A unified explanation of reference dependence, loss aversion, and the reflection effect

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Abstract

This paper offers a unified explanation of four prominent features of behavioral economics: reference dependence, loss aversion, concave utilities for gains, and convex utilities for losses. In this account, an agent chooses an alternative over the status quo only if each of the agent's candidate preferences approves the change. The set of candidates must be rich enough to include, for any subset of goods, (1) preferences that are more favorable to these goods than the other candidates and (2) preferences that are less favorable. Greater favorability towards a good obtains when the good's marginal rate of substitution relative to other goods is greater and diminishes less rapidly. Existing explanations of the same behavioral phenomena need to invoke several different psychological mechanisms and posit unobservable preferences.

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1 Introduction

Utility functions in prospect theory and related behavioral models display four distinct features. The first is reference dependence: utility either for a prize in a lottery or for a good consumed with certainty is defined relative to the reference or status quo point. The second is loss aversion: the marginal value of gains is smaller than the marginal value of losses. The third and fourth impose an asymmetric reflection effect on the curvature of utility: the value of gains relative to the reference point is concave while the value of losses is convex. See Figure 1 and Kahneman and Tversky (1979, 1984) and Tversky and Kahneman (1991). These features explain several hallmarks of behavioral economics: (a) the framing effects that stem from reference dependence; (b) the status quo bias and endowment effects that accompany loss aversion, for example, the willingness-to-accept–willingness-to-pay disparity; (c) risk-seeking for losses; and (d) risk-aversion for gains.[1]

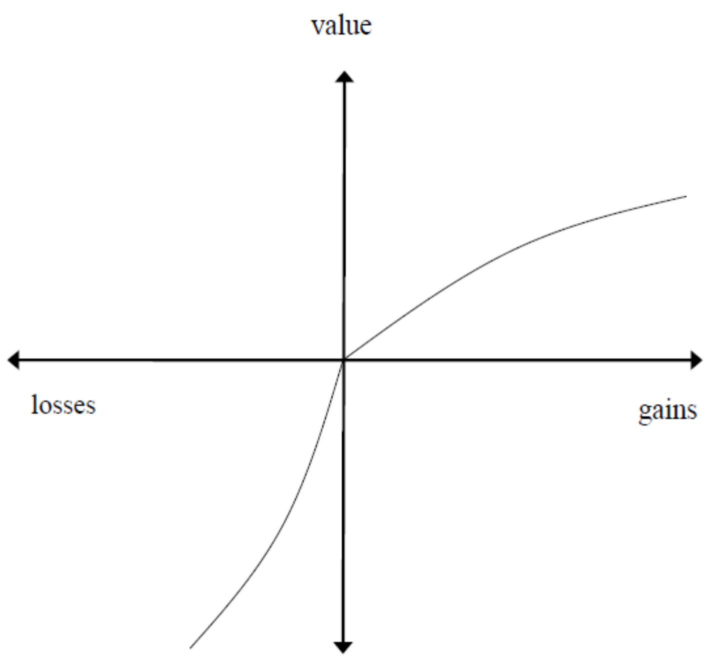

Figure 1: The prospect theory value function

A drawback of the four features of utility is that they appear to require four independent

[1]For early work on the theory and evidence for status quo bias, the endowment effect, and reference dependence, see Thaler (1980), Knetch and Sinden (1984, 1987), Samuelson and Zeckhauser (1988), Knetsch (1989), Kahneman et al. (1990)), Gul (1991), Sugden (2003), Köbberling and Wakker (2005), and Kőszegi and Rabin (2006). The WTA-WTP disparity was first identified in the theory of contingent valuation (see Carson (1997), Hammack and Brown (1974), and Hausman (1993)).

psychological mechanisms. This paper grounds the four features in a single mechanism, the partial judgments of an agent who is not sure how to evaluate consumption options. The agent will be endowed with a diverse set of evaluations or preferences and accepts an alternative to a reference or status quo point only when all of the evaluations recommend the change. The agent's worst-case analysis of the alternative will be decisive: if the worst-case evaluation of the change in each good's consumption, summed across goods, recommends the alternative then so will any other combination of evaluations. This focus on the worst case leads the agent to apply high-marginal utility evaluations to losses of a good and low-marginal utility evaluations to gains. Both loss aversion and reference dependence then appear even though the agent does not view losses to have greater intrinsic significance than gains. A similar asymmetry arises in the utility curvature of gains and losses: the concave utilities are worst-case for gains since marginal utility then falls further and the convex utilities are for similar reasons worst-case for losses. The four features of utility can therefore emerge as an endogenous and rational consequence of an agent's diversity of evaluations.[2]

A single family of evaluations will generate these properties of utility for all values of the decision-maker's reference point. Since the same set of preferences will govern the agent's decisions as the reference point changes, the model can assess an agent's welfare through time and make falsifiable predictions.[3] The transitivity of preferences, for example, can be checked. The alternative of assuming a distinct and complete set of preferences for every reference point, as in prospect theory, posits preference judgments that are difficult and sometimes impossible to observe and cannot make welfare comparisons.

Both the present account and the behavioral concepts of prospect theory try to explain a common set of decision-making regularities by departing from the classical model of rationality. The departure in this paper however is limited to just one point, the replacement of one evaluation with a set of evaluations. It is this feature that allows the theory to share some of the observational and welfare advantages of the classical model.

The decision-maker's diversity of evaluations will formally be a set of 'candidate preferences' over state-contingent or traditional goods. Given the unanimity rule that every

[2]Mandler (2004, 2005) gives a compatible explanation of the endowment effect and the WTA-WTP disparity though not of loss aversion, concavity for gains, or convexity for losses.

