Optimal Robust Double Auctions
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Abstract

In a stylized exchange economy with single, continuous good and quasi-linear utilities, we propose a novel double-auction format featuring two (forward and reverse) clock auctions, Vickrey-style payments, and carefully designed per-unit taxes. In the spirit of Ausubel (2004), we show that there is a sincere ex-post perfect equilibrium of the game and that the market-clearing price is the Walrasian equilibrium price in an economy with deformed utilities. Furthermore, we show how the clocks can be adjusted dynamically to minimize the informational spillover between the two auctions. Finally, we show that the said taxes can implement an optimal robust mechanism in the sense of ex-post IC and IR constraints but ex-ante objectives, such as efficiency or revenue, in private values setting similar to Lu and Robert (2001). Further tractability is achieved given quadratic utilities, allowing for comparisons with nearly efficient robust mechanisms of Andreyanov and Sadzik (2021).

JEL Classfication: D44

Keywords: optimal, robust, double auctions, exchange, taxation, mechanism design

1 Introduction

In the 2017 FCC auction (dubbed *Incentive Auction*) for spectrum licenses, the standard *forward auction* was combined with the novel *reverse auction* to acquire and repackage spectrum, historically dispersed over numerous small owners, see Ausubel et al. (2012, 2017) and Cramton et al. (2015). However, the success was tarnished by several instances of supply reduction.[1] Combined with the uniform-price nature of payments, this could lead to under-performance in terms of revenue, see Doraszelski et al. (2017).[2] These shortcomings set the stage for an auction format that would be in some sense strategy-proof and also revenue-maximizing, that is, robust and optimal.

In this paper, we devise a dynamic auction that works in a two-sided market and multi-unit demand and supply, in other words, a typical double-auction environment. This auction should promote sincere bidding and have the capacity to be optimal. We are primarily interested in optimality with independent private values and ex-post IC, IR, and market-clearing constraints. For simplicity, we model the trade of a perfectly divisible, homogeneous asset but a non-linear utility $u(q_i)$ from consuming the asset. We break down the problem into three logical steps.

In the first step, we lay down general auction rules. Since it is a dynamic auction, it will inevitably share multiple features with the Ausubel auction. To be precise, there will be two continuous Ausubel auctions: forward and reverse, in which all agents participate. Next, we describe activity rules and transactional prices. Then we introduce the per-unit (i.e., marginal) taxes, crucial to the optimality of the auction, paid on top of the baseline Vickrey-style payments.

We show that this game has a sincere ex-post equilibrium in the spirit of Ausubel (2004, 2006), see Proposition 1. Sincerity here means that agents submit demands as if they were price-takers but with a new utility $v(q)$ instead of the original $u(q)$. We will refer to the fictitious exchange environment with these new utilities as the *virtual economy*. The auction rules then act as a Walrasian tatonnement, leading straight towards an efficient outcome in the virtual economy.

The ex-post nature of the equilibrium offers much freedom in adjusting the two clocks. Indeed, the existence of the sincere equilibrium does not depend on the

[1] For example, OTA Broadcasting, a private equity firm, has sold less than half of their owned spectrum, some of which was acquired just before the Incentive Auction, see Ausubel et al. (2017) for details.

[2] Public officials were concerned about raising enough amount of revenue from incentive auctions, see Loertscher et al. (2015) footnote 30. Also internet link 1, internet link 2

fine details of the price dynamics. For example, one could move the clocks to almost continuously match supply with demand, as in McAfee (1992). However, the objective that this approach minimizes - the temporary mismatch of supply and demand - has no direct connection to either efficiency or revenue.

Instead, we suggest paying attention to the unconstrained flow of information between the forward and reverse auction, as it might support unwanted equilibria. Thus, we suggest moving the clocks to minimize the damaging informational spillover between the two auctions. To be precise, we show that a price path exists such that the number of agents for whom spillover occurs is monotonically decreasing until there is at most one such agent, see Proposition 2. Moreover, simple price dynamics can mimic this price path we refer to as adaptive price dynamics.

In the second step, we study a broad class of smooth (except for $q = 0$) optimization problems, equivalent to finding an efficient and robust direct mechanism in the virtual economy parametrized by one-dimensional private types. We call it a v-optimal mechanism. We show how and under what conditions our dynamic auction implements this direct mechanism, see Proposition 3.

In the third step, we derive the optimal (profit-maximizing) direct mechanism in the original economy, given type distributions and original utilities $u(q_i, \theta_i)$. We show that, under mild regularity conditions, it is precisely a v-optimal mechanism that can be implemented via our auction, see Proposition 4.

We pay special attention to the so-called worst-off types. Lu and Robert (2001), working on a similar mechanism with interim constraints, admits that two-sided trade creates difficulties beyond standard mechanism design. Indeed, the monotonicity of a trader's virtual valuation typically fails at the worst-off type, even if the distribution of types is regular. Moreover, if we switch to the ex-post constraints, the locus of the worst-off types is conditioned on other players' types, making it nearly untractable.

Despite the apparent complexity of the optimal mechanism, the implementation is relatively simple - two Ausubel auctions with a per-unit tax on top. We will refer to the anti-derivative w.r.t. q of the per-unit tax as the *integrated tax*. The optimal integrated tax has three key features. First, it conditions the clock price due to the ex-post nature of the mechanism. Second, it generically has a kink at zero. Third, they are typically concave (but less concave than the utility) on each side of the

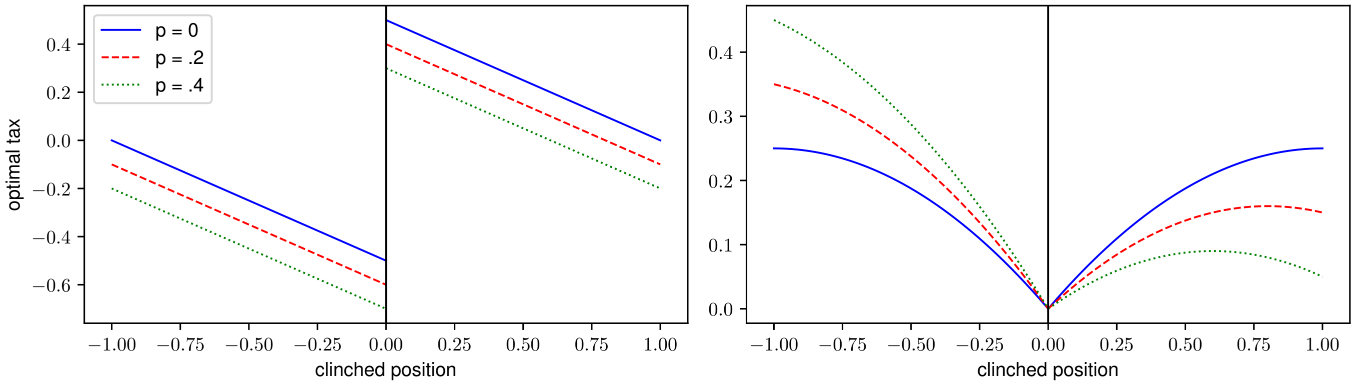

Figure 1: Per-unit (left figure) and integrated (right figure) optimal tax at different levels of the clock-price p, for a quadratic utility and uniform[-1,1] distribution.

kink.

The cusp shape of the integrated tax comes from two conflicting ideas. On the one hand, the auctioneer wants to exclude the traders whose contribution to exchange is minimal to exert pressure on the rest of the traders upwards. A convex kink "in the middle" guarantees precisely that. However, on the other hand, the auctioneer wants to minimize the distortion among the strongest buyers and sellers. Thus, the tax should flatten closer to the "shoulders."

In Andreyanov and Sadzik (2021), certain ad-hoc robust mechanisms were similarly implemented with taxes, albeit in a sealed-bid fashion. The implementation of the central mechanism, called σ-VCG, featured smooth, progressive taxes. To be precise, the integrated tax was equal to $\sigma q^2/2$. In the same paper, another robust mechanism was suggested, which can be thought of as a bid-ask spread.[3] This mechanism was implemented with an integrated tax of $\delta|q|$; thus, we will refer to it as δ-VCG.

A natural question is how much revenue the simpler but more practical σ-VCG and δ-VCG mechanisms yield relative to the optimal robust mechanism. To answer this question, we focus on a special case of ex-ante symmetric agents and quadratic utility. We consider two type distributions: uniform and logistic. In the large economy limit, we calculate expected revenue in closed form. For finite economies, however, we must rely on a combination of analytic results and Monte Carlo simulations.

The measurements in Table 1 imply that a quadratic tax is significantly under-

[3] See Example 8 in Andreyanov and Sadzik (2021)

	revenue from optimal		utility from efficient	
type distribution	optimal σ-VCG	optimal δ-VCG	optimal σ-VCG	optimal δ-VCG
uniform	50%	88.8%	75%	74.1%
logistic	64.8%	99.7%	75%	74.4%

Table 1: Percentage of optimal revenue and efficient surplus, achieved by the optimal σ-VCG and δ-VCG mechanisms in the large economy limit.

performing relative to the bid-ask spread in terms of revenue. Figure 2 also shows that σ-VCG is apparently dominated by δ in a much broader sense. Namely, for any σ-VCG mechanism, there exist a δ-VCG mechanism that yields the same expected utility but higher expected revenue. Moreover, the δ-VCG family of mechanisms is nearly identical to the Pareto frontier for the logistic distribution.

The paper is organized as follows. In Section 2 we explore the literature. In Section 3, we lay out a flexible design of the dynamic auction. In Section 4, we define a sincere equilibrium, and in Section 5, we discuss ways to minimize the informational spillover. In Section 6, we study the implementation of a broad class of mechanisms that we call v-optimal. In Section 7, we study profit-maximization and how it relates to v-optimality. Finally, in Section 8, we solve several examples with quadratic utility and conclude in Section 9.

The formal proofs for sections Sections 4 to 7 are contained in Appendices A to D.

2 Literature

Our paper is linked to several strands of the literature: the design of robust, optimal mechanisms and the design of practical auction rules.

The first strand is the classical literature on optimal mechanism design. The concept of virtualization, necessary for optimality, was developed independently by Mussa and Rosen (1978) and Myerson (1981). It was later generalized, among others, by Wilson (1985), Gresik and Satterthwaite (1989), Maskin and Riley (2000) and Lu and Robert (2001), to be used for two-sided and multi-unit environments. We add to this body of literature a non-linear utility and a small observation, see Lemma 2, that allows circumventing the non-monotonicity of virtual type.

