma}_t}.$$

Emissions can fall for only two reasons. First, emissions fall in response to an increase of the SCC for a reasons other than a mere increase of output. Possible reasons for such an increase include updates of the damage estimate or a better understanding of climate dynamics. Such an increase can also result from incorporating uncertainty into the model (Traeger 2018). These changes are essentially exogenous to the model. *The endogenous increase of the SCC over time as a result of growing consumption has no effect on emissions*; such growth equally increases the relative value of the climate and the value of emitting. Second, *the Hotelling rent increases endogenously over time, reducing emissions if fossil fuels are scarce.* The Hotelling rent can also fall exogenously as the result of a new and unexpected discovery of fossil fuel deposits. Section 5 discusses the endogenous response of the Hotelling rent to changes in the SCC, connecting the two drivers of an emission change.

4.3 DICE

The DICE structure is summarized by equations (9) and (11). The fossil fuel energy (emissions) elasticity is (see Appendix B.1)

$$\sigma_{Y,E}(\cdot) = \sigma_{G,E}\left(\frac{E_t}{E_t^{BAU}}\right) = -\frac{\frac{E_t}{E_t^{BAU}} f'\left(\frac{E_t}{E_t^{BAU}}\right)}{A_t^E - f\left(\frac{E_t}{E_t^{BAU}}\right)}, \tag{18}$$

turning equation (17) into an implicit equation. Figure 2 plots $\sigma_{Y,E}(\cdot)$ for different years. DICE's economy features a finite fossil fuel satiation level (BAU). At BAU, increasing emissions no longer increase production and $\sigma_{Y,E}(\cdot) = 0$. On the other end of the figure, at full abatement ($E_t = 0$), a relative increase of emissions, say 10% percent of a single ton of carbon, is a negligible quantitative increase and has no notable impact on overall production, again implying $\sigma_{Y,E}(\cdot) = 0$. At $\frac{E_t}{E_t^{BAU}} \approx \frac{1}{3}$ the output elasticity of emissions is maximal and $\sigma_{Y,E}(\cdot)$ is locally constant. Over time, technological progress makes output less sensitive to emissions and the magnitude of the elasticity drops everywhere.

By equation (17), the relative reduction of emissions resulting from a relative increase of the social cost now depends on the share of emissions that have already been abated. Emissions will be more responsive when we are close to BAU because reductions are cheap. Given prices are already high and emissions are already low, emissions will also be more responsive in relative terms once we get close to full abatement. Of course, in absolute terms, emission reductions are much more expensive at the low emissions end of the curve, DICE has convex abatement costs whose impact I am about to flesh out. DICE is formulated in terms of the abatement rate $\mu_t = 1 - \frac{E_t}{E_t^{BAU}}$, characterizing avoided emissions as a fraction of BAU emissions. I can express DICE's optimal emission control as a surprisingly simple closed-form expression (see Appendix B.1)

$$\mu_t = \left(\frac{\Gamma_t}{p_t^{back}[1 - D_t(T_{1,t})]}\right)^{\frac{1}{\theta_2 - 1}} \approx \sqrt{\frac{\Gamma_t}{p_t^{back}[1 - D_t(T_{1,t})]}}. \tag{19}$$

The approximation on the right uses DICE's parameter value $\theta_2 = 2.8$ and approximates the exponent $\frac{5}{9} \approx \frac{1}{2}$. The parameter θ_2 measures the convexity of abatement costs as a function of the abatement rate. The exogenous process p_t^{back} is part of Nordhaus' formulation of green technological progress (A_t^E) and specifies the backstop cost, i.e., the marginal cost of abating the last emission unit.

First, the optimal abatement rate increases in the total social cost Γ_t. In general, it is the sum of the social cost of carbon and the Hotelling rent. For optimal policy, DICE assumes that fossil fuels are not scarce so that $\Gamma_t = SCC_t$ and the *abatement rate is approximately proportional to the square root of the SCC*. Second, DICE's abatement rate increases in the part of green technological progress that decreases the optimal backstop price p_t^{back} (the other determinants of A_t^E cancel in the abatement rate). Third, the optimal abatement rate increases with climate damage and, thus, temperature. This effect is additional to climate change damages increasing the SCC. Equation (19)'s damage contribution is more subtle and results from damages driving a wedge between DICE's increase in emissions, which is proportional to gross production, and the resulting consumption benefit, which is net of climate damages. As a result of this wedge, damages amplify the SCC-measured costs relative to emission benefits.[13] Fourth, more convex abatement costs as measured by a larger θ_2 *increase* the abatement rate. This effect might seem surprising, but abatement costs in DICE contain the component $\mu_t^{\theta_2}$. Given that μ_t is smaller than unity, a higher θ_2 reduces the effective abatement costs for $\mu_t \in (0, 1)$. The more intuitive implication of the cost convexity is captured by the elasticity of the abatement rate w.r.t. the SCC, $\sigma_{\mu,\Gamma} = \frac{\partial \mu_t}{\partial \Gamma_t} \frac{\Gamma_t}{\mu_t} = \frac{1}{\theta_2 - 1}$(keeping temperature fix), which is time constant and indeed *falls* in the convexity of abatement costs. While an increase in DICE's cost-convexity parameter increases abatement, such an increase makes abatement less responsive to the SCC. Section 5 will develop further results governing DICE's emission response.

[13]This contribution relies on Nordhaus' assumption that emissions are proportional to gross output, i.e., a climate-induced output reduction does not reduce emissions. If emissions are proportional to net output, the damage term disappears from equation (19).

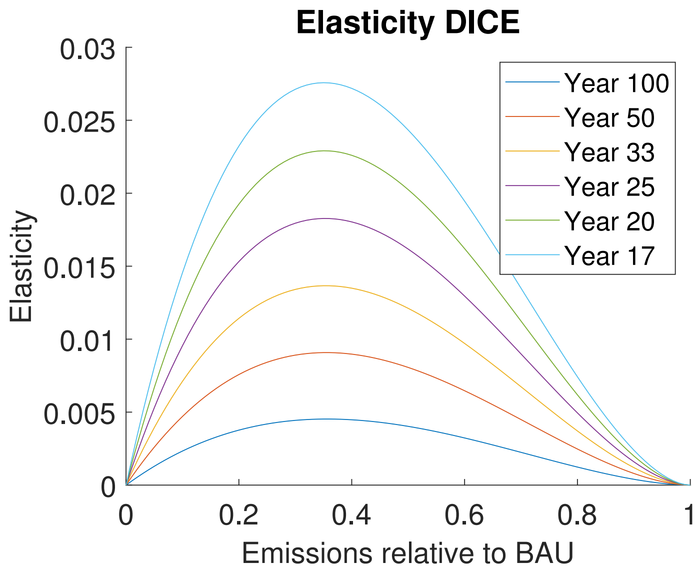

Figure 2: DICE's fossil fuel elasticity of production as a function of emissions relative to BAU. Evaluated for different years into the future.

4.4 Golosov et al. (2014)

In difference to DICE, Golosov et al. (2014) explicitly model, distinguish, and predict the use of oil and coal. Their solution has four distinctive features. First, annual coal use increases substantially under BAU, approximately 5-fold by the end of the century and almost 40-fold by 2200. Second, even in the optimal scenario, coal use still increases over the projected time horizon. Third, oil use drops substantially over time in both the BAU and the optimal scenario. Fourth, oil use hardly responds to the optimal policy (whereas coal use responds strongly). The discussion of the present section focuses on (i) a structural comparison between decarbonization in DICE and in the present model, (ii) the model's predicted increase in coal use and (iii) the divergence between coal and oil use (in both the BAU and the optimal scenarios). Section 5 will return to the model's response moving from BAU to the optimal policy.

Golosov et al. (2014) combine the final good Cobb-Douglas production (9) with the energy sector in equation (12). The resulting output elasticity of energy input i is $\sigma_{Y,E_i} = \nu a_i \big(\frac{E_{i,t}}{E_t}\big)^s$, where $E_t = (a_1 E_1^s + a_2 E_2^s + a_3 E_3^s)^{\frac{1}{s}}$ is the energy composite. Compared to the the simple Cobb-Douglas case, the elasticity ν is multiplied by the energy factor's CES-weight a_i and its physical share in the energy composite E_t. Figure 3 graphs the production elasticity of oil and coal over their share in the energy composite E_t. The solid lines represent Golosov et al.'s (2014) calibration, reflecting their calibrated elasticity of interfuel substitution of 0.95 (parameter $s = -0.058$). The dashed lines reflect a hypothetical scenario mentioned by the authors where the elasticity of interfuel substitution is 2 rather than 0.95. The production elasticity of coal assumes that the coal sector's labor input, which regulates coal use, is varied accordingly.

The first implication of their calibration with limited substitutability (negative s) is that full decarbonization is impossible. The production elasticity remains finite as either fossil input converges to zero, implying that the productivity of the first unit of a fossil resource is infinite. No finite SCC can retire either of the fossil fuels completely. While the limit is a technicality, the shape of the curves approaching zero show that the CES formulation makes substitution away from fossil fuels much harder than in DICE as we reach a low carbon economy.[14]

Second, as compared to DICE, Golosov et al.'s (2014) CES-formulation does not have a natural saturation level for fossil fuel use. A higher oil use always increases production. In the laissez faire, oil use is finite only because of intertemporal scarcity, and coal use is finite only because labor input to the coal sector is finite. No matter how high the fossil fuel use, the production elasticities of oil and coal remain finite.[15]

[14]Figure 3 plots the production elasticity over the share in the energy composite, whereas Figure 2 graphs the production elasticity over the fraction of BAU emissions. These graphs are somewhat directly comparable as emissions tend to zero. The unity has a somewhat different interpretation as I briefly discuss below. For a more detailed discussion and alternative graphical depictions I refer to the Appendix.

[15]This result holds as long as all energy inputs remain positive. A unit share of oil in the energy composite does not imply the exclusive use of oil, but, e.g., an equal share of oil, coal, and renewable energy. As discussed above, there is no BAU upper bound as for DICE. The expression $E_{oil,t}/E_t$ would tend to infinity if oil was the only energy source. In this case, the production elasticity converges to zero; this finding merely expresses that production with only one energy composite is not possible in the model.

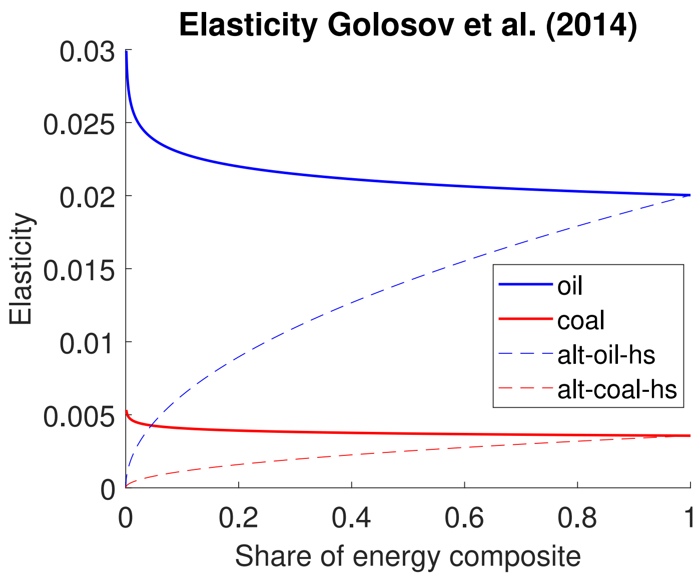

Figure 3: Golosov et al.'s (2014) fossil fuel elasticity of production for oil and coal. The horizontal axis varies the share oil or coal relative to the energy composite, i.e., $\frac{E_i}{E_t}$. The elasticities are constant over time. The solid lines represent Golosov et al.'s (2014) calibration (limited substitutability). The dashed lines depict a hypothetical scenario with a higher elasticity of interfuel substitution of 2.

