Asset Prices, Welfare Inequality, and Leverage*

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Abstract

Do rising asset prices make savers better off? The traditional way to answer this question is to study wealth inequality. This paper studies the effect of fundamental drivers of rising asset prices (a rise in patience, an increase in productivity, financial innovation, or a bubble driven by financial frictions) on top *welfare* inequality between super rich entrepreneurs (borrowers) and savers through *leverage*. Using a model with financial frictions, idiosyncratic risks, and unequal capital income, I show that different fundamental drivers of rising asset prices move wealth inequality and savers' welfare in different directions by affecting leverage differently. Given the rising asset prices, falling risk-free rates, and rising top wealth inequality observed in the U.S., the model suggests that the rising patience of the super-rich is the main driver of the trend, and therefore savers are worse off.

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1 Introduction

Increasing top wealth inequality and rising asset prices over the past few decades have raised important questions about their causes and effects. It has also led to policy debates on the impact of rising asset prices on welfare inequality. This paper answers the question: Do rising asset prices make savers better off?

In answering this question, the object of the inequality measure matters. Traditionally, focus has been on wealth inequality. However, the welfare effect of rising wealth inequality is rather obscure.[1] In response, a small but growing literature has emerged to study welfare inequality.[2] This paper takes one step further by studying the effect of *fundamental drivers* of rising asset prices on welfare inequality. It considers an environment in which there are two types of agents – entrepreneurs and savers – and two assets – productive capital and risk-free bond.[3] Only entrepreneurs can invest in productive capital and own private business. And I refer to entrepreneurs as the super-rich.[4] In equilibrium, entrepreneurs borrow from savers to finance their assets, that is, they use *leverage*. Therefore, there are two endogenous asset prices, the price of productive capital and the price of risk-free bond, i.e. the risk-free rate. In this environment, I analyze the impact of fundamental drivers of asset price changes on savers' welfare.

The source of changes in asset prices matter as well. I consider an economy where financial frictions limit risk-sharing and asset bubbles may occur. In this economy, there are four types of fundamental drivers of asset price changes: "patience"[5], productivity[6], financial

[1]Saez et al. [2021] and Cochrane [2020] hold two opposite views on whether rising asset prices benefit the rich or just "on paper".

[2]Fagereng et al. [2022] study the effect of rising asset prices on welfare inequality mainly in a partial equilibrium framework. Greenwald et al. [2021] study the effect of falling interest rates on wealth inequality and welfare inequality, but do not focus on studying various fundamental drivers of the falling interest rates.

[3]In the full model in section 4, there are three asset classes: private equity, public equity, and risk-free bond. Productive capital is the underlying asset of both private equity and public equity.

[4]In steady state, entrepreneurs are ensured to be richer than savers. I will discuss model details in later sections.

[5]That is, the time discount rate. I discuss the empirical interpretations of "patience" later in this section.

[6]That is, asset pay-offs in an endowment economy. The net present value of an asset increases when its productivity or future pay-offs increase.

innovation (or regulation)[7], and a bubble driven by financial frictions. I show that different fundamental drivers of asset price changes have different implications on wealth inequality and savers' welfare by affecting entrepreneurs' leverage differently. Financial frictions in my model play three important roles. First, they can explain asset price spikes and collapses in economic booms and recessions. Second, financial frictions can impact inequality by reducing allocation efficiency. Third, they also create bubbles when the market value of an asset exceeds its fundamental value (the net present value of all its future cash flows). Allowing for bubbles is important because recent studies have shown that the dotcom bubble and the Great Recession resulted in spikes in wealth inequality fluctuations (Gomez [2019]). An et al. [2022] provide empirical evidence that the burst of stock market bubbles has large redistribution effects. The gap in this literature is a lack of a general equilibrium framework that takes into account the effect of asset bubbles on inequality.

I first consider a deterministic two-period model with an exogenous interest rate and an endogenous asset price (the price of productive capital).[8] The asset price changes via two channels in this simple model: productivity channel and interest rate channel. I show that a rising asset price always increases wealth inequality. However, depending on which channel is at work, it can have different consequences for the welfare of savers. In one instance, higher asset price due to higher productivity directly increases wealth inequality because the super-rich hold all the assets whose value increase. Savers' welfare is not affected since they do not hold the asset. This is the productivity channel. In the other instance, higher asset price due to lower interest rate increases wealth inequality and benefits the super-rich who borrow at the expense of the savers. The magnitude of this welfare effect depends on agents' borrowing and lending positions. This is the interest rate channel. In a partial equilibrium analysis, I show that even though savers do not own assets, the effect of changing asset price

[7]Financial innovation in this paper refers to less severe financial frictions and/or a lower volatility of idiosyncratic risk, while financial regulation refers to the opposite of financial innovation.

[8]There are no bubbles in the deterministic two-period model.

spills over to them through the abovementioned interest rate channel.[9]

In order to understand how asset prices and leverage interact, and how they jointly affect wealth inequality and the welfare of savers, it is essential that they are both endogenously determined. The second step therefore is to endogenize the interest rate in the two-period model, which allows me to study the welfare effect of fundamental drivers of rising asset prices. I show that rising "patience" of super rich entrepreneurs increases asset prices, decreases leverage, increases wealth inequality, and makes savers worse off. When super rich entrepreneurs become more "patient", their saving demand increases, and they borrow less. In general equilibrium, the interest rate decreases and wealth inequality rises. As a result, rising "patience" of super rich entrepreneurs decreases the welfare of savers.[10]

Given the intuition from the two-period model, I subsequently enrich the analysis by considering three more important elements: uncertainty, bubbles, and endogenous feedback from wealth inequality. The motivation is three-fold. Uncertainty creates a risk premium that decreases asset prices, and generates precautionary saving motives that reduce leverage. By contrast, bubbles raise the value of an asset but reduce precautionary saving motives. Because these forces work in conflicting directions, they each have a different prediction for wealth inequality. The relative strength of the competing effects are determined by the fundamental drivers of rising asset prices in the richer model.

As the final step, I develop an infinite-horizon model with financial frictions, idiosyncratic risks and endogenous bubbles. I characterize the long-run level of wealth inequality and welfare both in an economy both with and without bubbles. I also discuss the fluctuations of wealth inequality and welfare in response to the four fundamental drivers of asset price changes: "patience", productivity, financial innovation, and a bubble driven by financial frictions.

[9]Using Norwegian data, Fagereng et al. [2022] show that the rich borrows against their private business and debt is an important asset class that accounts for welfare gains and losses.

[10]Since this experiment constitutes a change in the entrepreneurs' preferences, we cannot say whether they are better off or not.

The first fundamental driver, rising "patience" of entrepreneurs, increases asset prices, decreases leverage, increases wealth inequality. Savers are worse off. This is in an echo of the two-period model's results.

The second fundamental driver, an increase in productivity, increases asset prices as well as leverage, but has no impact on wealth inequality. However, savers benefit from higher productivity due to positive income effect. This result shows that even though wealth inequality does not change, rising asset prices can still have an effect on welfare.

Unlike conventional wisdom that the third or the fourth fundamental driver, financial innovation or a bubble, has the tendency to increase wealth inequality, I find instead that they reduce wealth inequality and increase the welfare of savers. This result can be explained intuitively in two ways. From a portfolio choice perspective, a bubble increases the market value of an asset[11] and financial innovation increases asset prices. Keeping leverage fixed, a bubble or financial innovation would directly increase wealth inequality since super rich entrepreneurs hold assets that are rising in value. This is the *price channel*. However the story is incomplete, because leverage changes as well. In fact, super rich entrepreneurs borrow more from savers when there is a bubble or financial innovation because their precautionary saving motives decrease. This is the *leverage channel*. In general equilibrium, this channel dominates the price channel.[12] From a risk perspective, by taking more risks and receiving a higher return, the super rich entrepreneurs accumulate more wealth relative to the savers. Financial innovation or a bubble reduces the total idiosyncratic risks in the economy which are entirely borne by the super-rich.[13] Consequently, wealth inequality decreases and savers are better off.

[11]To be precise, I use "market value of an asset" rather than "asset price" when there is a bubble. Recall that the market value of an asset is the sum of the fundamental value of an asset (net present value of all future cash flows) and the value of bubble.

[12]This theoretical result that both financial innovation and bubbles expand leverage is also consistent with empirical evidence on bubbles, credit cycles, and financial crisis (see Schularick and Taylor [2012], Jordà et al. [2015], Brunnermeier and Oehmke [2013] among others).

[13]I will elaborate on why bubbles reduce idiosyncratic risks later in the paper.

Through the lens of my theory, the observed rising asset prices, falling risk-free rates and rising top wealth inequality in the past few decades in the U.S. suggest that the rising "patience" of super rich entrepreneurs is the main driver of the trend, and savers are worse off.

To better understand this result, I discuss the theoretical intuition, the relevant empirical interpretations, and related results in the literature. The theoretical intuition is robust: When super rich entrepreneurs, who are borrowers, become more "patient", they borrow less. Their net worth increases and thus wealth inequality increases. Empirically, rising "patience" of entrepreneurs can be interpreted as a slowdown in firm dynamism (lower firm entry/exit rate), a change in demographic characteristics (growing dispersion of life expectancy by wealth groups). This result is also complementary to the literature where the time discount rate of the whole economy is considered to be the driver of increasing inequality.[14] My framework identifies the secular stagnation and the dispersion of demographic changes across wealth groups to be the most promising underlying changes in the economy that drive the joint trend of rising asset prices, declining interest rate, and increasing top wealth inequality.

Empirically, there are time periods in which asset prices and wealth inequality do not co-move. My theory suggests that this is due to productivity changes. The theory also suggests that in periods of negative comovement between asset prices and wealth inequality, there are changes in financial innovation or bubbles at work. In both these cases, rising asset prices make savers better off.

1.1 Literature review

This paper contributes to the growing literature on understanding the impact of rising asset prices on inequality, as well as the literature on asset bubbles driven by financial frictions.

Inequality has been extensively studied in a large and evolving literature. This paper

[14] Liu et al. [2022] argues that a reduction in time discount factor of the whole economy can help explain the rising profit share and declining productivity growth following the decline in the interest rate. Greenwald et al. [2021] argues that declining interest rates due to a combination of declining time discount rate, growth rate, and growth uncertainty can explain a large fraction of the increasing wealth inequality.

contributes to the recent studies on how asset prices impact wealth inequality, but with a focus on top inequality. Kuhn et al. [2020] show that asset prices are significant factors in wealth inequality in the US. Fagereng et al. [2019] show that capital gains play an important role in saving behavior. Albuquerque [2022] shows that portfolio changes matter for wealth inequality as well. Cioffi [2021] and Xavier [2021] study wealth inequality by incorporating heterogeneity in risk exposure and asset returns in partial equilibrium models. Gomez et al. [2016] studies the role of aggregate risk in shaping wealth inequality and asset prices. Gomez and Gouin-Bonenfant [2020] studies the long-run effect of low interest rate on wealth inequality. This paper is also related to the small but growing literature studying welfare inequality. Fagereng et al. [2022] and Greenwald et al. [2021] study wealth inequality and welfare inequality mainly in a partial equilibrium setting. Complementary to their result, I consider a general equilibrium framework with financial frictions, idiosyncratic risks, and unequal capital income and characterize the leverage channel with endogenous interest rate. I show that different fundamental drivers of rising asset prices affect wealth inequality and welfare differently through the leverage channel.

This paper also contributes to the literature on rational bubbles that have positive value due to financial frictions. I characterize a new type of bubble and study the effect of bubble on *inequality*. In my model, bubbles expands leverage by lowering precautionary saving motives in an economy with idiosyncratic risks. A long literature studies rational bubbles in line with Samuelson [1958] and Tirole [1985], such as Martin and Ventura [2012] on growth and Farhi and Tirole [2012] on liquidity among many others, see Martin and Ventura [2018] for a comprehensive survey. Recent works like Reis [2021] and Brunnermeier et al. [2022] show that bubbles can explain the high level of government debt that cannot be sustained by fiscal surplus. Miao and Wang [2018] studies stock price bubbles that relax the borrowing limit which is directly given by credit constraint. While in my model, there is no borrowing constraint and bubbles relax the limit on public stock market which is *indirectly* given by an equity constraint.

Finally, this model fits in the macro-finance literature highlighting financial frictions. The equity constraint in my model is in the same spirit as Brunnermeier and Sannikov [2014], where entrepreneurs have to keep some fraction of the firms as private equity because of agency problems. The idiosyncratic risks associated with private capital is related to Di Tella and Hall [2020]. I depart from them by differentiating private equities and public equities. In the basic model, I abstract from investment, aggregate risks, and labor. I acknowledge that these simplifications may be important inputs to the model and hope to develop extensions addressing them.

This paper is organized as follows. Section 2 provides motivating facts for key modelling elements, section 3 discusses the two-period model, section 4 sets up the full model and discusses fundamental equilibrium without bubbles, section 5 discusses bubble equilibrium, section 6 analyzes how changes in asset price affect inequality and welfare, and finally section 7 concludes.

2 Motivating facts

Motivated by the well-established fact in the literature that wealth inequality is rising and exploding at the top end in recent decades, as well as a long strand of research showing that asset valuations have also been rising in recent decades, I study how financial assets affect top inequality. This section provides some motivations and rationales for key modelling elements.

A focus on capital A large literature studies the driving forces of rising wealth inequality at the top. While differences in labor income and saving rates are considered important factors driving the exploding trend in many studies, De Nardi and Fella [2017] show that labor income differences only can not explain the wealth concentration at the top and Fagereng et al. [2019] show that saving rates only differ by wealth groups when capital gains are included. Rising asset prices and capital gains in recent decades have become the focus of a growing literature to understand the wealth concentration at the top. Benhabib et al. [2011] show theoretically that capital income risk, rather than labor income, drives the properties of right tail of wealth distribution. Thus, my model features an economy where capital income uncertainty is one of the key elements determining inequality.

Financial market structure and frictions It has been shown that there are systematic differences in portfolio compositions and rates of return along the wealth distribution: The super rich entrepreneurs group is characterized by a heavy portfolio share in high-return assets, especially private business equity, while the savers group holds mainly public equity, such as stock market index fund, and safe assets such as deposit (Fagereng et al. [2019],Kuhn et al. [2020], Martínez-Toledano [2020],Xavier [2021], and Albuquerque [2022]).[15] To capture such portfolio heterogeneity, I include three classes of assets in the model: private equity, public equity, and risk-free bond. I also assume restricted participation in equity market: savers cannot hold private equity, but can hold public equity *inactively*. Since the access to private equity market is the access to high-return assets, the restricted private equity market participation gives rise to return heterogeneity of different groups. The heterogeneous portfolios and returns arise in the model are consistent with the empirical facts discussed earlier. While I interpret the *inactive* participation of savers in public stock market as their holdings of public equity through pension, which account for a non-negligible proportion in the data for some countries, U.S. for example.

Idiosyncratic risk The important role of idiosyncratic risk in explaining the top wealth concentration has been studied both theoretically and empirically (Campbell et al. [2019], Gomez [2019],Benhabib et al. [2019], and Atkeson and Irie [2020]). Kartashova [2014] has documented that private equities on average earn a premium over public equities due to idiosyncratic risks and such return difference varies with economic fundamentals. Di Tella and Hall [2020] show that idiosyncratic risks affect the return of capital and create inefficient

[15] I focus on financial asset in this paper and do not consider housing explicitly.

recessions. I incorporate idiosyncratic risks associated with private equities as an important ingredient in the model for asset prices and inequality.

3 Two-period Model

In this section, I develop a two-period model with two agents, an entrepreneur and a saver. This two-period model is a simplified version of the full model in section 4. I start with partial equilibrium analysis with exogenous interest rate and endogenous asset price. I show that rising asset prices always increase wealth inequality and hurt the savers. I then proceed to general equilibrium analysis with endogenous interest rate. I show the main result that declining "impatience" of the entrepreneurs raises asset prices, increases wealth inequality, and hurts the savers. At the end of this section, I discuss the limitation of the two-period model.

3.1 Model Set-up

In the two-period model, time $t \in \{0, 1\}$. There are two agents, an entrepreneur and a saver, representing for the top 1 percent wealth group and the group in between top 1 and 10 percent of the wealth distribution.

Preferences Entrepreneur and saver differ in "patience". Saver's discount factor is denoted as $\frac{1}{1+\rho}$ and entrepreneur's discount factor is denoted as $\frac{1}{1+\rho^e}$, where ρ and ρ^e are discount rates of saver and entrepreneur respectively.[16] Entrepreneur is more "impatient" than saver, that is $\rho^e = \rho + \delta^e$ where δ^e captures the relative "impatience" of entrepreneur. One can intuitively think that entrepreneur and saver share a common discount rate ρ, but entrepreneur may die or become bankrupt in period $t = 1$ at a Poisson rate of δ^e.[17]

[16] I denote discount factors in this way to keep consistency with the full model in section 4

[17] This intuition can be proved when time is continuous and infinite. The assumption that entrepreneurs are less patient is standard in the macro-finance literature to prevent the entrepreneurs from eventually taking over all the wealth in the economy and becoming fully self-financed. See Kiyotaki and Moore [1997] and Bernanke et al. [1999] for example.