The second strand is the design of robust mechanisms for exchange markets, typically with increasingly many participants. The concept of robustness is in the sense of Wilson (1987), Bergemann and Morris (2005), and Chung and Ely (2007), mean-

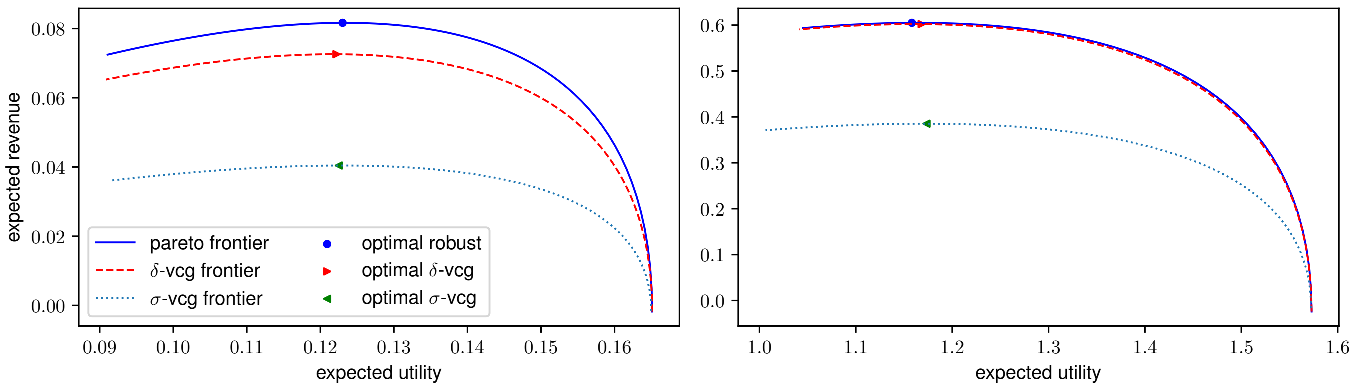

Figure 2: Welfare comparison of σ-VCG and δ-VCG mechanisms to the Pareto frontier, for $n = 100$ agents with quadratic utility and either uniform[-1,1] (left figure) or logistic (right figure) type distribution.

ing that the mechanism should work for all information structures, distributions, and beliefs. Here we can distinguish three approaches. The first classical approach is to find a mechanism with given properties, assuming that the type distribution is known. The second approach is to estimate the properties of the distribution using some kind of cross-validation, see Kojima and Yamashita (2017), or estimate it on the fly, see Loertscher and Marx (2020) and Loertscher and Mezzetti (2021). The third approach is consider the worst-case, relative to the maximized objective, scenario, see Brooks and Du (2021) and Suzdaltsev (2022). Our paper belongs to the first approach, which can be justified by saying that the distribution can always be estimated using a small randomly sampled group.

The third strand is the design of simple mechanisms when optimal mechanisms are impractical. For example, in Hart and Nisan (2017) it was argued that simple mechanisms for selling two goods could achieve a guaranteed fraction of the optimal revenue. In Andreyanov and Sadzik (2021), two families of simple mechanisms (σ-VCG and δ-VCG) were suggested for an exchange environment with multi-unit demands. In this paper, we give the means to compare them to the optimal mechanism and find that they often capture a significant portion of optimal revenue.

Our numerical exercises contribute to the long ongoing debate over the efficiency-revenue tradeoff in two-sided markets with private information on both sides. One of the oldest results in this area is the impossibility of budget surplus under efficient trade by Myerson and Satterthwaite (1983), meaning that full ex-post efficiency is very costly in terms of revenue. Another argument was made by Gresik and Satterthwaite (1989) that optimal mechanisms converge to efficiency at a quadratic rate, and in Lu and Robert (2001) that they converge to a simple bid-ask spread. Both results, however, rely on either unit demand or linear utility. With decreasing returns to scale, optimal mechanisms neither converge to efficiency nor to bid-ask spreads, which we confirm under quadratic utility. Furthermore, Loertscher et al. (2015) argues that the efficiency-revenue tradeoff is steeper in markets with two-sided private information compared to those with one-sided, meaning that full optimality might be too costly in terms of efficiency. It, therefore, seems natural to find a compromise between the fully optimal and efficient mechanisms. With our quadratic-utility model, we can trace the whole Pareto frontier. Interestingly, for logistic type distribution, the simple mechanisms based on bid-ask spreads (δ-VCG) almost reach that frontier.

Our paper also contributes to the expanding literature studying uniform-price and pay-as-bid auctions, see Back and Zender (1993), Ausubel et al. (2014) and Wang and Zender (2002). One of the main takeaways is that demand reduction with multi-unit demands can severely impact auction revenues. We show that one possible solution to the problem is a combination of a per-unit tax with a bid-ask spread. However, in our numerical exercises, the latter is disproportionally more important. Furthermore, Burkett and Woodward (2020) argues that there could also be low-revenue equilibria and suggests using reserve prices. Such "collusive-seeming" equilibria also emerge in our setting, but for a different reason: the inadvertent informational spillover between the two sides of the auction.

Finally, in the domain of robust auction design with private values, our paper is most similar to McAfee (1992) in its oral double-clock design and Ausubel (2004, 2006) in the clinching design of the payments. However, to our best knowledge, we are the first to characterize the optimal tax in the robust setting and to show that the price path can be optimized in terms of informational spillovers - a property unique to robust double auctions.

3 Dynamic auction

Our auction can be thought of as two copies: *forward* (i.e., ascending, buyers') and *reverse* (i.e., descending, sellers'); of the efficient dynamic auction of Ausubel (2004), with carefully crafted per-unit taxes on top of the baseline Vickrey-style payments. These additional payments are necessary to implement mechanisms other than efficient, for example, revenue-maximizing ones.

Our auction also resembles the recent Incentive auction used for spectrum bandwidth reallocation in its double-clock design. However, the payment rule is very different.

Forward, reverse auctions and clinching

There are two clock auctions that run either sequentially or in parallel. To distinguish between them, we will use superscript $+$ for the forward auction and $-$ for the reverse. We denote the *clock prices* in these auctions as p^+ and p^-.

Each player i participates in both auctions and, at any current price, submits a demand q_i^+ in the forward auction and q_i^- in the reverse auction. To be precise, in each round of either auction, the auctioneer first announces the clock price, and then bidders simultaneously and independently from each other respond with quantities.

The forward (reverse) auction starts at a low price p_0^+ (high price p_0^-) and then gradually raises (lowers) it. The price in the forward (reverse) auction stops at the first moment when the total demand becomes non-positive (non-negative). We will refer to this pair of, possibly different, clock prices as the *stop-off price*.

There is much freedom in how the auctioneer can move the clock prices towards each other. The exact instructions would depend on the auctioning style (discrete or continuous clocks) and also on the objectives of the auctioneer, which we will discuss later.

Following Ausubel (2004), at any clock-prices, we define *residual demands* as

$$q_{-i}^+ := -\sum_{j \neq i} q_j^+, \quad q_{-i}^- := -\sum_{j \neq i} q_j^-,$$

and *clinched demands* as

$$q_{i,c}^+ := \max(0, q_{-i}^+), \quad q_{i,c}^- := \min(0, q_{-i}^-).$$

Activity rules and transfers

Buyers and sellers can submit any demands as long as they satisfy two types of activity rules. First, demands in both auctions are non-increasing in their respective prices. Second, at any prices other than the stop-off ones, the agent's demand in the forward auction is no greater than her demand in the reverse auction. We do not impose any additional restrictions on the demands at this point.

Similar to Ausubel (2004), payments are made for the clinched units of the good and consist of two parts. The first part is standard - each incrementally clinched unit costs exactly the clock price at which it was clinched in the corresponding auction. The second part consists of marginal taxes $m\tau$ that depend on both the current price clock and the current position in clinched demands. Namely, agent i pays $m\tau(p^+, q^+_{i,c})$ for the unit clinched in the forward auction and $m\tau(p^-, q^-_{i,c})$ in the reverse.

Thus, agent i's total payment given final allocation q will therefore be equal to

$$\int_0^q p_{-i}(x) + m\tau_i(p_{-i}(x), x)dx,$$

where $p_{-i}(\cdot)$ is the *inverse residual demand curve* facing agent i.

It is worth mentioning that agents do not have direct control over the units that they clinch and the payments that they make or receive. However, they can affect the stop-off price.

Price dynamics

The two-sided nature of the environment requires us to decide on the order in which the price clocks will be moved. We refer to the protocol for switching between the two price clocks as *price dynamics*. We will consider two such protocols.

With *simple* price dynamics, we first fully advance the price clock in one of the two auctions: forward or reverse; until it hits the stop-off price. After that, we fully advance the price clock in the opposite auction. If the clock runs at a constant speed, the two auctions will end in a finite time.

While this is a remarkably naive approach, it is convenient for showing the existence of sincere equilibrium of the game, see Section 4. Moreover, the proof does not depend on the particular order in which the clock prices move as long as they are guaranteed to meet the stop-off price. We can therefore argue, albeit informally, that this result extends to all price dynamics that move the clocks monotonically and find the correct stop-off price.

A more sophisticated approach is to choose which clock to move based on the history of revealed demands. This can be done either at discrete times or continuously, depending on the nature of the price clock. Importantly, the clocks always have to move forward so that there will be a sincere equilibrium.

With price dynamics, which we refer to as *adaptive*, the clocks move according to the following rule: if the number of agents i for whom $q_i^+ > q_{-i}^-$, is greater (less) than the number of agents j for whom $q_j^- < q_{-j}^+$, move the forward (reverse) clock; otherwise, move either clock. These price dynamics are associated, albeit not perfectly, with a price path that optimizes the flow of information between the forward and reverse auctions, see Section 5.

Clearing rule

The *clearing rule* is a protocol for finalizing allocations and transfers at the moment when both forward and reverse auctions stop. If everyone plays continuously, then there will be an exact market clearing. However, because demands are allowed to jump, one can end up with a mismatch of supply and demand in the auction.

If such a mismatch happens, some of the most recent demands might require rationing. One such rationing procedure was proposed in Okamoto (2018), but there are others. In fact, any rationing procedure would suffice as long as each agent receives an allocation inside her true jump of demand at the stop-off price, and she is indifferent across the whole range of that jump.

4 Sincere bidding

We model quasilinear utilities $u_i(q_i) - t_i$, where t_i is the transfer and $u_i(q_i)$ is agent i's utility from holding q_i units of asset. We assume that u_i is continuously differentiable in q_i so that it has a well-defined derivative $mu_i(q) := \frac{\partial}{\partial q} u_i(q)$ everywhere. Let $\mathcal{U}_i$ denote the set of possible utility functions. Later, we will explain how $\mathcal{U}_i$ interacts with the scope of tax functions τ.

Central to the analysis of our auctions are *marginal taxes* $m\tau_i(p, q)$ and so-called *sincere demands* $d_i(p)$. These demands per se are not strategies but will capture most of the equilibrium behavior.

Definition 1. *A sincere demand $d_i(p)$ is defined as*

$$d_i(p) = \arg max_q \left[u_i(q) - \int_0^q m\tau_i(p,x)dx - pq \right].$$

Similar to Ausubel (2004, 2006), sincere demands are associated with a price-taking behavior, but with the taxes mixed into the utility. Indeed, one can think of the sincere demands as competitive demands given deformed utilities $u_i(q) - \int_0^q m\tau_i(p,x)dx$. We will refer to the $\int_0^q m\tau_i(p,x)dx$ term as the *integrated tax.*

Assuming that the marginal tax $m\tau_i(q,p)$ decreases strictly slower than marginal utility $mu_i(q)$ at all prices p, we can equivalently define the sincere demand as the unique solution to $p = mu_i(q) - m\tau_i(p,q)$.

For the purpose of this section, we will assume that at the starting prices p_0^+, p_0^-, there is no clinching and that the price clocks move according to the simple price dynamics. To be precise, first, the price clock in the forward auction moves at x-per-second speed until it reaches the stop-off price. Then, the price clock in the reverse auction does the same. The auction finishes after time $(p_0^- - p_0^+)x$ seconds has passed.