I now return to a discussion of Golosov et al.'s (2014) stylized findings governing the trajectories of coal and oil use over time. To ease interpretation, I spell out subindices 1 and 2 explicitly as oil and coal. Golosov et al. (2014) only optimize the oil input directly, for which equation (1) translates into

$$E_{oil,t} = \left(\frac{\nu a_{oil} Y_t^{net}}{HOT_{oil,t} + SCC_t}\right)^{\frac{1}{1-s}} E_t^{-\frac{s}{1-s}} = \left(\frac{\nu a_{oil}}{\tilde{\Gamma}_{oil,t}}\right)^{\frac{1}{1-s}} E_t^{-\frac{s}{1-s}}, \tag{20}$$

setting oil based-fossil fuel emissions in relation to the economy's total energy use. The authors do not explicitly optimize the coal and renewables inputs; these inputs are determined indirectly by the labor allocated to the corresponding sectors. Using the first order conditions for labor optimization instead of equation (1) delivers[16]

$$E_{coal,t} = \frac{\sigma_{Y,N_{coal,t}}(\boldsymbol{A}_t, \boldsymbol{N}_t^*, \boldsymbol{K}_t^*, \boldsymbol{E}_t^*)Y_t^{net}}{\frac{\omega_t}{A_{coal,t}} + \Gamma_{coal,t}} = \left. \frac{\nu \; a_{coal}}{\frac{1-a-\nu}{N_{0,t}A_{coal,t}} + \tilde{\Gamma}_{coal,t}} \right)^{\frac{1}{1-s}} E_t^{-\frac{s}{1-s}} . \tag{21}$$

The variable ω_t characterizes the wage (in consumption equivalents). The right side expresses the wage as a relation between labor productivity and employment in the final goods sector. For a given level of the energy composite, the level of coal-based fossil fuel emissions increases in the energy sector's dependence on coal. This dependence increases with a high CES-weight a_{coal} and with a low level of substitutability s (a low or negative s extracts a higher-index root of a fraction). The authors assume that there is no scarcity rent for coal, which seems particularly reasonable for an optimal emissions scenario. As a result, $\Gamma_{coal,t} = SCC_t$.

In the laissez-faire scenario, also $SCC_t = 0$ and the first term in brackets on the right of equation (21) becomes $\left(N_{0,t}A_{coal,t}\frac{\nu \; a_{coal}}{1-a-\nu}\right)$. The authors assume a labor augmenting technological progress of 2% annually in the coal sector. The model only features labor in the coal sector and coal use is proportional to effective labor. Thus, along with the effectiveness of labor, coal use grows. Given $s = -0.058$, the exponent on this bracket is approximately unity and the exponent on the second term, the energy composite, is small ($\frac{s}{1-s} \approx 0.05$). Thus, coal use in the laissez faire should grow at approximately 2% annually. Indeed, a 2% growth matches closely the observed 5-fold growth of coal use until the end of the century and a 40-fold increase until 2200.

It is insightful to analyze the ratio of equations (20) and (21) to understand the quickly diverging evolution of oil and coal use in the model

$$\frac{E_{oil,t}}{E_{coal,t}} = \left. \frac{a_{oil}}{a_{coal}} \; \frac{\frac{\omega_t}{A_{coal,t}} + SCC_t}{HOT_{i,t} + SCC_t} \right)^{\frac{1}{1-s}} . \tag{22}$$

[16]The r.h.s. results after substituting in the elasticity and solving the implicit equation (first equality) for $E_{t,coal}$. See Appendix B.2 for details. Golosov et al. (2014) present a slightly reformulated version of equation (21) for coal use, they do not present a corresponding analytic expression for oil use.

Golosov et al.'s (2014) oil sector lacks technological progress. Thus, the technological progress in the coal industry favors future coal use over future oil use. In addition, the oil sector is subject to the increasing Hotelling rent $HOT_{oil,t}$, which is absent in coal sector. Thus, intertemporal scarcity further amplifies the pattern that coal use dominates oil use in the future. Following the analoguous arguments of the discussion for coal, equation (20) also shows that oil use falls not only relative to coal use, but also in absolute terms. Instead of technological progress, the first term on the right side of equation (20) now reduces oil use as a result of an increasing Hotelling rent (per unit of output).

Both drivers of the divergence between oil and coal use remain active in the optimal scenario. Ceteris paribus, the positive SCC reduces the wegde between the two time paths, explaining why the divergence is substantially larger in the laissez fair scenario. That said, an increase in the SCC also changes the Hotelling rent, slightly complicating the situation. I will develop the argument more carefully in Section 5, where I also discuss the distinct responses of the two fossil fuels to an increase in the SCC.

4.5 CapCES: Capital in energy production and final and intermediate CES aggregation

The production system summarized by equations (13-15) employs capital production and introduces a variety of final goods with different energy dependencies. I use the model to explain why we are unlikely to observe Golosov et al.'s (2014) finding regarding the strong increase of coal use relative to oil in the close future, and why we have not observed it in the past either. I move straight to an analysis of the fossil-based emission ratio between oil and coal, which generalizes to

$$\frac{E_{oil,t}}{E_{coal,t}} = \frac{\sigma_{e,oil}(\cdot)}{\sigma_{e,coal}(\cdot)} \frac{a_{trans,t}}{a_{other,t}} \left(\frac{c_{trans,t}}{c_{other,t}}\right)^{s_t} \frac{1 + \left(\frac{e_{renew,t}}{e_{coal,t}}\right)^{\tilde{s}_{elec,t}}}{1 + \left(\frac{e_{renew,t}}{e_{oil,t}}\right)^{\tilde{s}_{trans,t}}} \frac{\beta SCC_t}{HOT_{oil,t} + SCC_t} \tag{23}$$

where $\sigma_{e,oil}(\cdot) = \frac{\partial g_{oil,t}(\cdot)}{\partial E_{oil}} \frac{E_{oil}}{e_{oil,t}}$ and $\sigma_{e,coal}(\cdot) = \frac{\partial g_{coal,t}(\cdot)}{\partial E_{coal}} \frac{E_{coal}}{e_{coal,t}}$. The other variables are defined in the model introduction on page 11.

The first term, $\frac{\sigma_{e,oil}(\cdot)}{\sigma_{e,coal}(\cdot)}$ reflects possible *technological progress* in the production of oil- versus coal-based energy. It generalizes Golosov et al.'s (2014) setting by permitting technological progress not only in their labor-based coal but also in the oil sector, which reduces (or eliminates) one of their two drivers favoring coal use over time. Moreover, technological progress no longer has to increase coal use per unit of labor, the main driver of the somewhat extreme growth of coal use in Golosov et al. (2014).

The next ratio $\frac{a_{trans,t}}{a_{other,t}}$ characterizes the *demand* weights for transport versus other consumption and their potential *shift over time*. Section 3 introduces these weights as reflecting the economy's consumption composition. These weights are relevant because the global expansion of the transport sector made it one of the faster growing emission sources over the past decades. In the simple and elegant setting of Golosov et al. (2014), oil use falls over time even in the business as usual scenario (if falls because of the Hotelling rent). Demand shifts play an important role in explaining why we have observed the opposite trend over the past decades with strongly increasing global oil consumption. The subsequent factor reflects the relative size of the two sectors and their change over time. For limited substitutability ($s < 1$), the ratio of the sector's sizes has only a moderated impact on the corresponding fossil use.[17]

With the *growing provision of renewable energy* $e_{renew,t}$, the ratios $\frac{e_{renew,t}}{e_{coal,t}}$ and $\frac{e_{renew,t}}{e_{oil,t}}$ increase over time. The respective increase is (power-)weighted by the corresponding *substitutability* parameters, which is substantially *lower* in the oil-based transport sector ($\tilde{s}_{trans,t} < \tilde{s}_{elec,t}$). The equation shows that the lower substitutability between fossil-fuel and renewables in the transport sector yet again increases oil use over time relative to coal.[18] This driver plays a limited role in explaining why past fossil use did not follow the pattern of Golosov et al. (2014), but it makes it even more likely that also the fossil-

[17]For complements ($s < 0$), the effect can even go into the opposite direction. However, the size of the sectors is endogenous and related to the weights and the degree of substitutability so the the corresponding ratio should be interpreted with a grain of salt.

[18]The renewable share in the electricity sector grows with a higher power in the numerator, increasing the oil share. In case $\tilde{s}_{trans,t} < 0$, the numerator not only grows slower, but it even falls.

based emissions in the closer future will not follow such a pattern. As batteries becomes cheaper and more efficient, the substitutability will increase and the corresponding factor will slowly cease to favor oil based energy and emissions.

Non-constant Elasticity of Fuel Substitution (N-CES). The discussed specification accounts for time changing substitutabilities, and for sectorial differences in the substitutability. Yet, even within a sector and at a given point in time, a constant elasticity of interfuel substitution is questionable. Estimates find almost ubiquitously that the substitutability between fossil fuels and renewable energy is limited. Moreover, the elasticity of substitution between coal and renewable electricity is substantially more limited than the substitutability between oil and renewable electricity. At current prices, indeed, it seems reasonable that oil and electricity are more substitutable as their energy costs are more on a par. However, under a sufficiently high carbon tax, the power sector can somewhat easily substitute coal-based electricity against renewable-based electricity. Acknowledging current issues with the volatility of renewable electricity, it still seems reasonable that once the relative price exceeds a certain threshold, substitutability of coal with renewables will be more price-responsive and the elasticity likely rises above unity. In contrast, the transport sector has clear technical limits in substituting oil against electricity. We are currently far from supplying the world with enough battery power to substitute gas powered by electric vehicles around the globe. Even more so, we currently lack the technology to electrify marine and air transport. Indeed, most projections of a green energy transition defy the local estimates of the respective elasticities of substitution between oil versus coal and electricity and suggest that coal use falls just as much if not more than oil use under an optimal policy scenario.[19]

[19] Coal also enters directly into chemical processes in the industrial sector. The most difficult (large-scale-) process to decarbonize is the coal use in the steel industry. Thus, we would expect that the current substitutability is low, followed by a somewhat high substitutability regime at prices where we can swap out the power sector, and then a hart to substitute regime for the final part of the coal-based emissions. For the last part, the most promising technology is carbon capture and storage (CCS). Once CCS is available, it will again be easier to eliminate coal-based emissions from stationary sources than to eliminate oil-based emissions form mobile sources in the transport sector.

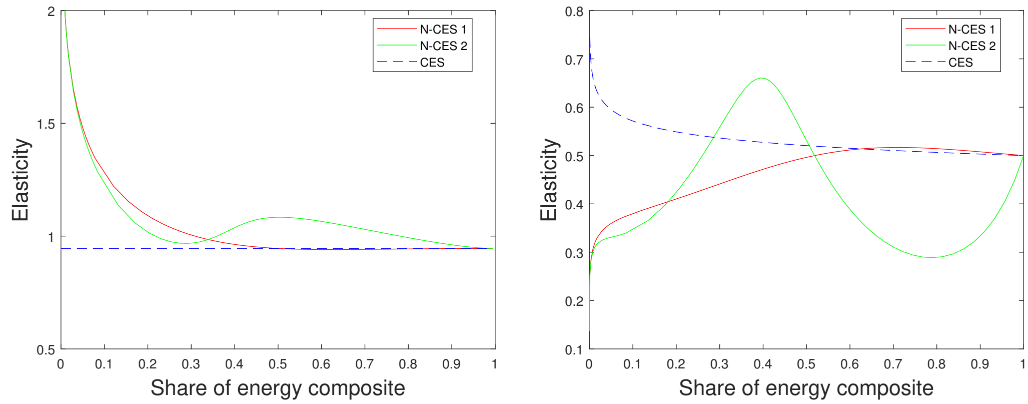

Figure 4: Non-constant elasticities of substition and the resulting production elasticity to emissions. Left: Two examples of non-constant interfuel substitutability. Right: Corresponding impact on the production elasticity to emissions from the corresponding sector, which plays a crucial role for the economy's response to climate policy.

Figure 4 illustrates the implications of a non-constant elasticity of interfuel substitution on the production elasticity of emissions from the corresponding sector. Unity corresponds to an equal energy share of two different energy inputs in a given sector. The first example (N-CES 1) assumes that the elasticity stays initially close the limited substitability calibrated by Golosov et al. (2014), but then increases above unity. The resulting production elasticity first increases slightly, then falls, and finally permits full decarbonization. The second example (N-CES 2) analyzes the case where the elasticity first increases into a domain of higher substitutability before hitting some technological obstacles that push back into the low substitutability domain, ultimatly permitting for full decarbonization. The corresponding production elasticity shows that even somewhat moderate ups and downs of the elastictity, in in the neighborhood of unity, can have major responses for the production elasticity of emissions and, thus, the economy's emission response to climate policy.