[3]The door is therefore open to behavioral welfare economics à la Bernheim and Rangel (2007, 2009).

candidate preference must approve a change, the agent will not be able to rank some pairs of alternatives; the preferences that result will therefore be incomplete as in Bewley (1986) where each candidate preference stems from a probability distribution. The incompleteness is mainly an advantage. When the agent's status quo consumption forms the agent's reference point, the model will not need to postulate an unobservable preference between non-status quo alternatives; instead the agent acts 'as if' the status quo is preferred.

The candidate preferences will satisfy assumptions that I pose initially using utility representations. The agent will specify, for each state s, a set of utilities for a good (or money) to be delivered at s, and each candidate preference is formed by a draw from these sets. The utilities for the good delivered at some s can be partially ordered by how favorably the functions evaluate the good: define utility U to be *more optimistic than* V if and only if U has greater marginal utility at 0 than V and V is more concave than U.[4] The utility U thus begins with greater marginal utility than V and U's marginal utility advantage thereafter can only become greater; for U increments of a good and risks are more appealing. The 'more optimistic than' label applies most straightforwardly to an agent who is unsure ex ante how desirable a good is; but it can also be read as shorthand for more favorable attitudes held with certainty.

The key assumption will be that the sets of utilities for goods at individual states are diverse in the sense that they include utility functions that are greatest and least according to the more-optimistic-than ordering. The worst-case utility function for increases in a good will then be the most concave utility with the smallest baseline marginal utility and the worst-case for decreases will be the most convex utility with the greatest baseline marginal utility. Since the unanimity (Pareto) aggregation of the agent's candidate preferences is driven by the worst-case utilities, the preference that emerges will be represented by utilities for gains and losses of a good that satisfy concavity for gains, convexity for losses, and loss aversion.

After the next section, which gives the utility version of the main result, I turn to a model of choice over uncertain acts that imposes assumptions on preferences. Although the same

[4] A function v is more concave than u if there is a concave transformation of u that generates v: see de Finetti (1949), Pratt (1964), Debreu (1976).

properties of utility will continue to hold, the switch from assumptions on marginal utilities to marginal rates of substitution leads to a curious exception when there are exactly three states of nature.

2 A two-good utility overview

Suppose, for concreteness, a decision-maker consumes just two goods, each delivered at a separate state of nature. The goods can also be interpreted as traditional physical commodities. The agent begins with a reference point or status quo consumption $\bar{x} = (\bar{x}_1, \bar{x}_2) \gg 0$ and for $i = 1, 2$ has a utility u_i for changes relative to $\bar{x}_i$ of consumption at state i. A change in consumption at state i must then lie in the interval $[-\bar{x}_i, \infty)$. A u_i of the type used in prospect theory satisfies the following properties.

Definition 1 *The function u_i is a **behavioral value function** if it is* (1) *strictly increasing and continuous,* (2) *strictly concave on* $[0, \infty)$*,* (3) *strictly convex on* $[-\bar{x}_i, 0]$*, and satisfies* (4) $u_i(0) = 0$ *and* (5) $D_-u_i(0) > D_+u_i(0)$.[5]

The concavity and convexity conditions (2) and (3) capture diminishing returns for gains and increasing returns for losses, both relative to $\bar{x}_i$. The requirement in (5) that the left derivative of u_i at 0 is greater than the right derivative of u_i at 0 entails loss aversion: the marginal value of gains is strictly smaller than the marginal value of losses. Due to (2) and (3), u_i is left and right differentiable at 0 (we allow $+\infty$ as a derivative) and hence (5) is a well-defined requirement.

The total utility change induced by an alternative $x = (x_1, x_2)$ will equal the sum $u_1(x_1 - \bar{x}_1) + u_2(x_2 - \bar{x}_2)$. A pair of behavioral value functions thus generates a *reference-dependent preference* $\succsim_{\bar{x}}$ over nonnegative bundles defined by

$$x \succsim_{\bar{x}} y \Longleftrightarrow u_1(x_1 - \bar{x}_1) + u_2(x_2 - \bar{x}_2) \geq u_1(y_1 - \bar{x}_1) + u_2(y_2 - \bar{x}_2).$$

See Tversky-Kahneman (1991) for example. Preferences over lotteries, perhaps with quantities of money as prizes, can be covered by interpreting $u_i(x_i - \bar{x}_i)$ as the product of the

[5] If f is a function from $A \subset \mathbb{R}$ into $\mathbb{R}$ then $D_-f(x)$ and $D_+f(x)$ will denote, respectively, the left and right derivatives of f evaluated at x.

probability of a prize x_i and its Bernoulli utility, e.g., $u_i(x_i - \bar{x}_i) = \pi_i v_i(x_i)$. This interpretation applies regardless of how probabilities are weighted, probability distributions are transformed, or utilities are integrated. When y equals the reference point $\bar{x}$, the preference $\succsim_{\bar{x}}$ approves a change from $\bar{x}$ to x, or $x \succsim_{\bar{x}} \bar{x}$, when $u_1(x_1 - \bar{x}_1) + u_2(x_2 - \bar{x}_2) \geq 0$.