Imagine for now that, on the equilibrium path, all bidders will play their sincere demands. Absent per-unit taxes will lead to an efficient allocation, as in the general equilibrium with the fictitious Walrasian auctioneer. Indeed,

$$p = mu_i(q), \quad i = 1, \ldots, n,$$

is both a system of sincere demands without taxes and the first-order condition for maximizing the sum of utilities.

To the contrary, with per-unit taxes, the allocation is no longer efficient given the original utilities u_i. However, it can be considered efficient given some other utilities v_i. We can reverse engineer these utilities, up to a constant, by solving the following system of first-order conditions.

$$\begin{cases} p = mu_i(q) - m\tau_i(p,q), & i = 1, \ldots, n \\ p = mv_i(q_i), & i = 1, \ldots, n. \end{cases}$$

In other words, $mv_i(q)$ is the graph of the set of points in the (q,p) space, where the first-order conditions are satisfied for the sincere demand.

We would like the v_i utilities not only to exist but also to be strictly concave so that the first-order approach is valid. We will also refer to the economy with utilities v_i as the *virtual economy.*

Linearizing this system around (p, q), we get that

$$mv'_{i,q}(q) = -\frac{m\tau'_{i,q}(p,q) - mu'_{i,q}(q)}{1 + m\tau'_{i,p}(p,q)},$$

at points of differentiability. This motivates the following joint assumption on the set of possible utilities and taxes.

Assumption 1. *For any i and any possible utility $u_i \in \mathcal{U}_i$,*

$$m\tau'_{i,q}(q) - mu'_{i,q}(q) > \varepsilon, \quad (1 + m\tau'_{i,q}(q,p))^{-1} > \varepsilon,$$

for all $(p, q) \in \mathbb{R} \times (\mathbb{R}\backslash\{0\})$ and some $\varepsilon > 0$.

In other words, the tax has to be *less concave than* the utility at all prices, and the marginal tax can not decrease too fast in the current price. This assumption guarantees that $mv_i(q)$ is strictly decreasing with a slope of at least ε^2, and thus there is exactly one efficient outcome in the virtual economy.

Abusing notation, denote player i's strategies at clock prices $p^+ \leqslant p^-$ as $q_i^+(p^+)$ and $q_i^-(p^-)$ but the history of his play at the previous price p as $q_i^+(p|p^+)$ in the forward auction and $q_i^-(p|p^-)$ in the reverse.

Definition 2. *A collection of strategies $\{q_i^+(\cdot), q_i^-(\cdot)\}_{i=1}^n$ is said to comprise a sincere ex-post perfect equilibrium if, at any point in time t, the following strategies in the forward and reverse auctions*

$$q_i^+(p^+) := \min(d_i(p^+), \inf_{p \leqslant p^+} q_i^+(p|p^+)), \tag{1}$$

$$q_i^-(p^-) := \max(d_i(p^-), \sup_{p \geqslant p^-} q_i^-(p|p^-)), \tag{2}$$

where $\{d_i(\cdot)\}_{i=1}^n$ are sincere demands, constitute a Nash equilibrium in the continuation game, had all private information been common knowledge.

Formulas (1) and (2) simply mean that on and off-equilibrium paths, the agent is playing his sincere demand, as long as the monotonicity activity rule is not binding. Otherwise, he keeps his latest demand (in the respective market) unchanged until he can play his sincere demand again. We will refer to this strategy as *sincere bidding.*

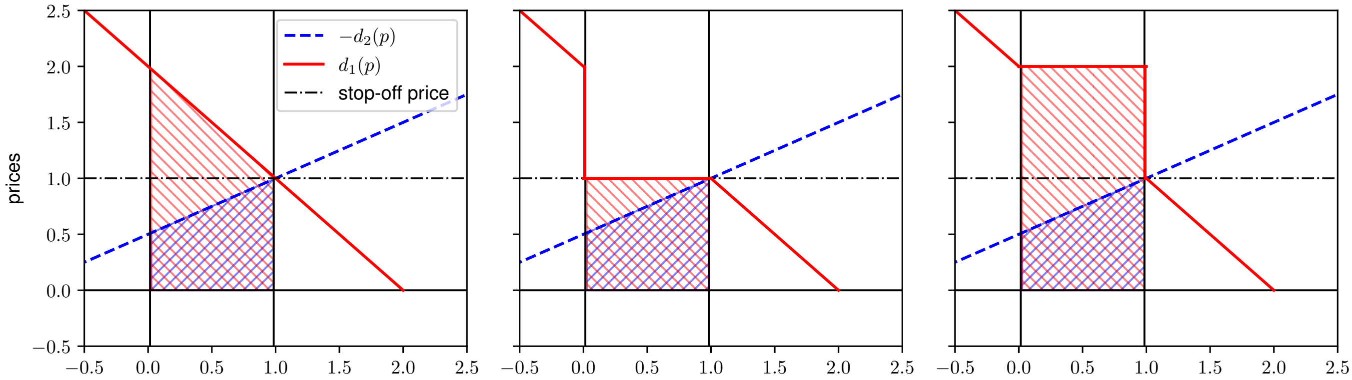

Figure 3: Payments in a sincere equilibrium (left), when player 1 advances her demand very late (middle), and very early (right).

We are now ready to state the first main result of our paper.

Proposition 1. *For any utilities and taxes satisfying Assumption 1,*

1. *the market clearing price and allocation under sincere bidding coincides with the Walrasian equilibrium in the virtual economy,*

2. *with simple price dynamics, sincere bidding is an ex-post perfect equilibrium.*

See Appendix A for formal proof.

It is worth noting that, with only a forward auction and with no taxes, the sincere play was weakly dominant if no information was revealed, see Ausubel (2004). The clock would advance at a constant speed, and the bidders would play sincerely.

Unfortunately, this is not true in the two-sided setting. The reason is that, by merely switching between the auctions, the auctioneer releases information that could be used for strategic purposes, see Examples 1 and 2.

Example 1. *With simple price dynamics and only the stop-off price revealed, sincere bidding is not weakly dominant.*

Consider two players $i = 1, 2$ with sincere demands $d_1(p) = 2 - p$, $d_2(p) = 1 - 2p$ that are common knowledge, and no additional taxes.

Under sincere bidding, the price in the forward auction starts as low as 0.5 and increases till the stop-off price of 1. The reverse auction starts with the clock price as high as $p = 2$ and falls down to 1. At the stop-off price, the first and second player's total clinches amount to 1 and −1, that is, first player is the buyer. The average prices are 0.75 and 1.5 correspondingly, see Figure 3 (left).

Consider now a modified strategy for player 1. Namely, if the stop-off price after the forward auction is less than 1, he plays sincerely in the reverse auction. Otherwise, he plays $\tilde{d}_1(p) = d_1(2) = 0$, that is, he advances his demand very early.

If player 2 proceeds with bidding sincerely, he will clinch everything at the stop-off price 1, see Figure 3 (right). If, however, she shifts her sincere demand to $\tilde{d}_2(p) = d_2(p) - \varepsilon$, for a small $\varepsilon > 0$, the stop-off price in the forward auction will be equal to $1 - \frac{\varepsilon}{2}$ and player 1 will then play sincerely in the reverse auction.

Consequently, the second player bids sincerely. Her loss due to the insidious actions of the first player in the reverse auction amounts to exactly 0.5, while by slightly shifting her demand, she can get arbitrarily close to her payoff in the sincere equilibrium. This completes the example.

One might say that weak dominance is a too strong equilibrium concept, and survival by eliminating weakly dominant strategies should be used instead, as was done in Ausubel (2004).

Example 2. *With simple price dynamics, and full history revealed at each point in time, there are strategies other than sincere bidding which survive iterated elimination of weakly dominated strategies.*

To build the counterexample, consider, again, sequential price dynamics and two players $i = 1, 2$ with sincere demands $d_1(p) = 2 - p, d_2(p) = 1 - 2p$.

Assume that both players bid sincerely in the forward auction, then the stop-off price would be equal to 1. At this point, both players know their final allocation, but the payments for player 2 are not yet known.

Consider now a modified strategy for player 1. Namely, in the reverse auction he plays $\tilde{d}_1(p^*)$, where p^* is the stop-off price. This means that as soon as the reverse auction starts, player 1 announces one demand to be equal to 1, thus effectively removing all the uncertainty from the auction, see Figure 3 (middle).

Clearly, this is an ex-post perfect equilibrium, and insincere play by player 1 in the reverse auction can never be eliminated, no matter what the original beliefs were, which completes the example.

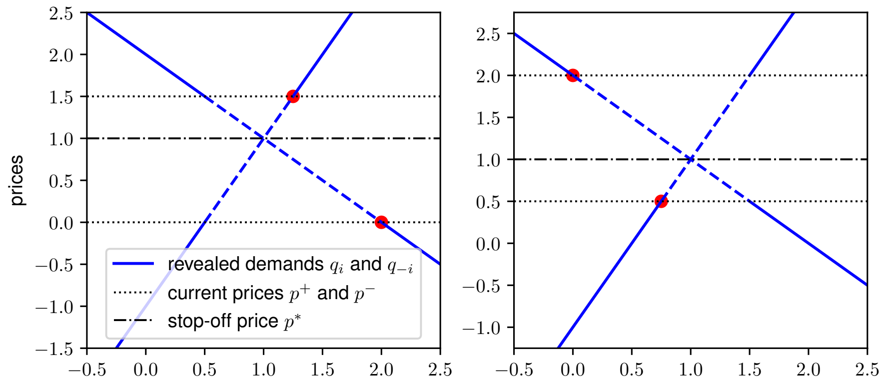

Figure 4: Informational spillover into forward auction (left figure) and into the reverse auction (right figure).

5 Informational spillover

The multiplicity of equilibria, discussed in the previous section, can be traced to a simple observation that by making the stop-off price known to the bidders, the auctioneer makes their incentives to reveal further true demand very weak. The premature revelation of the stop-off price is a consequence of a more general phenomenon: the inadvertent *informational spillover* between the two auctions.

For the purpose of this section, we assume that the auction is fully transparent.

Definition 3. *For agent i, and at clock prices $p^+ \leqslant p^-$ we say that there is informational spillover into the forward auction if $q^-_{-i}(p^-) < q^+_i(p^+)$, and into the reverse auction if $q^-_i(p^-) < q^+_{-i}(p^+)$.*

Imagine that at some point in time, the residual demand in the forward auction is ahead of agent i's sincere demand in the reverse auction, that is, $d_i(p^-) < q^+_{-i}(p^+)$, see Figure 4 (right). Then, assuming that all other agents bid sincerely, i can reveal any value between $[d_i(p^-), q^+_{-i}(p^+)]$ in the reverse auction without risking changing the stop-off price. Alternatively, he can keep ones demand unchanged for the range of prices $[d_i^{-1}(p^+), p^-]$ in the reverse auction. Thus, informational spillover allows supporting equilibria that are not sincere.