5 Endogenous Emission Response to Changes in the SCC

5.1 General Result

Thus far, I focused on the optimal emission trajectory over time and the share of oil versus coal-based fossil emissions for a given magnitude of the SCC. Updates to our assessment of climate damages or climate dynamics change the optimal carbon tax. Similarly, Traeger (2018) shows how uncertainty increases the current SCC. This section analyzes how an increase in the SCC affects optimal emissions, focusing on two crucial endogenously moving pieces. First, an increase in the SCC will restructure the economy. In the extreme case, an increase in the SCC could restructure the economy sufficiently to increase a fossil resource's optimal use despite the increase of its total social cost $\Gamma_{i,t}$. For example, by driving out the particularly dirty coal, an increase in the SCC can increase the productivity of the cleaner gas, which can lead to an overall increase of gas use despite a higher social cost. Second, an increase in the SCC reduces the scarcity of fossil fuels that respond normally, i.e., whose use drecreases with an increase of the SCC. If these fossil fuels have a scarcity value, then the implied reduction in scarcity implies a reduction of the Hotelling rent; this endogenous response crowds out the SCC increase in the total social cost $\Gamma_{i,t}$ of using the fossil fuel.

The subsequent proposition analyzes the emission response to a change in the SCC, arising from a change of the fundamental factors determining climate change, damages, or uncertainty. To avoid repeated usage of "$\tilde{\cdot}$", I re-define the fundamental part of the SCC as $\Sigma_t \equiv \widetilde{SCC}_t = \frac{SCC_t}{Y_t^{net}}$. Given a marginal change of Σ, I denote the resulting rates of change of the endogenous variables by $\hat{z} \equiv \frac{dz}{d\Sigma}\frac{1}{z}$ for $z \in \{SCC_t, E_t, Y_t, Y_t^{net}, \sigma_{Y,E_i}(\cdot), \widetilde{HOT}_{i,t}\}$. E.g., $S\hat{C}C_t = \hat{Y}_t^{net} + \widehat{\widetilde{SCC}}$. In general, a change in the severity of climate change affects a resource's scarcity value and the Hotelling rent changes as $H\hat{O}T_{i,0} = \hat{Y}_0^{net} + \widehat{\widetilde{HOT}}_{i,0}$.

Proposition 2 *A relative change $\widehat{\widetilde{SCC}}$ in the climate part of the SCC results* *in the relative emission change*

$$\hat{E}_{i,t} = -\widehat{SCC} + \gamma_{i,t}(\widehat{SCC} - \widehat{HOT}_{i,t}) + \hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \tag{24}$$

where $\gamma_{i,t} = \frac{HOT_{i,t}}{\Gamma_{i,t}}$ *denotes the Hotelling share of the total social cost. Moreover, the following conditions sign different determinants of the emission response in the analytic model, i.e. the starred versions of equations (1-6). Assume that fossil resource* i *has a positive Hotelling rent* $HOT_{i,0} > 0$ *and that* $\hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \leq (1 - \gamma_{i,t})\widehat{SCC}$ *for all* t*. Then* $\widehat{HOT}_{i,t} \leq 0$*. If also* $\hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \geq 0$ *for all* t*, then* $-\widehat{SCC} + \gamma_{i,t}(\widehat{SCC} - \widehat{HOT}_{i,t}) \leq 0$*. The same combination of statements holds in strict inequalities.*

Equation (24) follows from equation (17) and characterizes the emission response to a change in the SCC. The term $\widehat{SCC}$ is the relative ("percentage") increase of the SCC in response to a different assessment of climate dynamics or damages, or as a result of incorporating uncertainty. I now discuss how a ("marginal") 10% increase of the SCC ($\widehat{SCC} = 10\%$)affects emissions in the different model structures.

5.2 No resource scarcity

First, I discuss the case without resource scarcity. Given the abundancy of coal, this assumption is most reasonable in a model using a single aggregate fossil fuel like the simple Cobb-Douglas model or DICE. In this case, the share of the Hotelling rent in the total social cost is $\gamma_{i,t} = 0$ and $\hat{E}_{i,t} = \hat{\sigma}_{Y,E_i}(\cdot) - \widehat{SCC}$. In the **simple Cobb-Douglas** model, $\hat{\sigma}_{Y,E_i}(\cdot) = 0$ and

$$\hat{E}_{i,t} = -\widehat{SCC}.$$

Emissions fall by 10% if the climate part of the SCC increases by 10%. The simple Cobb-Douglas model sidesteps the emission consequences of the endogenous reorganization of the economy. In contrast, **DICE**'s emission elasticity of production changes endogenously. By equation (18), it (only) depends on emissions relative to BAU (or the abatement rate). Figure 2 shows that, if emissions are close to BAU, the reorganization of the DICE economy "counteracts" the 10% increase of climate change part of the SCC. Emissions are only reduced by $\hat{E}_{i,t} = -\widehat{\widetilde{SCC}} + \hat{\sigma}_{Y,E_i}(\cdot)$, where $\hat{\sigma}_{Y,E_i}(\cdot) > 0$. However, as discussed in section 4.3, falling emissions will eventually lead to a falling production elasticity. If emissions drop to approximately one third of potential emissions the elasticity is constant. Once emissions are reduced beyond this critical level, the production elasticity "reinforces" the increase of the (climate part of) the SCC and emissions fall by more than 10%.

It is insightful to square this finding with equation (19) for DICE's optimal abatement rate. This equation implies[20]

$$\hat{\mu}_t = \frac{1}{\theta_2 - 1}(\widehat{\widetilde{SCC}} + \hat{Y}_t) \approx \frac{1}{2}(\widehat{\widetilde{SCC}} + \hat{Y}_t). \tag{25}$$

A 10% increase of the SCC increases the abatement rate by approximately 5% independently of the prevailing abatement level (unless limited by full abatement).[21] Is this finding in line with equation (24)? Given that $E_t^{\hat{BAU}} = \hat{Y}_t$, the definition of the abatement rate implies

$$\hat{E}_{i,t} = -\frac{\mu_t}{1-\mu_t}\hat{\mu}_t + \hat{Y}_t = -\frac{\mu_t}{1-\mu_t}\frac{1}{\theta_2 - 1}(\widehat{\widetilde{SCC}} + \hat{Y}_t) + \hat{Y}_t. \tag{26}$$

For $\mu_t \approx \frac{2}{3}$ and $\frac{1}{\theta_2 - 1} \approx \frac{1}{2}$ the relation implies $\hat{E}_{i,t} = -\widehat{\widetilde{SCC}}$. Indeed, we obtained this result earlier based on equation (24) and Figure 2, observing that $\hat{\sigma}_t \approx 0$ for $\mu_t \approx \frac{2}{3}$. Equation (26) adds more structure to the earlier reasoning. A low abatement rate μ_t goes along with a low (base-) carbon tax. A given percentage increase on a low carbon tax only implies a small absolute increase and has less of an impact on emissions. A high abatement rate μ_t implies that emissions are already low (and taxes are high). Increasing $\widetilde{SCC}$ by 10% will reduce emissions by more than 10% unless – as in the simple Cobb-Douglas

[20]The process p_t^{back} is exogenous. The total social cost is limited to $S\hat{C}C_t = \widehat{\widetilde{SCC}} + \hat{Y}_t^{net}$ and the result follows recognizing that $\hat{Y}_t^{net} - (\widehat{1 - D(T_t)}) = \hat{Y}_t$.

[21]In the current period temperature is fix and $\hat{Y}_0^{net} = \hat{Y}_0$ so that the right of equation (25) is simply $\frac{1}{2}S\hat{C}C_0$. More generally, the negative response of $\hat{Y}_t$ resulting from an increase in $\widehat{\widetilde{SCC}}$ will be much smaller than $\widehat{\widetilde{SCC}}$. Likely in the magnitude the right side already approximated $\frac{5}{9} \gtrsim \frac{1}{2}$.

case – the marginal productivity of the first emission unit was infinitely high. In DICE it is not, and absolute emissions drop to zero at a finite tax level.

Golosov et al. (2014)'s production elasticity of oil is $\sigma_{Y,oil} = \nu a_{oil}\left(\frac{E_{oil,t}}{E_t}\right)^s$. Given the author's complementarity assumption of energy sources ($s < 0$), *the economy's endogenous "restructuring" always has a stabilizing effect of fossil use*, a falling share of oil in energy production implies an increase of its productivity elasticity. As oil consumption goes to zero, $\sigma_{Y,oil}$ converges to infinity, contrasting with the earlier models where $\sigma_{Y,E}$ is constant in the simple Cobb-Douglas economy and approaches zero in DICE as fossil use goes to zero. The models stabilization effect increases with the CES weight of the fossil resource. Comparing Golosov et al.'s (2014) BAU and optimal scenarios, coal use drops substantially whereas oil use is hardly affected by the optimal carbon tax. The present reasoning suggests one cause of this finding; the authors' CES weight on oil ($a_{oil} = 0.5008$) is almost six times that of coal ($a_{coal} = 0.08916$). This weight difference is driven by the price ratio of oil to coal.[22]. Hereby, the cost of oil is fully assigned to the Hotelling rent, which leads to the second reason why oil responds so much stronger than coal in Golosov et al.'s (2014).

5.3 Resource scarcity

Let me now assume that fossil resource i has a positive Hotelling rent $HOT_{i,0} > 0$. Then, the previously discussed emission reduction is counteracted by at least one of *two additional terms* in equation (24). *First*, $\gamma_{i,t}\widehat{\widehat{SCC}}$ captures that an increase of the (climate part of the) SCC only captures part of the total social cost of carbon Γ_t. Let the total social cost be composed in equal parts of SCC and scarcity value ($\gamma_{i,t} = \frac{1}{2}$). Then, the (direct) net response of emissions to $\widehat{\widehat{SCC}}$ in equation (24) is only $\frac{1}{2}\widehat{\widehat{SCC}}$. *The fact that the SCC is only part of the total social cost always reduces the direct emission response.* Second, the scarcity (Hotelling) part of the resource's total social cost responds

[22] I note that the relation holds *despite* the different carbon content, not because of it as stated on 60 of the manuscript. The authors reasoning contains a sign mistake, most likely resulting from overlooking that $s < 0$. See end of Appendix B.2 for details.

as well. Let me assume that fossil resource i's productivity increase is limited to $\hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \leq (1-\gamma_{i,t})\widehat{\widehat{SCC}}$ for all t. Say $\widehat{\widehat{SCC}} = 10\%$ and the SCC's share is 50%, then the reorganization of the economy should increase the production elasticity of the resource by less than 5%. In the simple-Cobb-Douglas case $\hat{\sigma}_{Y,E_i}(\cdot) = 0$ and the assumption is always met. In the case of DICE, the response of the production elasticity $\hat{\sigma}_{Y,E_i}\big(\frac{E_t}{E_t^{BAU}}\big)$ is initially positive and eventually negative. Thus, the condition will always be satisfied for abatement exceeding some critical level. This condition is sufficient (but not necessary) for the resource's shadow value to fall in response to an increase of the carbon tax. Then, also *the second novel contribution in equation* (2)*, the term* $-\hat{\varphi}_{R,i,0}\widehat{\widehat{SCC}} \geq 0$ *counteracts the emission reduction.* The intuition is simple. The economy has been using less of the resource today because of its scarcity value and the desire to save the resource for tomorrow. As a consequence of more severe climate change, the usefulness of the resource falls and so does its scarcity value. The reduction in scarcity value crowds out part of the SCC's increase.