Preference theories based on behavioral value functions are open to two objections on the grounds of parsimony. First, they invoke several different psychological mechanisms: two different utility curvatures, a discontinuity in the marginal value of gains and losses, and the reference dependence implied by letting choices be determined by gains and losses. Second, a distinct preference relation will arise for each reference point $\bar{x}$ and most of these preferences will be unverifiable: when $\bar{x}$ is an agent's status quo holding, only the agent's choices between $\bar{x}$ and other alternatives can be observed. It is even problematic to infer that an agent who refuses to switch from $\bar{x}$ to an alternative y prefers $\bar{x}$, that is, to infer $\bar{x} \succsim_{\bar{x}} y$: a refusal to switch may indicate inertia rather a preference judgment.

There is also no welfare connection among the $\succsim_{\bar{x}}$. If an agent willingly moves from a status quo $\bar{x}$ to a new status quo y and then to a third option z, we do not know if the agent is better or worse off with z than with $\bar{x}$.

In the alternative theory in this paper a decision-maker has a fixed set of candidate preferences or utilities and the agent prefers y over $\bar{x}$ only when the candidates unanimously endorse a move from $\bar{x}$ to y. It will then be rational for the agent to evaluate a change in the consumption of a good using the worst-case utility for that good. As we will see, the worst case for a decrease in consumption will be the candidate utility function that is most convex and with the greatest baseline marginal utility and hence the most willing to bear risk. For an increase, the worst case will be the utility that is most concave and with the smallest baseline marginal utility. This endogenous determination of the worst case will follow from the unanimity rule, not because the pain of a loss is inherently more acute than the pleasure of a gain. The set of candidate utilities moreover does not vary with the reference point; one set of evaluations rules throughout.

To derive these results from candidate preferences and utilities that are as conventional as possible, I will assume that each of the agent's candidate utility functions for consumption at state i will be a $U_i : \mathbb{R}_+ \to \mathbb{R}$ that is strictly increasing and differentiable at 0. A utility

U_i thus has the standard domain of all nonnegative bundles and I use a capital letter to underscore the difference compared to u_i.

Let $\mathcal{U}_i$ be the agent's set of candidate utilities for consumption at state i and $\mathcal{U} \subset \mathcal{U}_1 \times \mathcal{U}_2$ the agent's set of candidate utility profiles, which will be fixed henceforth. The agent's *unanimity or Pareto preference* $\trianglerighteq$ is then given by $x \trianglerighteq y$ if and only if

$$U_1(x_1) + U_2(x_2) \geq U_1(y_1) + U_2(y_2) \text{ for all } (U_1, U_2) \in \mathcal{U}.$$

Although $\trianglerighteq$ is incomplete, the incompleteness makes no practical difference for comparisons between a status-quo reference point $\bar{x}$ and an alternative y: if when $\bar{x}$ and y are not ranked by $\trianglerighteq$ the agent sticks with the status quo then the agent will act 'as if' $\bar{x}$ is preferred to y. For decisions relative to the status quo, therefore, we could declare $\bar{x} \trianglerighteq y$ to hold whenever $y \trianglerighteq \bar{x}$ fails to obtain. This partial completion of $\trianglerighteq$ would posit additional preference judgments only at *some* of the junctures where prospect and similar theories take the same step.

The model will hinge on the presence of utilities in $\mathcal{U}_i$ that are more optimistic or favorable towards consumption at state i than the other functions in $\mathcal{U}_i$ and on utilities in $\mathcal{U}_i$ that are more pessimistic. One utility function is considered more optimistic than another if its marginal utility is greater and its marginal utility diminishes less rapidly. The concept of 'more concave than' ties these two requirements of optimism together: if U has greater marginal utility at 0 than V and if V is more concave than U then V's marginal utility will never catch up: it will remain smaller than U's marginal utility at all consumption levels greater than 0. (A function $V : \mathbb{R}_+ \to \mathbb{R}$ is *more concave than* $U : \mathbb{R}_+ \to \mathbb{R}$ if there exists a strictly increasing and concave $f : U(\mathbb{R}_+) \to \mathbb{R}$ such that $V = f \circ U$.) This more-concave view of when marginal utility diminishes more rapidly enjoys the advantage that it does not require U and V to be everywhere differentiable.

Greater optimism is represented by a *more-optimistic-than* partial order R on the differentiable-at-0 functions from $\mathbb{R}_+$ to $\mathbb{R}$ that is defined by:

$$U R V \iff \begin{array}{l} \text{(i) } DU(0) \geq DV(0), \text{ and} \\ \text{(ii) } V \text{ is more concave than } U \end{array}.$$

So when URV holds, U assigns higher marginal utility than V to a good at its lowest possible consumption level and marginal utility thereafter can only become even greater for U than

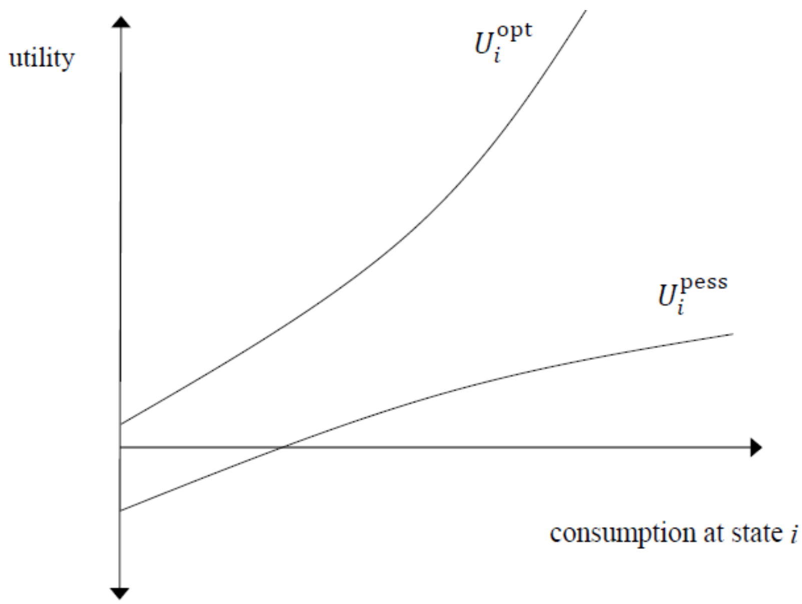

Figure 2: Optimistic and pessimisstic utility functions

for V. See Figure 2.