For consistency with later sections, I assume logarithmic utility for both entrepreneur and saver.[18]

Technology There is productive capital in fixed supply K. Capital K produces aK units of consumption good at time $t = 1$, where a is the productivity of capital and is an exogenous parameter. Entrepreneur and saver are endowed with W_0^e and W_0^s respectively at time $t = 0$.

Financial assets There are productive capital K in fixed supply and risk-free bond B in zero net supply in the economy. One can also think of capital as an asset like stock since it delivers cash flows over time and a is the dividend paid out per unit of the asset. Denote q as the price of per unit capital K and $1 + r^f$ as the (gross) return of the risk-free bond B. I refer to q as capital price or asset price interchangeably and r^f as the (net) interest rate.

Financial market structure Importantly, saver is restricted from stock market participation and entrepreneur holds all the capital in the economy.[19] At time $t = 0$, entrepreneur and saver cannot trade the productive capital K due to market segmentation, but can trade risk-free bond B freely.

Optimization problems Now I can write optimization problems of entrepreneur and saver. The entrepreneur's problem is as follows:

$$
\begin{aligned}
V^e = \max_{C_0^e, C_1^e, K_1, B^e} \quad & U(C_0^e) + \frac{1}{1+\rho^e} U(C_1^e) \\
\text{s.t.} \quad & C_0^e + qK_1 + B^e = qK + W_0^e \\
& C_1^e = aK_1 + (1 + r^f)B^e
\end{aligned}
\tag{1}
$$

where C_t^e is entrepreneur's consumption at time t, K_t is entrepreneur's capital holding at time t, and B^e is entrepreneur's bond holding at time 0. Entrepreneur chooses how much

[18]The main result and key mechanism does not depend on utility function forms.

[19]I leave the distinction between private equity and public equity for later sections.

capital to hold at $t = 1$, how much to invest in bond at time $t = 0$, and the consumption plan over time. Since $t = 1$ is the final period, agents consume everything that they own at $t = 1$.

Similarly, saver's optimization problem is as follos:

$$\begin{aligned} V^s = \max_{C_0^s, C_1^s, B^s} \quad & U(C_0^s) + \frac{1}{1+\rho} U(C_1^s) \\ \text{s.t.} \quad & C_0^s + B^s = W_0^s \\ & C_1^s = (1 + r^f) B^s \end{aligned} \tag{2}$$

where C_t^s is saver's consumption at time t, B^s is saver's bond holding at time 0. Without access to capital, saver simply chooses how much to invest in bond to smooth consumption over time.

Lemma 1 (Asset price) *The price of capital q is given by the following asset pricing equation*

$$q = \frac{a}{1 + r^f} \tag{3}$$

Proof. From first-order conditions of entrepreneur's problem. See appendix. ■

From the asset pricing equation (3), one can see that changes of asset price come from *two channels*. The first channel is *interest rate*, that is, changes of interest rate r^f. The second channel is *productivity*, that is, changes of productivity a.

Wealth inequality Interested in how asset price changes affect inequality, I introduce wealth inequality. Denote η_t as entrepreneur's wealth share at time t. Entrepreneur's wealth share η_0 at $t = 0$ is exogenous. While entrepreneur's wealth share η_1 at $t = 1$ is endogenous and as follows

$$\eta_1 = \frac{qK_1 + B^e}{qK_1} = \frac{aK_1 + B^e(1 + r^f)}{aK_1 + (B^e + B^s)(1 + r^f)} \tag{4}$$

In the two-agent model, I refer to entrepreneur's wealth share as *wealth inequality.* A rise in entrepreneur's wealth share is an increase in wealth inequality.

3.2 Partial equilibrium

Proposition 1 (Partial equilibrium) *In the partial equilibrium analysis where the interest rate r^f is exogenous and asset price q is endogenous, entrepreneur's wealth share tomorrow η_1 responds to changes of asset price q as follows:*

$$\frac{d\eta_1}{dq} > 0 \tag{5}$$

And saver's welfare V^s respond to changes of asset price q as follows[20]*:*

$$\begin{aligned} \frac{\partial V^s}{\partial a} &= 0 \\ \frac{\partial V^s}{\partial r^f} &= \frac{1}{1+\rho} U'(C_1^s) \underbrace{B^s}_{>0} > 0 \end{aligned} \tag{6}$$

where B^s is saver's bond holding, K_1 is entrepreneur's asset holding at $t = 1$, r^f is the (net) interest rate, and a is productivity.

Proof. See appendix. ■

The interest rate (r^f) channel and productivity (a) channel of asset price changes emerge for welfare changes as well.

From (6), one can see that saver's welfare is affected by the change of interest rate r^f, not by the change of productivity a. Since saver can not hold any capital but can trade risk-free bond with entrepreneur freely. The saver's welfare is only affected through the *interest rate channel*.

I summarize the partial equilibrium results as follows: The rising asset price $q = \frac{a}{1+r^f}$ due to a lower interest rate r^f increases wealth inequality and hurts the saver who now saves at a lower interest rate.

[20]Here I am using the market clearing condition for capital and the exogenous interest rate is such that entrepreneur is a borrower.

3.3 General equilibrium

Partial equilibrium analysis shows that even though the saver does not own the asset, the effect of rising asset prices spills over to them through leverage and interest rates. I proceed to link asset price changes and leverage in general equilibrium by imposing market clearing conditions for risk-free bonds to have endogenous interest rate.

Market clearing For capital with fixed supply K and risk-free bond with net zero supply:

$$K_1 = K \tag{7}$$

$$B^s + B^e = 0 \tag{8}$$

Proposition 2 (General equilibrium) *In general equilibrium, I solve for asset price q, wealth inequality tomorrow η_1, and welfare of the saver V^s, given exogenous parameters of the model $\{a, \rho^e, \rho, K, W_0^e, W_0^s\}$ where a is productivity, ρ^e and ρ are discount rates of entrepreneur and saver, W_0^e and W_0^s are the endowment of entrepreneur and saver today, and K is the total supply of capital. I study the comparative statics with respect to entrepreneur's relative "impatience" (a decrease in $\delta^e = \rho^e - \rho$),*

1. *Asset price q and interest rate r^f:*

$$\frac{\partial q}{\partial \delta^e} < 0 \tag{9}$$

$$\frac{\partial r^f}{\partial \delta^e} > 0 \tag{10}$$

2. *Wealth inequality (entrepreneur's wealth share) tomorrow η_1:*

$$\frac{\partial \eta_1}{\partial \delta^e} < 0 \tag{11}$$

3. Saver's welfare V^s is as follows:

$$\frac{\partial V^s}{\partial \delta^e} > 0 \tag{12}$$

Proof. See appendix. ■

A decline in entrepreneur's relative "impatience" (a decrease in $\delta^e = \rho^e - \rho$) increases entrepreneur's saving demand, decreases the interest rate r^f, and increases asset price q. Wealth inequality tomorrow increases because the saver now faces a lower interest rate r^f.

A decline in entrepreneur's relative "impatience" (a decrease in δ^e) hurts the saver due to lower interest rate. Using envelop theorem, one can write the welfare changes as follows,

$$\frac{\partial V^s}{\partial \delta^e} = \frac{1}{1+\rho} U'(C_1^s) \underbrace{B^s}_{>0} \underbrace{\frac{\partial r^f}{\partial \delta^e}}_{>0} > 0$$

where $B^s > 0$ is the saving of saver and $B^e < 0$ is the borrowing of entrepreneur.

I summarize the general equilibrium results as follows: A decline in entrepreneur's relative "impatience" (a decrease in δ^e) raises asset price, increases wealth inequality tomorrow, and hurts the saver.

3.4 What's missing

While the analysis is sharp and intuitive in the simple two-period model, some important elements that affect asset prices, inequality and welfare are still missing. I discuss three key elements in this section: uncertainty, bubbles, and endogenous feedback from wealth inequality.

Uncertainty A volatile economic environment makes it difficult to ignore uncertainty. Uncertainty creates precautionary saving motive and requires a risk premium. Previous studies have shown that precautionary saving motive is important for determining consumption plans and asset prices are heavily influenced by risk premium.

Uncertainty can result in conflicting direct effect and indirect effect on asset prices, wealth inequality and welfare. A lower level of uncertainty leads to a lower risk premium, which directly increases asset prices. A lower level of uncertainty also leads to a lower level of precautionary saving motive, which increases the interest rate and decreases entrepreneur's borrowing. The direct effect of less uncertainty tends to increase wealth inequality and the welfare of the rich, whereas the indirect effect tends to decrease them.

Bubble A bubble emerges when the market value of an asset exceeds its fundamental value, which is the net present value of all future cash flows. In the two-period model, I cannot identify how bubbles affect asset prices and inequality differently from the fundamental value of an asset.

Bubbles also result in conflicting direct effect and indirect on asset prices, wealth inequality and welfare. A rise in the value of bubble directly increases the market value of an asset. While it also increases interest rate because of inter-temporal substitution effect. The increased interest rate indirectly decreases asset prices. The direct effect of a bubble tends to increase wealth inequality and the welfare of the rich, while the indirect effect tends to decrease them.

Endogenous feedback from wealth inequality An important mechanism that is missing in the two-period model is the endogenous feedback from wealth inequality η to the economy. When there are more than two periods, wealth inequality becomes an endogenous state variable that asset prices and people's consumption-saving plans depend on. The endogenous feedback from wealth inequality in a multi-period model will turn out to be important in section 4.

In the following section, I develop an infinite-horizon model that takes into account uncertainty and endogenous bubbles and characterize the endogenous feedback from wealth inequality to the economy.

4 Full model

In this section, I develop a continuous-time infinite-horizon version of the two-period model in section 3 with a few key deviations:[21]

1. There are two groups of agents, entrepreneurs $i \in [0,1]$ and savers $j \in [0,1]$.
2. There are public equities and private equities on financial market.
3. There is uncertainty in the economy.
4. Asset bubbles are endogenously formed.

4.1 Full Model Set-up

Preferences There are two groups of agents, entrepreneurs and savers. Both groups of agents have logarithmic utility for tractability. Entrepreneurs (discount rate $\rho^e = \rho + \delta^e$) are more "impatient" than savers (discount rate ρ).

Technology Entrepreneurs and savers live in an endowment economy with productive capital (or a tree).[22] Per unit capital produces a units of output and a is productivity. Only entrepreneurs can manage private capital and private capital is exposed to idiosyncratic risks. When managed by any individual entrepreneur i, private capital k_t^i evolves according to the following Ito process

$$\frac{dk_t^i}{k_t^i} = g dt + \tilde{\sigma} d\tilde{Z}_{i,t} \tag{13}$$

where g is the expected growth rate of capital, and $\tilde{\sigma}$ is the volatility of idiosyncratic risk $d\tilde{Z}_{i,t}$. Idiosyncratic shock $d\tilde{Z}_{i,t}$ is specific to each entrepreneur i. The idiosyncratic shocks $d\tilde{Z}_{i,t}$, $\forall i$ are independent and they cancel out in the aggregate, i.e. $\int_0^1 d\tilde{Z}_{i,t} = 0$. One can think of each entrepreneur as running a private firm using capital to produce. Entrepreneurs

[21] Time is set to be continuous for tractability.
[22] Or we are in an AK-production economy with infinite investment cost.

can buy and sell capital on the market. The price of capital per unit is denoted as q_t, which is an endogenous process. Postulate the process for capital price q_t which I will solve for as follows,

$$\frac{dq_t}{q_t} = \mu_t^q dt$$

The capital price q_t does not carry any idiosyncratic risk because it is determined in the aggregate.

Financial market The financial market consists of private equities, public equities, and risk-free bonds.

Entrepreneurs can issue outside equities to a public stock market but there is a maximum $1-\underline{\chi}$ fraction of capital that can be promised as outside equity due to agency frictions arising from incentive problems. That is, entrepreneurs must hold at least $\underline{\chi}$ fraction of the value of their private firm as private equity. This constraint is also known as the "skin in the game".[23] The value of outside equity issued by entrepreneur i is denoted as $V_t^{oe,i} = (1-\chi_{it})q_t k_t^i$, where $1 - \chi_{it}$ is the fraction of capital issued as outside equity by entrepreneur i.

Idiosyncratic risks cancel out after the public stock market pools outside equities together. And the diversified outside equities form a stock market index fund free from idiosyncratic risks, S&P 500 index fund for example. I refer to public equity and the stock market index fund interchangeably. Savers are restricted from holding private capital but they can hold the stock market index fund in an *inactive* way.[24] One can interpret this assumption as the savers hold the stock market index fund through pension. The value of the stock market index fund (public equity) is denoted as V_t^{mf} and will be determined in equilibrium.

The fraction of public equity held by savers is denoted as $1 - \kappa$, where κ is a parameter

[23]This type of equity constraint is wildly used in the macro-fiance literature, micro-founded in the corporate finance literature, and receives supportive empirical evidence.

[24]Allowing savers to optimally choose their portfolio share in public equity will create indeterminacy between leverage and public equity holding. However, all other results (asset prices, wealth inequality, bubbles) remains the same.

of the model.[25] Using the market clearing condition for public equity, entrepreneurs hold κ fraction of the public equity in equilibrium.

Both entrepreneurs and savers can trade risk-free bond freely. Risk-free bond is in zero net supply and is denoted as B_t.

Figure 1 shows the balance sheets of entrepreneurs and savers respectively and figure 2 shows the financial market structure.

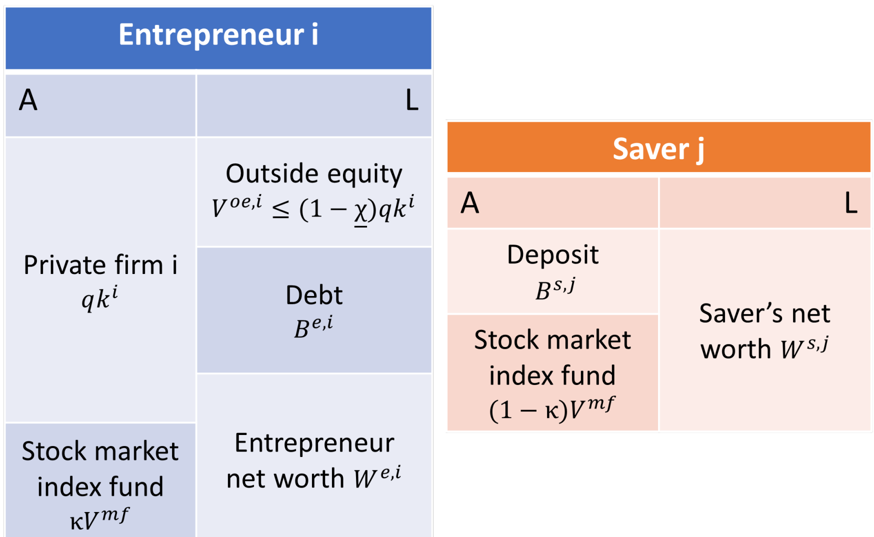

Figure 1: Balance sheets of entrepreneur and saver

Asset returns I introduce notations for asset returns which are *endogenous* processes. The return of capital when held by entrepreneur i, i.e. private equity, is denoted as follows

$$\begin{aligned} dr_t^{k,i} &= \underbrace{\frac{a}{q_t}}_{\text{dividend yield}} dt + \underbrace{\frac{d(q_t k_t)}{q_t k_t}}_{\text{capital gain}} \\ &= (\frac{a}{q_t} + g + \mu_t^q +)dt + \tilde{\sigma} d\tilde{Z}_t^i \end{aligned} \tag{14}$$

[25]I restrict the value of κ such that $\kappa > \max\{\frac{\underline{\chi}}{1-\underline{\chi}}(\frac{\tilde{\sigma}}{\sqrt{\delta^e}} - 1), \underline{\chi}(1 - \frac{\tilde{\sigma}}{\sqrt{\delta^e}})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho+\sqrt{\delta^e}}\frac{\rho}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho}\}, \frac{\underline{\chi}}{1-\underline{\chi}}(\frac{\tilde{\sigma}}{\delta^e} - 1)\}$. I discuss this restriction in appendix.