The drop of the resource's shadow value in response to a tighter climate policy relates closely to a large literature on the so-called "green paradox" (Sinn 2012, van der Ploeg & Withagen 2012, Jensen et al. 2015). The present paper embeds some of the corresponding finding into a general integrated assessment model. For example, a green "paradox" arises if a new technology is expected to reduce or replace carbon fuel usage in the medium to long-run future ($\sigma_{Y,E_i}(\boldsymbol{A}_t)$ lower for some $t > T$). Then, future demand and, thus, $HOT_{i,t}$ fall with respect to a world in which such a technology is absent (or not expected), and near-term emissions under the optimal tax tend to be higher in a world with the cleaner expected technology. As the previous paragraph points out, this response is optimal if the carbon tax is set at the right level. Other examples of the green paradox build more directly on the suboptimality of policy. For example, let the policy maker announce a higher carbon tax for the future without taxing the fossil fuel immediately. Then, the future reduction in scarcity lowers $HOT_{i,t}$ already today while the optimal carbon tax is not yet in place. As a result, the total (private) cost of the fossil resource falls and emissions increase further above the optimal trajectory than without the announcement. Similarly, if a large fraction of countries sets the optimal carbon tax, the resulting reduction in $HOT_{i,t}$ makes it more attractive for a non-participating country to emit. This paragraph is merely a teaser for a large literature on the topic and Proposition 2 embeds some of this discussion into a full-fledged integrated assessment model of climate change. It also emphasizes how some degree of crowding out is optimal. Resources with a positive Hotelling rent have a current use value that is higher than their extraction costs. If we set carbon taxes to their optimal level, their use value will be still exceed the SCC, or in case the scarcity value drops to zero the resource's use value will still equal the SCC. Before the SCC increase, the scarcity value "helped" climate policy to be a little closer to where it should have been. Another literature on supply side policy tries to use "Hotelling's help" to reduce emissions in settings where we cannot reach agreements on optimal taxation. In principle, an extreme (negative) response of a resource's shadow value to an increase in the (climate part of the) SCC can even trigger a net increase in emissions. At the present level of generality it is hard to rule out such a scenario. ACE's general production system permits that a small change in the SCC can make a huge difference in equilibrium productivities and flip a resource from being scarce to being replaced in most periods. Then, the resource's scarcity value drops to zero, but it might still be used for a few periods and present emissions from the resource increase. Proposition 2 shows that the additional assumption $\hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \geq 0$ is sufficient (but not necessary) to rule out such an extreme response. The assumption holds for simple Cobb-Douglas but only holds initially in DICE (up to an abatement rate of $\mu_t \approx \frac{2}{3}$). For models with a single aggregate fossil resource it is unlikely (yet not impossible) that present emissions increase in response to an optimal shift in climate policy. In a more detailed model of the energy sector it is somewhat more likely that emissions of a single fossil fuel can increase in response to a higher SCC. For example, a higher SCC can trigger the phasing out of the particularly dirty coal, increasing usage of the currently more expensive but cleaner gas.

Returning to Golosov et al.'s (2014) results, the discussion helps to further understand their finding that the introduction of the first best carbon tax hardly changes oil use but substantially reduces coal consumption. Their model prices coal by its extraction costs, whereas oil consumption is costly only because of its Hotelling rent. For oil, $\gamma_{oil,t}$ is initially one and then decreases as the SCC increases. In addition, all of the oil price was assigned to the Hotelling rent, making $\varphi_{R,oil,0}$ large and the oil price particularly responsive to a reduction in oil scarcity. In contrast, $\gamma_{coal,t} = 0$ for coal and the two r.h.s. terms counteracting the emission reduction in (2) are absent. In addition, the first counteracting term discussed above is smaller as a result of the CES-weights.

6 Conclusions

The paper discusses how different production structures of integrated assessment models affect the resulting CO_2 emissions that drive global warming. It also discusses the interaction of carbon taxation and scarcity rents in driving and countering mitigation efforts. DICE lumps different fossil fuels together and relies on a set of implicit equations specifying the carbon intensity of production. Despite DICE's apparent complexity, the evolution of emissions over time is mostly driven by DICE's assumption about the (monotonically falling) backstop price and it's response to the SCC is mostly driven by a single cost-convexity parameter. The implied production elasticity to fossil fuel emissions is concave over emissions, single peaked, falls over time, and allows for both saturation in fossil use and full decarbonization of the economy.

Golosov et al. (2014) distinguish oil and coal-based CO_2 emissions. The underlying CES-structure of energy use implies a time-constant production elasticity of fossil fuel inputs that is not concave and exhibits a natural satiation point in oil use nor permits full decarbonization of the economy. Golosov et al.'s (2014) numeric simulations find increasing coal use even in the optimal scenario and a steeply decreasing use of oil that is mostly unresponsive to optimal policy. I explain these results based on the models differential treatment of technological progress and rents in the oil and coal sectors. The discussion suggests that these findings are somewhat special, and I suggest a more general model that combines the more promising aspects of both DICE and Golosov et al. (2014). The model keeps analytic tractability and permits a direct discussion of the different drivers of emissions. Adding capital to the energy sectors does not necessarily have a major impact. In contrast, I suggest various changes in the treatment of interfuel substitutability and technological progress that have a major impact for describing current stylized facts about fossil use as well as more realistic predictions of future emission scenarios in low complexity and analytically tractable IAMs. These changes include both time changing interfuel substitutabilities and a non-contant elasticity of substitution aggregator that can navigate more realistic transformation scenarios where technical constraints and possibilities suggest that local estimates of elasticities do not extend globally.

Finally, I discuss how the different models respond to changes in the carbon price by restructuring (or not) the corresponding economies and how these results are affected by changing resource scarcities. The paper's treatment simultaneously applies to generic IAMs as well as to analytically tractable IAMs with explicit closed-form solutions for the optimal carbon tax. The paper does not attempt (and cannot) replace complex IAMs such as WITCH, REMIND, or other models with similar detail. It addresses features and issues of low complexity models that are widely employed as a result of their simpler analytic or numeric tractability. It explains their output and suggests how to improve such models in ways better reflecting the stylized facts underlying energy use and transition, while keeping a tractable model.

References

Acemoglu, D., Hemous, D., Barrage, L., Aghion, P. et al. (2019), Climate change, directed innovation, and energy transition: The long-run consequences of the shale gas revolution, *in* '2019 Meeting Papers', number 1302, Society for Economic Dynamics.

Anderson, E., Brock, W., Hansen, L. P. & Sanstad, A. H. (2014), 'Robust analytical and computational explorations of coupled economic-climate models with carbon-climate response', *RDCEP Working Paper 13-05* .

Arrow, K., Chenery, H., Minhas, B. & Solow, R. (1961), 'Capital–labor substitution and economic efficiency', *The Review of Economics and Statistics* **43**(3), 225–248.

Bosetti, V., Carraro, C., Galeotti, M., Massetti, E. & Tavoni, M. (2006), 'A world induced technical change hybrid model', *The Energy Journal* (Special Issue# 2).

Bretschger, L., Bretschger, L., Pattakou, A. & Pattakou, A. (2019), 'As bad as it gets: How climate damage functions affect growth and the social cost of carbon', *Environmental and Resource Economics* **72**(1), 5–26.

Brock, W. A. & Mirman, L. J. (1972), 'Optimal economic growth and uncertainty: The discounted case', *Journal of Economic Theory* **4**, 479–513.

Brock, W. & Xepapadeas, A. (2017), 'Climate change policy under polar amplification', *European Economic Review* **99**, 93 – 112. Combating Climate Change. Lessons from Macroeconomics, Political Economy and Public Finance.

Dietz, S. & Venmans, F. (2018), 'Cumulative carbon emissions and economic policy: in search of general principles', *Grantham Working Paper 283* .

Edenhofer, O., Bauer, N. & Kriegler, E. (2005), 'The impact of technological change on climate protection and welfare: Insights from the model mind', *Ecological Economics* **54**(2-3), 277–292.

Emmerling, J., Drouet, L., Reis, L. A., Bevione, M., Berger, L., Bosetti, V., Carrara, S., Cian, E. D., D'Aertrycke, G. D. M., Longden, T., Malpede, M., Marangoni, G., Sferra, F., Tavoni, M., Witajewski-Baltvilks, J. & Havlik, P. (2016), 'The witch 2016 model documentation and implementation of the shared socioeconomic pathways', *FEEM Note di Lavoro* (42.2016).

Epstein, L. G. & Zin, S. E. (1989), 'Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework', *Econometrica* **57**(4), 937–69.

Gerlagh, R. & Liski, M. (2018*a*), 'Carbon prices for the next hundred years', *The Economic Journal* **128**(609), 728–757.

Gerlagh, R. & Liski, M. (2018*b*), 'Consistent climate policies', *Journal of the European Economic Association* **16**(1), 1–44.

Golosov, M., Hassler, J., Krusell, P. & Tsyvinski, A. (2014), 'Optimal taxes on fossil fuel in general equilibrium', *Econometrica* **82**(1), 41–88.

Hambel, C., Kraft, H. & Schwartz, E. S. (2018), 'The carbon abatement game', *SSRN 3122261* .

Hassler, J. & Krusell, P. (2012), 'Economics and climate change: Integrated assessment in a multi-region world', *Journal of the European Economic Association* **10**(5), 974–1000.

Hassler, J., Krusell, P. & Olovsson, C. (2019), Directed technical change as a response to natural-resource scarcity, Technical report, Sveriges Riksbank Working Paper Series 375.

Hassler, J., Krusell, P., Olovsson, C. & Reiter, M. (2018), 'Integrated assessment in a multi-region world with multiple energy sources and endogenous technical change', *Working Paper* .

Heal, G. (1984), Interactions between economy and climate: A framework for policy design under uncertainty, *in* V. K. Smith & A. D. White, eds, 'Advances in Applied Microeconomics', Vol. 3, JAI Press, Greenwich, CT, pp. 151–168.

Hoel, M. & Karp, L. (2002), 'Taxes versus quotas for a stock pollutant', *Resource and Energy Economics* **24**, 367–384.

Howard, P. H. & Sterner, T. (2017), 'Few and not so far between: A meta-analysis of climate damage estimates', *Environmental and Resource Economics* **68**, 197–225.

Iverson, T. & Karp, L. (2020), 'Carbon taxes and climate commitment with non-constant time preference', *The Review of economic studies* .

Jensen, S., Mohlin, K., Pittel, K. & Sterner, T. (2015), 'An introduction to the green paradox: The unintended consequences of climate policies', *Review of Environmental Economics and Policy* **9**(2), 246–265.

Karp, L. (2017), 'Provision of a public good with altruistic overlapping generations and many triibes', *The Economic Journal* **127**(607), 2641–2664.

Karp, L. & Traeger, C. P. (2021), 'Smart cap'.

Karp, L. & Zhang, J. (2006), 'Regulation with anticipated learning about environmental damages', *Journal of Environmental Economics and Management* **51**, 259–279.

Karp, L. & Zhang, J. (2012), 'Taxes versus quantities for a stock pollutant with endogenous abatement costs and asymmetric information', *Economic Theory* **49**, 371–409.

Karydas, C. & Xepapadeas, A. (2019), 'Pricing climate change risks: Capm with rare disasters and stochastic probabilities', *CER-ETH Working Paper Series Working Paper* **19**, 311.

Kaya, A., Csala, D. & Sgouridis, S. (2017), 'Constant elasticity of substitution functions for energy modeling in general equilibrium integrated assessment models: a critical review and recommendations', *Climatic Change* **145**(1), 27–40.

Luderer, G., Bosetti, V., Jakob, M., Leimbach, M., Steckel, J. C., Waisman, H. & Edenhofer, O. (2012), 'The economics of decarbonizing the energy systemresults and insights from the recipe model intercomparison', *Climatic Change* **114**(1), 9–37.

Luderer, G., Leimbach, M., Bauer, N., Kriegler, E., Baumstark, L., Bertram, C., Giannousakis, A., Hilaire, J., Klein, D., Levesque, A., Mouratiadou, I., Pehl, M., Pietzcker, R., Piontek, F., Roming, N., Schultes, A., Schwanitz, V. J. & Strefler, J. (2021), Description of the remind model (version 1.6), Technical report, SSRN Working Paper 2697070.

Ma, C. & Stern, D. I. (2016), 'Long-run estimates of interfuel and interfactor elasticities', *Resource and Energy Economics* **46**, 114–130.

Manne, A., Mendelsohn, R. & Richels, R. (1995), 'Merge: A model for evaluating regional and global effects of ghg reduction policies', *Energy policy* **23**(1), 17–34.

Newell, R. G. & Pizer, W. A. (2003), 'Regulating stock externalities under uncertainty', *Journal of Environmental Economics and Management* **45**, 416–432.

Nordhaus, W. D. (2017), 'Revisiting the social cost of carbon', *Proceedings of the National Academy of Sciences* **114**(7), 1518–1523.

Paltsev, S., Reilly, J. M., Jacoby, H. D., Eckaus, R. S., McFarland, J. R., Sarofim, M. C., Asadoorian, M. O. & Babiker, M. H. (2005), The mit emissions prediction and policy analysis (eppa) model: version 4, Technical report, MIT Joint Program on the Science and Policy of Global Change.

Pindyck, R. S. (2020), 'The social cost of carbon revisited', *Journal of Environmental Economics and Management* **94(C)**, 140–160.

Sinn, H.-W. (2012), *The Green Paradox*, MIT Press, Cambridge.

Stern, D. I. (2012), 'Interfuel substitution: a meta-analysis', *Journal of Economic Surveys* **26**(2), 307–331.

Traeger, C. P. (2018), Analytic climate economy, Technical Report 3307622, SSRN Working Paper.

Traeger, C. P. (2019), 'Capturing intrinsic risk aversion', *SSRN Working Paper 3462905* .

Traeger, C. P. (2021), 'ACE – Analytic Climate Economy', *American Economic Journal: Economic Policy* . conditional acceptance.