Assumptions For $i = 1, 2$,

1. $\mathcal{U}_i$ contains a R-greatest or 'most optimistic' function and a R-least or 'most pessimistic' function, that is, a U_i^{opt} and U_i^{pess} such that $U_i^{\text{opt}} R\, U_i$ and $U_i R\, U_i^{\text{pess}}$ for all $U_i \in \mathcal{U}_i$,

2. $\mathcal{U}_i$ contains a strictly concave function and a strictly convex function, and

3. $(U_i^{\text{opt}}, U_{-i}^{\text{pess}})$ and $(U_i^{\text{pess}}, U_{-i}^{\text{opt}})$ are both elements of $\mathcal{U}$.

Under Assumptions 1 and 2, U_i^{opt} must be strictly convex and U_i^{pess} must be strictly concave. For example, since due to 2 there is a strictly concave $U_i \in \mathcal{U}_i$ and, due to 1, $U_i R\, U_i^{\text{pess}}$, there exists a strictly increasing and concave f such that $U_i^{\text{pess}} = f \circ U_i$ which implies that U_i^{pess} is strictly concave.

Assumptions 1 and 2 are cardinal: if they hold and a common increasing transformation is applied to the functions in $\mathcal{U}_i$ then the assumptions will continue to hold only if the transformation is affine. The diversity assumption of section 3.2 will provide an ordinal, preference-based replacement.

Assumption 3 states that the set of candidate utility profiles $\mathcal{U} \subset \mathcal{U}_1 \times \mathcal{U}_2$ is rich enough to include the boundary profiles of utilities that are maximally optimistic towards one good

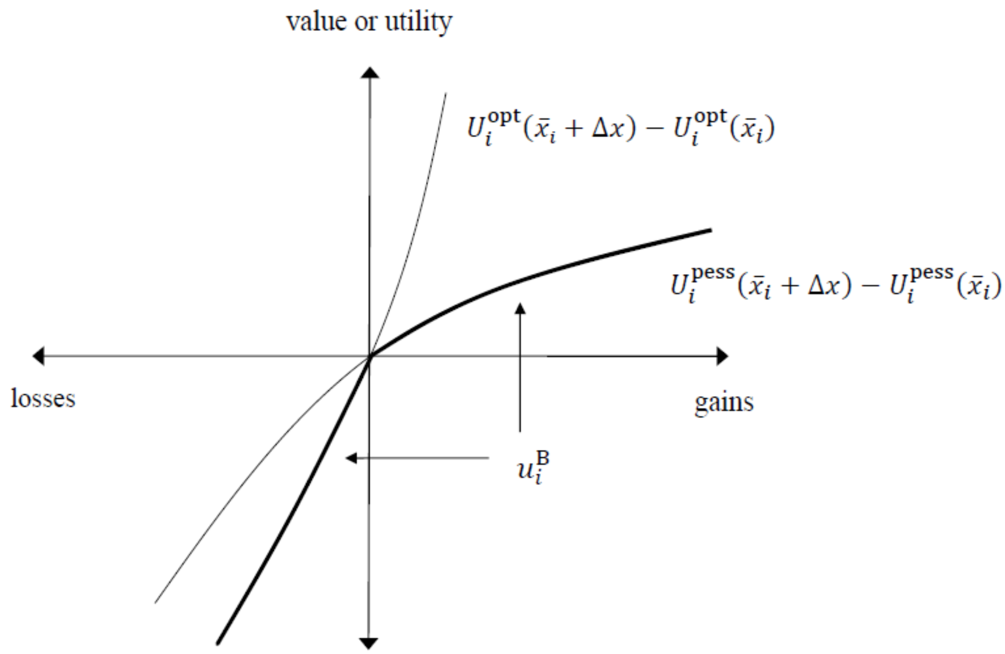

Figure 3: The behavioral value function derived from U_i^{opt} and U_i^{pess}

and maximally pessimistic towards the other good. Assumption 3 will be used only for the less important part of the Theorem to come ('only if').

We will show that, regardless of how the reference point $\bar{x}$ is set, the unanimity preference $\trianglerighteq$ delivers the same predictions for decisions relative to $\bar{x}$ as does a reference-dependent preference $\succsim_{\bar{x}}$ that is generated by a pair of behavioral value functions. The functions that will do the job are derived from the extreme optimism and pessimism utilities given by Assumption 1. Given a reference point $\bar{x} \gg 0$, define for each state i a behavioral value function u_i^{B} by

$$u_i^{\text{B}}(\Delta x_i) = \begin{cases} U_i^{\text{opt}}(\bar{x}_i + \Delta x_i) - U_i^{\text{opt}}(\bar{x}_i) & \text{if } \; -\bar{x}_i \leq \Delta x_i < 0 \\ U_i^{\text{pess}}(\bar{x}_i + \Delta x_i) - U_i^{\text{pess}}(\bar{x}_i) & \text{if } \;\; \Delta x_i \geq 0 \end{cases}.$$

See Figure 3. Since U_i^{opt} is strictly increasing and strictly convex and U_i^{pess} is strictly increasing and strictly concave, u_i^{B} has the required continuity, increasingness, and concavity-convexity properties. Moreover, our assumption that $DU_i^{\text{opt}}(0) \geq DU_i^{\text{pess}}(0)$, the strict convexity of U_i^{opt}, and the strict concavity of U_i^{pess} together imply that $DU_i^{\text{opt}}(\bar{x}_i) > DU_i^{\text{pess}}(\bar{x}_i)$ and hence $D_-u_i^{\text{B}}(0) > D_+u_i^{\text{B}}(0)$. So each u_i^{B} is indeed a behavioral value function.