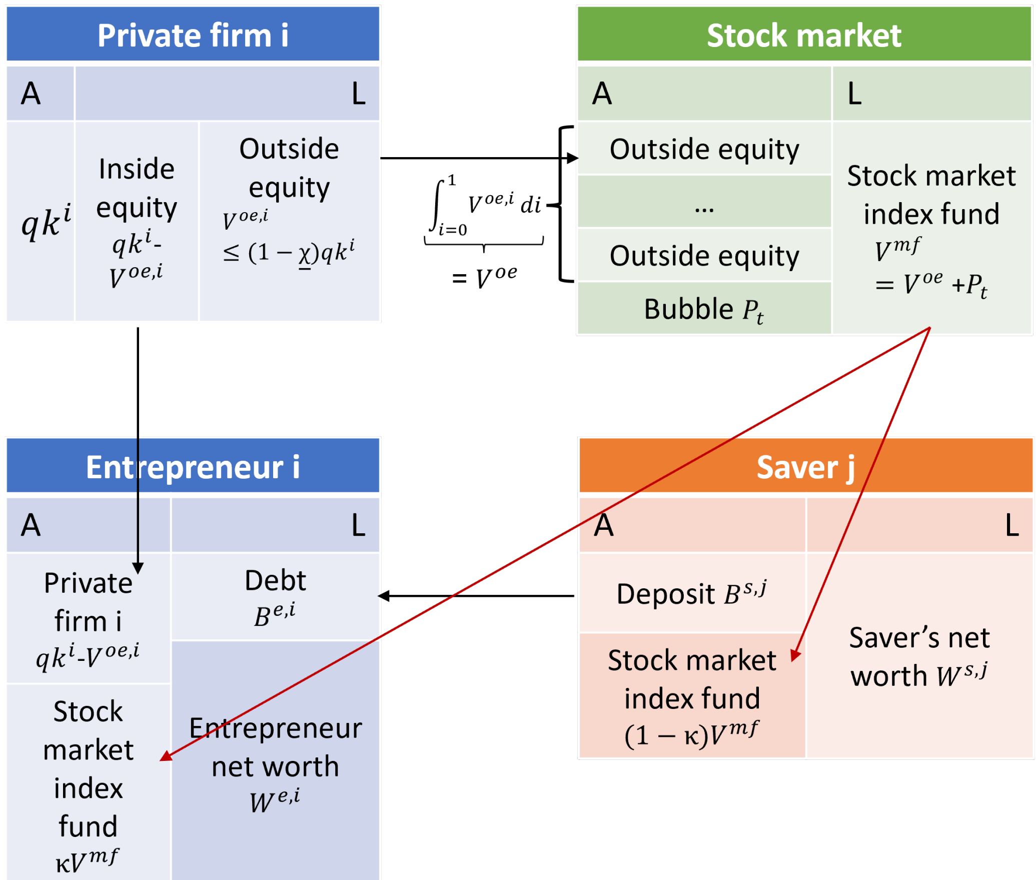

Figure 2: Financial market structure

The return of outside equity issued by entrepreneur i is

$$dr_t^{oe,i} = \mathbb{E}[dr_t^{oe,i}] + \tilde{\sigma} d\tilde{Z}_t^i$$

where the expected return of outside equity $\mathbb{E}[dr_t^{oe,i}]$ is determined in equilibrium. Outside equity has the same risk characteristic as inside equity but may have a different expected return due to the equity constraint. Without the equity constraint, the expected return of outside equity should equal the return of inside equity, $\mathbb{E}[dr_t^{oe,i}] = \mathbb{E}[dr_t^{k,i}]$. However, when the equity constraint binds, the expected return of outside equity is lower than the return of inside equity $\mathbb{E}[dr_t^{oe,i}] < \mathbb{E}[dr_t^{k,i}]$.

Finally, the return of public equity (the stock market index fund) is denoted as dr_t^{mf}, and the return of risk-free bond is denoted as $\frac{dB_t}{B_t} = r_t^f dt$. The return of public equity and risk-free rate are the same for each individual.

Wealth and portfolio shares I introduce notations for wealth and portfolio shares, which will be determined in equilibrium. Denote the wealth of entrepreneur i as $W_t^{e,i}$ and the wealth of saver j as $W_t^{h,j}$. The portfolio share of capital of entrepreneur i is denoted as $\theta_t^{k,i} = \frac{q_t k_t^i}{W_t^{e,i}}$. Outside equity's portfolio share of an entrepreneur's wealth is denoted as $\theta_t^{oe,i} = \frac{-(1-\chi_{it})q_t k_t^i}{W_t^{e,i}}$.[26] The portfolio share of public equity (the stock market index fund) held by entrepreneur i is denoted as $\theta_t^{mf,i}$. And the portfolio share of public equity (the stock market index fund) held by saver j is denoted as $\alpha_t^{mf,i} = \frac{(1-\kappa)V_t^{mf}}{W_t^{s,j}}$. Entrepreneurs can optimally choose their portfolio share in public equity $\theta_t^{mf,i}$, while savers take their portfolio share in public equity $\alpha_t^{mf,i} = \frac{(1-\kappa)V_t^{mf}}{W_t^{s,j}}$ as given.

Wealth inequality Define entrepreneurs' wealth share as

$$\eta_t = \frac{W_t^e}{W_t^e + W_t^s} \tag{15}$$

where $W_t^e = \int_{i=0}^1 W_t^{e,i}\, di$ and $W_t^s = \int_{j=0}^1 W_t^{s,j}\, dj$ are the aggregate wealth of entrepreneurs and savers at time t respectively. I refer to η_t as *wealth inequality*. If entrepreneurs' wealth share increases, wealth inequality increases.

Optimization problems The optimization problem for entrepreneur i is as follows:

$$\begin{aligned} \max_{\{c_t^{e,i}, \theta_t^{k,i}, \theta_t^{oe,i}, \theta_t^{mf}\}_{t=0}^{\infty}} \quad & \mathbb{E}\left[\int_0^\infty e^{-\rho^e t} \log c_t^{e,i} dt\right] \\ \text{s.t.} \quad & \frac{dW_t^{e,i}}{W_t^{e,i}} = r_t^f dt + \theta_t^{k,i}(dr_t^{k,i} - r_t^f dt) + \theta_t^{oe,i}(dr_t^{oe,i} - r_t^f dt) + \theta_t^{mf,i}(dr_t^{mf} - r_t^f dt) - \frac{c_t^{e,i}}{W_t^{e,i}} dt \\ & -\theta_t^{oe,i} \leq (1 - \underline{\chi})\theta_t^{k,i} \end{aligned} \tag{16}$$

[26] I use a minus sign because outside equities are issued by entrepreneurs.

An entrepreneur optimally choose the consumption plan and portfolio shares of private (inside) equity, outside equity, public equity, and risk-free bond, taking the returns of the assets as given.

The optimization problem for saver j is as follows[27]:

$$\begin{aligned} \max_{\{c_t^{s,j}\}_{t=0}^{\infty}} \quad & \int_0^{\infty} e^{-\rho t} \log c_t^{s,j} dt \\ \text{s.t.} \quad & \frac{dW_t^{s,j}}{W_t^{s,j}} = r_t^f dt + \alpha_t^{mf,j}(dr_t^{mf} - r_t^f dt) - \frac{c_t^{s,j}}{W_t^{s,j}} dt \end{aligned} \tag{17}$$

A saver optimally choose the consumption and saving plan, taking the risk-free rate as given. I leave the HJB equations for optimization problems and first-order conditions in appendix.

Market clearing condition The market for consumption clears as follows

$$C_t^e + C_t^s = aK_t \tag{18}$$

Equation (18) is the market clearing condition for consumption good. The left-hand side of (18) is the total demand of consumption good in the economy, where $C_t^e = \int_i c_t^{e,i} \, di$ and $C_t^s = \int_j c_t^{s,j} \, dj$ are the aggregate consumption of entrepreneurs and savers respectively. The right-hand side of (18) is the total supply of consumption good in the economy, where $K_t = \int_i k_t^i \, di$ is the aggregate capital and total production at time t in the economy is aK_t.

Definition of bubble I define bubble as follows

$$P_t = V_t^{mf} - \int_i V_t^{oe,i} \, di \tag{19}$$

Recall that V_t^{mf} is the total value of public equity which is held by entrepreneurs, or market value. The second term $\int_i V_t^{oe,i} \, di$ the value of all outside equities issued by en-

[27]Note that savers do not carry any idiosyncratic risks because the stock market index fund diversifies idiosyncratic risks.

trepreneurs, that is, the fundamental value of public equity. On the left-hand side, P_t is the wedge between the market value and fundamental value of public equity which can arise endogenously. I refer to P_t as the value of *bubble*, which will be determined in equilibrium.

For the value of bubble, I will also work with $p_t \equiv \frac{P_t}{(1-\chi_t)K_t}$ for easier mathematical expressions. Postulate a process for p_t which I will solve for,

$$\frac{dp_t}{p_t} = \mu_t^p dt$$

Note that the process of p_t does not contain idiosyncratic risks, because the value of bubble is determined in the aggregate.

For the rest of the paper, I use superscripts $\{f, b\}$ to distinguish variables in fundamental equilibrium (defined in the following section 4.2) and bubble equilibrium (defined in section 5.2) when necessary. And I use an overline to denote variables in steady states.

4.2 Fundamental Equilibrium

In this section, I focus on and solve for fundamental equilibrium which is defined later. I define and solve for bubble equilibrium in section 5.

Definition 1 (Fundamental equilibrium) *A fundamental equilibrium is a process of capital price q_t, a process of outside equity return dr_t^{oe}, a process of risk-free rate r_t^f, a process of public equity return dr_t^{mf}, and a process of entrepreneurs' wealth share η_t^f, given exogenous parameters of the model $\{\delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, g\}$, such that*

1. *entrepreneurs solve optimization problem* (16)

2. *savers solve optimization problem* (17)

3. *consumption good's market clears* (18)

4. *the value of bubble is zero, $P_t = 0$*

Key equations for fundamental equilibrium I provide some key equations for solving the fundamental equilibrium with intuition and leave the technical details in appendix.

From first-order conditions with respect to consumption, entrepreneurs and savers optimally consume a constant fraction of their wealth with logarithmic utility. The constants are their discount rates ρ and r:

$$\frac{c_t^{e,i}}{W_t^{e,i}} = \underbrace{\rho + \delta^e}_{\rho^e}, \quad \frac{c_t^{s,j}}{W_t^{s,j}} = \rho \tag{20}$$

The expected return of outside equity $\mathbb{E}[dr_t^{oe,i}]$ is pivotal for equilibrium because the maximum issuance of outside equity is limited by the equity constraint. In fundamental equilibrium, the equity constraint binds, it follows that

$$\mathbb{E}[dr_t^{k,i}] > \mathbb{E}[dr_t^{oe,i}] = dr_t^{mf} \tag{21}$$

The first inequality $\mathbb{E}[dr_t^{k,i}] > \mathbb{E}[dr_t^{oe,i}]$ shows that the expected return of private equity (inside equity) is higher than the expected return of outside equity. This is because of the binding equity constraint. The second equality $\mathbb{E}[dr_t^{oe,i}] = dr_t^{mf}$ shows that the expected return of outside equity is equal to the return of public equity. This comes directly from equation (19) when the value of bubble is zero ($P_t = 0$). As one will see in next section, equation (21) only holds true in fundamental equilibrium and changes in bubble equilibrium. Entrepreneurs' wealth share, i.e. wealth inequality, which is the important endogenous state variable of the model, evolves as follows,

$$\frac{d\eta_t^f}{\eta_t^f} = (1 - \eta_t^f)\left(-\delta^e + \left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2\right) dt \tag{22}$$

Equation (22) shows the decisive forces for wealth inequality: the patience gap between entrepreneurs and savers δ^e and the effect of uncertainty $\left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2$. Entrepreneurs are more impatient than savers, they consume a larger fraction of their wealth than savers, $\delta^e > 0$.

The patience gap decreases entrepreneurs' wealth share. While entrepreneurs earn a higher return that compensates for the idiosyncratic risks associated with their private capital, $\left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2 > 0$. The risk premium increases entrepreneurs' wealth share, thus wealth inequality. Another interpretation of $\left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2$ is entrepreneurs' precautionary saving motive.[28] With a higher level of precautionary saving motive, entrepreneurs borrow less, which increases their wealth share and wealth inequality.

I solve for the steady state[29] of fundamental equilibrium and summarize the results in the following proposition.

Proposition 3 (Fundamental equilibrium steady state) *In steady state of fundamental equilibrium, capital price, wealth inequality, and the value of bubble are as follows*

$$\overline{q}^f = \frac{a}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{23}$$

$$\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} \tag{24}$$

$$\overline{p}^f = 0 \tag{25}$$

where ρ and r are discount rates of entrepreneurs and savers respectively, a is the productivity of capital, $\underline{\chi}$ is the minimum fraction of the private firm that must be kept by entrepreneurs as inside equity, and $\tilde{\sigma}$ is the volatility of idiosyncratic risks associated with private capital.

For a non-degenerate wealth distribution in steady state where entrepreneurs are wealthier than savers, the following parameter restriction is required:

$$\frac{1}{2} < \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} < 1 \tag{26}$$

[28] In this model, risk premium of capital and precautionary saving motive happen to be the same, because the only source of uncertainty in the economy is the idiosyncratic risk associated with private capital.

[29] Or a balanced growth path with respect to aggregate capital which is the only source of growth in the economy. Steady-state is regarding to the lower-case variables that are scaled to be per unit of capital.

Risk premium of capital and risk-free rate in steady state are as follows

$$\frac{\mathbb{E}[d\overline{r}^{k,f,i} - \overline{r}^f dt]}{dt} = \underline{\chi}\tilde{\sigma}\sqrt{\delta^e} \tag{27}$$

$$\overline{r}^f = \rho + g \tag{28}$$

where g is the expected growth rate of capital.

Proof. See appendix. ■

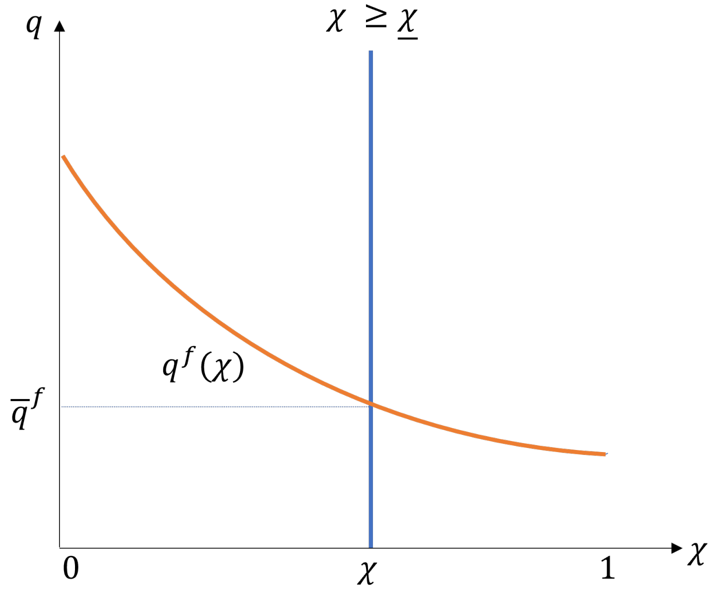

Figure 3: Capital price in fundamental equilibrium steady state

Figure 3 showed how capital price is determined by capital structure (inside equity and outside equity) in fundamental equilibrium steady state. The x-axis, χ, is the fraction of the private firm that is kept as inside equity. The y-axis, q, is the price of capital. In equilibrium, entrepreneurs would like to issue maximum amount of outside equity to offload idiosyncratic risks. The equity constraint always binds, $\chi = \underline{\chi}$. Capital price q^f is decreasing in χ because the more inside equity kept, the more idiosyncratic risk, the higher risk premium of capital, and thus lower price.

4.3 Connection with the two-period model

In this section, I connect the full model with the two-period model in section 3.