Valentini, E. & Vitale, P. (2019), 'Optimal climate policy for a pessimistic social planner', *Environmental and Resource Economics* **72**(2), 411–443.

van der Ploeg, F. (2018), 'The safe carbon budget', *Climatic change* **147**(1), 47–59.

van der Ploeg, F. & Withagen, C. (2012), 'Is there really a green paradox?', *Journal of Environmental Economics and Management* **64**(3), 342–363.

Weil, P. (1990), 'Nonexpected utility in macroeconomics', *The Quarterly Journal of Economics* **105**(1), 29–42.

Appendix

A Proofs and Calculations of Emission Response

A.1 Proof of Proposition 1

Using the general Bellman equation (7)

$$V_t(K_t, \boldsymbol{T}_t, \boldsymbol{M}_t, \boldsymbol{R}_t) = \max_{\boldsymbol{C}_t, \boldsymbol{N_t}, \boldsymbol{K}_t, \boldsymbol{E_t}} u(\boldsymbol{C}_t) + \beta V_{t+1}(K_{t+1}, \boldsymbol{T}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}) \ .$$

results in the first order condition (FOC) for consumption

$$u'(C_t) = \beta \frac{\partial V_{t+1}(\cdot)}{\partial K_{t+1}}.$$

As announced in Footnote 2, I will first show the slightly more general version where damages are not assumed to be multiplicatively separabel. Let the generic production function be $\tilde{F}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, \boldsymbol{T}_t)$, containing temperature as an explicit argument.

The FOC for fossil-based energy input E_i ($i \in \{1, ..., I^d\}$ is

$$\beta \frac{\partial V_{t+1}(\cdot)}{\partial K_{t+1}} \frac{\partial \tilde{F}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, \boldsymbol{T}_t)}{\partial E_{i,t}} + \beta \frac{\partial V_{t+1}(\cdot)}{\partial M_{t+1}} - \beta \frac{\partial V_{t+1}(\cdot)}{\partial R_{t+1}}$$

$$\Rightarrow \frac{\partial F(\cdot)}{\partial E_{i,t}} = -\frac{\beta \frac{\partial V_{t+1}(\cdot)}{\partial M_{t+1}}}{\beta \frac{\partial V_{t+1}(\cdot)}{\partial K_{t+1}}} + \frac{\beta \frac{\partial V_{t+1}(\cdot)}{\partial R_{t+1}}}{\beta \frac{\partial V_{t+1}(\cdot)}{\partial K_{t+1}}} = -\frac{\beta \frac{\partial V_{t+1}(\cdot)}{\partial M_{t+1}}}{u'(C_t)} + \frac{\beta \frac{\partial V_{t+1}(\cdot)}{\partial R_{t+1}}}{u'(C_t)}$$

where I used the FOC for consumption. Substituting in the defining equation of the SCC and the Hotelling rent and rearranging gives

$$\frac{\partial F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, \boldsymbol{T}_t)}{\partial E_{i,t}} = SCC_t + HOT_{i,t} = \Gamma_{i,t}$$

$$\Rightarrow \tilde{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, \boldsymbol{T}_t) \equiv \frac{\partial F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, \boldsymbol{T}_t)}{\partial E_{i,t}} \frac{E_t}{Y_t} = \Gamma_t \frac{E_t}{Y_t}$$

$$\Rightarrow E_t = \frac{Y_t \ \tilde{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, \boldsymbol{T}_t)}{\Gamma_t}$$

which is a general version of the statement in the proposition not relying on multiplicative damages. In the case of multiplicative damages, $\tilde{\sigma}_{Y,E_{i,t}}(\cdot) = \frac{\partial F(\cdot)(1-D(T_{1,t})}{\partial E_{i,t}} \frac{E_{i,t}}{Y_t} = \frac{\partial F(\cdot)}{\partial E_{i,t}} \frac{E_{i,t}}{Y_t}(1 - D(T_{1,t})) = \sigma_{Y,E_i}(1 - D(T_{1,t}))$ so that

$$E_t = \frac{Y_t^{net}\ \sigma_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t)}{\Gamma_{i,t}} = \frac{\sigma_{Y,E_i}(\cdot)}{\tilde{\Gamma}_{i,t}}.$$

A.2 Proof of Proposition 2

It is $SCC_t = Y_t^{net}\widetilde{SCC}_t$. I define the rates of change resulting from a change of $\widetilde{SCC}$ as $\hat{x} \equiv \frac{dx}{d\widetilde{SCC}}\frac{1}{x}$ for $x \in \{Y_t^{net}, \sigma_{Y,E_i}, \Gamma_t, SCC_t, HOT_{i,t}, \varphi_{R,i,0}, \widetilde{SCC}\}$. These changes are endogenous. Equation (17) delivers

$$\hat{E}_{i,t} = \hat{Y}_t^{net} + \hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \hat{\Gamma}_{i,t}.$$

It is $\Gamma_{i,t} = HOT_{i,t} + SCC_t = Y_t^{net}(\widetilde{HOT}_{i,t} + \widetilde{SCC}_t)$. Therefore,

$$\hat{\Gamma}_{i,t} = \hat{Y}_t^{net} + \frac{\frac{d\widetilde{HOT}_{i,t}}{d\widetilde{SCC}_t} + \frac{d\widetilde{SCC}_t}{d\widetilde{SCC}_t}}{\widetilde{HOT}_{i,t} + \widetilde{SCC}_t} = \hat{Y}_t^{net} + \frac{\widehat{\widetilde{HOT}}_{i,t}\widetilde{HOT}_{i,t} + \widehat{\widetilde{SCC}}_t\widetilde{SCC}_t}{\widetilde{HOT}_{i,t} + \widetilde{SCC}_t}$$

$$= \hat{Y}_t^{net} + \gamma_{i,t}\widehat{\widetilde{HOT}}_{i,t} + (1 - \gamma_{i,t})\widehat{\widetilde{SCC}}_t$$

where $\gamma_{i,t} = \frac{HOT_{i,t}}{\Gamma_{i,t}} = \frac{\widetilde{HOT}_{i,t}}{\tilde{\Gamma}_{i,t}}$. Therefore,

$$\hat{E}_{i,t} = \hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \widehat{\widetilde{SCC}} + \gamma_{i,t}(\widehat{\widetilde{SCC}} - \widehat{\widetilde{HOT}}_{i,t})$$

as stated in the proposition.

The second part of the proof requires additional structure that is, in particular, satisfied in the fully analytic model. Traeger (2021) shows that the value function solving the fully analytic model is of the form

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_t.$$

Moreover, the consumption rate is constant and and marginal utility is

As a result, I find

$$SCC_t = -\frac{\beta\frac{\partial V_{t+1}(K_{t+1}, \boldsymbol{T}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1})}{\partial M_{t+1}}}{u'(C_t)} = -\beta\varphi_{M,1}C_t = -\beta x^* \varphi_{M,1} Y_t^{net}$$

given logarithmic utility and the model's time-constant consumption rate x^*. Again, I note that in comparison to Traeger (2021), I have an additional β in the definition of the SCC because I define the SCC as the cost of an additional unit of emission in the current period rather than an instantaneous increase of atmospheric carbon in the current period. Similarly, I find that $HOT_{i,t} = \frac{\beta \frac{\partial V_{t+1}(\cdot)}{\partial R_{i,t+1}}}{u'(c_t)} = \beta \varphi_{R,i,t} x^* Y_t^{net}$. Using Traeger's (2021) result that the Hoteling rent in utils increases with the inverse of the utility discount factor, $\varphi_{R,i,t} = \beta^{-t} \varphi_{R,i,0}$, I find

$$\widetilde{SCC}_t = \beta x^* \varphi_{M,1} \quad \text{and} \quad \widetilde{HOT}_{i,t} = \beta^{1-t} x^* \varphi_{R,i,0}$$

Assuming the current change of $\widetilde{SCC}_0$, I have

$$\widehat{\widetilde{SCC}} \equiv \widehat{\widetilde{SCC}}_t = \widehat{\widetilde{SCC}}_0 \quad \text{and} \quad \widehat{\widetilde{HOT}}_{i,t} = \hat{\varphi}_{R,i,0}$$

where $\hat{\varphi}_{R,i,0}$ is the endogenous response of the current shadow value of the resource (in utils). The time aggregated change of emissions and, thus, use of the fossil fuel is

$$\sum_t dE^*_{i,t} = \sum_t E_t \Big(\hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \frac{HOT_{i,t} \hat{\varphi}_{R,i,0} + SCC_t \widehat{\widetilde{SCC}}}{HOT_{i,t} + SCC_t} \Big).$$

Given the Hotelling rent is positive, the resource constraint is binding and[23]

$$0 = \sum_t E_t \Big(\hat{\sigma}_{Y,E_i}(\cdot) - \frac{HOT_{i,t}}{HOT_{i,t} + SCC_t} \hat{\varphi}_{R,i,0} - \frac{SCC_t}{HOT_{i,t} + SCC_t} \widehat{\widetilde{SCC}} \Big). \tag{A.1}$$

Assume that

$$\hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \leq \frac{SCC_t}{HOT_{i,t} + SCC_t} \widehat{\widetilde{SCC}} \quad \forall t. \tag{A.2}$$

This assumption is equivalent to

$$\hat{\sigma}_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \leq \frac{\frac{\partial SCC_t}{\partial \widetilde{SCC}_t}}{\Gamma_{i,t}} = \frac{SCC_t}{\widetilde{SCC}\ \Gamma_{i,t}}$$

[23] If the contstraint flips from binding to non-binding due to a non-marginal change it has to hold $\sum_t dE^*_{i,t} \leq 0$.Then, the resource is no longer scarce after the change and the Hotelling rent drops to zero. I assumed a marginal change for simplicity.

noting that $\frac{\partial SCC_t}{\partial \widetilde{SCC}_t} = Y_t^{net} = \frac{SCC_t}{\widetilde{SCC}_t} = SCC_t\widehat{\widetilde{SCC}}_t$ (note $\widehat{\widetilde{SCC}}_t = \frac{1}{\widetilde{SCC}_t}$). Under assumption (A.2), the first and last term in equation (A.1)'s bracket result in a negative contribution. Then, equation (A.1) can only be satisfied if there exists t^* such that

$$\frac{HOT_{i,t^*}}{HOT_{i,t^*} + SCC_t^*}\hat{\varphi}_{R,i,0} \leq 0 \quad \Leftrightarrow \quad \hat{\varphi}_{R,i,0} \leq 0, \tag{A.3}$$

where the equivalence holds because the SCC and the total social cost are positive. If such t^* did not exist, the right side of equation (A.1) would be strictly negative. Making inequality (A.2) strict also renders inequality (A.3) strict.

Now assume that $\hat{\sigma}_{Y,E_i}(\cdot) \geq 0$. Then equation (A.1) can only be satisfied if there exists t^{**} such that

$$\frac{HOT_{i,t^{**}}\hat{\varphi}_{R,i,0} + SCC_{t^{**}}\widehat{\widetilde{SCC}}}{HOT_{i,t^{**}} + SCC_{t^{**}}} \geq 0 \quad \Leftrightarrow \quad \varphi_{R,i,0}\beta^{1-t^{**}}x^*\hat{\varphi}_{R,i,0} - \beta x^*\varphi_{M,1}\widehat{\widetilde{SCC}} \geq 0$$

because the total social cost and global net output are positive. We know that $\hat{\varphi}_{R,i,0} \leq 0$ and $-\beta x^*\varphi_{M,1}\widehat{\widetilde{SCC}} \geq 0$. Moreover, β^{-t} is increasing in time ($\beta < 1$) and all other terms are constant. Thus, the latter condition holds in some period only if it holds in the first period. Therefore,

$$HOT_{i,0}\hat{\varphi}_{R,i,0} \geq -SCC_0\widehat{\widetilde{SCC}},$$

which delivers the proposition's statement governing the Hotelling rent together with equation (A.3). Again, note that the assumption $\hat{\sigma}_{Y,E_i}(\cdot) > 0$ also renders the subsequent inequalities strict.