Since, relative to the other candidate utilities, U_i^{opt} has the greatest marginal utility at 0 and is more convex, U_i^{opt} gives a more negative evaluation of reductions in x_i than any other candidate $U_i \in \mathcal{U}_i$:

$$U_i^{\text{opt}}(\bar{x}_i + \Delta x_i) - U_i^{\text{opt}}(\bar{x}_i) \leq U_i(\bar{x}_i + \Delta x_i) - U_i(\bar{x}_i) \text{ when } \Delta x_i \leq 0.$$

Similarly, since U_i^{pess} has the smallest marginal utility at 0 and is more concave than the agent's other candidate utilities, U_i^{pess} gives a more negative evaluation of increases in x_i:

$$U_i^{\text{pess}}(\bar{x}_i + \Delta x_i) - U_i^{\text{pess}}(\bar{x}_i) \leq U_i(\bar{x}_i + \Delta x_i) - U_i(\bar{x}_i) \text{ when } \Delta x_i \geq 0.$$

Now consider a consumption change, say $\Delta x_1 \leq 0$ and $\Delta x_2 \geq 0$ for concreteness, that is approved by $(u_1^{\text{B}}, u_2^{\text{B}})$. That is, $u_1^{\text{B}}(\Delta x_1) + u_2^{\text{B}}(\Delta x_2) \geq 0$ or equivalently

$$U_1^{\text{opt}}(\bar{x}_i + \Delta x_i) - U_1^{\text{opt}}(\bar{x}_i) + U_2^{\text{pess}}(\bar{x}_i + \Delta x_i) - U_2^{\text{pess}}(\bar{x}_i) \geq 0.$$

It follows that all of the candidate utility pairs in $\mathcal{U}$ also approve the change:

$$U_1(\bar{x}_i + \Delta x_i) - U_1(\bar{x}_i) + U_2(\bar{x}_i + \Delta x_i) - U_2(\bar{x}_i) \geq 0.$$

The change is therefore $\trianglerighteq$ or unanimity preferred. Conversely, since $(U_1^{\text{opt}}, U_2^{\text{pess}})$ and $(U_1^{\text{pess}}, U_2^{\text{opt}})$ are both in $\mathcal{U}$, any consumption change that is unanimity preferred must be approved by $(u_1^{\text{B}}, u_2^{\text{B}})$. These arguments lead to the following result.

Theorem 1 *For each reference point $\bar{x} \gg 0$, there exist behavioral value functions $(u_1^{\text{B}}, u_2^{\text{B}})$ such that, for all $\Delta x \in [-\bar{x}_1, \infty) \times [-\bar{x}_2, \infty)$,*

$$\bar{x} + \Delta x \trianglerighteq \bar{x} \textit{ if and only if } u_1^{\text{B}}(\Delta x_1) + u_2^{\text{B}}(\Delta x_2) \geq 0.$$

The proof of Theorem 1 is in the Appendix.

Theorem 1 implicitly compares the concision of a unanimity preference $\trianglerighteq$ and a pair of behavioral value functions. The two theories have different advantages. If the reference point is fixed, one pair of behavioral value functions – the $(u_1^{\text{B}}, u_2^{\text{B}})$ we have specified – can represent all of the consumption changes that are approved by all of the utility pairs in $\mathcal{U}$ and provides a succinct summary of the agent's choices. But if the goal is a single account that applies regardless of the reference point – perhaps to identify the agent's intertemporal welfare judgments – then the advantage shifts to the unanimity preference. One preference then ties together all of the agent's choices.

3 Diverse preferences over uncertain acts

The analysis will now place assumptions on preferences over acts rather than on utilities and allow arbitrarily many states or goods. After a preliminary on expected utility representations, I introduce orderings on preferences that define when an agent is more favorable or optimistic towards consumption at one state relative to another state. As before, an agent with a set of preferences makes decisions relative to a reference point using a unanimity rule, and if the set of preferences is sufficiently rich with respect to the optimism orderings, then increases in goods will be evaluated by low-marginal utility concave utility functions and decreases by high-marginal utility convex utility functions, leading to reference dependence, loss aversion, and the reflection effect. Once again a single set of preferences will apply to all reference points.

3.1 Classical preferences over acts

Let $\Omega = \{1, ..., S\}$ be a finite set of states with $S \geq 2$. Any $x_s \in \mathbb{R}_+$ is the good or prize delivered at state s, for example a quantity of money. An act is therefore a $(x_1, ..., x_S) \in \mathbb{R}_+^\Omega$. An act can also be viewed as a bundle of traditional goods. The subjective probabilities that the agent implicitly assigns to states will be unchanging; we can however accommodate lotteries that can deliver quantities of money with finely-graded probabilities by letting S be large.

Let $\succsim$ be a preference relation on $\mathbb{R}_+^\Omega$ with asymmetric part $\succ$ and symmetric part $\sim$.