The question that we want to answer is whether there exists a price path that, in some sense, minimizes informational spillover. The simple price dynamics can not give us this property. Indeed, after fully advancing the clock in the forward auction, the residual demand there is no less than the sincere demand in the reverse auction for every player, and strictly so for strictly decreasing demands. Moreover,

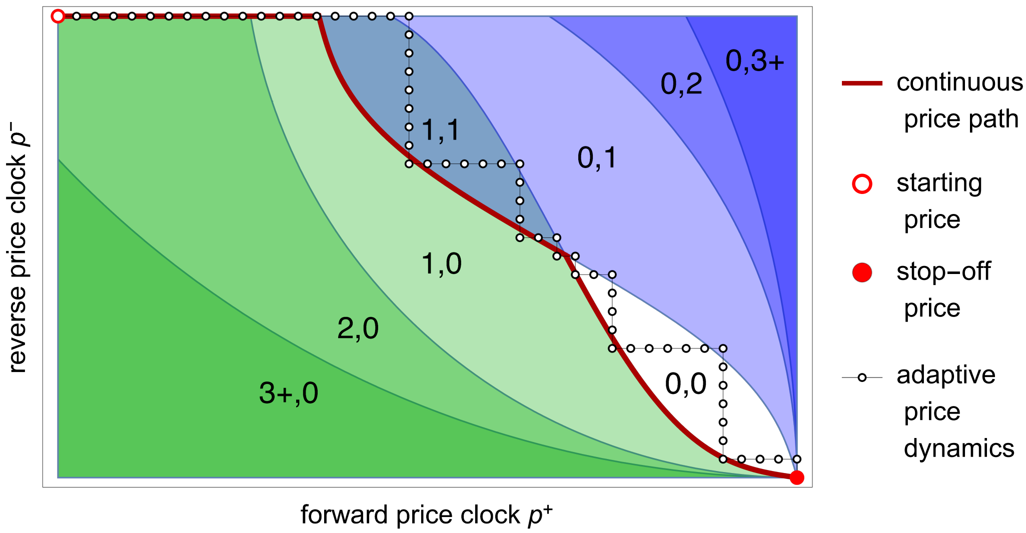

Figure 5: Illustration of the price path associated with adaptive price dynamics, and the spillover-minimizing continuous price path in Proposition 2.

for $n = 2$, it is simply impossible to rule out informational spillover with generic demands. However, we can try to make the number of agents that experience informational spillover as small as possible.

This approach is motivated by the following lemma:

Lemma 1. *For any clock prices $p^+ \leqslant p^-$: if there is spillover into both auctions, then there is spillover for exactly one agent.*

According to this lemma, the number of agents experiencing spillovers at any point in time is far from arbitrary. Represented by a pair of numbers, it can only be one of the following: $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, $(2+, 0)$, $(0, 2+)$; where $x+$ stands for "x and more", see Figure 5 for a stylized illustration. Moreover, when the numbers are $(1, 1)$, the same agent experiences spillover on both sides.

With this structure at hand, we can show that, for any collection of well-behaved sincere demands, a price path exists with special properties. Namely, along this path, the number of agents with spillovers decreases till there is at most one such agent, and it stays that way.

To make the result sharp, we put a few technical assumptions on the sincere demands and treat them as known.

Assumption 2. *Let i) agents play continuous and (weakly) monotone sincere demands $\{q_i(\cdot)\}_{i=1}^n$, ii) there exist a stop-off price p^* such that $\sum q_i(p^*) = 0$.*

Proposition 2. *Under Assumption 2, for any starting prices* $p_0^+ \leqslant p_0^-$*, there exist a (weakly) monotone path* $p_+(t), p_-(t)$ *connecting* (p_0^+, p_0^-) *with* (p^*, p^*) *continuously. The path consists of two parts. In the first part, the number of agents experiencing informational spillover decreases monotonically until there is, at most, one such agent. In the second part, there is still at most one such agent.*

See Appendix B for formal proof.

How does this help with the design of the auction? If we could find a realistic rule that mimics the aforementioned price path, it could be considered superior to other rules, such as simple price dynamics. The adaptive price dynamics does precisely that by moving the clocks in a way that balances the number of spillovers in the forward and reverse auctions, see Figure 5

Of course, one could say that adaptive price dynamics does more - it tries to reduce the number of spillovers to 0. However, there is no guarantee that the set of prices for which there are no spillovers is connected nor that it reaches the stop-off price. Thus, we can only guarantee to monotonically reduce the number of agents for whom spillover takes place to at most 1, but not 0.

6 Direct mechanisms

This section complements the double clock auction with taxes by providing an *optimal* taxation function. To achieve this goal, we add more structure to the agents' preferences: single-dimensional types and single-crossing preferences. We consider a flexible class of designer objectives, which covers expected profit maximization and near-efficiency as special cases. First, we derive an optimal direct mechanism for this class of objectives. Then, we derive the taxation function that achieves the same allocation and payoffs in the sincere equilibrium of our dynamic auction.

Single-dimensional types

We model agent i's preferences as a single-dimensional, private type $\theta_i \in \mathbb{R}$. Thus, agent i's payoff with type θ_i, asset allocation $q_i \in \mathbb{R}$ and money transfer $t_i \in \mathbb{R}$, is

$$u_i(\theta_i, q_i) - t_i.$$

We will refer to the whole profile of types as θ and the profile of types other than agent i as θ_{-i}. We begin with a minimal set of assumptions that are typically used in the mechanism design literature.

Assumption 3. *θ_i is independently distributed with a strictly increasing CDF F_i, $u_i(\theta_i, q_i)$ is twice continuously differentiable and strictly single crossing ($\frac{\partial^2}{\partial\theta_i \partial q_i} u_i(\theta_i, q_i) > 0$), for all i and θ_i, q_i in the support.*

We focus on direct mechanisms with a truth-telling equilibrium, invoking the revelation principle. A direct mechanism (q, t) consists of an allocation rule $q : \mathbb{R}^n \to \mathbb{R}^n$ and a transfer rule $t : \mathbb{R}^n \to \mathbb{R}^n$. A direct mechanism must satisfy the incentive compatibility (IC) and individual rationality (IR) constraints so that the agents play a truth-telling equilibrium. In this paper, we require that both IC and IR constraints are satisfied ex-post, that is, at each type profile on the type space, as in Andreyanov and Sadzik (2021), rather than on average, as in Lu and Robert (2001). Formally, they are defined as below.

Definition 4. *A direct mechanism (q, t) satisfies the ex-post IC and IR constraint if it satisfies the following inequalities.*

IC: $u_i(\theta_i, q(\theta_i, \theta_{-i})) - t(\theta_i, \theta_{-i}) \geqslant u_i(\theta_i, q(\theta_i', \theta_{-i})) - t(\theta_i', \theta_{-i})$,

IR: $u_i(\theta_i, q(\theta_i, \theta_{-i})) - t(\theta_i, \theta_{-i}) \geqslant u_i(\theta_i, 0)$.

for all i and all θ in the support.

Denote by $\tilde{u}_i(\theta_i, q_i)$ the net (i.e. added relative to the autarky), utility of agent i:

$$\tilde{u}_i(\theta_i, q_i) = u_i(\theta_i, q_i) - u_i(\theta_i, 0).$$

A standard mechanism-design argument tells, see e.g. Milgrom and Segal (2002), that under Assumption 3, a direct mechanism (q, t) is ex-post IC if and only if: $q_i(\theta_i, \theta_{-i})$ is weakly increasing in θ_i and the *envelope conditions* hold:

$$t_i(\theta_i, \theta_{-i}) = \tilde{u}_i(\theta_i, q_i(\theta_i, \theta_{-i})) - \tilde{u}_i(\theta_i', q_i(\theta_i', \theta_{-i})) - \int_{\theta_i'}^{\theta_i} \frac{\partial}{\partial \theta_i} \tilde{u}_i(x, q_i(x, \theta_{-i})) dx, \quad (3)$$

for all i and $\theta_i, \theta_i', \theta_{-i}$ in the support.

Net surplus and worst-off-types

Another convenient way to describe an ex-post IC direct mechanism - is in terms of the agent's net equilibrium payoff $\tilde{s}_i(\theta_i, \theta_{-i})$ that we refer to as her *net surplus*:

$$\tilde{s}_i(\theta_i, \theta_{-i}) = \tilde{u}_i(\theta_i, q_i(\theta_i, \theta_{-i})) - t_i(\theta_i, \theta_{-i}) = \max_{\theta_i'} \{\tilde{u}_i(\theta_i, q_i(\theta_i', \theta_{-i})) - t_i(\theta_i', \theta_{-i})\}.$$

Furthermore, let $wot(\theta_{-i})$ denote the set of *worst-off types*, and $tet(\theta_{-i})$ denote the set of *types excluded from trade*, of agent i in a mechanism:

$$wot(\theta_{-i}) = arg\min_{\theta_i'} \tilde{s}_i(\theta_i', \theta_{-i}), \quad tet(\theta_{-i}) = \{\theta_i' : q_i(\theta_i', \theta_{-i}) = 0\}.$$

Lemma 2. *Under Assumption 3, in an ex-post IC direct mechanism* (q, t)*,*

$$tet(\theta_{-i}) \subset wot(\theta_{-i}).$$

The above lemma allows to recast the envelope conditions (3):

$$\tilde{s}_i(\theta_i, \theta_{-i}) = \inf_{\theta'} \tilde{s}_i(\theta_i', \theta_{-i}) + \int_{\theta_i^*}^{\theta_i} \frac{\partial}{\partial \theta_i} \tilde{u}_i(x, q_i(x, \theta_{-i}))dx, \quad \forall \theta_i^* \in tet(\theta_{-i}), \tag{4}$$

for any i and θ_{-i} in the support such that $tet(\theta_{-i})$ is non-empty.

v-optimality

We are interested in a broad class of ex-post IC and IR direct mechanisms, which we will refer to as *v-optimal.*

Consider a collection of functions $v_i(\theta_i, q)$, which can be interpreted as individual contributions of each agent to a certain social utility. We wish to maximize this social utility subject to the *market clearing* constraint $\sum_{i=1}^n q_i = 0$, also ex-post, that is, satisfied for all types in the support. Additionally, we normalize each agent's payoff at the worst-off type to be equal to her payoff in the autarky.

Definition 5. *A v-optimal direct mechanism* (q, t) *maximizes*

$$\iiint_{\mathbb{R}^n} \left[\sum_{i=1}^n v_i(\theta_i, q_i)\right] \prod_j dF_j(\theta_j) \tag{5}$$

subject to the constraints: $q_i(\theta_i, \theta_{-i})$ *is weakly increasing in* θ_i*, envelope conditions* (3)*, market clearing and* $\inf_{\theta_i'} \tilde{s}(\theta_i', \theta_{-i}) = 0$*, for all* i *and* θ_i, θ_{-i} *in the support.*

While not fully general, this formulation covers a number of important families of mechanisms. In particular, three such families have been studied before. The first family, studied in Gresik and Satterthwaite (1989), Lu and Robert (2001), in the context of Bayesian IC and IR constraints, can be informally defined via $v_i = (1-\alpha)u_i + \alpha t_i$, and can be thought of as a convex combination of efficient and profit-maximizing mechanisms. The second and third families, studied in Andreyanov and Sadzik (2021), are $v_i = u_i - \sigma q_i^2/2$ and $v_i = u_i - \delta|q|$. They can be thought of as nearly efficient mechanisms, capable of balancing the budget ex-post through controlled demand reduction. By coincidence, if the utility is quadratic: $u_i(\theta_i, q) = \theta_i q - \mu q^2/2$, the second family also contains (for $\sigma = \frac{\mu}{n-2}$) the uniform-price double auction, studied, among others, in Kyle (1989) and Rostek and Weretka (2012).