Proposition 4 (Connection with the two-period model) *Assume there is no uncertainty, $\tilde{\sigma} = 0$, and the expected growth rate of capital $g = 0$, the partial equilibrium and general equilibrium results in the two-period model in section 3 in the full model are recovered as follows,*

1. *In partial equilibrium with an exogenous constant risk-free rate r^f and endogenous asset price q^f:*

   (a) *Asset price q^f and wealth inequality η_t^f are*

$$q^f = \frac{a}{r^f} \tag{29}$$

$$\eta_t^f = 1 - \frac{B_t}{q^f K_t} \tag{30}$$

   (b) *Wealth inequality η_t^f changes with respect to asset price q^f as follows*

$$\frac{d\eta_t^f}{dq^f} > 0 \tag{31}$$

   (c) *Savers' welfare $V^{s,f}$ change with respect to asset price q^f as follows*

$$\frac{dV^{s,f}}{dq} = \int_0^\infty -e^{-\rho t} U'(C_t^s) \rho K_t q^f \frac{d\eta_t^f}{dq} \, dt < 0 \tag{32}$$

2. *Solving for general equilibrium with endogenous risk-free rate r_t^f and asset price q_t^f:*

   (a) *capital price q_t^f,*

$$q_t^f = \frac{a}{\delta^e \eta_t^f + \rho} \tag{33}$$

$$\frac{\partial q_t^f}{\partial \delta^e} < 0 \tag{34}$$

(b) risk-free rate r_t^f,

$$r_t^f = \frac{(\rho + \delta^e)\delta^e \eta_t^f}{\delta^e \eta_t^f + \rho} + \rho \tag{35}$$

(c) wealth inequality η_t^f,

$$\eta_t^f = \frac{1}{e^{\delta^e t + \log \frac{1-\eta_0}{\eta_0}} + 1} \tag{36}$$

$$\frac{\partial \eta_t^f}{\partial \delta^e} < 0 \tag{37}$$

(d) and welfare of savers $V^{s,f}$,

$$V^{s,f} = \int_0^\infty e^{-\rho t} \log(\rho(1 - \eta_t^f) q_t^f K_t)\, dt \tag{38}$$

$$\frac{\partial (V^{s,f}\big|_0^\infty)}{\partial \delta^e} > 0 \tag{39}$$

where a is the productivity of capital, $\rho + \delta^e$ and ρ are discount rates of entrepreneurs and savers respectively, δ^e is the relative "impatience" of entrepreneurs, and η_0 is the initial wealth inequality.

Proof. See appendix. ■

Consistency with the two-period model Partial equilibrium results of the infinite-horizon model are consistent with the two-period model (see proposition 1): rising asset prices always increases wealth inequality and hurt the savers.

General equilibrium results of the infinite-horizon model are also consistent with the two-period model: Declining relative "impatience" of entrepreneurs (a decrease in δ^e) raises asset price q_t, increases wealth inequality η_t, and decreases welfare of savers $V^{s,f}$.

5 Bubble equilibrium

I start this section by providing intuitions for bubbles. Then I define and solve for bubble equilibrium, and compare bubble equilibrium with fundamental equilibrium. I finish this section by discussing the effect of bubble on inequality.

5.1 Intuition for bubbles

Before solving for bubble equilibrium, I provide some intuition why there can be a bubble on public equity by identifying critical reasons in my model, as well as making an analogy to the existing literature.

There are three critical reasons in my model for bubbles to exist. First, idiosyncratic risks are diversified away in the aggregate on public stock market. Entrepreneurs are willing to offload their idiosyncratic risks by issuing outside equities of their private firms to the public stock market. Second, there is an equity constraint that limits outside equity issuance. Entrepreneurs still carry some idiosyncratic risks that create precautionary saving motive for them. Third, there is a trade-off for impatient entrepreneurs between borrowing from savers to consume and leveraging up idiosyncratic risks on their wealth which increases their precautionary saving motive. Bubbles on public equity reduce entrepreneurs' precautionary saving motive because the value of bubble does not carry idiosyncratic risks. Impatient entrepreneurs can thus borrow more from savers compared to when there is no bubble.

I make an analogy and show the subtle difference between bubbles in my model and bubbles that have positive value because they directly relax some constraint (see Kocherlakota [2009] and Miao and Wang [2018] for example). Recall that in my model there is an equity constraint that limits how much outside equities entrepreneurs can issue. However, this constraint also *indirectly* limits the supply of public equity. The value of bubble increases the supply of public equity, which breaks the indirect limitation of (outside) equity constraint on public equity.

$$\underbrace{V_t^{mf}}_{\text{value of public equity}} \leq \underbrace{(1-\underline{\chi})q_t K_t}_{\text{limit on outside equity}} + \underbrace{P_t}_{\text{value of bubble}} \tag{40}$$

As shown in equation (40), bubbles relax the indirect limit on public equity supply but not the direct limit on outside equity issuance.

Bubbles in this paper are stable as in the literature on rational bubbles driven by financial frictions. There is no *endogenous* switches from fundamental equilibrium to bubble equilibrium. However, as in the literature, one can think of there is an *exogenous* probability π for fundamental equilibrium to realize and a probability $1-\pi$ for bubble equilibrium to realize. In the baseline model, I do not talk about stochastic bubbles but simply compare fundamental equilibrium and bubble equilibrium.

5.2 Bubble equilibrium

Definition 2 (bubble equilibrium) *A bubble equilibrium is a capital price process q_t, a process of outside equity return dr_t^{oe}, a process of risk-free rate r_t^f, a process of public equity return dr_t^{mf}, and a process of entrepreneurs' wealth share η_t^b, given exogenous parameters of the model $\{\delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, g\}$, such that*

1. *entrepreneurs solve optimization problem* (16)

2. *savers solve optimization problem* (17)

3. *consumption good's market clears* (18)

4. *the value of bubble is positive, $P_t > 0$*

5. *the equilibrium is trembling-hand perfect.*[30]

[30] I use the trembling-hand perfect equilibrium as an equilibrium refinement to resolve indeterminacy. See the following section and appendix for details.

Key equations for bubble equilibrium savers and entrepreneurs have the same optimal consumption plans as constant fractions of their wealth in bubble equilibrium as in fundamental equilibrium (20).

However, the pivotal equation for the expected return of outside equity in bubble equilibrium becomes

$$\mathbb{E}[dr_t^{k,i}] = \mathbb{E}[dr_t^{oe,i}] > dr_t^{mf} \tag{41}$$

In equation (41), the first equality $\mathbb{E}[dr_t^{k,i}] = \mathbb{E}[dr_t^{oe,i}]$ shows that the expected return of outside equity $\mathbb{E}[dr_t^{oe,i}]$ is equal to the expected return of private equity $\mathbb{E}[dr_t^{k,i}]$. Entrepreneurs are indifferent between inside equities and outside equities, and the equity constraint is not binding.[31] The second equality $\mathbb{E}[dr_t^{oe,i}] > dr_t^{mf}$ shows that the expected return of outside equity is higher than the return of public equity. This is because outside equities earn a risk-premium for idiosyncratic risks. Figure 4 showed the value of bubble and capital price is determined by capital structure in bubble equilibrium steady state.

Recall that in fundamental equilibrium, equation (21) shows that the expected return of outside equity is lower than the expected return of private equity due to the binding equity constraint and equal to the return of public equity, $\mathbb{E}[dr_t^{k,i}] > \mathbb{E}[dr_t^{oe,i}] = dr_t^{mf}$. The comparison between equation (21) and (41) shows the origin of the bubble: the wedge between the return of public equity dr_t^{mf} and the expected return of outside equity $\mathbb{E}[dr_t^{oe,i}]$. It is exactly this wedge in return that creates a wedge in the market value and the fundamental value of public equity, $P_t > 0$, as shown in equation (19). Figure 5 showed the supply and demand for public equity in bubble equilibrium steady state.

The evolution of wealth inequality, i.e. entrepreneurs' wealth share, in bubble equilibrium is as follows,

$$\frac{d\eta_t^b}{\eta_t^b} = (1-\eta_t^b)\left(-\delta^e + \left(\frac{\underline{\chi} q_t \tilde{\sigma}}{\eta_t^b[\underline{\chi} q_t + (1-\underline{\chi})(q_t + p_t)]} \right)^2 \right) dt \tag{42}$$

[31]The issuance of outside equity is thus indeterminate without equilibrium refinement.

Similar intuition as in fundamental equilibrium applies to the wealth share evolution in bubble equilibrium: Impatience of entrepreneurs decrease their wealth share, $\delta^e > 0$. While the risk premium of idiosyncratic risks associated with private capital increases entrepreneurs' wealth share, $\left(\frac{\underline{\chi} q_t \tilde{\sigma}}{\eta_t^b[\underline{\chi} q_t + (1-\underline{\chi})(q_t+p_t)]}\right)^2 > 0$. Note that given the same level of wealth inequality η_t, the risk premium or the precautionary saving motive is *lower* in bubble equilibrium compared to fundamental equilibrium. To achieve the same level of risk premium or the precautionary saving motive as in fundamental equilibrium, a *lower* level of wealth inequality is needed in bubble equilibrium.

I solve for the steady state of bubble equilibrium and summarize the results in the following proposition.

Proposition 5 (Bubble equilibrium steady state) *In steady state, capital price, wealth inequality, and the value of bubble are as follows*

$$\overline{q}^b = \frac{a}{\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{43}$$

$$\overline{\eta}^b = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e} + (1-\underline{\chi})\frac{\tilde{\sigma}\delta^e}{\rho}} \tag{44}$$

$$\overline{p}^b = \frac{a}{\rho} - \overline{q}^b \tag{45}$$

where $\rho + \delta^e$ and ρ are discount rates of entrepreneurs and savers respectively, a is the productivity of capital, $\underline{\chi}$ is the minimum fraction of the private firm that must be kept by entrepreneurs as inside equity, and $\tilde{\sigma}$ is the volatility of idiosyncratic risks associated with private capital.

For a non-degenerate steady state wealth distribution, the following parameter restriction is required:

$$[\underline{\chi} - (1-\underline{\chi})\frac{\delta^e}{\rho}]\tilde{\sigma} < \sqrt{\delta^e} \tag{46}$$

Risk premium of capital and risk-free rate in steady-state are as follows

$$\frac{\mathbb{E}[d\overline{r}^{k,b,i} - \overline{r}^f dt]}{dt} = \tilde{\sigma}\sqrt{\delta^e} \tag{47}$$

$$\overline{r}^f = \rho + g \tag{48}$$

where g is the expected growth rate of capital.

Proof. See appendix. ■

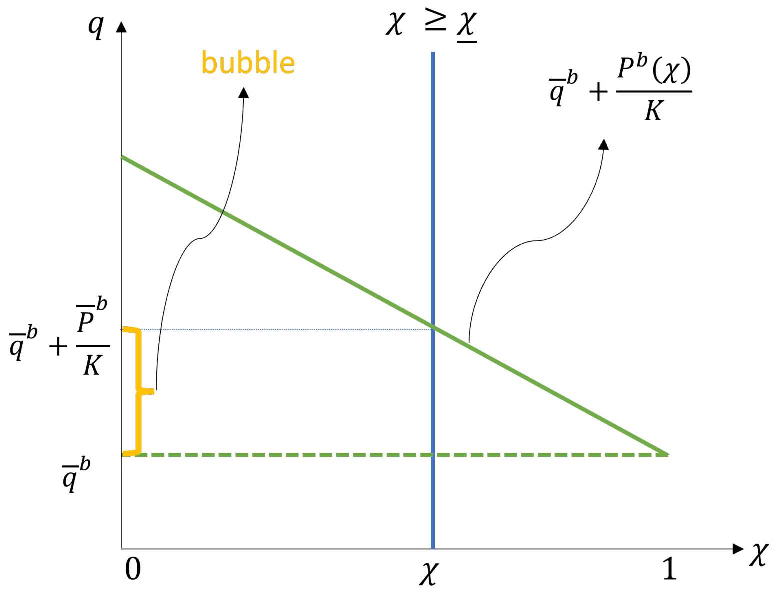

Figure 4: Capital price in bubble equilibrium steady state

Figure 4 shows how the value of bubble is determined by capital structure in bubble equilibrium steady state. And figure 5 shows the comparison between fundamental equilibrium and bubble equilibrium. The x-axis, χ, is the fraction of the private firm that is kept as inside equity. The y-axis, q, is price. Note that in bubble equilibrium, price of private capital $\overline{q}^b$ does not depend on capital structure. Because entrepreneurs are indifferent between inside equity and outside equity in bubble equilibrium. The value of bubble $\frac{P^b}{K}$ is linearly decreasing in χ.

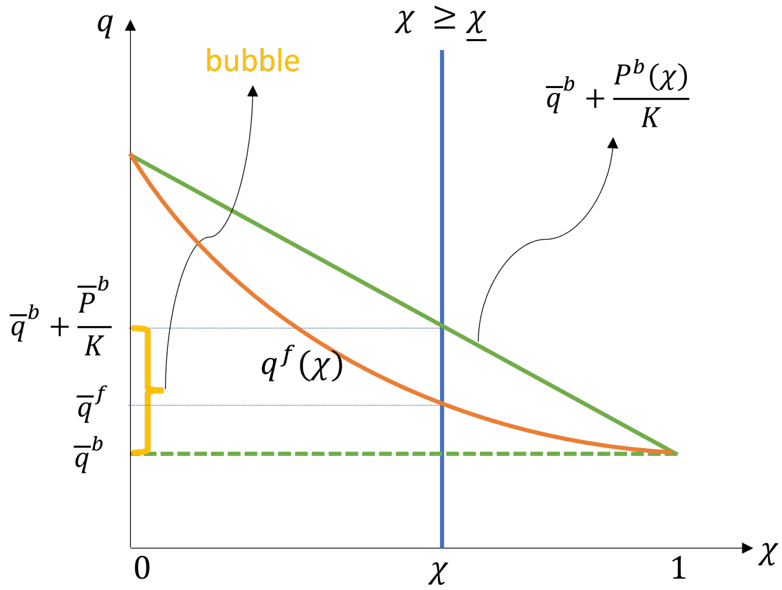

Figure 5: Comparison of capital price in bubble and fundamental equilibrium

5.3 Comparison of fundamental and bubble equilibrium

I compare steady states of bubble equilibrium and fundamental equilibrium in the following proposition, and discuss the mechanism with intuition.

Proposition 6 (Comparison of fundamental and bubble equilibrium) *Comparing steady states of fundamental and bubble equilibrium as follows,*

1. *Capital price:*

$$\overline{q}^b < \overline{q}^f \tag{49}$$

2. *Total wealth in the economy:*

$$\overline{q}^b K_t + \overline{P}_t > \overline{q}^f K_t \tag{50}$$

3. *Risk-free bond issued by entrepreneurs:*

$$\overline{B}_t^b > \overline{B}_t^f \tag{51}$$

4. Wealth inequality:

$$\overline{\eta}^b < \overline{\eta}^f \tag{52}$$

5. Risk premium on capital:

$$\frac{\mathbb{E}[d\overline{r}^{k,b,i} - \overline{r}^f dt]}{dt} > \frac{\mathbb{E}[d\overline{r}^{k,f,i} - \overline{r}^f dt]}{dt} \tag{53}$$

Equation (49) shows that capital price is lower in bubble equilibrium than in fundamental equilibrium. In fundamental equilibrium, the expected return of outside equity equals the return of public equity (21). In bubble equilibrium, the expected return of outside equity equals the return of inside equity, which is higher than the return of public equity (41). Higher expected return of outside equity lowers capital price in bubble equilibrium. As a corollary, the total amount of idiosyncratic risks associated with private capital is reduced in bubble equilibrium, $\overline{q}^b k_t^i \tilde{\sigma} < \overline{q}^f k_t^i \tilde{\sigma}$.

Equation (50) shows that total wealth is higher in bubble equilibrium than in fundamental equilibrium, since bubbles reduce the total amount of idiosyncratic risks that cannot be diversified which is the key friction in the economy.

Equation (51) shows entrepreneurs borrow more from savers in bubble equilibrium than in fundamental equilibrium. Entrepreneurs' precautionary saving motive decreases in bubble equilibrium as the value of bubble does not carry idiosyncratic risks.

Equation (52) shows that wealth inequality is *lower* in bubble equilibrium than in fundamental equilibrium. This result seems counter-intuitive at the first sight. I provide two intuitive explanations. If one looks at the balance sheet of entrepreneurs (see figure 1), their total asset value is indeed higher in bubble equilibrium. This is the *price channel*. Nevertheless, on the liability side, entrepreneurs' borrowing also increases in bubble equilibrium. A bubble increases the total value of entrepreneurs' assets and decreases their precautionary saving motive. Entrepreneurs borrow more from savers in bubble equilibrium than in fundamental equilibrium. This is the *leverage channel*. In equilibrium, leverage channel dominates. As a result, wealth inequality is lower in bubble equilibrium. Another explanation is to look at entrepreneurs' wealth accumulation process (42). Entrepreneurs accumulate their wealth by taking idiosyncratic risks and earning a higher return. As bubbles reduce the total amount of idiosyncratic risks in the economy, entrepreneurs' wealth share decreases. Thus wealth inequality decreases.

Equation (53) shows that risk premium on capital is higher in bubble equilibrium than in fundamental equilibrium. Bubble equilibrium is achieved when entrepreneurs are willing to hold a bubble. In order to clear the equity market, the expected return of capital is higher and the risk premium on capital is higher in bubble equilibrium.

5.4 Safety and liquidity effect of bubble on wealth inequality

After solving for both fundamental equilibrium and bubble equilibrium, I am able to delve deeper into how bubbles affect inequality. In this section, I study some extreme cases and provide more intuition.

A bubble increases the *safety* of the economy by reducing idiosyncratic risks and increases the *liquidity* of the economy by allowing entrepreneurs to borrow more. As a result, wealth inequality decreases. I examine some extreme cases in the following propositions to identify the safety effect and the liquidity effect of bubble on wealth inequality.