A1. Weak order: $\succsim$ is complete and transitive.

A2. Independence: for all $\Theta \subset \Omega$, $(x_s)_{s\in\Theta}$, $(y_s)_{s\in\Theta} \in \mathbb{R}_+^\Theta$, and $(z_s)_{s\in\Omega\setminus\Theta}, (z'_s)_{s\in\Omega\setminus\Theta} \in \mathbb{R}_+^{\Omega\setminus\Theta}$,

$$((x_s)_{s\in\Theta}, (z_s)_{s\in\Omega\setminus\Theta}) \succsim ((y_s)_{s\in\Theta}, (z_s)_{s\in\Omega\setminus\Theta}) \text{ if and only if }$$
$$((x_s)_{s\in\Theta}, (z'_s)_{s\in\Omega\setminus\Theta}) \succsim ((y_s)_{s\in\Theta}, (z'_s)_{s\in\Omega\setminus\Theta}).$$

A3. Continuity: for each $x \in \mathbb{R}_+^\Omega$, the sets $\{y \in \mathbb{R}_+^\Omega : y \succ x\}$ and $\{y \in \mathbb{R}_+^\Omega : x \succ y\}$ are open relative to $\mathbb{R}_+^\Omega$.

A4. Monotonicity: for all $x, y \in \mathbb{R}_+^\Omega$, if $x \geq y$ and $x \neq y$ then $x \succ y$.

The independence assumption A2 states that the preference relation on $\mathbb{R}_+^{\Theta}$ induced by $\succsim$ when the $\Omega \setminus \Theta$ entries of acts are fixed is independent of the levels $(z_s)_{s \in \Omega \setminus \Theta}$ at which those entries are fixed.

Proposition 1 *If $\succsim$ satisfies A1-A4 and $S \geq 3$ then there exist continuous and strictly increasing functions $U_s : \mathbb{R}_+ \to \mathbb{R}$ for $s \in \Omega$ such that, for all $x, y \in \mathbb{R}_+^{\Omega}$,*

$$x \succsim y \Leftrightarrow \sum_{s \in \Omega} U_s(x_s) \geq \sum_{s \in \Omega} U_s(y_s).$$

The array of functions $(U_s)_{s \in \Omega}$ is unique up to an increasing affine transformation.

A $(U_s)_{s \in \Omega}$ that satisfies the $\Leftrightarrow$ in Proposition 1 is a *representation of* $\succsim$. To interpret $x, y \in \mathbb{R}_+^{\Omega}$ as lotteries, we can as in the previous section think of each U_s as equal to the product $\pi_s V_s$ where $\pi_s > 0$ is the probability of state s and V_s is the Bernoulli utility of prizes at s.

Since A4 implies that each state is 'essential' in the language of Debreu (1960), that paper proves all of Proposition 1 except the increasingness of the U_s, and that property follows straightforwardly from A4.

3.2 The ordering of preferences

As with the more-optimistic-than ordering of utilities in section 2, a preference $\succsim$ must pass two tests to be judged more favorable to consumption at a state s than another preference $\succsim'$. Optimism and pessimism will again designate favorable and unfavorable attitudes towards goods, possibly because the agent does not know how desirable consumption at a state is.

To pass the first test, $\succsim$ must have a greater marginal rate of substitution than $\succsim'$ between consumption at s and consumption at a different state s' when consumption at s is near 0. Unlike comparisons of the marginal utilities of a good, a behavioral comparison of the value of consumption at s must be made relative to consumption of another good.

To pass the second test, the marginal rate of substitution between s and s' must diminish more rapidly for $\succsim$ than for $\succsim'$ as consumption increases at s. In principle, this test could simply check which preference has a utility for s that is more concave as a function of consumption at s. But to avoid using utility to compare preferences, I will apply an alternative and simpler behavioral criterion, due to Baillon, Driesen, and Wakker (2012), that is equivalent to defining the more pessimistic preference to be the one with the more concave utility for s.

Throughout the remainder of the paper, I assume without further remark that each preference $\succsim$ satisfies A1-A4 and has a representation.

We begin by defining the marginal rate of substitution between consumption at s and s' when consumption x_s at state s is small. Let x_s and $x_{s'}(x_s, \succsim)$ be quantities of consumption at states s and s' that are $\succsim$-indifferent to the quantities 0 and $y_{s'}$ when consumption at all other states remains fixed. That is, for the acts x and y given in the table below, $x_{s'}(x_s, \succsim)$ is defined by the requirement $x \sim y$.

	s	s'	$s'' \neq s, s'$
x	x_s	$x_{s'}(x_s, \succsim)$	$w_{s''}$
y	0	$y_{s'}$	$w_{s''}$

For readability, the notation suppresses the dependence of $x_{s'}(x_s, \succsim)$ on $y_{s'}$. Since the utility for each good is continuous and strictly increasing, there exists a unique $x_{s'}(x_s, \succsim)$ that satisfies this definition when x_s is sufficiently near 0 and $x_{s'}(x_s, \succsim)$ is of course independent of the $w_{s''}$ for $s'' \in \Omega \setminus \{s, s'\}$.