We place a few technical assumptions on the auctioneer's objective v, which ensure that the v-optimal mechanism is a solution to a smooth (with a notable exception of $q = 0$) and convex optimization problem.

Assumption 4. *$v_i(\theta_i, q_i)$ is twice continuously differentiable, strictly concave in q_i and strictly single crossing, for all i, $q_i \neq 0$, and θ_i in the support.*

With a slight abuse of notation, let $mv_i(\theta_i, q_i)$ denote $\frac{\partial}{\partial q_i} v_i(\theta_i, q_i)$ at points of differentiability, and the sub-gradient of v_i otherwise. Likewise, let $mu_i(\theta_i, q_i)$ denote $\frac{\partial}{\partial q_i} u_i(\theta_i, q_i)$. We now move on to characterize v-optimal direct mechanisms.

Monotonicity of allocation

As is common in the literature, we first attempt to solve a relaxed problem - where we drop the monotonicity constraint so that we can solve for the allocation q, pointwise, and then check if it is monotone. A version of the Kuhn-Tucker theorem ensures that the Lagrangian method applies in the relaxed problem, yielding the first order conditions:

$$p(\theta) \in mv_i(\theta_i, q_i(\theta)), \qquad \sum_{i=1}^{n} q_i(\theta) = 0, \tag{6}$$

where $p(\theta) \in \mathbb{R}$ is the Lagrange multiplier.[4] By strict concavity of the v_i functions, $q_i(\theta)$ is single-valued.

Below we verify that the solution to the relaxed problem is indeed monotone. For any $q \neq 0$, we may linearize (6) around (p, q) as below

$$mv'_{i,\theta_i} + mv'_{i,q_i} q'_{i,\theta_i} = p'_{\theta_i}, \quad mv'_{j,q_j} q'_{j,\theta_i} = p'_{\theta_i}, \quad j \neq i. \tag{7}$$

We can then solve for the slopes using market clearing

$$p'_{\theta_i} = \frac{mv'_{i,\theta_i}}{mv'_{i,q_i}} \left(\sum \frac{1}{mv'_{k,q_k}}\right)^{-1}, \quad q'_{j,\theta_i} = \frac{mv'_{i,\theta_i}}{mv'_{j,q_j}} \left(\frac{1/mv'_{j,q_j}}{\sum 1/mv'_{k,q_k}} - \mathbb{I}(j = i) \right). \tag{8}$$

[4] For example, Theorem 1 and 2 in Luenberger 1969, 217p and 221p.

Clearly, under strict single crossing and strict concavity of the v_i functions, the allocation of any agent is strictly increasing in her type.[5] We would like to further strengthen this property by uniformly bounding the slopes of mv_i.

Assumption 5. *$mv'_{i,\theta} > \varepsilon$, and $-1/mv'_{i,q} > \delta$, for all i and $\theta_i, \theta_{-i}, q_i$ in the support, and some $\varepsilon, \delta > 0$.*

We can therefore bound the slope of the allocation function from below:

$$q'_{j,\theta_j} = mv'_{j,\theta_j} \cdot \frac{(-1/mv'_{j,q_j}) \cdot (\sum_{k \neq j} -1/mv'_{k,q_k})}{(-1/mv'_{j,q_j}) + (\sum_{k \neq j} -1/mv'_{k,q_k})} \geqslant \frac{n-1}{n} \varepsilon \delta. \tag{9}$$

Consequently, one can invert the allocation function with respect to own type everywhere except $q = 0$. We will refer to it as an *inverse allocation function* $q_i^{-1}(x, \theta_{-i})$, defined on the $(\mathbb{R} \backslash \{0\}) \times \mathbb{R}^{n-1}$ domain.

Existence of the type excluded from trade

Before we proceed to the first main result of this section, there is one technical observation that we need to make. Namely, we would like that, in a v-optimal mechanism, there exists a type excluded from trade, that is, $tet(\theta_{-i})$ is nonempty. Together with the *taxation principle*, this will allows to once again recast the envelope conditions using the inverse allocation function.

Lemma 3. *Under Assumptions 3 to 5, in a v-optimal mechanism (q, t), $tet(\theta_{-i})$ is nonempty, and the transfers can be written as:*

$$\tilde{t}_i(q, \theta_{-i}) = \int_0^q mu_i(q_i^{-1}(x, \theta_{-i}), x) dx, \tag{10}$$

for all i and θ_{-i} in the support.

For the exposition, we provide two alternative versions of Assumption 5 that would ensure the existence of types excluded from trade, see Section C.3. The first version requires that all v_i, F_i are identical, and can be used with compact support. The second version requires that for any $p \in \mathbb{R}$ there exists a type θ_i in the support such that $p \in mv_i(\theta_i, 0)$.

[5] The allocation function is also strictly decreasing in types of others, and the market clearing price is strictly increasing in all types.

Taxation scheme

We are now ready to derive the taxation scheme associated with the auction described in Section 3, which would match the one in our v-optimal mechanism. According to the rules of the auction, the payments consist of two parts: the Vickrey-style payments and the integrated (along the residual supply curve) marginal taxes

$$\tilde{\tilde{t}}_i(q, \theta_{-i}) = \int_0^q m\tau(p_{-i}(x), \theta_{-i}) + p_{-i}(x)dx, \tag{11}$$

where $p_{-i}(x)$ is the residual supply curve facing agent i.

If we could set the marginal tax equal to the wedge between mu_i and mv_i at the desired allocation, the agents would essentially perceive v_i as their true utility. It only remains to do it for every realization of types.

Definition 6. *Set the marginal tax $m\tau_i(p, q) = x$, where $(x, \hat{\theta})$ solves*

$$\begin{cases} x = mu_i(\hat{\theta}, q) - mv_i(\hat{\theta}, q), \\ p = mv_i(\hat{\theta}, q), \end{cases} \tag{12}$$

for all p, q in the support.

We refer to the solution $\hat{\theta}_i(p, q)$ to the system of equations (12), as the *fixed-point type*. It is correctly defined on the $\mathbb{R} \times (\mathbb{R} \backslash \{0\})$ domain, and so is the marginal tax.

We are ready to formulate our second main result.

Proposition 3. *Under Assumptions 3 to 5, the sincere equilibrium in the double clock auction with the marginal tax $m\tau(p, q)$ defined by (12) achieves the same allocation and transfer as in the v-optimal mechanism.*

The proof proceeds by observing, quite mechanically, the equivalence between the transfers $\tilde{t}_i(\theta_i, q_i)$ and $\tilde{\tilde{t}}_i(\theta_i, q_i)$, see Appendix C.

Finally, by linearizing (12) around (p, q), we can derive the slopes of the fixed-point type and the marginal tax

$$\hat{\theta}'_p = 1/mv'_\theta, \quad \hat{\theta}'_q = -mv'_q/mv'_\theta,$$
$$m\tau'_p = mu'_q\hat{\theta}'_p - 1, \quad m\tau'_q = mu'_\theta\hat{\theta}'_q + mu'_q.$$

Corollary 1. *Under Assumptions 3 to 5, the marginal tax $m\tau_i$ is continuously* *differentiable for all $q \neq 0$ and satisfies*

$$m\tau_q' - mu_q' > 0, \quad m\tau_p' + 1 > 0.$$

In other words, the (integrated) tax is less concave than the utility, and the marginal tax can not respond to the change in price too fast.

7 Revenue maximization

We now move on to the special case of interest, which is revenue maximization. Ignoring the monotonicity constraint, we will attempt to maximize the average transfer

$$\iint\limits_{\mathbb{R}^{n-1}} \sum_{i=1}^{n} \left[\int_{\mathbb{R}} (\tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i}))\, dF_i(\theta_i) \right] \prod_{j \neq i} dF_j(\theta_j) \tag{13}$$

subject to market clearing and envelope conditions. Naturally, in the revenue-maximizing mechanism, any leftover surplus can be extracted via translation of monetary transfers, therefore, $\inf_{\theta_i'} \tilde{s}_i(\theta_i', \theta_{-i}) = 0$.

Before we proceed with a classic Myersonian trick, there is one more assumption that we have to make, related to the integrability of net surplus, which is necessary for integration by parts on the whole real line. [6]

Assumption 6. *$\sum_{i=1}^{n} \tilde{u}_i(\theta_i, q_i) \leqslant C(\theta_i)$ for any $q : \sum_{i=1}^{n} q_i = 0$ and some function $C(x)$, such that $\int C(x) dF_i(x) < \infty$.*

Although this assumption is very weak, from it follows that the expected net surplus in the exchange economy is finite. To see the importance of this observation, note that even with simple quadratic models as in Section 8, the utility is not bounded on $\mathbb{R}$, and thus the expected net surplus is not obviously bounded.

Lemma 4. *Under Assumptions 3, 4 and 6:*

$$\int \tilde{s}_i(z, \theta_{-i}) dF_i(z) < \infty$$

for all θ_{-i} in the support.

With this at hand, we split the integral of the net surplus at the type excluded from trade (which is guaranteed to exist) and apply integration by parts to each of

[6] Riemann if F is continuous, or, more generally, Stiltjes

the two halves. This gives us the following equivalence

$$\int_{\mathbb{R}} \left(\tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i})\right) dF_i(\theta_i) = \int_{\mathbb{R}} J_i(\theta_i, q_i) dF_i(\theta_i) \tag{14}$$

$$J_i(\theta_i, q_i) = \tilde{u}(\theta_i, q_i) - \frac{\mathbb{I}(q_i > 0) - F(\theta_i)}{f(\theta_i)} \frac{\partial}{\partial \theta} \tilde{u}(\theta_i, q). \tag{15}$$

We will refer to J_i as the *virtual utility*.

It is worth noting that the virtual utility, in this particular form, is continuous in both allocation and type. Indeed, the only potential source of discontinuity is the indicator function $\mathbb{I}(q_i > 0)$ multiplied by $\frac{\partial}{\partial \theta}\tilde{u}(\theta_i, q)$, which is zero at $q = 0$. Thus, there is no jump at $q_i = 0$. Instead, there is a concave kink. Had we not used Lemma 2, the virtual utility would be discontinuous in type.

It remains to check whether the premise of the Proposition 3 is satisfied so that we can also claim the implementation of the profit-maximizing mechanism. One simple way to achieve this - is to put more assumptions on the true utilities u_i.

Assumption 7. *F_i is log-concave and*

$$u'''_{\theta qq}(\theta_i, q) \cdot sgn(q) < 0, \quad u'''_{\theta\theta q}(\theta_i, q) \cdot sgn(q) > 0,$$

for all i, $q \neq 0$, and θ_i in the support.

This assumption guarantees that Assumption 4 is satisfied for $v_i = J_i$.

This leads to the following proposition.

Proposition 4. *Under Assumptions 3, 4 and 6, the profit-maximizing mechanism is v-optimal with v_i equal to the virtual utility J_i.*

See Appendix D for formal proof.

Corollary 2. *Under Assumptions 3, 6 and 7, the virtual utility J_i is twice continuously differentiable, strictly concave in q_i and strictly single crossing, for all i, $q_i \neq 0$, and θ_i in the support.*

Likewise, even stronger restrictions on the utility u_i can guarantee that Assumption 5, or its alternative versions, are satisfied for $v_i = J_i$.