Proposition 7 (Safety effect) *I examine the steady states of two extreme cases with no uncertainty,*

1. *In an economy with no equity constraint ($\underline{\chi} = 0$): price of capital q, wealth inequality η, and value of bubble P in fundamental equilibrium and bubble equilibrium are as follows,*

$$\overline{q}^f = \frac{a}{\rho} \qquad \overline{q}^b = \frac{a}{\tilde{\sigma}\sqrt{\delta^e} + \rho}$$

$$\overline{\eta}^f = 0 \qquad \overline{\eta}^b = 0$$

$$\overline{P}_t^f = 0 \qquad \overline{P}_t^b = (\frac{a}{\rho} - \overline{q}^b)K_t$$

and

$$\lim_{\underline{\chi}\to 0}\left(\frac{\overline{\eta}^b}{\overline{\eta}^f}\right)=\frac{1}{1+(1-\underline{\chi})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}}=\frac{1}{1+\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}}$$

2. *In an economy with no idiosyncratic risk ($\tilde{\sigma}=0$): price of capital q, wealth inequality η, and value of bubble P in fundamental equilibrium and bubble equilibrium are as follows,*

$$\overline{q}^f=\overline{q}^b=\frac{a}{\rho}$$
$$\overline{\eta}^f=\overline{\eta}^b=0$$
$$\overline{P}_t^f=\overline{P}_t^b=0$$

and

$$\lim_{\tilde{\sigma}\to 0}\left(\frac{\overline{\eta}^b}{\overline{\eta}^f}\right)=\frac{1}{1+(1-\underline{\chi})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}}=1$$

where ρ and r are discount rates of entrepreneurs and savers respectively, $\underline{\chi}$ is the minimum fraction of the private capital that must be kept by entrepreneurs, $\tilde{\sigma}$ is the volatility of idiosyncratic risks, a is the productivity of capital, and K_t is the aggregate capital in the economy at time t.

In both cases, there are no idiosyncratic risks in equilibrium. The fundamental equilibrium in both cases are the same. Entrepreneurs can borrow against all their wealth, so liquidity is perfect in both cases. The value of bubble P differs. While the value of bubble is zero when idiosyncratic risk is zero, $\tilde{\sigma}=0$, the bubble can still sustain a positive value in the case of $\underline{\chi}=0$. The wealth share of entrepreneurs become 0 in steady state in both cases. The level of wealth inequality, which is first-order, does not seem to be affected by safety.

However, in both cases, η converges to zero, but at different rates,

$$\lim_{\underline{\chi}\to 0}\left(\frac{\overline{\eta}^b}{\overline{\eta}^f}\right) = \frac{1}{1+(1-\underline{\chi})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}} = \frac{1}{1+\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}} \tag{54}$$

$$\lim_{\tilde{\sigma}\to 0}\left(\frac{\overline{\eta}^b}{\overline{\eta}^f}\right) = \frac{1}{1+(1-\underline{\chi})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}} = 1 \tag{55}$$

This comparison captures the subtle *second-order safety effect* of bubble on wealth inequality.

Proposition 8 (Liquidity effect) *I examine the steady states of two extreme cases with maximum level of uncertainty and frictions,*

1. *In an economy with no public stock market ($\underline{\chi} = 1$): price of capital q, wealth inequality η, and value of bubble P in fundamental equilibrium and bubble equilibrium are as follows,*

$$\overline{q}^f = \overline{q}^b = \frac{a}{\tilde{\sigma}\sqrt{\delta^e}+\rho}$$
$$\overline{\eta}^f = \overline{\eta}^b = \frac{\tilde{\sigma}}{\sqrt{\delta^e}}$$
$$\overline{P}^f_t = \overline{P}^b_t = 0$$

2. *In an economy with infinite volatility of idiosyncratic risks ($\tilde{\sigma} = +\infty$): price of capital q, wealth inequality η, and value of bubble P in fundamental equilibrium and bubble equilibrium are as follows,*

$$\overline{q}^f = 0 \qquad \overline{q}^b = 0$$
$$\overline{\eta}^f = 1 \qquad \overline{\eta}^b = \frac{\underline{\chi}\rho}{(1-\underline{\chi})\delta^e}$$
$$\overline{P}^f_t = 0 \qquad \overline{P}^b_t = (1-\underline{\chi})\frac{a}{\rho}K_t$$

where $\rho + \delta^e$ and ρ are discount rates of entrepreneurs and savers respectively, δ^e is the *relative "impatience" of entrepreneurs, $\underline{\chi}$ is the minimum fraction of the private capital that must be kept by entrepreneurs, $\tilde{\sigma}$ is the volatility of idiosyncratic risks, a is the productivity of capital, and K_t is the aggregate capital in the economy at time t.*

In the case of $\underline{\chi} = 1$, there is no public stock market access and idiosyncratic risks can not be diversified. As idiosyncratic risk goes up, $\tilde{\sigma} > \sqrt{\delta^e}$, the liquidity in the economy stops and all the wealth are held by entrepreneurs, $\overline{\eta}^f = \overline{\eta}^b = 1$. The value of bubble is zero, $\overline{P}_t^b = 0$. In the case of $\underline{\chi} < 1$, the public stock market operates to the extent of $1 - \underline{\chi} > 0$, and the bubble has a positive value, $P_t^b > 0$. In fundamental economy, as idiosyncratic risk goes up, $\underline{\chi}\tilde{\sigma} > \sqrt{\delta^e}$, all the wealth are held by entrepreneurs, $\overline{\eta}^f = 1$, and liquidity in the economy stops. However, there can still be liquidity in the bubble economy even as idiosyncratic risk goes to infinity, $\tilde{\sigma} \to +\infty$,

$$\lim_{\tilde{\sigma}\to+\infty} \underbrace{\frac{\tilde{\sigma}}{\sqrt{\delta^e}}}_{\text{wealth inequality } \overline{\eta}^b \text{ in the case of } \underline{\chi}=1} = 1 \tag{56}$$

$$\lim_{\tilde{\sigma}\to+\infty} \underbrace{\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e} + (1-\underline{\chi})\frac{\tilde{\sigma}(\delta^e)}{\rho}}}_{\text{wealth inequality } \overline{\eta}^b \text{ in the case of } \underline{\chi}<1} = \frac{\underline{\chi}\rho}{(1-\underline{\chi})\delta^e} \tag{57}$$

and

$$\frac{\underline{\chi}\rho}{(1-\underline{\chi})\delta^e} < 1 \quad \text{if} \quad \underline{\chi}\rho < (1-\underline{\chi})\delta^e \tag{58}$$

This comparison captures the *first-order liquidity effect* of bubble on wealth inequality.

In general cases, safety effect and liquidity effect work together. By looking at extreme cases in this section, I try to provide sharper intuition for these effects. Also note that $\underline{\chi} = 1$ case corresponds to the two-period model in section 3. In this case, the value of the endogenous bubble is zero. The public stock market that pools outside equities is the key to form endogenous bubbles that have positive value.

6 Inequality and welfare

In this section, I analyze the effect of rising asset prices due to different shocks on inequality measured in wealth, consumption and welfare both in fundamental equilibrium and bubble equilibrium.

Recall that I defined $\delta^e = \rho^e - \rho$ which captures the relative "impatience" of entrepreneurs to savers. One can interpret δ^e as the entry/exit or the motality/fertility rate of the entrepreneurs and ρ as a common discount rate shared by all agents in the economy.

I will work with a set of parameters $\{\delta^e, \rho, \tilde{\sigma}, \underline{\chi}, g, a\}$, where besides δ^e and ρ, $\underline{\chi}$ is the minimum fraction of the private capital that must be kept by entrepreneurs, $\tilde{\sigma}$ is the volatility of idiosyncratic risks, a is the productivity of capital, and g is the growth rate of aggregate capital in the economy. The shocks are categorized into three types: the relative "impatience" of entrepreneurs (δ^e), financial innovation and regulation ($\underline{\chi}\tilde{\sigma}$), productivity ($a$), as well as a bubble ($p_t > 0$). I will focus on steady-state analysis.

6.1 Level of inequality

In this section, I compare the steady-state level of wealth inequality, consumption inequality, and welfare in fundamental equilibrium and bubble equilibrium.

Wealth inequality Recall that wealth inequality is defined as entrepreneurs' wealth share relative to total wealth in the economy:

$$\eta_t = \frac{W_t^e}{W_t^e + W_t^s}$$

Since entrepreneurs and savers consume a constant fraction of their wealth with logarithmic utility, and the constants are their discount rates $\rho + \delta^e$ and ρ.[32] Interestingly, if we define consumption inequality as the aggregate consumption of entrepreneurs relative to the

[32]Using logarithmic utility allows me to characterize equilibrium in closed-form. However, the leverage channel remains with more general utility functions.

total consumption of the economy, it does not always move in the same direction as wealth inequality. Because consumption inequality is affected by discount rates as well as wealth inequality. Keeping the discount rates fixed, consumption inequality increases when wealth inequality increases. When there are shocks to discount rates, consumption inequality is not only directly affected by changes in discount rates themselves, but also indirectly affected by changes in wealth inequality. I expands on consumption inequality in the appendix.

Welfare The value function of savers in fundamental economy as an example for illustration is as follows

$$V^{s,f} = \int_0^\infty e^{-\rho t} \left(\underbrace{\log(1-\eta_t^f)}_{\text{wealth share}} + \underbrace{\log \rho}_{\text{consumption rate}} + \underbrace{\log\left(q_t^f K_t\right)}_{\text{total wealth}} \right) dt \tag{59}$$

As shown in equation (59), savers' welfare is affected by their wealth share, consumption rate, and total wealth in the economy. Savers do not carry any risk, so their welfare is not affected by precautionary saving motive.

The value functions for savers starting from steady state in fundamental economy and bubble economy are denoted as $\overline{V}^{s,f}$ and $\overline{V}^{s,b}$ respectively.

Proposition 9 (Level of inequality) *Comparing the level of wealth inequality, consumption inequality and welfare,*

1. *Wealth inequality:*

$$\overline{\eta}^b < \overline{\eta}^f \tag{60}$$

2. *Leverage:*

$$\overline{B}_t^b > \overline{B}_t^f \tag{61}$$

3. *Savers' welfare:*

$$\overline{V}^{s,b} > \overline{V}^{s,f} \tag{62}$$

Proof. See appendix. ■

Wealth inequality *decreases*, and welfare of the savers *increases* in bubble equilibrium compared to fundamental equilibrium in steady state.

The decrease in wealth inequality is discussed in proposition 6. Consumption inequality also decreases as there is no discount rate shocks.

Leverage increases in bubble equilibrium as the positive value of bubble decreases the precautionary saving motive of entrepreneurs, as shown in equation (42) and discussed in proposition 6.

Savers are better off in steady state of bubble equilibrium compared to fundamental equilibrium. The increase in savers' welfare follows from the decreased consumption inequality in bubble equilibrium.

6.2 Rising asset prices on inequality and welfare

I consider small changes of parameters in steady state and ignore the transition to new steady state which happens fast. The following proposition states how rising asset prices due to different shocks affect wealth inequality and welfare in a fundamental economy and is presented in Figure 6.

Proposition 10 *I identify the effect of rising asset prices due to different shocks on inequality in fundamental equilibrium steady state as follows,*

1. *When the relative "impatience" of entrepreneurs (δ^e) decreases, asset price q^f rises, wealth inequality η^f increases. Savers' welfare $V^{s,f}$ always decreases.*

$$d\delta^e < 0 \implies \begin{cases} dq^f > 0 \\ dB^f < 0 \quad d\eta^f > 0 \\ dV^{s,f} < 0 \end{cases}$$

2. *When there is financial innovation ($\underline{\chi}\tilde{\sigma}$ decreases), asset price q^f rises, wealth inequality*

η^f decreases. Savers' welfare $V^{s,f}$ always increases.

$$d(\underline{\chi}\tilde{\sigma}) < 0 \implies \begin{cases} dq^f > 0 \\ dB^f > 0 \quad d\eta^f < 0 \\ dV^{s,f} > 0 \end{cases}$$

3. *When productivity increases (a increases), asset price q^f rises, wealth inequality η^f do not change. Savers' welfare $V^{s,f}$ increases.*

$$da > 0 \implies \begin{cases} dq^f > 0 \\ dB^f > 0 \quad d\eta^f = 0 \\ dV^{s,f} > 0 \end{cases}$$

Proof. See appendix. ■

Asset price (public equity) ↑	Leverage	Wealth inequality	welfare of savers
Entrepreneurs' relative "impatience" ↓	↓	↑	☹
Financial innovation	↑	↓	☺
Productivity ↑	↑	—	☺

Figure 6: Rising asset price on inequality in fundamental equilibrium

As figure 6 shows, we can theoretically identify the exact shock to the asset price q^f by looking at the co-movement of the asset price q^f and wealth inequality $\overline{\eta}^f$, and conclude how the rising asset price q^f affect savers' welfare.

Relative "impatience" of entrepreneurs When entrepreneurs become less impatient relative to savers (δ^e decreases), they consume a smaller fraction of their wealth everyday which directly decreases their consumption share, however, they also borrow less from savers (leverage B^f decreases) which increases their wealth share and indirectly increases their consumption share. As entrepreneurs are wealthier than savers $\overline{\eta}^f > \frac{1}{2}$, the indirect effect dominates the direct effect, the consumption inequality increases. As a result, the welfare of savers decreases because their consumption falls.

Through the lens of my theory, the main trend of rising asset prices and rising wealth inequality in the past a few decades suggests that the relative "impatience" of the super rich entrepreneurs to savers declines. In this case, savers are worse off.

Financial innovation Financial innovation (a decrease in $\underline{\chi}\tilde{\sigma}$) increases the asset value and reduces the precautionary saving motive of entrepreneurs. Entrepreneurs borrow more from savers than before (leverage B^f increases). As a result, wealth inequality and consumption inequality decrease. The welfare of savers increases as their consumption increases.

This is a general equilibrium result. On one hand, financial innovation raises asset prices and increases the asset value of the super rich entrepreneurs. This is the *price channel*. On the other hand, it reduces the precautionary saving motive of the entrepreneurs who are borrowers, and they borrow more from the savers. This is the *leverage channel*. In equilibrium, leverage channel dominates. Wealth inequality and consumption inequality decreases. Savers are better off.

Productivity An increase in productivity a increases the asset price q^f and leverage B^f, but not wealth inequality nor consumption inequality in fundamental economy[33]. However, savers benefit from higher productivity due to a positive income effect. This result shows that even though wealth inequality and consumption inequality do not change, rising asset prices can still have welfare effect.

[33]This result holds true for CRRA utility functions

6.3 Rising asset prices on inequality in bubble economy

The following proposition states how rising asset prices due to different shocks affect inequality in bubble economy and is presented in Figure 7.

Proposition 11 *I identify the effect of rising asset prices due to different shocks on inequality in bubble equilibrium steady state as follows,*

1. *When the relative "impatience" of entrepreneurs (δ^e) decreases, the price of private equity q^b rises, the price of public equity $(q^b + p^b)$ does not change, wealth inequality η^b increases. Savers' welfare $V^{s,b}$ decreases.*

$$d\delta^e < 0 \implies \begin{cases} dq^b > 0 \quad d(q^b + p^b) = 0 \\ dB^f < 0 \qquad d\eta^b > 0 \\ dV^{s,b} < 0 \end{cases}$$

2. *When there is financial innovation ($\underline{\chi}\tilde{\sigma}$ decreases), the price of private equity q^b rises, the price of public equity $(q^b + p^b)$ does not change, wealth inequality η^b decreases. Savers' welfare $V^{s,b}$ increases.*

$$d(\underline{\chi}\tilde{\sigma}) < 0 \implies \begin{cases} dq^b > 0 \quad d(q^b + p^b) = 0 \\ dB^f > 0 \qquad d\eta^b < 0 \\ dV^{s,b} > 0 \end{cases}$$

3. *When productivity increases (a increases), the price of private equity q^b rises, the price of public equity $(q^b + p^b)$ rises, wealth inequality η^b do not change. Savers' welfare $V^{s,b}$ increases.*

$$da > 0 \implies \begin{cases} dq^b > 0 \quad d(q^b + p^b) > 0 \\ dB^f > 0 \qquad d\eta^b = 0 \\ dV^{s,b} > 0 \end{cases}$$

private equity price↑	public equity price	Leverage	Wealth inequality	welfare of savers
Entrepreneurs' relative "impatience" ↓	—	↓	↑	☹
Financial innovation	—	↑	↓	☺
Productivity ↑	↑	↑	—	☺

Figure 7: Rising asset price on inequality in bubble equilibrium

Proof. See appendix. ■

In bubble economy, as figure 7 shows, we can theoretically identify the exact shock to asset prices q^b and $(q^b + p^b)$ by looking at the co-movement of the price of private equity q^b, the price of public equity $(q^b + p^b)$, and wealth inequality $\overline{\eta}^b$, and conclude how the rising asset prices affect savers' welfare.