We can measure the gain from consuming x_s rather than 0 at s by the compensating change in consumption at s', $|x_{s'}(x_s, \succsim) - y_{s'}|$. The 'discrete' marginal rate of substitution between s and s' at x_s is the ratio of this change to the change in consumption at s:

$$\frac{|x_{s'}(x_s, \succsim) - y_{s'}|}{x_s - 0} = \frac{y_{s'} - x_{s'}(x_s, \succsim)}{x_s}.$$

For the first test of greater optimism, a greater marginal rate of substitution must hold at the baseline, where consumption at s approaches 0, and we therefore compare for two different preferences the limit of the above ratio as x_s converges to 0. Define the *marginal rate of substitution between s and s' to be greater for $\succsim$ than for $\succsim'$* if, for all $y_{s'} > 0$, these limits exist for $\succsim$ and $\succsim'$ and

$$\lim_{x_s \to 0+} \frac{y_{s'} - x_{s'}(x_s, \succsim)}{x_s} \geq \lim_{x_s \to 0+} \frac{y_{s'} - x_{s'}(x_s, \succsim')}{x_s}.$$

In the second test, the more optimistic preference must have a marginal utility for s or

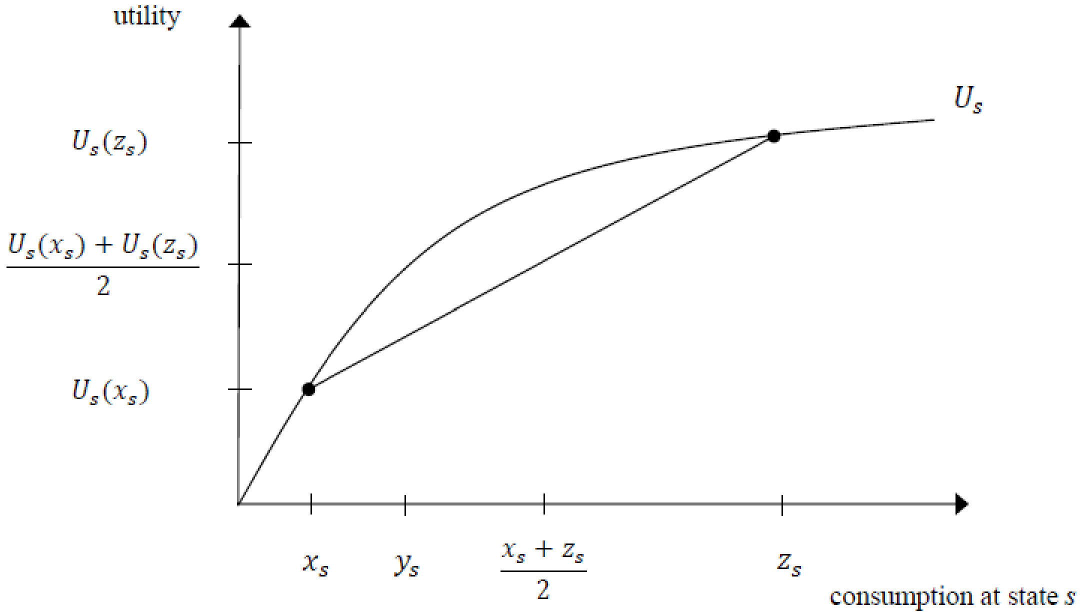

Figure 4: A concave utility and a s-midpoint

a marginal rate of substitution between s and s' that diminishes less rapidly than the less optimistic preference. To apply the equivalent, behavioral criterion in Baillon et al. (2012), define y_s to be a *s-midpoint between x_s and z_s for* $\succsim$ if there exists a state s', nonnegative numbers a and b, and $w_{s''} \geq 0$ for $s'' \in \Omega \setminus \{s, s'\}$ such that, for each of the tables below, the acts defined by the rows are $\succsim$-indifferent:

s	s'	$s'' \neq s, s'$
x_s	a	$w_{s''}$
y_s	b	$w_{s''}$

s	s'	$s'' \neq s, s'$
y_s	a	$w_{s''}$
z_s	b	$w_{s''}$

.

So when y_s is a s-midpoint between x_s and z_s the benefit of the increment $x_s - y_s$ of consumption at s is equivalent to the benefit of the increment $y_s - z_s$: each increment is exactly counterbalanced by the increment $a - b$ in consumption at s'. See Figure 4. The Proposition below essentially follows Baillon et al. and the proof in the Appendix is akin to theirs.[6]

Proposition 2 *Let $\succsim$ be represented by $(U_s)_{s \in \Omega}$. Then, for $s \in \Omega$, U_s is concave if and only if, for every $x_s, y_s, z_s \geq 0$ such that y_s is a s-midpoint between x_s and z_s for $\succsim$,*

[6]The proofs of Proposition 2 and of Proposition 3 are more direct since our treatment avoids any explicit introduction of probabilities.

$$y_s \leq \frac{x_s + z_s}{2}.$$

Moreover U_s is strictly concave if and only if the above inequality is always strict. The same conclusions hold if 'convex' replaces 'concave' and $\geq$ replaces $\leq$ above.

A name for preferences where the above inequality is strict will be useful. If, for some $\succsim$, the $\leq$ in Proposition 2 can be replaced by $<$ for every $x_s, y_s, z_s \geq 0$ that satisfies (1) and (2) then $\succsim$ has *strictly concave s-midpoints*. Similarly, if the $\leq$ can be replaced by $>$ for every $x_s, y_s, z_s \geq 0$ then $\succsim$ has *strictly convex s-midpoints*.

Let $\succsim$ and $\succsim'$ be two preferences and let s be a state. Define $\succsim$ to have *lesser s-midpoints* than $\succsim'$ if, for all $x_s, y_s, y'_s, z_s \geq 0$ such that y_s is a s-midpoint between x_s and z_s for $\succsim$ and y'_s is a s-midpoint between x_s and z_s for $\succsim'$, we have $y_s \leq y'_s$. This definition and the next Proposition also follow Baillon et al. (2012) and I provide a similar proof in the Appendix.