8 Symmetric quadratic model

This section illustrates our methodology in a symmetric model where each agent i has the following quadratic utility function.

$$u_i(\theta_i, q_i) = \theta_i q_i - \frac{\mu}{2} q_i^2,$$

for some known $\mu > 0$. We consider two log-concave distributions of private types θ_i: uniform on the $[-1, 1]$ interval and logistic (i.e., with full support). The above specification provides additional tractability and allows for comparison across different mechanisms.

8.1 Pareto frontier

We now solve for the mechanism that maximizes a linear combination of expected revenue and efficiency, in other words, finds the Pareto frontier. Following the arguments in Section 6, we have to maximize $\sum_i J_{\alpha,i}(\theta_i, q_i)$ over q, subject to the market clearing constraint $\sum q_i = 0$, where

$$J_\alpha(\theta_i, q_i) = q_i \left[\varphi_\alpha(\theta_i) \cdot \mathbb{I}(q_i > 0) + \psi_\alpha(\theta_i) \cdot \mathbb{I}(q_i \leqslant 0)\right] - \frac{\mu}{2} q_i^2,$$

pointwise, where $\varphi_\alpha(\theta) = \theta_i - \alpha \frac{1-F(\theta_i)}{f(\theta_i)}$ and $\psi_\alpha(\theta) = \theta + \alpha \frac{F(\theta_i)}{f(\theta_i)}$ are monotone functions, as long as F is log-concave.

To identify the optimal allocation, we must find a Lagrange multiplier $p(\theta)$ such that the market clears and the first-order conditions hold. This leads to the following solution

$$\begin{aligned} d_i(p|\theta_i) &= \mu^{-1} \left[(\varphi(\theta_i) - p) \cdot \mathbb{I}(q > 0) + (\psi(\theta_i) - p) \cdot \mathbb{I}(q < 0)\right] = \\ &= \mu^{-1} \left[\min(0,\ \psi(\theta_i) - p) + \max(0,\ \varphi(\theta_i) - p)\right], \end{aligned}$$

for each agent i, which will also be her sincere demand in the auction implementation. The Lagrange multiplier $p(\theta)$ is then the root of $\sum_{i=1}^n d_i(p|\theta_i)/n$, that is, the average sincere demand.

Finally, the marginal tax $m\tau$ and the fixed-point type $\hat{\theta}$ solve the system of equations (12) and thus

$$\hat{\theta}(p, q) = \varphi_\alpha^{-1}(\mu q + p) \cdot \mathbb{I}(q > 0) + \psi_\alpha^{-1}(\mu q + p) \cdot \mathbb{I}(q < 0) \quad (16)$$

$$m\tau(p, q) = \hat{\theta}(p, q) - (\mu q + p) \quad (17)$$

Proposition 5. *In the symmetric quadratic model with a log-concave distribution F, the optimal mechanism is implemented via marginal taxes* (17).

Since the worst-off types are in the interior of the type space, the transfers can be formally written out, conditional on the value of the Lagrange multiplier.

$$t_i(q_i|\theta_{-i}) = \begin{cases} \int_0^q \left(\varphi_\alpha^{-1}(\mu z + p_{-i}(z|\theta_{-i})) - \mu z\right) dz, & q > 0 \\ \int_0^q \left(\psi_\alpha^{-1}(\mu z + p_{-i}(z|\theta_{-i})) - \mu z\right) dz, & q < 0 \end{cases}, \tag{18}$$

where $p_{-i}(z|\theta_{-i})$ is the inverse residual supply curve. Despite a relatively simple mechanism implementation, its explicit characterization is rather difficult in a finite economy, even for standard distributions.

8.1.1 Uniform distribution

When the distribution is uniform, $\varphi_1(\theta) = 2\theta - 1$, $\psi_1(\theta) = 2\theta + 1$, thus

$$d_i(p|\theta_i) = \mu^{-1}\left[\min(0,\ 2\theta_i + 1 - p) + \max(0,\ 2\theta_i - 1 - p)\right],$$

and the marginal tax is

$$m\tau(p,q) = \frac{-\mu q - p - 1}{2} + \mathbb{I}(q > 0).$$

Note that the number of agents excluded from trade depends on the location of the root of the average demand curve and thus can not be easily characterized.

For example, for just $n = 3$ agents, the exclusion region follows an elaborate pattern, see Figure 6. When all three types are close to each other (a light grey area), nobody is trading. Next, with two significantly opposing types and a third in the middle (a dark grey area), only opposing types are trading with each other. Finally, when two types oppose the third, all three players are trading (black area).

When the number of players grows, the pattern becomes more complicated. However, the root of the average demand curve will converge in the probability limit, which is equal to 0. Thus, the limit exclusion region will be simply $\theta_i \in [-1/2, 1/2]$.

8.1.2 Logistic distribution

For a logistic distribution, $\varphi_1(\theta) = \theta - 1 - e^{-\theta}$, $\psi_1(\theta) = \theta + 1 + e^{\theta}$, thus

$$d_i(p|\theta_i) = \mu^{-1}\left[\min(0,\ \theta_i + 1 + e^{\theta} - p) + \max(0,\ \theta_i - 1 - e^{-\theta} - p)\right],$$

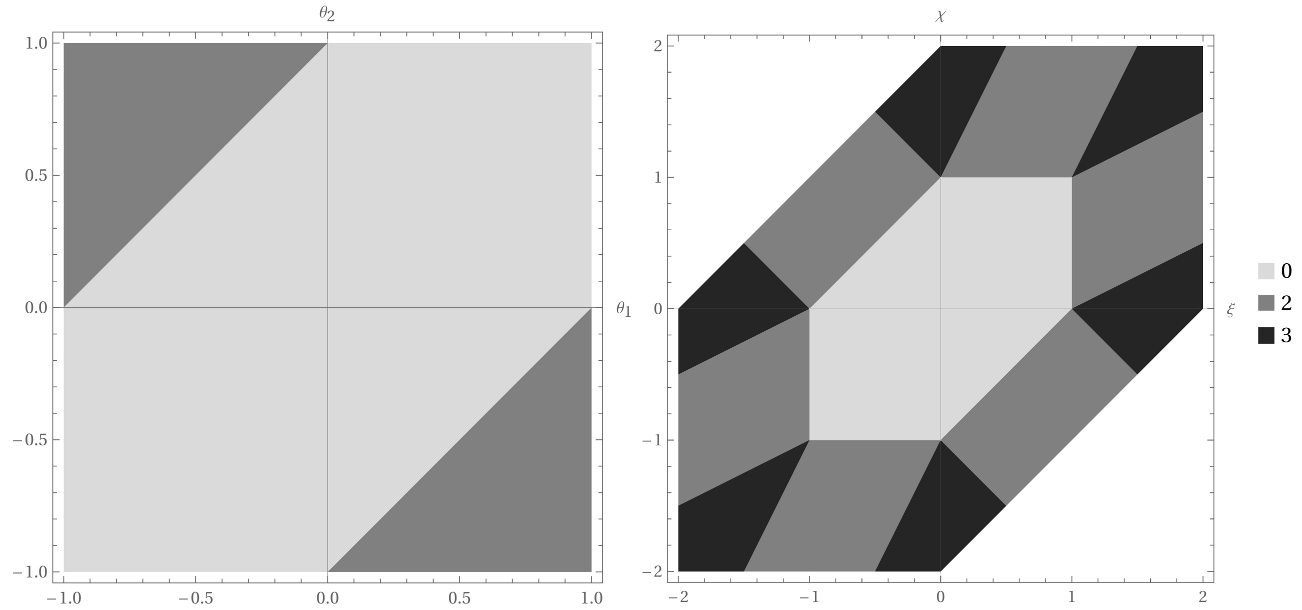

Figure 6: Exclusion region for two (left) players and three (right) players for a uniform[-1,1] distribution of types. The latter is in the coordinates $\xi = \theta_1 - \theta_3$, $\chi = \theta_2 - \theta_3$, independent from the value of θ_3.

and the marginal tax is

$$m\tau(p, q) = sgn(q) \cdot \left[1 + \omega(e^{-1-sgn(q)\cdot(\mu q+p)})\right].$$

The exclusion region for $n = 3$ follows a pattern similar to that of the uniform distribution, see Figure 7. In the limiting economy, again, the root of the average demand curve will be equal to 0. Thus the limit exclusion region is simply $\theta_i \in [-1 - \omega(1/e), 1 + \omega(1/e)]$, where $\omega(z)$ is the product-logarithm function.

8.2 σVCG mechanisms

Our first benchmark is smooth nearly-efficient mechanisms in Andreyanov and Sadzik (2021) called σ-VCG mechanisms, which can be thought of as an attempt to control demand reduction explicitly.

One way to define this mechanism is the maximizer of $\sum_i J_{\sigma,i}(\theta_i, q_i)$ over q, subject to the market clearing constraint $\sum q_i = 0$, where

$$J_{\sigma,i}(\theta_i, q_i) = \theta_i q_i - \frac{\mu + \sigma}{2} q_i^2.$$

The ex-post allocation and transfer in this mechanism can be derived:

$$q_i = \frac{n-1}{n}\frac{\theta_i - \bar{\theta}_{-i}}{\mu + \sigma}, \quad t_i(q_i) = \bar{\theta}_{-i} q + \frac{\mu + n\sigma}{2(n-1)} q^2,$$

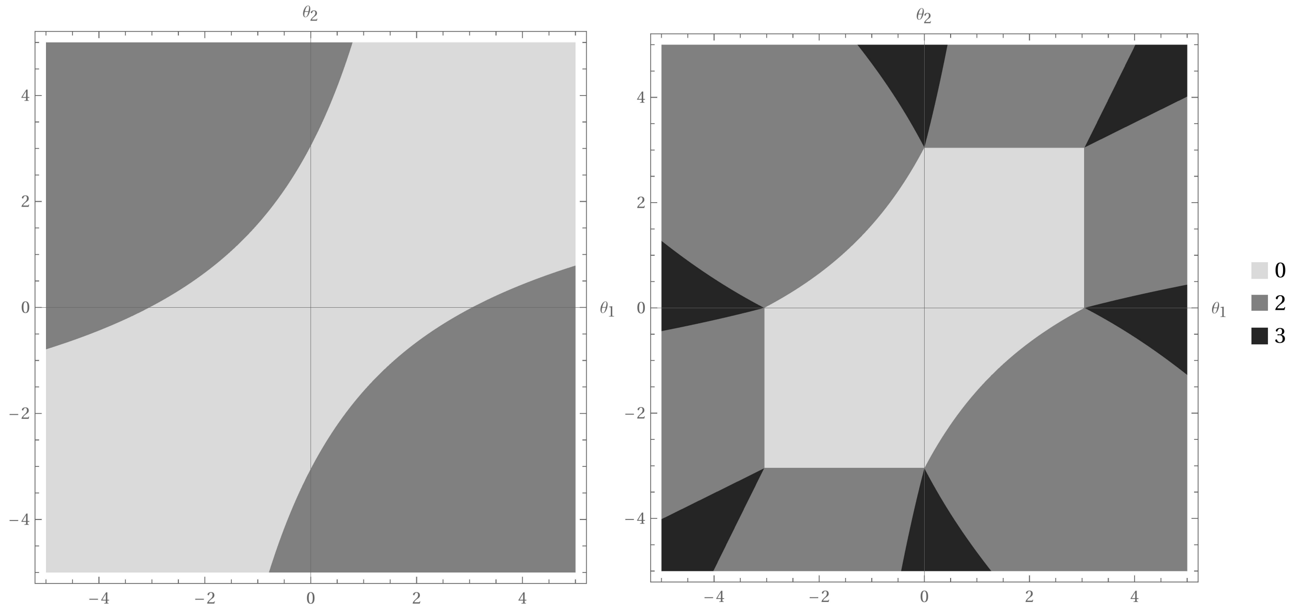

Figure 7: Exclusion region for two (left figure) players and three (right figure) players for a logistic distribution of types. The latter is in the coordinates θ_1, θ_2 for a fixed $\theta_3 = 0$.

where $\bar{\theta}_{-i} = \frac{1}{n-1}\sum_{j\neq i}\theta_j$ is the average type other than agent i's type.