Public equity In bubble economy, the price of private equity q^b and public equity (q^b+p^b) differ and they do not necessarily move together. The price of public equity (per unit of capital) in steady state is $(\overline{q}^b + \overline{p}^b) = \frac{a}{\rho}$. Only changes to productivity a and the common impatience of the economy ρ affect the price of public equity. In order to identify the shock in bubble economy, we need to look at both the changes of private equity price q^b and public equity price $(q^b + p^b)$.

6.4 The effect of bubble on comparative statics of inequality

In this section, I compare the comparative statics of inequality in fundamental economy and bubble economy.

Proposition 12 (Financial shocks) *I consider comparative statics with respect to the volatility of idiosyncratic risk and the tightness of the equity constraint* $\{\tilde{\sigma}, \underline{\chi}\}$*:*

1. *Wealth inequality in response to financial shocks:*

$$\frac{\partial \overline{\eta}^f}{\partial \tilde{\sigma}} > \frac{\partial \overline{\eta}^b}{\partial \tilde{\sigma}} > 0$$

$$\frac{\partial \overline{\eta}^f}{\partial \underline{\chi}} > 0, \quad \frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} > 0$$

and

$$\frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} < \frac{\partial \overline{\eta}^f}{\partial \underline{\chi}} \iff \frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}(1-\underline{\chi})^2 + 2(1-\underline{\chi}) - 1 > 0$$

2. *Savers' welfare in response to financial shocks:*

$$\frac{\partial \overline{V}^{s,f}}{\partial \tilde{\sigma}} < \frac{\partial \overline{V}^{s,b}}{\partial \tilde{\sigma}} < 0$$

$$\frac{\partial \overline{V}^{s,f}}{\partial \underline{\chi}} < 0, \quad \frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} < 0$$

and

$$\frac{\partial \overline{V}^{s,f}}{\partial \underline{\chi}} < \frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} \iff \frac{\tilde{\sigma}}{\delta}\underline{\chi}^2 - 2\underline{\chi} + 1 > 0$$

Proof. See appendix. ■

In bubble equilibrium, the total value of asset that do not carry idiosyncratic risks is higher and the idiosyncratic risks on entrepreneurs' balance sheet is lower than in fundamental equilibrium. Hence, the effect of financial innovation on wealth inequality is mitigated by a higher value of safe asset in bubble economy in the first place.

In steady state of a bubble economy, the risk-premium earned by entrepreneurs decreases as the idiosyncratic risk $\tilde{\sigma}$ decreases, the value of bubble relative to the price of private equity also decreases, which further mitigates the decreasing effect of lower idiosyncratic risk on wealth inequality.

For a decreases of the tightness of the equity constraint $\underline{\chi}$, risk-premium decreases, while the value of bubble relative to the price of private equity increases, which amplifies the decreasing effect of lower idiosyncratic risk on wealth inequality.

Proposition 13 (Discount rate shock) *I consider comparative statics with respect to the relative "impatience" of entrepreneurs* $\{\delta^e\}$,

1. *Wealth inequality in response to discount rate shocks:*

$$\frac{\partial \overline{\eta}^f}{\partial \delta^e} < \frac{\partial \overline{\eta}^b}{\partial \delta^e} < 0$$

2. *Welfare in response to discount rate shocks:*

$$\frac{\partial \overline{V}^{s,f}}{\partial \delta^e} < \frac{\partial \overline{V}^{s,b}}{\partial \delta^e} \iff \frac{-1}{1-\overline{\eta}^{C,f}} \frac{\partial \overline{\eta}^{C,f}}{\partial \delta^e} < \frac{-1}{1-\overline{\eta}^{C,b}} \frac{\partial \overline{\eta}^{C,b}}{\partial \delta^e}$$

where $\overline{\eta}^C = \frac{(\rho+\delta^e)\overline{\eta}}{\delta^e\overline{\eta}+\rho}$ *is the consumption inequality.*

Proof. See appendix. ■

A decrease in the relative "impatience" of entrepreneurs (δ^e) in steady state increases wealth inequality more in fundamental equilibrium than in bubble equilibrium, because the value of bubble "buffers" for savers. A lower δ^e directly decreases consumption inequality, while the increasing wealth inequality indirectly increases consumption inequality. Both in fundamental economy and bubble economy, a decrease of δ^e decreases consumption inequality as well as the welfare of savers. Whereas the relative magnitude of decreasing δ^e on savers' welfare in the two types of economies depends on parameters of the model.

7 Conclusion

This paper studies the effect or fundamental drivers of rising asset prices on top inequality through the leverage channel. Through the lens of my theory, the observed rising asset prices and rising wealth inequality at the top end in the past a few decades suggest that the declining relative impatience of the super rich entrepreneurs is the main driver of the trend. Savers are worse off.

Taking advantage a stylized model for clear mechanism and sharp intuition, this paper also leaves opportunities for future research on extensions of the model, optimal stabilization policies, as well as empirical and quantitative exercises.

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A Technical details

Appendix to be completed.

A.1 Two-period model

First-order conditions

$$1 + r^f = \frac{(1+\rho^e)U'(C_0^e)}{U'(C_1^e)} = \frac{(1+\rho)U'(C_0^s)}{U'(C_1^s)} \tag{63}$$

$$q = \frac{aU'(C_1^e)}{(1+\rho)U'(C_0^e)} = \frac{a}{1+r^f} \tag{64}$$

$$B^s = \frac{1}{2+\rho} W_0^s \tag{65}$$

Wealth inequality tomorrow:

$$\eta_1 = \frac{aK + B^e(1+r^f)}{aK + B^e(1+r^f) + B^s(1+r^f)} = \frac{1}{1 + \frac{B^s}{qK+B^e}} \tag{66}$$

Using market clearing condition for bonds and capital,

$$1 + r^f = \frac{(1+\rho^e)(2+\rho)aK}{W_0^s(2+\rho^e) + W_0^e(2+\rho)}$$

$$q = \frac{a}{1+r^f}$$

$$\eta_1 = 1 - \frac{W_0^s(1+\rho^e)}{W_0^s(2+\rho^e) + W_0^e(2+\rho)}$$

Consumption of entrepreneurs and savers are

$$C_0^e = W_0^e + \frac{1}{2+\rho} W_0^s \tag{67}$$

$$C_1^e = \eta_1 aK \tag{68}$$

$$C_0^s = \frac{1+\rho}{2+\rho} W_0^s \tag{69}$$

$$C_1^s = (1-\eta_1)aK \tag{70}$$

where $\rho^e = \rho + \delta^e$. Comparative statics with respect to δ^e:

$$\frac{\partial \eta_1}{\partial \delta^e} < 0 \tag{71}$$

$$\frac{\partial V^s}{\partial \delta^e} = \frac{1}{1+\rho} U'(C_1^s) \underbrace{B^s}_{>0} \underbrace{\frac{\partial r^f}{\partial \delta^e}}_{>0} > 0 \tag{72}$$

A.2 Full model

The HJB equation for entrepreneurs' problem (16) is

$$\rho^e V^{e,i}(W^{e,i}) = \max_{\{c_t^{e,i},\theta_t^{k,i},\theta_t^{oe,i},\theta_t^{mf}\}_{t=0}^{\infty}} \left\{ \log c^{e,i} + V'(W^{e,i})W^{e,i}\mu_t^{w,e,i} + \frac{1}{2}V''(W^{e,i})(W^{e,i}\tilde{\pi}^{w,e,i})^2 + \lambda_t^i \left[(1-\underline{\chi})\theta_t^{k,i} + \theta_t^{oe,i}\right] \right\}$$

where

$$\frac{dW^{e,i}}{W^{e,i}} = \mu_t^{w,e,i}dt + \tilde{\pi}^{w,e,i}d\tilde{Z}_t^i \tag{73}$$

$$\mu_t^{w,e,i} = \frac{-c^{e,i}}{W^{e,i}} + r_t^f + \theta_t^{k,i}\frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f dt\right]}{dt} + \theta_t^{oe,i}\frac{\mathbb{E}\left[dr_t^{oe,i} - r_t^f dt\right]}{dt} + \theta_t^{mf}\frac{\mathbb{E}\left[dr_t^{mf} - r_t^f dt\right]}{dt} \tag{74}$$

$$\tilde{\pi}^{w,e,i} = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma} \tag{75}$$

Guess a value function $V^{E,i}(W^{e,i}) = \gamma_t + \rho^e \log W_t^{e,i}$ and take first order conditions, we have

$$c_t^{e,i} = \rho^e W_t^{e,i} \tag{76}$$

$$\frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f dt\right]}{dt} = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 - \lambda_t^i(1-\underline{\chi}) \tag{77}$$

$$\frac{\mathbb{E}\left[dr_t^{oe,i} - r_t^f dt\right]}{dt} = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 - \lambda_t^i \tag{78}$$

$$\frac{\mathbb{E}\left[dr_t^{mf} - r_t^f dt\right]}{dt} = 0 \tag{79}$$

The HJB equation for savers' problem (17) is

$$\rho V^{s,j}(W^{s,j}) = \max_{\{c^{s,j}\}_{t=0}^{\infty}} \left\{ \log c^{s,j} + V'(W^{s,j})W^{s,j}\mu_t^{w,s,j} \right\}$$

where

$$\frac{dW^{s,j}}{W^{s,j}} = \mu_t^{w,s,j}dt \tag{80}$$

$$\mu_t^{w,s,j} = \frac{-c^{s,j}}{W^{s,j}} + r_t^f + \alpha_t^{mf}\frac{dr_t^{mf} - r_t^f dt}{dt} \tag{81}$$

(82)

Guess a value function $V^{s,j}(W^{s,j}) = \beta_t + \rho \log W_t^{s,j}$ and take first order conditions, we have

$$c_t^{s,j} = \rho W_t^{s,j} \tag{83}$$

Note that savers do not carry any idiosyncratic risks because the stock market index fund diversifies idiosyncratic risks.

A.2.1 Fundamental equilibrium

Market clearing conditions for fundamental economy:

$$aK_t = C_t^e + C_t^s = \rho^e W_t^e + \rho W_t^s = (\rho^e \eta_t^f + \rho(1-\eta_t^f))q_t K_t \tag{84}$$

$$V_t^{mf} = V_t^{oe} \tag{85}$$

Public equity is exactly the pooled outside equities issued by entrepreneurs, so we have the return of public equity equals to the expected return of outside equity.

$$dr_t^{mf} = \mathbb{E}_t\left[dr_t^{oe,i}\right] \tag{86}$$

And since the public equity does not carry idiosyncratic risks, the return of public equity also equals to the risk-free rate in equilibrium,

$$\mathbb{E}_t\left[dr_t^{oe,i}\right] = dr_t^{mf} = r_t^f dt \tag{87}$$

From entrepreneurs' first order conditions, we have

$$\lambda_t^i = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 > 0$$

so the equity constraint binds, we have

$$\theta_t^{k,i} + \theta_t^{oe,i} = \underline{\chi}\theta_t^{k,i} \tag{88}$$

A.2.2 Steady state of fundamental economy

In fundamental economy, the total wealth in the economy is $q_t K_t$. The value of bubbble $P_t = 0$. Define entrepreneurs' wealth share as $\eta_t = \frac{W_t^e}{W_t^e + W_t^s} = \frac{W_t^e}{q_t K_t}$.

From first order conditions, we have asset pricing equation for capital

$$\frac{\mathbb{E}[dr_t^{k,i} - r_t^f dt]}{dt} = \frac{(\underline{\chi}\tilde{\sigma})^2}{\eta_t^f} \tag{89}$$

In equilibrium we also have

$$dr_t^{mf} = \mathbb{E}_t\left[dr_t^{oe,i}\right] = r_t^f dt \tag{90}$$

Optimal consumption ratio with logarithmic utilities: $\frac{c_t^e}{W_t^e} = \rho^e$ and $\frac{C_t^s}{W_t^s} = \rho$.

We can now derive the evolution of entrepreneurs wealth share using Ito's lemma,

$$\frac{d\eta_t^f}{\eta_t^f} = \underbrace{(1-\eta_t^f)\left(-\delta^e + \left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2\right)}_{\equiv \mu_t^{\eta,f}} dt \tag{91}$$

Consumption good's market clearing condition

$$aK_t = C_t^e + C_t^s = \rho^e W_t^e + \rho W_t^s = (\rho^e \eta_t^f + \rho(1-\eta_t^f))q_t K_t \tag{92}$$

from which we can solve for capital price q_t as a function of entrepreneurs' wealth share η_t^f,

$$q_t^f = \frac{a}{\delta^e \eta_t^f + \rho} \tag{93}$$

The risk-free rate is given by

$$r_t^f = \rho + \mu_t^{c,s,f} = \rho^e + \mu_t^{c,e,f} - \underbrace{\left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2}_{\text{precautionary saving motive}} \tag{94}$$

where $\mu_t^{c,s,f}$ and $\mu_t^{c,e,f}$ are the growth rates of savers' consumption and entrepreneurs' consumption respectively. Since in equilibrium we have $C_t^e = \rho^e \eta_t^f q_t K_t$ and $C_t^s = \rho(1-\eta_t^f)q_t K_t$, we have

$$\frac{dc_t^{e,i}}{c_t^{e,i}} = \mu_t^{c,e,f} dt + \tilde{\pi}^{c,e,i} d\tilde{Z}_t^i = \big(\mu_t^{\eta,f} + \mu_t^q + g\big)dt + +\tilde{\pi}^{c,e,i} d\tilde{Z}_t^i \tag{95}$$

$$\frac{dc_t^{s,j}}{c_t^{s,j}} = \mu_t^{c,s,f} dt = \big(-\frac{\eta_t^f}{(1-\eta_t^f)}\mu_t^{\eta,f} + \mu_t^q + g\big)dt \tag{96}$$

Solving for steady state, that is when $\mu_t^{\eta,f} = 0$ and $\mu_t^q = 0$. Combining (22) and (92), we have

$$\overline{q}^f = \frac{a}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{97}$$

$$\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} \tag{98}$$

$$\overline{p}^f = 0 \tag{99}$$

Here we focus on a non-degenerate steady state wealth distribution and requires that

$$\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} < 1 \tag{100}$$

And at the steady state we have

$$\overline{r}^f = \rho + g \tag{101}$$

and we also have the risk premium of capital at the steady state

$$\frac{\mathbb{E}[d\overline{r}^{k,i} - \overline{r}^f dt]}{dt} = \frac{(\underline{\chi}\tilde{\sigma})^2}{\overline{\eta}^f} = \underline{\chi}\tilde{\sigma}\sqrt{\delta^e} \tag{102}$$

A.2.3 Connection with two-period model: a special deterministic case

Initial conditions: Entrepreneurs have initial wealth $W_0^e = \eta_0 aK$ and savers have initial wealth $W_0^s = (1-\eta_0)aK$.