B Calculations of Model Specifications

B.1 Appendix DICE

In DICE (any version) production is of the form

$$Y_t = F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t) = A_tK_t^{\kappa}N_t^{1-\kappa}\left[1 - \Lambda_t(\cdot)\right] \tag{B.1}$$

where $\Lambda_t(\cdot)$ are the abatement costs. DICE uses the abatement rate as the direct control variable and abatement costs are a function of the abatement rate. The abatement rate is $\mu_t = 1 - \epsilon_t$, where $\epsilon_t = \frac{E_t}{E_t^{BAU}}$, as can be verified from DICE's emission equation

$$E_t = (1 - \mu_t) E_t^{BAU} \quad \Rightarrow \quad \epsilon_t = 1 - \mu_t = \frac{E_t}{E_t^{BAU}}$$
$$\text{with } E_t^{BAU} = \sigma_t A_t K_t^{\kappa} N_t^{1-\kappa}, \tag{B.2}$$

where σ_t is an exogenous process reflecting (exogenous) decarbonization of production. The subsequent derivation will use equation (B.2) further below, where $A_t K_t^{\kappa} N_t^{1-\kappa}$ cancels between this equation and Y_t^{net}, substantially simplifying the formula for the abatement rate and rendering it independent of $A_t K_t^{\kappa} N_t^{1-\kappa}$. However, the structural model assumptions require treating E_t^{BAU} as exogenous. That slightly inconsistent treatment of DICE's business as usual emissions makes the result a (close) approximation to the true DICE model.

DICE's abatement costs take the form

$$\Lambda_t(\cdot) = \theta_{1,t}(1 - \epsilon_t)^{\theta_2} \tag{B.3}$$

where $\theta_{1,t}$ is an exogenous process reducing abatement costs over time and $\theta_2 = 2.8$. Inserting equation (B.3) back into equation (B.1) results in equation (9) stated in the main text

$$Y_t = F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t) = A_t K_t^{\kappa} N_t^{\eta} \, G\left(\boldsymbol{A}_t^E, \boldsymbol{N}_t^E, \boldsymbol{E}_t\right).$$

with $\eta = 1 - \kappa$ and

$$G\left(A_t^E, E_t^{BAU}, E_t\right) = 1 - \theta_{1,t}\left(1 - \tfrac{E_t}{E_t^{BAU}}\right)^{\theta_2} = 1 - \theta_{1,t}\left(1 - \epsilon_t\right)^{\theta_2}$$
$$= 1 - \tfrac{1}{A_t^E} f\left(\tfrac{E_t}{E_t^{BAU}}\right) \quad \text{with} \quad A_t^E = \theta_{1,t}^{-1} \quad \text{and} \quad f(\epsilon_t) = \left(1 - \epsilon_t\right)^{\theta_2},$$

delivering equation (11).

Then the elasticity of production w.r.t. emissions (fossil-fuel-measured energy) is

$$\sigma_{Y,E}(\cdot) = \sigma_{G,E}(\cdot) = \frac{\partial G\left(A_t^E, E_t^{BAU}, E_t\right)}{\partial E_t} \frac{E_t}{G\left(A_t^E, E_t^{BAU}, E_t\right)}$$

$$= -\frac{E_t \frac{1}{A_t^E} \frac{1}{E_T^{BAU}} f'\left(\frac{E_t}{E_t^{BAU}}\right)}{1 - \frac{1}{A_t^E} f\left(\frac{E_t}{E_t^{BAU}}\right)} = -\frac{\frac{E_t}{E_T^{BAU}} f'\left(\frac{E_t}{E_t^{BAU}}\right)}{A_t^E - f\left(\frac{E_t}{E_t^{BAU}}\right)} = -\frac{\epsilon_t f'\left(\epsilon_t\right)}{A_t^E - f\left(\epsilon_t\right)}.$$

The first order condition for emissions becomes

$$\epsilon_t = \frac{1}{E_t^{BAU}} \frac{Y_t^{net} \sigma_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t^*, \boldsymbol{K}_t^*, \boldsymbol{E}_t^*)}{HOT_{i,t} + \beta SCC_t}$$

$$= -\frac{A_t K_t^\kappa N_t^\eta [1 - D_t(T_{1,t})] G\left(\boldsymbol{A}_t^E, \boldsymbol{N}_t^E, \boldsymbol{E}_t\right)}{E_t^{BAU}(HOT_{i,t} + \beta SCC_t)} \frac{\epsilon_t f'\left(\epsilon_t\right)}{A_t^E - f\left(\epsilon_t\right)}$$

$$= -\frac{A_t K_t^\kappa N_t^\eta [1 - D_t(T_{1,t})]}{HOT_{i,t} + \beta SCC_t} \frac{\epsilon_t f'\left(\epsilon_t\right)}{A_t^E E_t^{BAU}} = -\frac{[1 - D_t(T_{1,t})]}{HOT_{i,t} + \beta SCC_t} \frac{\epsilon_t f'\left(\epsilon_t\right)}{A_t^E \sigma_t},$$

where I substituted in DICE's formula for BAU emissions stated as equation (B.2). Solving the equation for the emission rate results in

$$\epsilon_t = f'^{-1}\left(-\left(HOT_{i,t} + \beta SCC_t\right) \frac{A_t^E \sigma_t}{1 - D_t(T_{1,t})}\right).$$

Using the specific functional form of DICE $f(\epsilon_t) = (1 - \epsilon_t)^{\theta_2}$ I find

$$f'(\epsilon_t) = -\theta_2 \left(1 - \epsilon_t\right)^{\theta_2 - 1} \quad \Rightarrow \quad f'^{-1}(z) = 1 - \left(-\frac{z}{\theta_2}\right)^{\frac{1}{\theta_2 - 1}}$$

and

$$\epsilon_t = 1 - \left(\frac{\Gamma_t A_t^E \sigma_t}{\theta_2 [1 - D_t(T_{1,t})]}\right)^{\frac{1}{\theta_2 - 1}} = 1 - \left(\frac{\Gamma_t A_t^E \sigma_t}{2.8\ [1 - D_t(T_{1,t})]}\right)^{\frac{5}{9}}$$

where I used DICE's value $\theta_2 = 2.8$. Expressed in terms of the abatement rate

$$\mu_t = \left(\frac{\Gamma_t\ A_t^E\ \sigma_t}{2.8\ [1 - D_t(T_{1,t})]}\right)^{\frac{1}{\theta_2 - 1}} \approx \sqrt{\frac{\Gamma_t\ A_t^E\ \sigma_t}{2.8[1 - D_t(T_{1,t})]}},$$

where I used the approximation $\frac{5}{9} \approx \frac{1}{2}$. Finally, DICE assumes that

$$\theta_{1,t} = A_t^{E^{-1}} = \frac{p_t^{back}\ \sigma_t}{\theta_2}$$

so that the expression simplifies further to

$$\mu_t = \left(\frac{\Gamma_t}{p_t^{back}[1 - D_t(T_{1,t})]}\right)^{\frac{1}{\theta_2 - 1}} \approx \sqrt{\frac{\Gamma_t}{p_t^{back}[1 - D_t(T_{1,t})]}},$$

using once again the approximation $\frac{5}{9} \approx \frac{1}{2}$ in the final step.

I THINK THE original was wrong only having D, NOT 1-D

DICE's marginal benefits from emissions are

$$MB_t^{DICE}(E_t) = \frac{\partial F}{\partial E_t} = (A_t L_T)^{1-\kappa} K_t^{\kappa} \left(1 - D(T_t)\right) \Psi_t \Big[1 - \frac{E_t}{E_t^{BAU}}\Big]^{1.8} \frac{2.8}{E_t^{BAU}}.$$

This marginal benefit curve is falling, convex, and for $E_t \to E_t^{BAU}$ the curve and its slope both approach zero, as is easily observed by taking the according derivatives.

Not conditioning on the SCC. Deriving "from scratch" using FOC

$$(1 + \beta\varphi_k) \frac{\frac{\partial F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\partial E_{i,t}}}{F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)} = \beta(\varphi_{R,i,t+1} - \varphi_{M,1}) \tag{B.4}$$

$$\Leftrightarrow E_{i,t} = \frac{(1 + \beta\varphi_k)\sigma_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\beta(\varphi_{R,i,t+1} - \varphi_{M,1})}$$

Let $\tilde{\Gamma}_{i,t} = \frac{(\varphi_{R,i,t} - \beta\varphi_{M,1})}{1+\beta\varphi_k} = (\varphi_{R,i,t} - \beta\varphi_{M,1})x^* = \frac{HOT_{i,t}}{Y_t^{net}} + \beta\frac{SCC}{Y_t^{net}}$. NOTE: Should probably be called $\tilde{\Gamma}_{i,t+1}$. The β results from the (model's) delay of emissions entering the atmosphere. Then, by the first order condition (B.4), I find

$$\frac{\frac{\partial F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\partial E_{i,t}}}{F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)} = \tilde{\Gamma}_{i,t}$$

$$\Leftrightarrow \frac{\theta_{1,t}\theta_2\mu_t^{\theta_2 - 1}\frac{1}{E_t^{BAU}}}{1 - \theta_{1,t}\mu_t^{\theta_2}} = \tilde{\Gamma}_{i,t}$$

$$\Leftrightarrow \theta_{1,t}\frac{\theta_2}{E_t^{BAU}}\mu_t^{\theta_2 - 1} = \tilde{\Gamma}_{i,t}\left(1 - \theta_{1,t}\mu_t^{\theta_2}\right)$$

$$\Leftrightarrow \theta_{1,t}\mu_t^{\theta_2 - 1}\left(\frac{\theta_2}{E_t^{BAU}} + \mu_t\tilde{\Gamma}_{i,t}\right) = \tilde{\Gamma}_{i,t}$$

Using $\theta_{1,t} = A_t^{E^{-1}} = \frac{p_t^{back}\ \sigma_t}{\theta_2}$ and $E_t^{BAU} = \sigma_t A_t K_t^{\kappa} N_t^{1-\kappa} = \sigma_t Y_t^{gross}$ (i.e. without abatement expenditure and damages) the equation can alternatively be written in terms of the backstop price p_t^{back} as

$$p_t^{back}\sigma_t\mu_t^{\theta_2-1}\left(\frac{1}{E_t^{BAU}}+\frac{\mu_t}{\theta_2}\tilde{\Gamma}_{i,t}\right)=\tilde{\Gamma}_{i,t}$$

$$\Leftrightarrow p_t^{back}\mu_t^{\theta_2-1}\left(1+\frac{\mu_t}{\theta_2}\tilde{\Gamma}_{i,t}E_t^{BAU}\right)=Y_t^{gross}\tilde{\Gamma}_{i,t}$$

$$\Leftrightarrow p_t^{back}\mu_t^{\theta_2-1}\left(1+\frac{\mu_t}{\theta_2}\sigma_t\tilde{\Gamma}_{i,t}Y_t^{gross}\right)=Y_t^{gross}\tilde{\Gamma}_{i,t}$$

The equation does not have an explicit solution. However, there is a closed-form solution specifying full abatement. Transforming the equation to

$$p_t^{back}\mu_t^{\theta_2-1}=Y_t^{gross}\tilde{\Gamma}_{i,t}\left(1-\frac{\mu_t}{\theta_2}\sigma_t\right)$$

$$\Leftrightarrow \frac{p_t^{back}\mu_t^{\theta_2-1}}{1-\frac{\mu_t}{\theta_2}\sigma_t}=Y_t^{gross}\tilde{\Gamma}_{i,t}$$

shows that the left side increases strictly in μ_t. Therefore, full abatement ($\mu_t = 1$) holds if

$$Y_t^{gross}\tilde{\Gamma}_{i,t}\geq\frac{p_t^{back}}{1-\frac{\sigma_t}{\theta_2}}\qquad\text{or}\qquad\tilde{\Gamma}_{i,t}\geq\frac{p_t^{back}}{\left(1-\frac{\sigma_t}{\theta_2}\right)Y_t^{gross}}=\frac{p_t^{back}}{Y_t^{gross}-\frac{E_t^{BAU}}{\theta_2}}.$$

$$\Leftrightarrow (1-\Lambda(1))\left(HOT_{i,t}+\beta SCC\right)\geq\frac{p_t^{back}}{1-\frac{\sigma_t}{\theta_2}}\qquad\text{or}$$

$$HOT_{i,t}+\beta SCC\geq\frac{p_t^{back}}{\left(1-\frac{\sigma_t}{\theta_2}\right)(1-\theta_{1,t})}.$$

B.2 Emission Derivations Golosov et al. (2014)

$a_{oil} = \kappa_1 = 0.5008$

$a_{coal} = \kappa_2 = 0.08916$

Golosov et al. (2014) combine Cobb-Douglas final good production from equation (9) with the energy sector

$$G\left(E_1, A_2N_2, A_3N_3\right) = \left(\kappa_1 E_1^s + \kappa_2 (A_2N_2)^s + \kappa_3 (A_3N_3)^s\right)^{\frac{\nu}{s}} = \left(\kappa_1 E_1^s + \kappa_2 E_2^s + \kappa_3 E_3^s\right)^{\frac{\nu}{s}}$$

which gives rise to the elasticities of production

$$\sigma_{Y,E_i} = \nu a_i \left(\frac{E_{i,t}}{E_t}\right)^s \quad \text{with energy composite} \quad E_t = \left(a_1 E_1^s + a_2 E_2^s + a_3 E_3^s\right)^{\frac{1}{s}}.$$

Golosov et al. (2014) only optimize the oil input directly, for which equation (1) translates into

$$E_{oil,t} = \frac{\nu a_{oil} \left(\frac{E_{oil,t}}{E_t}\right)^s Y_t^{net}}{HOT_{oil,t} + \beta SCC_t} \quad \Rightarrow \quad E_{oil,t}^{1-s} = \frac{\nu a_{oil} E_t^{-s} Y_t^{net}}{HOT_{oil,t} + \beta SCC_t}$$

$$\Rightarrow E_{oil,t} = \left. \frac{\nu a_{oil}}{\tilde{\Gamma}_{oil,t}} \right)^{\frac{1}{1-s}} E_t^{-\frac{s}{1-s}} \quad \text{where } \tilde{\Gamma}_{oil,t} = \frac{HOT_{oil,t} + \beta SCC_t}{Y_t^{net}}$$

delivering equation (20) stated in the main text.