Since both transfer and utility are quadratic in types, we can compute their expected values given the variance of the type distribution:

$$\mathbb{E}t_i = \frac{(n-2)\sigma - \mu}{2n(\mu+\sigma)^2}\mathbb{V}\theta_i, \quad \mathbb{E}u_i = \frac{(n-1)(\mu+2\sigma)}{2n(\mu+\sigma)^2}\mathbb{V}\theta_i,$$

since $\mathbb{E}\bar{\theta}_{-i} = \mathbb{E}\theta_i$, $\mathbb{E}(\bar{\theta}_{-i})^2 = \frac{\mathbb{E}\theta_i^2 + (n-2)(\mathbb{E}\theta_i)^2}{n-1}$ and $\mathbb{V}\theta = \mathbb{E}\theta_i^2 - (\mathbb{E}\theta_i)^2$. Naturally, for a uniform-price double auction ($\sigma = \frac{\mu}{n-2}$), the expected payment is equal to zero, while for the efficient mechanism ($\sigma = 0$), it is negative.

Finally, the maximum expected transfer over σ-VCG mechanisms is attained at $\sigma = \frac{n\mu}{n-2}$, and is equal to $\frac{(n-2)^2}{n(n-1)} \cdot \frac{\mathbb{V}\theta}{8\mu}$, while the utility is equal to $\frac{(n-2)(3n-2)}{n(n-1)} \cdot \frac{\mathbb{V}\theta}{8\mu}$.

8.3 δVCG mechanisms

Our second benchmark is non-smooth nearly-efficient mechanisms in Andreyanov and Sadzik (2021), which can be thought of as a bid-ask spread of size 2δ, which we refer to as δ-VCG mechanisms.

One way to define this mechanism is the maximizer of $\sum_i J_{\delta,i}(\theta_i, q_i)$ over q, subject to the market clearing constraint $\sum q_i = 0$, where

$$J_{\delta,i}(\theta_i, q_i) = q_i \left[\varphi_\delta(\theta_i) \cdot \mathbb{I}(q_i > 0) + \psi_\delta(\theta_i) \cdot \mathbb{I}(q_i \leqslant 0)\right] - \frac{\mu}{2} q_i^2,$$

pointwise, where $\varphi_\delta(\theta) = \theta_i - \delta$ and $\psi_\delta(\theta) = \theta + \delta$. The rest of the algorithm is identical to the one used for revenue maximization, so we have to rely on Monte Carlo simulations for finite economies.

In the limit economy, however, there is no supply reduction, so the agent's demand is equal to $(\theta_i - \delta)/\mu$ if he turns out to be a buyer, and $(\theta_i + \delta)/\mu$ if he turns out to be a seller. Moreover, for symmetric distributions, the limit of the equilibrium price will be equal to 0, so buyers will pay a per-unit price of δ, while sellers will get a per-unit price of $-\delta$. Thus, we can compute the expected payment and utility:

$$\mathbb{E}t_i = 2\delta \int_\delta^{F^{-1}(1)} \left[\frac{x-\delta}{\mu}\right] dF(x)$$

$$\mathbb{E}u_i = 2 \int_\delta^{F^{-1}(1)} \left[x\frac{x-\delta}{\mu} - \frac{\mu}{2}(\frac{x-\delta}{\mu})^2\right] dF(x)$$

which can be easily maximized over δ, for any given distribution.

9 Conclusion

We have studied an optimal robust mechanism in a double-auction environment similar to Lu and Robert (2001). While the direct mechanism is rather complicated, the associated implementation is simple - two Ausubel auctions: forward and reverse, with the price clocks running towards each other, much like in the celebrated Incentive Auction.

The clock nature of the auction can be thought of as a means to solve for the Lagrangean multiplier in the optimization problem, where the sum of virtual utilities is maximized subject to market clearing constraints, as long as these virtual utilities are concave. Moreover, the task of finding the worst-off types necessary for explicitly characterizing the direct mechanism is implicitly solved in the equilibrium of the auction.

The virtualization of utilities in a model with single-dimensional types is achieved by introducing a marginal tax that depends on the player's current (clinched) position and the clock price but not on the positions of other players. A hallmark feature of this tax is that it combines a relatively standard exclusion of the weakest traders with a tax scheme that can be explicitly computed.

The equilibrium is sincere in a sense similar to Ausubel (2004), but players submit demands as if their utilities were replaced with virtual utilities. However, the two-sided nature of the auction poses new challenges. In particular, the informational spillover between the forward and reverse auctions makes it impossible to eliminate all non-sincere equilibria if the auction is fully transparent. However, it is possible to move the clocks to minimize this spillover.

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Appendix A Proofs for Section 4

A.1 Proof of Proposition 1.

Observe first that, in every subgame, along the conjectured equilibrium path, the revealed demands in the sincere ex-post equilibrium coincide with the sincere demands $\{d_i(p)\}_{i=1}^n$ which, together with the market-clearing condition fully characterize the outcome (price and allocation) of the game.

We will now prove that *the market clearing price and allocation under sincere bidding coincide with the Walrasian equilibrium in the virtual economy.*

Suppose that players bid by the sincere strategy in the clock auction. Then auction outcomes are characterized by the first-order condition for the sincere demands

$$p = mu(q_i) - m\tau(p, q_i), \quad i = 1, \ldots, n.$$

By definition of mv_i, the first-order condition above can also be expressed as

$$p = mv_i(q_i), \quad i = 1, \ldots, n,$$

which are the first order conditions for Walrasian demands in the economy with utilities v_i. Since the second-order conditions in both cases are satisfied by Assumption 1, the first-order conditions show that the market clearing price and allocation in both equilibria are the same.

We will now prove that *sincere bidding is an ex-post perfect equilibrium.* The proof considers two cases regarding the prior history of play in the clock auction.

First, we examine agents' incentives on the equilibrium path of play, where the demands revealed before the subgame was all sincere.

Assuming that all but player j continue to play sincerely, bidder j's payoff is path-independent in the following sense. At any counterfactual allocation q, the stop-off price $p_{-j}(q)$ is uniquely defined by the sincere demands of other players. Moreover, her payment to the auctioneer is equal to the marginal tax $m\tau_i(p_{-j}(x), x)$ plus $p_{-j}(x)$, integrated over $x \in [0, q]$. The optimality condition is, therefore

$$\begin{cases} mu_j(q) = m\tau_j(p, q) + p \\ p = mv_i(q_i), \quad i = 1, \ldots, n, \quad i \neq j \end{cases}$$

which yields $p = mv_i(q_j)$ by the definition of the v_i functions. Player j's payoff is aligned with the social surplus in the virtual economy, which is maximized by playing sincerely. Thus, sincere play is a nash equilibrium of the subgame.

Second, we consider incentives off the equilibrium path, where traders reported non-sincerely before this subgame.

Assuming that all but player j continue to play sincerely, bidder j's payoff is path-independent, but her actions are constrained by the demands revealed before the subgame. However, these payoffs are monotonically decreasing in the distance from the conjectured allocation. Thus, she finds it optimal to play as close to the sincere demand as possible. Therefore, sincere play is a nash equilibrium of the subgame.

Appendix B Proofs for Section 5

B.1 Proof of Lemma 1

Proof. To the contrary, assume that at some prices $p^+ \leqslant p^-$, there is informational spillover into both auctions, and, at the same time, there is informational spillover for more than one agent. This means that there exist two agents $i \neq j$ such that:

$$q_{-i}^-(p^-) < q_i^+(p^+), \quad q_j^-(p^-) < q_{-j}^+(p^+).$$

Using the definition of the residual demands, we can pair these inequalities:

$$-\sum_{k \neq i,j} q_k^-(p^-) < q_i^+(p^+) + q_j^-(p^-) < -\sum_{k \neq i,j} q_k^+(p^+)$$

which contradicts the fact that $q_k^-(p^-) \leqslant q_k^+(p^+)$ for all k. □

B.2 Proof of Proposition 2

Proof. Consider the domain of prices $(p^+, p^-) \in [p_0^+, p^*] \times [p^*, p_0^-]$ and denote the subset of prices that have x spillovers into the forward and y spillovers into the reverse auctions by $S_{x,y}$.

Observe first that $S_{0,2+}$ and $S_{2+,0}$ do not intersect by Lemma 1, thus any point in the domain belongs to either $S_{0,1+}$, $S_{1+,0}$, $S_{1,1}$ or $S_{0,0}$.

The price path connecting (p_0^+, p_0^-) with (p^*, p^*) will have two parts. The first part is a straight line, and the second part goes along the boundary of either $S_{0,1+}$ or $S_{1+,0}$, see Figure 5. To construct the price path, consider three cases.

Case 1: If the starting prices are in $S_{1+,0}$, we first advance the forward clock till it reaches the boundary of $S_{1+,0}$. After that we move along the path $(\tilde{p}^+(p), p)$ where

$$\tilde{p}^+(p) = \sup_{x \in [p_0^-, p^*]} x : (x, p) \in S_{1+,0}.$$

Case 2: If the starting prices are in $S_{0,1+}$, we first advance the reverse clock till it reaches the boundary of $S_{0,1+}$. After that we move along the path $(p, \tilde{p}^-(p))$ where

$$\tilde{p}^-(p) = \inf_{x \in [p^*, p_0^+]} x : (p, x) \in S_{0,1+}.$$

Case 3: If the starting prices (p_0^+, p_0^-) are in $S_{1,1}$, any of the aforementioned trajectories will work. Finally, the starting prices can not be in $S_{0,0}$ by assumption.

We argue that along the first part of the trajectory, the number of agents experiencing spillovers is weakly decreasing. Indeed, on the one hand, advancing the forward (reverse) clock does not increase the number of spillovers in the forward (reverse) auction. On the other hand, the number of spillovers in the reverse (forward) auction is fixed at 0 by construction.

The function $\tilde{p}^+(.)$ does not have to be continuous. However, if it is monotone, we can connect the (at most countably many) points of discontinuity to obtain a monotone and continuous path $p^+(t), p^-(t)$. It remains to show that $\tilde{p}^+(.)$ is weakly monotone and that, along this path, the number of agents that experience spillover is at most one.