In this case, we have in general equilibrium:

$$\eta_t^f = \frac{1}{e^{\delta^e t + c} + 1} \tag{103}$$

$$r_t^f = \frac{(\rho + \delta^e)\delta^e \eta_t^f}{\delta^e \eta_t^f + \rho} + \rho \tag{104}$$

where $c = \log \frac{1-\eta_0}{\eta_0}$ and I set $c = 1$ hereafter without loss of generality. we have

$$\frac{\partial \eta_t^f}{\partial \delta^e} = -t(1-\eta_t^f)\eta_t^f < 0 \tag{105}$$

and

$$\int_0^t r_s^f ds = -\ln\left(\rho e^{\delta^e t} + \rho + \delta^e\right) + (\rho + \delta^e)t + \ln(2\rho + \delta^e) \tag{106}$$

$$\frac{\partial r_t^f}{\partial \delta^e} = \frac{\eta_t^f[(\delta^e)^2\eta_t^f - \rho(\rho+\delta^e)\delta^e(1-\eta_t^f)t + 2\rho\delta^e]}{[\delta^e\eta_t^f + \rho]^2} \tag{107}$$

$$\frac{\partial(\int_0^t r_s^f ds)}{\partial \delta^e} = \frac{(\rho+\delta^e)t - 1}{\rho e^{\delta^e t} + \rho + \delta^e} + \frac{1}{2\rho + \delta^e} > 0 \tag{108}$$

$$V^s = \frac{\log C_0^s}{\rho} + \int_0^\infty e^{-\rho t} \underbrace{\int_0^t \frac{\rho(\delta^e)\eta_s^{W,f}}{\delta^e \eta_s^{W,f} + \rho} ds}_{R_t^h} dt \tag{109}$$

$$R_t^h = \delta^e t - \ln\left(\frac{\rho e^{\delta^e t} + \rho + \delta^e}{\rho + \delta^e}\right) + c_1 \tag{110}$$

$$\frac{\partial R_t^h}{\partial \delta^e} = \frac{\rho t}{\rho e^{\delta^e t} + \rho + \delta^e} + \frac{\rho e^{\delta^e t}}{\rho + \delta^e(\rho e^{\delta^e t} + \rho + \delta^e)} - \frac{\rho}{(\rho+\delta^e)(2\rho+\delta^e)} \geq 0 \tag{111}$$

In general equilibrium with endogenous risk-free rate r_t^f, wealth inequality η_t^f changes respond to changes of asset pay-off $\{a\}$ and relative "impatience" of entrepreneurs $\{\delta^e\}$ as follows,

$$\frac{\partial \eta_t^f}{\partial a} = 0; \quad \frac{\partial \eta_t^f}{\partial \delta^e} < 0$$

Savers' welfare $V^{s,f}$ responds to changes of asset pay-off $\{a\}$ and relative "impatience" of entrepreneurs $\{\delta^e\}$ as follows,

$$\frac{\partial (V^{s,f}\big|_0^\infty)}{\partial a} > 0$$

and

$$\frac{\partial (V^{s,f}\big|_0^\infty)}{\partial \delta^e} = \int_0^\infty e^{-\rho t} U'(C_t^s) \rho \underbrace{B_t \frac{\partial (\int_0^t r_s^f ds)}{\partial \delta^e}}_{>0} dt > 0 \tag{112}$$

A.3 Full model: bubble equilibrium

A.3.1 Intuition for bubble

As shown in figure 2, the value of the stock market index fund is

$$V_t^{mf} = \underbrace{(1-\chi_t) q_t K_t}_{\text{Value of outside equities}, V_t^{oe}} + \underbrace{P_t}_{\text{Bubble}} \tag{113}$$

where $1-\chi_t$ is the fraction of capital issued as outside equity in the aggregate. Since entrepreneurs are identical before idiosyncratic risk is realized, we have $\chi_{it} = \chi_t$. The value of outside equities is denoted as $V_t^{oe} = (1-\chi_t) q_t K_t$ and the value of bubble is denoted as P_t, which is also an endogenous process.

Since $\chi_t = \chi_{it} \geq \underline{\chi}$, we have

$$V_t^{oe} \leq (1-\underline{\chi}) q_t K_t$$

If there is no bubble, we simply have outside equity market clears as

$$V_t^{oe} = V_t^{mf}$$

The value of stock market index fund held by entrepreneurs is also constrained as

$$V_t^{oe} = V_t^{mf} \leq (1-\underline{\chi}) q_t K_t \tag{114}$$

A bubble with positive value relaxes this constraint for the stock market index fund because now we have

$$V_t^{mf} = V_t^{oe} + P_t$$

and the constraint (114) becomes

$$V_t^{mf} \leq (1-\underline{\chi}) q_t K_t + P_t \tag{115}$$

Bubbles relax the indirect limit on public equity due to the skin-in-the-game constraint.

A.3.2 Bubble equilibrium

Market clearing conditions for bubble economy:

$$aK_t = (\rho^e \eta_t^b + \rho(1-\eta_t^b))(q_t K_t + P_t) \tag{116}$$

$$V_t^{mf} = V_t^{oe} + P_t \tag{117}$$

Since $V_t^{mf} = V_t^{oe} + P_t$, the public equity is the pooled outside equity plus the bubble, the return of public equity is now

$$dr_t^{mf} = \frac{V_t^{oe}}{V_t^{mf}} \mathbb{E}_t\left[dr_t^{oe,i}\right] + \frac{P_t}{V_t^{mf}} \frac{dP_t}{P_t} \tag{118}$$

The public equity does not carry any idiosyncratic risk, so we have in equilibrium

$$dr_t^{mf} = r_t^f dt \tag{119}$$

We need to use other equilibrium conditions to determine the return of outside equity and the amount of outside equity issuance. And in equilibrium, we have the return of outside equity equals to the return of inside equity.

$$\frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f dt\right]}{dt} = \frac{\mathbb{E}\left[dr_t^{oe} - r_t^f dt\right]}{dt} = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 \tag{120}$$

and it follows that

$$\lambda_t^i = 0 \tag{121}$$

which implies χ_t can be any value in $[\underline{\chi}, 1]$.

Equilibrium refinement To determine the amount of outside equity issuance, we perturb the bubble equilibrium by allowing "trembling hands" of agents. Assume that there is $\epsilon > 0$ chance that agents play for the fundamental equilibrium and $1-\epsilon$ chance for the bubble equilibrium. We have

$$\frac{\mathbb{E}\left[dr_t^{oe} - r_t^f dt\right]}{dt} = (1-\epsilon)(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 \tag{122}$$

which implies

$$\lambda_t^i = \epsilon(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 > 0 \tag{123}$$

and the equity constraint binds, $\chi_t = \underline{\chi}$. The return of capital is as follows

$$\begin{aligned}\frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f dt\right]}{dt} &= (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 - (1-\underline{\chi})\epsilon(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 \\ &= (1-\epsilon+\underline{\chi}\epsilon)(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2\end{aligned} \tag{124}$$

Taking the limit of χ_t as $\epsilon \to 0$, we have

$$\lim_{\epsilon \to 0} \chi_t = \underline{\chi} \tag{125}$$

As long as there is a positive possibility of the fundamental equilibrium, we can not find a sequence of mixed strategies converging to $\chi_t \neq \underline{\chi}$. And we focus on the trembling-hand perfect equilibrium where $\chi_t = \underline{\chi}$ for the following analysis. This trembling-hand perfect equilibrium creates the highest value of bubble.

A.3.3 Steady state in bubble equilibrium

In bubble economy, the total wealth of the economy is $q_t K_t + P_t = q_t K_t + (1 - \underline{\chi}) p_t K_t$.

From entrepreneur's optimization problem, we have asset pricing equation for capital

$$\frac{\mathbb{E}[dr_t^{k,i} - r_t^f dt]}{dt} = \underbrace{\frac{q_t \underline{\chi} \tilde{\sigma}}{\eta_t^b [\underline{\chi} q_t + (1 - \underline{\chi})(p_t + q_t)]}}_{\text{price of risk}} \underbrace{\tilde{\sigma}}_{\text{risk}} \tag{126}$$

In equilibrium, we still have

$$dr_t^{mf} = r_t^f dt \tag{127}$$

as these assets are all risk-free.

And to determine the value of bubble, we write the return of public equity as

$$\begin{aligned} dr_t^{mf} &= \underbrace{\frac{(1 - \underline{\chi}) a K_t}{V_t^{mf}}}_{\text{dividend yield}} dt + \underbrace{\frac{dV_t^{mf}}{V_t^{mf}}}_{\text{capital gain}} = \frac{a}{p_t + q_t} dt + \frac{d((p_t + q_t) K_t)}{(p_t + q_t) K_t} \\ &= \left(\frac{a + p_t \mu_t^p + q_t \mu_t^q}{p_t + q_t} + g \right) dt \end{aligned} \tag{128}$$

where $1 - \underline{\chi}$ is the fraction of capital issued as outside equity, and V_t^{mf} is the value of mutual fund.

And we have consumption good market clearing condition

$$a K_t = (\rho^e \eta_t^b + \rho(1 - \eta_t^b))(q_t K_t + P_t) \tag{129}$$

Derive the evolution of entrepreneurs' wealth share in the bubbly economy, we have

$$\frac{d\eta_t^b}{\eta_t^b} = \underbrace{(1 - \eta_t^b) \left(-\delta^e + \left(\frac{\underline{\chi} q_t \tilde{\sigma}}{\eta_t^b [\underline{\chi} q_t + (1 - \underline{\chi})(q_t + p_t)]} \right)^2 \right)}_{\equiv \mu_t^{\eta,b}} dt \tag{130}$$

The risk-free rate is given by

$$r_t^f = \rho + \mu_t^{c,s,b} = \rho^e + \mu_t^{c,e,b} - \underbrace{\left(\frac{\underline{\chi} q_t \tilde{\sigma}}{\eta_t^b [\underline{\chi} q_t + (1-\underline{\chi})(q_t + p_t)]} \right)^2}_{\text{precautionary saving motive}} \tag{131}$$

where $\mu_t^{c,s,b}$ and $\mu_t^{c,e,b}$ are the growth rates of savers' consumption and entrepreneurs' consumption respectively. Since in equilibrium we have $C_t^e = \rho^e \eta_t^b (q_t K_t + P_t)$ and $C_t^s = \rho(1-\eta_t^b)(q_t K_t + P_t)$, we have

$$\frac{dc_t^{e,i}}{c_t^{e,i}} = \mu_t^{c,e,b} dt + \tilde{\pi}^{c,e,i} d\tilde{Z}_t^i = \big(\mu_t^{\eta,b} + \frac{(1-\underline{\chi}) p_t \mu_t^p + q_t \mu_t^q}{q_t + (1-\underline{\chi}) p_t} + g \big) dt + + \tilde{\pi}^{c,e,i} d\tilde{Z}_t^i \tag{132}$$

$$\frac{dc_t^{s,j}}{c_t^{s,j}} = \mu_t^{c,s,b} dt = \big(- \frac{\eta_t^b}{(1-\eta_t^b)} \mu_t^{\eta,b} + \frac{(1-\underline{\chi}) p_t \mu_t^p + q_t \mu_t^q}{q_t + (1-\underline{\chi}) p_t} + g \big) dt \tag{133}$$

We solve for prices q_t and p_t as functions of the state variable η_t^b. Combining (126), (127), (128), and (129), and use Ito's lemma, we have

$$\frac{a}{q_t} + \mu_t^q - \frac{a + p_t \mu_t^p + q_t \mu_t^q}{p_t + q_t} = \frac{\underline{\chi} q_t (\tilde{\sigma})^2}{\eta_t^b [\underline{\chi} q_t + (1-\underline{\chi})(p_t + q_t)]} \tag{134}$$

$$a = (\rho^e \eta_t^b + \rho(1-\eta_t^b))(q_t + (1-\underline{\chi}) p_t) \tag{135}$$

$$\mu_t^q = q'(\eta_t^b) \eta_t^b \mu_t^{\eta,b} \tag{136}$$

$$\mu_t^p = p'(\eta_t^b) \eta_t^b \mu_t^{\eta,b} \tag{137}$$

Solving for steady state, that is, $\mu_t^{\eta,b} = 0$, $\mu_t^q = 0$, and $\mu_t^p = 0$, we have

$$\overline{q}^b = \frac{a}{\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{138}$$

$$\overline{\eta}^b = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e} + (1-\underline{\chi}) \frac{\tilde{\sigma}\delta^e}{\rho}} \tag{139}$$

$$\overline{p} = \frac{a}{\rho} - \overline{q}^b \tag{140}$$

And at the steady state we have risk-free rate

$$\overline{r}^f = \rho + g \tag{141}$$

we also have the risk premium of capital at the steady state

$$\frac{\mathbb{E}[d\overline{r}^{k,i} - \overline{r}^f dt]}{dt} = \frac{\underline{\chi}\overline{q}(\tilde{\sigma})^2}{\overline{\eta}_t^b [\underline{\chi}\overline{q} + (1-\underline{\chi})(\overline{p} + \overline{q})]} = \tilde{\sigma}\sqrt{\delta^e} \tag{142}$$

Note that at the steady state, the risk-free rate in the bubble economy is the same as in the fundamental economy. But capital price, wealth inequality and the amount of borrowing and lending are different.

For a non-degenerate wealth distribution, we need

$$[\underline{\chi} - (1-\underline{\chi})\frac{\delta^e}{\rho}]\tilde{\sigma} < \sqrt{\delta^e} \tag{143}$$

A.3.4 Parameter restriction on κ

The first restriction

$$\kappa > \frac{\underline{\chi}}{1-\underline{\chi}}(\frac{\tilde{\sigma}}{\sqrt{\delta^e}} - 1) \tag{144}$$

is to make sure that entrepreneurs are borrowers. Given that $\tilde{\sigma} < \frac{\sqrt{\delta^e}}{\underline{\chi}}$ (for non-degenerate wealth distribution), the right hand side of equation (144) is strictly smaller than 1.

And the second restriction

$$\kappa > \underline{\chi}(1 - \frac{\tilde{\sigma}}{\sqrt{\delta^e}})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho + \sqrt{\delta^e}}\frac{\rho}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{145}$$

is to ensure that entrepreneurs borrow more in steady state of bubble equilibrium than in fundamental equilibrium. One can see that the right hand side of equation (145) is strictly smaller than 1.

A.4 Inequality and welfare

Recall that

$$\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}}$$

we can derive wealth share changes with respect to parameters:

$$\frac{\partial\overline{\eta}^f}{\partial\tilde{\sigma}} = \frac{\underline{\chi}}{\sqrt{\delta^e}} \tag{146}$$

$$\frac{\partial\overline{\eta}^f}{\partial\delta^e} = -\frac{\underline{\chi}\tilde{\sigma}}{2\delta^e\sqrt{\delta^e}} \tag{147}$$

$$\frac{\partial\overline{\eta}^f}{\partial\underline{\chi}} = \frac{\tilde{\sigma}}{\sqrt{\delta^e}} \tag{148}$$

$$\frac{\partial\overline{\eta}^f}{\partial\tilde{\sigma}} > 0; \quad \frac{\partial\overline{\eta}^f}{\partial\delta} < 0; \quad \frac{\partial\overline{\eta}^f}{\partial\underline{\chi}} > 0$$

Recall that in the bubble economy, the entrepreneurs' wealth share at steady state is

$$\overline{\eta}^b = \frac{1}{\frac{\sqrt{\delta^e}}{\underline{\chi}\tilde{\sigma}} + \frac{(1-\underline{\chi})}{\underline{\chi}}\frac{\delta^e}{\rho}}$$

and we can derive the comparative statics of wealth inequality with respect to parameters

$$\frac{\partial \overline{\eta}^b}{\partial \tilde{\sigma}} = \frac{\underline{\chi}\sqrt{\delta^e}}{(\sqrt{\delta^e} + (1-\underline{\chi})\frac{\delta}{\rho}\tilde{\sigma})^2} \tag{149}$$

$$\frac{\partial \overline{\eta}^b}{\partial \delta} = -\frac{\underline{\chi}\tilde{\sigma}(\frac{1}{2\sqrt{\delta^e}} + (1-\underline{\chi})\frac{\tilde{\sigma}}{\rho})}{(\sqrt{\delta^e} + (1-\underline{\chi})\frac{\delta^e}{\rho}\tilde{\sigma})^2} \tag{150}$$

$$\frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} = \frac{\rho\tilde{\sigma}(\rho\sqrt{\delta^e} + \delta^e\tilde{\sigma})}{(\rho\sqrt{\delta^e} + (1-\underline{\chi})\delta^e\tilde{\sigma})^2} \tag{151}$$

$$\frac{\partial \overline{\eta}^b}{\partial \tilde{\sigma}} > 0; \quad \frac{\partial \overline{\eta}^b}{\partial \delta} < 0; \quad \frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} > 0$$

Comparing with fundamental economy, we have

$$\frac{\partial \overline{\eta}^f}{\partial \tilde{\sigma}} > \frac{\partial \overline{\eta}^b}{\partial \tilde{\sigma}} > 0$$

$$\frac{\partial \overline{\eta}^f}{\partial \delta^e} < \frac{\partial \overline{\eta}^b}{\partial \delta^e} < 0$$

$$0 < \frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} < \frac{\partial \overline{\eta}^f}{\partial \underline{\chi}} \quad \text{if} \quad \frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}(1-\underline{\chi})^2 + 2(1-\underline{\chi}) - 1 > 0$$

Consumption inequality is different from wealth inequality as entrepreneurs and savers have different consumption rates.