Golosov et al. (2014) do not optimize the coal input directly. Instead, they assume that coal use is fully determined by labor input in the coal sector. Thus, the optimal coal use results from labor optimization after recognizing that equation (**??**) has to be modified replacing E_2 by A_2N_2. Then, the relevant terms on the r.h.s. Bellman equation for labor optimization become

$$(1+\beta\varphi_k) \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) + \lambda_t^N \left(1 - \textstyle\sum_{i=0}^3 N_{i,t}\right)$$
$$+\beta\varphi_{R,coal,t+1}\left(R_{coal,t} - A_{2,t}N_{2,t}\right) + \beta \boldsymbol{\varphi}_M^\top \left(\boldsymbol{\Phi M}_t + \left(E_{oil,t} + A_{2,t}N_{2,t}\right)\boldsymbol{e}_1\right)$$

giving rise to the FOC

$$N_{2,t} = \frac{(1+\beta\varphi_k)\sigma_{Y,N_2}(\boldsymbol{A}_t, \boldsymbol{N}_t^*, \boldsymbol{K}_t^*, \boldsymbol{E}_t^*)}{\lambda_t^N + \beta A_{2,t}\varphi_{R,coal,t+1} - \beta A_{2,t}\varphi_{M,1}} = \frac{(1+\beta\varphi_k)\nu a_{coal}\left(\frac{E_{coal,t}}{E_t}\right)^s}{\lambda_t^N + \beta A_{2,t}\varphi_{R,coal,t+1} - \beta A_{2,t}\varphi_{M,1}}$$

$$\Rightarrow E_{coal,t} = A_{2,t}N_{2,t} = \frac{\nu a_{coal}\left(\frac{E_{coal,t}}{E_t}\right)^s Y_t^{net}}{\left(\frac{\lambda_t^N}{A_{2,t}} + \beta\varphi_{R,coal,t+1} - \beta\varphi_{M,1}\right)\frac{Y_t^{net}}{(1+\beta\varphi_k)}}.$$

Defining $\omega_t^N = \lambda_t^N x^* Y_t^{net}$ as the wage in consumption equivalents I find

$$E_{coal,t} = \left(\frac{\nu a_{coal} Y_t^{net}}{\frac{\omega_t^N}{A_{coal,t}} + \Gamma_t} \right)^{\frac{1}{1-s}} E_t^{-\frac{s}{1-s}}. \tag{B.5}$$

Using the FOC for labor in the final goods sector, I find the relation

$$N_{0,t} = \frac{(1+\beta\varphi_k)\sigma_{Y,N_0}(\cdot)}{\lambda_t^N} = \frac{(1+\beta\varphi_k)(1-\kappa-\nu)}{\lambda_t^N} = \frac{1-\kappa-\nu}{\omega_t^N} Y_t^{net}$$

giving rise to the alternative formulation

$$E_{coal,t} = \left(\frac{\nu a_{coal} Y_t^{net}}{\frac{1-\kappa-\nu}{N_{0,t} A_{coal,t}} Y_t^{net} + \Gamma_t} \right)^{\frac{1}{1-s}} E_t^{-\frac{s}{1-s}} = \left(\frac{\nu a_{coal}}{\frac{1-\kappa-\nu}{N_{0,t} A_{coal,t}} + \tilde{\Gamma}_t} \right)^{\frac{1}{1-s}} E_t^{-\frac{s}{1-s}},$$

which is the r.h.s. version of equation (21). Recognizing that Golosov et al. (2014) assume the absence of a coal rent implies $\Gamma_t = \beta SCC$; dividing equation equation (20) by equation (B.5) delivers the energy use ratio stated in equation (22) of the main text.

B.3 Emission Derivations CapCES model

The elasticity production w.r.t. consumption good c_l is

$$\begin{aligned} \sigma_{Y,c_l}(\cdot) &= \frac{\partial Y_t(\cdot)}{\partial c_{l,t}} \frac{c_{l,t}}{Y_t} = \frac{\bar{s}}{s_t} \Big(\sum_l a_{l,t} c_{l,t}^s t \Big)^{\frac{\bar{s}}{s_t}-1} a_{l,t} \; s_t \; c_{l,t}^{s_t-1} \frac{c_{l,t}}{Y_t} \\ &= \bar{s} \frac{a_{l,t} \; c_{l,t}^{s_t}}{\sum_l a_{l,t} c_{l,t}^s t} = \bar{s} \; a_{l,t} \left(\frac{c_{l,t}}{Y_t} \right)^{s_t}. \end{aligned} \tag{B.6}$$

The $\bar{s}$ results from the additional exponent and the remaining part is the usual elasticity resulting from a CES aggregator. The Cobb-Douglas production function of an individual good exhibits the simple constant elasticity $\sigma_{c_l,d_l}(\cdot) = \frac{\partial c_{l,t}(\cdot)}{\partial d_{l,t}} \frac{d_{l,t}}{c_{l,t}} = \nu$ with respect to intermediate d_l. The intermediate d_l is another CES aggregator giving rise to the energy input elasticity

$$\sigma_{d_l,e_i}(\cdot) = \frac{\partial d_{l,t}(\cdot)}{\partial e_{i,t}} \frac{e_{i,t}}{d_{l,t}} = \tilde{a}_{l,i,t} \left(\frac{e_{i,t}}{d_{l,t}} \right)^{\tilde{s}_{l,t}},$$

where I explicitly introduced the CES-weights $\tilde{a}_{l,i,t}$ for the intermediate good aggregator $d_{l,t} = \left(\sum_i \tilde{a}_{l,i,t} e_{i,t}^{\tilde{s}_{l,t}}\right)^{\frac{1}{\tilde{s}_{l,t}}}$ as promised in footnote 7. Given the general functional form $g_{i,t}$ for energy production from a given fossil (or renewable) input, I merely define $\sigma_{e,i}(\cdot) = \frac{\partial g_{i,t}(\cdot)}{\partial E_i}\frac{E_i}{e_i}$. Using these results, the elasticity of production w.r.t. fossil input i becomes

$$\begin{aligned}
\sigma_{Y,E_i}(\cdot) &= \frac{\partial Y_t(\cdot)}{\partial E_{i,t}}\frac{E_{i,t}}{Y_t} = \sum_l \frac{\partial Y_t(\cdot)}{\partial c_{l,t}}\frac{c_{l,t}}{Y_t}\frac{\partial c_{l,t}(\cdot)}{\partial d_{l,t}}\frac{d_{l,t}}{c_{l,t}}\frac{\partial d_{l,t}(\cdot)}{\partial e_{i,t}}\frac{e_{i,t}}{d_{l,t}}\frac{\partial g_{i,t}(\cdot)}{\partial E_i}\frac{E_i}{e_i} \\
&= \textstyle\sum_l \sigma_{Y,c_l}(\cdot)\sigma_{c_l,d_l}(\cdot)\sigma_{d_l,e_i}(\cdot)\sigma_{e,i}(\cdot) \\
&= \bar{s}\ \nu \sum_l a_{l,t}\left(\frac{c_{l,t}}{Y_t}\right)^{s_t}\tilde{a}_{l,i,t}\left(\frac{e_{i,t}}{d_{l,t}}\right)^{\tilde{s}_{l,t}}\sigma_{e,i}(\cdot)
\end{aligned}$$

$$\Rightarrow E_{i,t} = \quad Y_t^{net}\frac{\bar{s}\ \nu \sum_l a_{l,t}\left(\frac{c_{l,t}}{Y_t}\right)^{s_t}\tilde{a}_{l,i,t}\left(\frac{e_{i,t}}{d_{l,t}}\right)^{\tilde{s}_{l,t}}\sigma_{e,i}(\cdot)}{HOT_{i,t}+\beta SCC}. \tag{B.7}$$

by Proposition 1. In the example, oil is only used only in the transport sector. Therefore, for $i = oil$ the sum is degenerate with $l = trans$. Similarly coal is only used in the other sectors, so that for $i = coal$ the sum is degenerate with $l = other$. Finally, by assumption, $HOT_{coal,t} = 0$. Thus, dividing equation (B.7) for oil ($i = oil$ and $l = trans$) over equation (B.7) for coal ($i = coal$ and $l = other$) delivers

$$\frac{E_{oil}}{E_{coal}} = \frac{\sigma_{e,oil}(\cdot)}{\sigma_{e,coal}(\cdot)}\frac{a_{trans,t}}{a_{other,t}}\left(\frac{c_{trans,t}}{c_{other,t}}\right)^{s_t}\frac{\tilde{a}_{trans,oil,t}}{\tilde{a}_{other,coal,t}}\frac{\left(\frac{e_{oil,t}}{d_{trans,t}}\right)^{\tilde{s}_{trans,t}}}{\left(\frac{e_{coal,t}}{d_{other,t}}\right)^{\tilde{s}_{other,t}}}\frac{\beta SCC}{HOT_{oil,t}+\beta SCC}$$

The intermediate transport uses oil and renewables

$$\begin{aligned}
&d_{trans,t} = \left(\tilde{a}_{trans,oil,t}e_{oil,t}^{\tilde{s}_{trans,t}} + \tilde{a}_{trans,renew,t}e_{renew,t}^{\tilde{s}_{trans,t}}\right)^{\frac{1}{\tilde{s}_{trans,t}}} \\
\Rightarrow &\left(\frac{e_{oil,t}}{d_{trans,t}}\right)^{\tilde{s}_{trans,t}} = \tilde{a}_{trans,oil,t}^{-1}\left(1+\frac{\tilde{a}_{trans,renew,t}}{\tilde{a}_{trans,oil,t}}\left(\frac{e_{renew,t}}{e_{oil,t}}\right)^{\tilde{s}_{trans,t}}\right)^{-1},
\end{aligned}$$

and similarly the intermediate for "other consumption" uses coal and renewables and it holds

$$\left(\frac{e_{coal,t}}{d_{other,t}}\right)^{\tilde{s}_{other,t}} = \tilde{a}_{other,coal,t}^{-1}\left(1+\frac{\tilde{a}_{other,renew,t}}{\tilde{a}_{other,coal,t}}\left(\frac{e_{renew,t}}{e_{coal,t}}\right)^{\tilde{s}_{other,t}}\right)^{-1}.$$

Thus, the expression for the fossil input ratio becomes

$$\frac{E_{oil}}{E_{coal}} = \frac{\sigma_{e,oil}(\cdot)}{\sigma_{e,coal}(\cdot)} \frac{a_{trans,t}}{a_{other,t}} \left(\frac{c_{trans,t}}{c_{other,t}}\right)^{s_t} \frac{1 + \frac{\tilde{a}_{other,renew,t}}{\tilde{a}_{other,coal,t}} \left(\frac{e_{renew,t}}{e_{coal,t}}\right)^{\tilde{s}_{elec,t}}}{1 + \frac{\tilde{a}_{trans,renew,t}}{\tilde{a}_{trans,oil,t}} \left(\frac{e_{renew,t}}{e_{oil,t}}\right)^{\tilde{s}_{trans,t}}}$$
$$\frac{\beta SCC}{HOT_{oil,t} + \beta SCC}.$$

Under the main text's assumption that the intermediate CES-weights are $\tilde{a}_{l,i,t} = 1$ for all l, i, the result implies equation (23) stated in the main text.