Monotonicity: Assume that $\tilde{p}^+(p_1^-) = p_1^+$, that is, (p_1^+, p_1^-) belongs to the closure of $S_{1+,0}$. Now, pick any $p_2^- < p_1^-$. When the clock prices move from (p_1^+, p_1^-) to (p_1^+, p_2^-), the number of spillovers in the reverse auction can not increase, while the number of spillovers in the forward auction is already at 0. Thus, (p_1^+, p_2^-) belongs to the closure of $S_{1+,0}$ as well, thus $\tilde{p}^+(p_2^-) \geqslant p_1^+$. Consequently, $\tilde{p}^+(p)$ is weakly monotone.

Finally, observe that $S_{1+,0}$ does not intersect with $S_{0,2}$ by Lemma 1. Thus, it can only share a boundary with $S_{1,1}$, $S_{0,1}$ and $S_{0,0}$. In either case, the number of agents experiencing spillovers is at most one.

□

Appendix C Proofs for Section 6

C.1 Proof of Lemma 2

Fix θ_{-i} and consider two mutually exclusive cases. Suppose first that the set of types excluded from trade is empty. Then the claim holds trivially.

Suppose that it is not empty. Let $\hat{\theta}_i$ be a type excluded from trade. By definition of net utility, $\frac{\partial}{\partial \theta_i}\tilde{u}_i(\hat{\theta}_i, q_i(\hat{\theta}_i, \theta_{-i})) = 0$.

Next, the net surplus functions $\tilde{s}_i$ are absolutely continuous, a.e. differentiable and

$$\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i-, \theta_{-i}) \leqslant \frac{\partial}{\partial \theta_i}\tilde{u}_i(\hat{\theta}_i, q_i(\hat{\theta}_i, \theta_{-i})) \leqslant \frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i+, \theta_{-i}),$$

where $\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i-, \theta_{-i})$ and $\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i+, \theta_{-i})$ are left-hand and right-hand partial derivatives respectively, see Theorems 1,2 in Milgrom and Segal (2002).

Next, at points of differentiability we can write:

$$\frac{\partial}{\partial \theta_i}\tilde{s}_i(\theta) = \frac{\partial}{\partial \theta_i}\tilde{u}_i(\theta_i, q_i(\theta)) = \int_0^{q(\theta)} \frac{\partial}{\partial \theta_i \partial q_i} u_i(\theta, x)dx,$$

thus $\tilde{s}_i$ is convex in θ_i by monotonicity of q_i in θ_i and single-crossing of u_i.

Finally, since $[\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i-, \theta_{-i}), \frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i+, \theta_{-i})]$ contains 0 at the type excluded from trade, by the necessary first-order conditions, $\hat{\theta}_i$ is also the worst-off type.

C.2 Proof of Lemma 3

Equation (9) shows that $q_i(\theta_i, \theta_{-i})$ is continuous in θ_i and bounds it's slope away from zero. Thus, $q_i(\theta_i, \theta_{-i})$ is guaranteed to cross 0 at some type $\theta_i \in \mathbb{R}$, in other words, $tet(\theta_{-i})$ is non-empty, for any θ_{-i} in the support. Next, by formula (4)

$$t_i(\theta) = \tilde{u}(\theta_i, q(\theta)) - \tilde{s}(\theta_i, q(\theta)) =$$
$$= \tilde{u}(\theta_i, q(\theta)) - \inf_{\theta_i} \tilde{s}_i(\theta_i, \theta_{-i}) - \int_{\theta*}^{\theta_i} \frac{\partial}{\partial \theta}\tilde{u}(x, q(x, \theta_{-i}))dx$$

where $\theta^* \in tet(\theta_{-i})$. Recalling that, in a v-optimal mechanism, $\inf_{\theta'} \tilde{s}_i(\theta'_i, \theta_{-i}) = 0$

$$t_i(\theta) = \int_{\theta*}^{\theta_i} \frac{d}{dx}\tilde{u}(x, q(x, \theta_{-i}))dx - \int_{\theta*}^{\theta_i} \frac{\partial}{\partial \theta}\tilde{u}(x, q(x, \theta_{-i}))dx =$$
$$= \int_{\theta*}^{\theta_i} \frac{\partial}{\partial q}\tilde{u}(x, q_i(x, \theta_{-i}))dq_i(x, \theta_{-i}) = \int_{\theta*}^{\theta_i} mu_i(x, q_i(x, \theta_{-i}))dq_i(x, \theta_{-i}).$$

Finally, we get formula (10) via monotone change of variables from x to $q_i^{-1}(x, \theta_{-i})$.

C.3 Proofs of Lemma 3 with alternative versions of Assumption 5

Version 1: v_i are identical, F_i are identical.

Proof. To the contrary, assume that for some realization of types θ_{-i}, trader i only trades strictly positive quantities. Define a type $z = \min_{j \neq i} \theta_j$, and observe that it belongs to the support of each agent. Consequently, we can say that $q_i(z, \theta_{-i}) > 0$.

Furthermore, the allocation can not decrease if we lower the types of traders $j \neq i$. Consequently, we can say that $q_i(z, \ldots, z) > 0$. But this can not be true because any $p \in mv_i(z, 0)$ solves the first-order conditions in the symmetric case. □

Version 2: for any i and $p \in \mathbb{R}$, there exist a type z in the support such that $p \in mv_i(z, 0)$.

Proof. Pick a trader i, and fix a profile of types θ_{-i}. Next, consider the economy without trader i, that is, solve a system of first-order conditions

$$mv_j(\theta_j, \tilde{q}_j) = \tilde{p}, \quad \forall j \neq i, \quad \sum_{j \neq i} \tilde{q}_j = 0.$$

This solution exists for some $\tilde{p}$.

Next, pick a type z in the support, such that $\tilde{p} = mv_i(z, 0)$. By construction, i is excluded from trade in the original economy with the profile of types (z, θ_{-i}). □

C.4 Proof of Proposition 3

We want to prove that agents face the same menus $t_i(q)$ in both the auction and the optimal mechanism. For that, it suffices to show that the integrand in (10) coincides with the one in (11) for any $q(\theta) \neq 0$.

Using the left-hand side of (12) we first write that

$$m\tau(p_{-i}(x), x) + p_{-i}(x) = mu_i(\hat{\theta}_i(p_{-i}(x), x), x).$$

Second, we combine the right-hand side of (12) with the definition of the residual supply curve

$$\begin{cases} p_{-i}(x) = mv_i(\hat{\theta}_i(p_{-i}(x), x), x) \\ p_{-i}(x) = mv_j(\theta_j, q_j), \quad \forall j \neq i \\ x + \sum_{j\neq i} q_j = 0. \end{cases}$$

The latter can be recognized as the system of first-order conditions for the optimal mechanism, given that x is the allocation of agent i and θ_{-i} are the types of others. Thus $\hat{\theta}_i(p_{-i}(x), x)$ and $q^{-1}(x, \theta_{-i})$ coincide and, therefore,

$$mu_i(\hat{\theta}_i(p_{-i}(x), x), x) = mu_i(q^{-1}(x, \theta_{-i}), x),$$

which completes the proof.

Appendix D Proofs for Section 7

D.1 Proof of Lemma 4

The boundedness of the expected net surplus comes from the fact that, on the one hand, the net surplus is nonnegative by IR, and on the other hand, the sum of net surpluses can not exceed the sum of net utilities at the efficient allocation

$$\tilde{s}_i(\theta_i, \theta_{-i}) \geqslant 0, \quad \sum \tilde{s}_i(\theta_i, \theta_{-i}) \leqslant C(\theta_i),$$

therefore $\tilde{s}_i(\theta_i, \theta_{-i}) \leqslant C(\theta_i)$ for any θ in the support, and thus $\int \tilde{s}_i(z, \theta_{-i}) dF_i(z)$ is majorized by $\int C(z) dF_i(z) < \infty$.

D.2 Proof of Proposition 4

Recall that our objective is

$$\iint\limits_{\mathbb{R}^{n-1}} \sum_{i=1}^{n} \left[\int_{\mathbb{R}} \left(\tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i})) \right) dF_i(\theta_i) \right] \prod_{j\neq i} dF_j(\theta_j), \tag{19}$$

where $\tilde{s}_i(\theta_i, \theta_{-i}) = \int_{\theta^*}^{\theta_i} \tilde{u}_1'(x, q(x, \theta_{-i}))dx$ and $\theta^* \in tet(\theta_{-i})$, that is, θ^* is one of the types excluded from trade by Lemma 2, and also one of the worst-off types, that is, $\tilde{s}_i(\theta_i^*, \theta_{-i}) = 0$. Since $\tilde{s}_i$ integrable by Lemma 4 and θ^* is finite by Lemma 3, we can use the following representation:

$$\begin{aligned} \int_{\mathbb{R}} \tilde{s}_i(\theta_i, \theta_{-i}) dF(\theta_i) &= \lim_{N\to\infty} \int_{-N}^{N} \tilde{s}_i(\theta_i, \theta_{-i}) dF_i(\theta_i) \\ &= \int_{-N}^{\theta^*} \tilde{s}(\theta_i, \theta_{-i}) dF(\theta_i) + \int_{\theta^*}^{N} \tilde{s}(\theta_i, \theta_{-i}) d(F(\theta_i) - 1) + o(N; \theta_{-i}), \end{aligned}$$

for any N sufficiently large. The remainder term vanishes in the limit, $\lim_{N\to\infty} o(N;\theta_{-i}) = 0$ due to the definition of improper integral. Integrating the first two terms by parts, we get that

$$\begin{aligned}\int_{\mathbb{R}} \tilde{s}(\theta_i,\theta_{-i})dF(\theta_i) &= \int_{-N}^{\theta^*} \tilde{u}_1'(\theta_i, q(\theta_i,\theta_{-i}))F(\theta_i)d\theta_i \\ &+ \int_{\theta^*}^{N} \tilde{u}_1'(\theta_i, q(\theta_i,\theta_{-i}))(F(\theta_i)-1)d\theta_i + o(N;\theta_{-i}) \\ &= \int_{\mathbb{R}} \frac{\mathbb{I}(q_i>0)-F(\theta_i)}{f(\theta_i)}\tilde{u}_1'(\theta_i,q)dF(\theta_i)\end{aligned}$$

Note that $\mathbb{I}(\theta_i > \theta_i^*) = \mathbb{I}(q_i > 0)$ since q_i is monotone in θ_i and θ_i^* is a type excluded from trade. Plugging it into (19) gives us the virtual value J_i.

We next need to show that the virtual value J is concave and single-crossing to use the first-order approach.

$$\begin{aligned}\frac{\partial^2 J}{\partial\theta\partial q} &= \frac{\partial^2 \tilde{u}}{\partial\theta\partial q} - \frac{\partial}{\partial\theta}\left(\frac{\mathbb{I}(q_i>0)-F(\theta_i)}{f(\theta_i)}\right)\frac{\partial^2\tilde{u}}{\partial\theta\partial q} - \frac{\mathbb{I}(q_i>0)-F(\theta_i)}{f(\theta_i)}\frac{\partial^3}{\partial\theta^2\partial q}\tilde{u} > 0, \\ \frac{\partial^2 J}{\partial q^2} &= \frac{\partial^2\tilde{u}}{\partial q^2} - \frac{\mathbb{I}(q_i>0)-F(\theta_i)}{f(\theta_i)}\frac{\partial^3}{\partial\theta\partial q^2}\tilde{u} < 0.\end{aligned}$$

Both properties are guaranteed by Assumption 7.