In fundamental economy, the aggregate consumption of entrepreneurs and savers are as follows,

$$\overline{C}^{e,f} = (\rho + \delta^e)\overline{\eta}^f \overline{q}^f K_t = \frac{(\rho + \delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho)} aK_t \tag{152}$$

$$\overline{C}^{s,f} = \rho(1 - \overline{\eta}^f)\overline{q}^f K_t = \left(1 - \frac{(\rho + \delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho)}\right) aK_t \tag{153}$$

Define consumption share as

$$\overline{\eta}^{C,f} = \frac{(\rho + \delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho)} \tag{154}$$

and we have

$$\frac{\partial \overline{\eta}^{C,f}}{\partial \tilde{\sigma}} = \frac{\underline{\chi}\rho(\rho+\delta^e)}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{155}$$

$$\frac{\partial \overline{\eta}^{C,f}}{\partial \delta} = \frac{\underline{\chi}\tilde{\sigma}r(\delta-2\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}-\rho)}{2\delta\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{156}$$

$$\frac{\partial \overline{\eta}^{C,f}}{\partial \underline{\chi}} = \frac{\rho(\rho+\delta^e)\tilde{\sigma}}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{157}$$

we require that steady state wealth inequality $\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} > \frac{1}{2}$. We have

$$\frac{\partial \overline{\eta}^{C,f}}{\partial \tilde{\sigma}} > 0; \quad \frac{\partial \overline{\eta}^{C,f}}{\partial \delta} < 0; \quad \frac{\partial \overline{\eta}^{C,f}}{\partial \underline{\chi}} > 0$$

In bubble economy, the consumption of entrepreneurs and savers are as follows,

$$\overline{C}_t^{e,b} = (\rho+\delta^e)\overline{\eta}^b(\overline{q}^b K_t + \overline{P}_t) = \frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+r)} aK_t \tag{158}$$

$$\overline{C}_t^{s,b} = \rho(1-\overline{\eta}^b)(\overline{q}^b K_t + \overline{P}_t) = \left(1 - \frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+r)}\right) aK_t \tag{159}$$

we have

$$\overline{C}^{e,b} < \overline{C}^{e,f}$$
$$\overline{C}^{s,b} > \overline{C}^{s,f}$$

Define consumption shares as

$$\overline{\eta}^{C,b} = \frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)} \tag{160}$$

and comparative statics with respect to parameters

$$\frac{\partial \overline{\eta}^{C,b}}{\partial \tilde{\sigma}} = \frac{\underline{\chi}\rho(\rho+\delta^e)}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{161}$$

$$\frac{\partial \eta^{C,b}}{\partial \delta} = \frac{\underline{\chi}\tilde{\sigma}\rho(\delta^e-2\tilde{\sigma}\sqrt{\delta^e}-\rho)}{2\delta\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{162}$$

$$\frac{\partial \overline{\eta}^{C,b}}{\partial \underline{\chi}} = \frac{(\rho+\delta^e)\tilde{\sigma}}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)} \tag{163}$$

$$\frac{\partial \eta^{C,b}}{\partial \tilde{\sigma}} > 0; \quad \frac{\partial \eta^{C,b}}{\partial \delta^e} < 0; \quad \frac{\partial \eta^{C,b}}{\partial \underline{\chi}} > 0$$

Note that $\frac{\partial \eta^{C,b}}{\partial \delta^e} < 0$ because entrepreneurs are richer than savers in fundamental equilibrium which requires $\underline{\chi}\tilde{\sigma} > \frac{1}{2}$.

The value function of savers in fundamental economy is as follows

$$V^{s,f} = \int_0^\infty e^{-\rho t}\left(\underbrace{\log(1-\eta_t^f)}_{\text{wealth distribution}} + \underbrace{\log\rho}_{\text{consumption rate}} + \underbrace{\log\left(q_t^f K_t\right)}_{\text{total wealth}} \right) dt \tag{164}$$

The value function of savers in bubble economy is as follows

$$V^{s,b} = \int_0^\infty e^{-\rho t}\left(\underbrace{\log(1-\eta_t^b)}_{\text{wealth distribution}} + \underbrace{\log\rho}_{\text{consumption rate}} + \underbrace{\log(q_t^b K_t + P_t)}_{\text{total wealth}} \right) dt \tag{165}$$

And the value function of savers starting from fundamental equilibrium steady state:

$$\begin{aligned} \overline{V}^{s,f} &= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log(1-\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}}) + \log\frac{a\rho}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho} + \log K_0 + \frac{g}{\rho}\right) dt\right] \\ &= \frac{\log(1-\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}}) + \log\frac{a\rho}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho} + \log K_0 + \frac{g}{\rho}}{\rho} \end{aligned} \tag{166}$$

We can derive savers' welfare changes with respect to parameters:

$$\frac{\partial \overline{V}^{s,f}}{\partial\tilde{\sigma}} < 0; \quad \frac{\partial \overline{V}^{s,f}}{\partial\delta^e} > 0; \quad \frac{\partial \overline{V}^{s,f}}{\partial\underline{\chi}} < 0$$

For discount shocks, we have

$$\frac{\partial \overline{V}^{s,f}}{\partial\tilde{\sigma}} = -\frac{1}{\rho}\left(\frac{1}{\frac{\sqrt{\delta^e}}{\underline{\chi}} - \tilde{\sigma}} + \frac{1}{\tilde{\sigma} + \frac{\rho}{\underline{\chi}\sqrt{\delta^e}}}\right) < 0 \tag{167}$$

For financial shocks, we have

$$\frac{\partial \overline{V}^{s,f}}{\partial\underline{\chi}} = -\frac{1}{\rho}\left(\frac{1}{\frac{\sqrt{\delta^e}}{\tilde{\sigma}} - \underline{\chi}} + \frac{1}{\underline{\chi} + \frac{\rho}{\tilde{\sigma}\sqrt{\delta^e}}}\right) < 0 \tag{168}$$

The value function of savers starting from steady state of bubble equilibrium,

$$\begin{aligned} \overline{V}^{s,b} &= \int_0^\infty e^{-\rho t}\left(\log\left(1-\frac{\underline{\chi}\tilde{\sigma}(\rho+\delta^e)}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)}\right) + \log aK_0 + \frac{g}{\rho}\right) dt \\ &= \frac{\log\left(1-\frac{\underline{\chi}\tilde{\sigma}(\rho+\delta^e)}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)}\right) + \log aK_0 + \frac{g}{\rho}}{\rho} \end{aligned} \tag{169}$$

The existence of bubble benefits savers since

$$\overline{V}^{s,b} > \overline{V}^{s,f} \tag{170}$$

Welfare responses to shocks also differ from the fundamental economy. We have savers' welfare changes with respect to parameters

$$\frac{\partial \overline{V}^{s,b}}{\partial \tilde{\sigma}} < 0; \quad \frac{\partial \overline{V}^{s,b}}{\partial \delta^e} > 0; \quad \frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} < 0 \tag{171}$$

For financial shock $\tilde{\sigma}$, we have

$$\frac{\partial \overline{V}^{s,b}}{\partial \tilde{\sigma}} = \frac{1}{\rho}\left(\frac{1}{\frac{\sqrt{\delta^e}\rho}{(1-\underline{\chi})\delta^e - \underline{\chi}\rho} + \tilde{\sigma}} - \frac{1}{\tilde{\sigma} + \frac{\rho}{\sqrt{\delta^e}}}\right) < 0 \tag{172}$$

Comparing with fundamental economy,

$$\frac{\partial \overline{V}^{E,b}}{\partial \tilde{\sigma}} > \frac{\partial \overline{V}^{E,f}}{\partial \tilde{\sigma}} < 0$$
$$\frac{\partial \overline{V}^{s,f}}{\partial \tilde{\sigma}} < \frac{\partial \overline{V}^{s,b}}{\partial \tilde{\sigma}} < 0$$

For financial shock, we have

$$\frac{\partial \overline{V}^{E,b}}{\partial \underline{\chi}} = \frac{1}{(\rho + \delta^e)}\frac{1}{\underline{\chi}} > 0$$
$$\frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} = -\frac{1}{\rho}\frac{\tilde{\sigma}(\rho + \delta^e)}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e} + \rho) - \underline{\chi}\tilde{\sigma}(\rho + \delta^e)} < 0$$

Comparing with fundamental economy,

$$\frac{\partial \overline{V}^{s,f}}{\partial \underline{\chi}} < \frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} < 0 \quad \text{if} \quad \frac{\tilde{\sigma}}{\delta^e}\underline{\chi}^2 - 2\underline{\chi} + 1 < 0 \tag{173}$$

A.5 Transition dynamics

The evolution of wealth inequality is as follows,

$$\frac{d\eta_t^f}{\eta_t^f} = (1 - \eta_t^f)\left(-\delta^e + \left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2\right)dt \tag{174}$$

There is strong asymmetry in the transition dynamics due to *leverage* effect of entrepreneurs, as shown by figure 8. The increase of wealth inequality is much faster than the decrease. This asymmetry due to leverage can help explain the rapid increase of top wealth inequality documented in the literature.

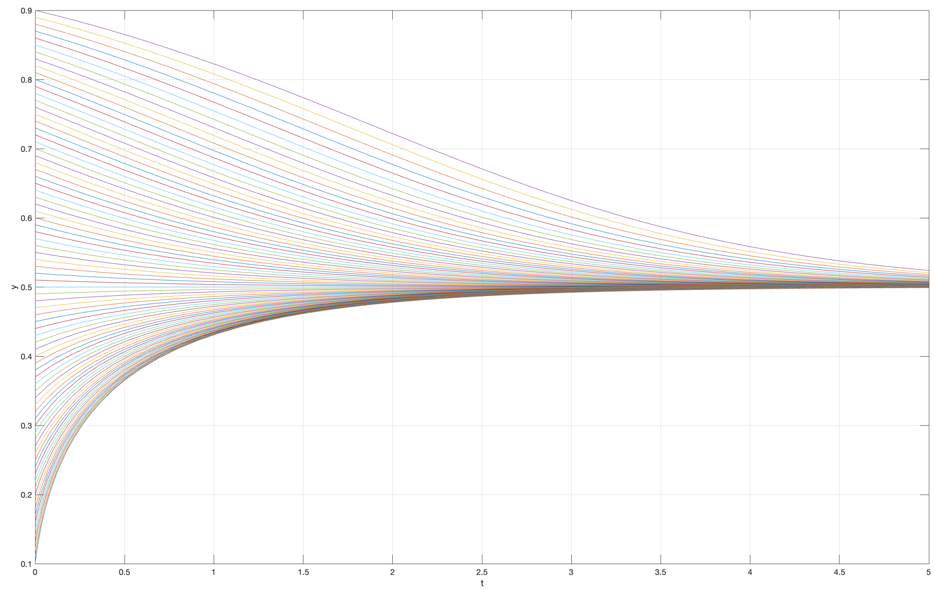

Figure 8: Transition dynamics

A.6 Extension: CRRA utility

One can characterize the closed-form steady states with general CRRA utilities. Denote γ^e and γ as the coefficient of relative risk aversion for entrepreneurs and savers respectively.

At steady state, we have

$$\overline{r}^f = \rho + \gamma g \tag{175}$$

$$\overline{\eta}^f = \sqrt{\frac{\gamma^e(\gamma^e+1)}{2}} \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{(\gamma^e-\gamma)g+\delta^e}} \tag{176}$$

$$\overline{q}^f = \frac{a}{\rho + (\gamma-1)g + (\frac{2}{\gamma^e+1}\delta^e + (\gamma^e-\gamma)g)\overline{\eta}^f} \tag{177}$$

and risk-free rate

$$\begin{aligned}\overline{r}^f &= \rho^e + \gamma^e g - \frac{\gamma^e(\gamma^e+1)}{2}\left(\frac{\underline{\chi}\tilde{\sigma}}{\overline{\eta}^f}\right)^2 \\ &= \rho + \gamma g\end{aligned} \tag{178}$$

Consumption to wealth ratio

$$\frac{\overline{c}^e}{\overline{W}^e} = \rho + (\gamma^e - 1)g + \frac{2}{\gamma^e+1}\delta^e \tag{179}$$

$$\frac{\overline{c}^s}{\overline{W}^s} = \rho + (\gamma - 1)g \tag{180}$$

And similarly for bubble equilibrium. In the case of $\gamma^e = \gamma > 1$, we have the main result still holds.

A.7 Extension: investment and labor

In this extension, I characterize the steady state of the economy with investment and labor.

Assume the evolution of private capital is

$$\frac{dk_t^i}{k_t^i} = (\Psi(\iota_t) - \phi)dt + \tilde{\sigma} d\tilde{Z}_{i,t} \tag{181}$$

where $\Psi(\iota_t)$ is the investment function (increasing and concave in ι_t) and ϕ is the depreciation rate of capital. Specify that

$$\Psi(\iota_t) = \frac{\log(\psi(\iota_t - \phi) + 1)}{\psi} \tag{182}$$

where ψ is the adjustment cost of investment. We are back at the endowment economy as $\phi \to \infty$. At steady state, investment and depreciation of capital cancel each other.

Entrepreneurs have the same preference as before, however, savers also supply labor $l_{j,t}$. Saver j's preference is

$$\int_0^\infty e^{-\rho t} \left(\log c_t^{s,j} - \frac{l_{j,t}^{1+\frac{1}{\nu}}}{1 + \frac{1}{\nu}} \right) dt \tag{183}$$

where ν is the Frisch elasticity.

Production function of entrepreneur i is given by

$$y_{it} = a k_{it}^\alpha l_{it}^{1-\alpha} \tag{184}$$

Since production function is constant return to scale, one can get the aggregate production function of the economy as

$$y_t = a k_t^\alpha l_t^{1-\alpha} \tag{185}$$

where k_t and l_t are aggregate capital and labor supply.

Savers' problem can still be mapped into a standard portfolio choice problem

$$\begin{aligned} \max_{\{c_t^{s,j}, l_{j,t}\}_{t=0}^\infty} \quad & \int_0^\infty e^{-\rho t} \left(\log c_t^{s,j} - \frac{l_{j,t}^{1+\frac{1}{\nu}}}{1 + \frac{1}{\nu}} \right) dt \\ \text{s.t.} \quad & \int_0^\infty \xi_{j,t} c_t^{s,j} \, dt \leq W_0^{s,j} + \int_0^\infty \xi_{j,t} w_t l_{j,t} dt \end{aligned} \tag{186}$$

where $\xi_{j,t}$ is the stochastic discount factor of saver j, and $W_0^{s,j}$ is the endowed wealth of saver j at time $t = 0$.

Since labor does not affect capital market risk-taking and all entrepreneurs use the same wage, we can drop the subscripts and write wage as the marginal product of labor,

$$w_t = a(1-\alpha)k_t^{\alpha} l_t^{-\alpha} \tag{187}$$

From first-order conditions from savers, we have

$$l_t^{\frac{1}{\nu}} = \frac{w_t}{c_t^s} = \frac{1-\alpha}{(1-\eta_t^{c,f})l_t} \tag{188}$$

where $\eta_t^{c,f} = \frac{(\rho+\delta^e)\eta_t^f}{\delta^e \eta_t^f + \rho}$ is the consumption share of entrepreneurs.

Investment is maximized as entrepreneurs maximize the expected return of their private firm and it does not affect risk-taking,

$$\max_{\iota_t}\{\frac{y_t - w_t l_t - \iota_t k_t}{q_t k_t} + \Psi(\iota) - \phi + \text{risk premium}\} \tag{189}$$

we have

$$\Psi'(\iota_t) = \frac{1}{q_t} \tag{190}$$

that is,

$$\iota_t = \frac{q_t - 1}{\psi} + \phi \tag{191}$$

The evolution of wealth inequality η_t^f does not change since neither investment nor labor affect patience or risk-taking.

We also have the market clearing condition for consumption good,

$$(\delta^e \eta_t^f + \rho)q_t = a k_t^{\alpha-1} l_t^{1-\alpha} - (\frac{q_t - 1}{\psi} + \phi) \tag{192}$$

Since $q_t = 1$ at steady state, we can solve for steady-state level of labor and capital

$$\bar{l} = \left(\frac{1-\alpha}{1-\overline{\eta}^c}\right)^{\frac{\nu}{1+\nu}} \tag{193}$$

$$\overline{k} = a^{\frac{1}{1-\alpha}} \left(\frac{1-\alpha}{1-\overline{\eta}^c}\right)^{\frac{\nu}{1+\nu}} (\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho + \phi)^{\frac{-1}{1-\alpha}} = \left(\frac{a}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho + \phi}\right)^{\frac{1}{1-\alpha}} \bar{l} \tag{194}$$

Savers' consumption at steady state is given by

$$\begin{aligned} \overline{c}^s &= (1-\overline{\eta}^c) a \overline{k}^{\alpha} \bar{l}^{1-\alpha} \\ &= a(1-\overline{\eta}^c)^{\frac{1}{1+\nu}} \left(\frac{a}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho + \phi}\right)^{\frac{\alpha}{1-\alpha}} \\ &= \frac{\rho(1 - \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}})}{(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho)^{\frac{1+\alpha\nu}{(1+\nu)(1-\alpha)}}} \left(\frac{1}{1 + \frac{\phi}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho}}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned} \tag{195}$$

A sufficient condition (given $\overline{\eta}^f > \frac{1}{2}$) for $\frac{\partial \overline{c}^s}{\partial \delta^e} < 0$ is

$$2\alpha\nu \leq \nu - \alpha \tag{196}$$

If $\alpha = \frac{1}{3}$, this condition implies that the Frisch elasticity $\nu \geq 1$.

When entrepreneurs get more "patient" ($\delta^e \downarrow$), labor l increases, capital k increases, capital to labor ratio $\frac{k}{l}$ increases, and wage w increases. Savers not only consume less, but also supply more labor.

One can show that with CRRA utilities, condition (196) is still a sufficient condition.