A note on Golosov et al.'s (2014) reasoning on carbon content and relative magnitude of the CES-weights. On page 60 of the manuscript, the authors state that "In reality, coal is a 'dirtier' energy source than oil: it produces more carbon emissions per energy unit produced. Since E_{oil} and E_{coal} are in the same units (carbon amount emitted) in the model, therefore, in a realistic calibration one should choose $a_{oil} > a_{coal}$", where I replaced their indices κ_1 and κ_2 using my labels for the CES-weights. Golosov et al.'s (2014) energy composite is the power mean of the individual energy sources, each of which is measured in units of carbon. Let me explicitly introduce fossil fuel specific conversion factors from carbon content to energy. Then, $Energy_i = \gamma_i E_i$ with $\gamma_{oil} > \gamma_{coal}$ because oil delivers more energy per emission unit. The Energy composite is of the form

$$\begin{aligned} Energy_t &= Mean(Energy_{oil}, Energy_{coal}, Energy_{renew}) \\ &= \left(w_{oil}(\gamma_{oil} E_{oil})^s + w_{coal}(\gamma_{coal} E_{coal})^s + w_{renew}(\gamma_{renew} E_{renew})^s\right)^{\frac{1}{s}} \\ &= \Big(\underbrace{w_{oil}\gamma_{oil}^s}_{\equiv a_{oil}} E_{oil}^s + \underbrace{w_{coal}\gamma_{coal}^s}_{\equiv a_{coal}} E_{coal}^s + \underbrace{w_{renew}\gamma_{renew}^s}_{\equiv a_{renew}} E_{renew}^s\Big)^{\frac{1}{s}}, \end{aligned} \tag{B.8}$$

where the w_i represent possible weights for other reasons but energy content. Without these additional weights, it is immediate from equation (B.8) that $s < 0$ implies $\gamma_{oil} > \gamma_{coal} \Rightarrow a_{oil} < a_{coal}$. If the energy sources are complements and oil gives more energy for the emissions-buck, the effective CES-weight on emissions from oil should be lower, not higher. This mistake in the reasoning does not affect their price-based calibration of the relative magnitudes and present comment is merely a clarification.

C Calibration Notes

For the calibration, we aim for two models. I suggest three models below, where the first two are alternative options. We leave calibrating the third for later, but should already include garthering the data needed for the calibration.

1. Model with two sectors, a

   - stylized "transport" sector which is really a catch all for oil use (also some renewable) and
   - a catch all for coal use (and also some renewable).

   This structure makes a lot of the calibration equations easier, but it is a bit more difficult to match the somewhat abstract sectors to real-world sectors and data because we will have to split all GDP in into a fossil and renewable, and an oil and renewable sector, which raises a bit of an ambiguity for how to deal with sectors that employ both or none.

2. Model with two sectors,

   - actual transport sector and it uses only oil and renewables (note, there's still some coal in transport but we would put it into the other sector)
   - the "other" sector which captures the remaining GDP and all three energy inputs, oil, coal, renewable.

3. More complex and more realistic 4 sector model:

   - sectors: transport, power, industrial, other
   - energy sources: oil, coal, renewables, and gas
   - we'll see based on the data what sector uses which energy sources, cutting less important ones out in a given sector and keeping the more important ones.

C.1 Equations for the calibration

We rely on the CapCES model, for now with constant elasticity of substitution that we take as given. We take SCC and Hotelling as given. For the global model we already looked for the SCC and its likely negligibly small. We look for an estimate of the Hotelling rent for oil and possibly gas. Hotelling rent for coal is probably zero. The generic calibration algorithm should work for any s, Hotelling rent, SCC. A play around value for s could be $s = -0.05$, which is Golosov et al.'s (2014) value (it's a bit high). We will eventually let s fall over time. Note that the interpretation of

- E_i is as the raw energy source,

for oil and coal it is the oil and coal in the ground (measured in CO_2 content). The only value is a potential scarcity rent. We will interpret the

- $e_{i,t} = g_{i,t}(\cdot)$ as the traded commodities.

This way we can use those price levels. Then the functions $g_{i,t}$ capture capital and labor use in the extraction sector and the function producing $c_{l,t}$ takes the labor and capital in the transport sector, power sector, and similar.[24]

Equation (B.6) specifies emission use for the general model

$$E_{i,t} = Y_t^{net} \frac{\bar{s}\ \nu \sum_l a_{l,t} \left(\frac{c_{l,t}}{Y_t}\right)^{s_t} \tilde{a}_{l,i,t} \left(\frac{e_{i,t}}{d_{l,t}}\right)^{\tilde{s}_{l,t}} \sigma_{e,i}(\cdot)}{HOT_{i,t} + \beta SCC}$$

The sum over l sums over the different sectors that employ the particular energy source i (likely all for renewables). The particular equation

$$\frac{E_{oil}}{E_{coal}} = \frac{\sigma_{e,oil}(\cdot)}{\sigma_{e,coal}(\cdot)} \frac{a_{trans,t}}{a_{other,t}} \left(\frac{c_{trans,t}}{c_{other,t}}\right)^{s_t} \frac{\tilde{a}_{trans,oil,t}}{\tilde{a}_{other,coal,t}} \frac{\left(\frac{e_{oil,t}}{d_{trans,t}}\right)^{\tilde{s}_{trans,t}}}{\left(\frac{e_{coal,t}}{d_{other,t}}\right)^{\tilde{s}_{other,t}}} \frac{SCC_t}{HOT_{oil,t} + SCC_t}$$

only holds for the first model version.

[24] It is not obvious how well this interpretation will fit with the functional forms and data and we might have to revise, but let's try it this way.

We obtain relative prices using the already calculated elasticities (actuall σ_{Y,e_i} here rather than σ_{Y,E_i} given we said the traded energy good with non-trivial price is e_i) to find

$$\frac{\partial Y_t(\cdot)}{\partial e_{i,t}} = \sigma_{Y,e_i}(\cdot)\frac{Y_t}{e_{i,t}} = \bar{s}\ \nu \sum_l a_{l,t}\left(\frac{c_{l,t}}{Y_t}\right)^{s_t} \tilde{a}_{l,i,t}\left(\frac{e_{i,t}}{d_{l,t}}\right)^{\tilde{s}_{l,t}} \frac{Y_t}{e_{i,t}}$$

and therefore we have the relative prices

$$\frac{\frac{\partial Y_t(\cdot)}{\partial e_{i,t}}}{\frac{\partial Y_t(\cdot)}{\partial e_{j,t}}} = \frac{\sum_l a_{l,t}\left(\frac{c_{l,t}}{Y_t}\right)^{s_t} \tilde{a}_{l,i,t}\left(e_{i,t}\right)^{\tilde{s}_{l,t}-1}\frac{1}{d_{l,t}^{\tilde{s}_{l,t}}}}{\sum_l a_{l,t}\left(\frac{c_{l,t}}{Y_t}\right)^{s_t} \tilde{a}_{l,j,t}\left(e_{j,t}\right)^{\tilde{s}_{l,t}-1}\frac{1}{d_{l,t}^{\tilde{s}_{l,t}}}}$$

At the sectorial level the relative price expression is much simpler, we obtain

$$\frac{\partial Y_t(\cdot)}{\partial c_{l,t}} = \sigma_{Y,c_l}(\cdot)\frac{Y_t}{c_{l,t}} = \bar{s}\ a_{l,t}\left(\frac{c_{l,t}}{Y_t}\right)^{s_t} \frac{Y_t}{c_{l,t}}$$

by equation (B.6) and, thus,

$$\frac{\frac{\partial Y_t(\cdot)}{\partial c_{l,t}}}{\frac{\partial Y_t(\cdot)}{\partial c_{k,t}}} = \frac{a_{l,t}}{a_{k,t}}\left(\frac{c_{l,t}}{c_{k,t}}\right)^{s_t-1}.$$

If we had obvious relative prices, this equation should make it easy to calibrate the a's. However, I think relative prices at the sectorial level are not obvious and we probably want to base the calibration on quantities. The tricky part is that we have a somewhat simplified model where the agent does not explicitly optimize over transport versus other consumption (and similar for other sectors). I think we should still be able to derive a quantity based calibration equation for the a's using consumption share of transport versus other consumption (and similar for other models). I leave it to you to try and think about it more carefully. If it does not work, the underlying ACE model transforms the present model into an equivalent model where the agent has log-CES preferences over the different consumption goods. Here, the a's directly correspond to the agent's demand, the s to the agent's substitutability in consumption, and the agent explicitly optimizes over each of the different consumption goods. So we can always resort to that more detailed model.

Note that the marginal product of e.g. renewable energy $\frac{\partial Y}{\partial e_{ren}}$ has to coincide across different sectors, which might be helpful for the calibration.

Finally, we have to specify the currently general functions given in equation (13)

$$e_{i,t} = g_{i,t}(A_{i,t}, K_{i,t}, N_{i,t}, E_{i,t})$$

$$\text{satisfying } \; g_{i,t}(A_{i,t}, \gamma K_{i,t}, N_{i,t}, E_{i,t}) = \gamma^{\tilde{\alpha}} g_{i,t}(A_{i,t}, K_{i,t}, N_{i,t}, E_{i,t}),$$

At least for coal (no Hotelling rent) and renewables, we have to introduce a cost function that leads to a natural saturation point, e.g.,

$$g_{i,t}(A_{i,t}, K_{i,t}, N_{i,t}, E_{i,t}) = A_{i,t} K_{i,t}^{\tilde{\alpha}} N_{i,t}^{1-\tilde{\alpha}-\tilde{\nu}} \left(\tilde{A}_t E_{i,t} - d E_{i,t}^2 \right)^{\tilde{\nu}} . \tag{C.1}$$

where $\tilde{A}_t$ is also technological progress permitting increasing amounts of renewables (and d a constant). An alternative and maybe slightly nicer version to calibrate would simply use a sector of the form

$$g_{i,t}(A_{i,t}, K_{i,t}, N_{i,t}, E_{i,t}) = A_{i,t} K_{i,t}^{\tilde{\alpha}} N_{i,t}^{1-\tilde{\alpha}-\tilde{\nu}} E_{i,t}^{\tilde{\nu}}$$

and then add a macro style cost to using resources by multiplying gross production in equation (1) with a cost term

$$Y_t = F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t)[1 - h(E_{1,t}, E_{2,t}, E_{3,t})]$$

where $h(E_{1,t}, E_{2,t}, E_{3,t})$, e.g. of the form $h(E_{1,t}, E_{2,t}, E_{3,t}) = cost_1(E_{1,t}) + cost_2(E_{2,t}) + cost_3(E_{3,t})$, specifies cost of resource use as a fraction of output. Such a model is feasible and would still go along with the analytic solution, but it would make the calculations of the production elasticity (and substiution elasticity) of the fossil fuels a bit more messy (haven't checked how much more messy). Maybe that's worth it. Maybe we can map these costs somewhat reasonably into the form of equation (C.1) or something similar that shows up only at the $g_{i,t}$ level (where it has to satisfy the homogeneity of capital of degree $\tilde{\alpha}$. The advantage of such a formulation is also that we can interpret the value of e_i as the price of renewables which would no longer work if we added macro-style costs to the final output function.

The form cannot avoid that there is unit substitutability between capital and the energy input, but we could avoid a similar and (even) less realistic substitubility between labor and the energy input implied by equation (C.1). E.g., we could use something like

$$g_{i,t}(A_{i,t}, K_{i,t}, N_{i,t}, E_{i,t}) = A_{i,t} K_{i,t}^{\tilde{\alpha}} \left(\tilde{A}_t N_{i,t}^{\gamma_1} E_{i,t}^{\gamma_2} - d_2 E_{i,t}^{\gamma_3} - d_3 N_{i,t}^{\gamma_4} \right). \tag{C.2}$$

where the γ's are designed to make the function slope down. Probably we only want 2 free parameters, but without knowing the coefficients I realized its not obvious that e.g. $\left(d_1 N_{i,t}^{1-\tilde{\alpha}-\tilde{\nu}} E_{i,t}^{\tilde{\nu}} - d_2 E_{i,t}^{2\tilde{\nu}} - d_3 N_{i,t}^{2(1-\tilde{\alpha}-\tilde{\nu})} \right)$ would in fact alway slope down if N and E are increased in the right proportion. I think a version of equation (C.2) would be my preferred starting point.