Proposition 3 *Let $\succsim$ and $\succsim'$ be represented by $(U_s)_{s\in\Omega}$ and $(U'_s)_{s\in\Omega}$. For each $s \in \Omega$, $\succsim$ has lesser s-midpoints than $\succsim'$ if and only if U_s is more concave than U'_s.*

An agent whose marginal utility for a good diminishes more rapidly in a representation $(U_s)_{s\in\Omega}$ – or that has a more rapidly diminishing MRS – is thus willing to accept a smaller quantity of the good as a replacement for a dispersed distribution of the good.

The component definitions of greater optimism are now complete.

Definition 2 *Let s and s' be states and $\succsim$ and $\succsim'$ be preferences with representations $(U_s)_{s\in\Omega}$ and $(U'_s)_{s\in\Omega}$. Then $\succsim$* ***is more optimistic than $\succsim'$ towards consumption at s relative to s'*** *or $\succsim R_{s,s'} \succsim'$ if:*

- *the marginal rate of substitution between s and s' is greater for $\succsim$ than for $\succsim'$,*
- *$\succsim'$ has lesser s-midpoints than $\succsim$.*

When convenient, I refer to the $\succsim'$ above as *more pessimistic than* $\succsim$.

We can now state the analogues to Assumptions 1-3 in section 2. First, for each partition of states into two cells, the set of candidate preferences must be diverse enough to contain a preference that is, relative to other candidate preferences, more optimistic towards consumption in the first cell relative to consumption in the second cell and more pessimistic towards consumption in the second cell relative to consumption in the first cell. Second, for any state s, the set of candidate preferences must contain both a preference that has strictly concave s-midpoints (hence with a representation that has a strictly concave utility for consumption at s) and a preference that has strictly convex s-midpoints (hence with a representation that has a strictly convex utility for consumption at s).

Definition 3 *A set of candidate preferences $\mathcal{P}$ is **diverse** if*

1. *for every nonempty $\Theta \subset \Omega$ with $\Theta \neq \Omega$, there exists a $\succsim^{\Theta} \in \mathcal{P}$ such that, for all $s \in \Theta$, $s' \in \Omega \setminus \Theta$, and $\succsim \in \mathcal{P}$,*

$$\succsim^{\Theta} R_{s,s'} \succsim \text{ and } \succsim R_{s',s} \succsim^{\Theta},$$

2. *for each $s \in \Omega$ there exist $\succsim_s^{\text{sconcave}}, \succsim_s^{\text{sconvex}} \in \mathcal{P}$ such that $\succsim_s^{\text{sconcave}}$ has strictly concave s-midpoints and $\succsim_s^{\text{sconvex}}$ has strictly convex s-midpoints.*

Diversity thus requires the set of candidate preferences to contain, for each set of states Θ, an 'extreme' $\succsim^{\Theta}$ that optimism-dominates all other candidates with respect to consumption at states in Θ and is optimism-dominated by all other candidates with respect to consumption at states outside of Θ.

The simplest way to construct a set of diverse preferences is to mimic section 2. Begin with a family of utility functions $\mathcal{U}_s$ for each s that has maximal and minimal elements: a most convex and strictly convex utility with marginal utility at 0 greater than any other function in $\mathcal{U}_s$ and a most concave and strictly concave utility with marginal utility at 0 less than any other function in $\mathcal{U}_s$. Then for each draw of $U_s \in \mathcal{U}_s$ for $s \in \Omega$, let there be a preference in $\mathcal{P}$ that is represented by the utility $\sum_{s \in \Omega} U_s$ defined on $\mathbb{R}_+^{\Omega}$.

4 Value functions from diverse preferences

Given a set of candidate preferences $\mathcal{P}$, let $\trianglerighteq$ be the unanimity preference defined by $x \trianglerighteq y$ if and only if $x \succsim y$ for all $\succsim \in \mathcal{P}$.

Theorem 2 *Suppose $\mathcal{P}$ is diverse and $S \neq 3$ and let $y \in \mathbb{R}^{\Omega}_{++}$ be the reference point. Then there exist behavioral value functions u^{B}_s for $s \in \Omega$ such that, for each alternative $x \in \mathbb{R}^{\Omega}_{+}$,*

$$x \trianglerighteq y \text{ if and only if } \sum\nolimits_{s\in\Omega} u^{\mathrm{B}}_s(x_s - y_s) \geq 0.$$

Thus the single unanimity preference $\trianglerighteq$ represents the same behavior relative to a reference point as a preference in prospect theory, regardless of how the reference point is set. If the reference point is fixed, this comprehensiveness brings no empirical advantage. But when an agent chooses repeatedly, a unanimity preference can unify a range of disparate behavioral deviations from classical rationality under one system of judgments.

The bulk of the proof of Theorem 2 establishes the following representation result using an induction argument: there exists an array of functions $(U^{\mathrm{opt}}_s, U^{\mathrm{pess}}_s)_{s\in\Omega}$ such that, for *every* nonempty $\Theta \subsetneq \Omega$, $\left((U^{\mathrm{opt}}_s)_{s\in\Theta}, (U^{\mathrm{pess}}_s)_{s\in\Omega\setminus\Theta}\right)$ represents the extreme preference $\succsim^{\Theta}$ given by Definition 3. The function U^{opt}_s moreover will be more optimistic than any U_s in a representation of another preference in $\mathcal{P}$ and U^{pess}_s will be more pessimistic than any U_s in a representation of another preference in $\mathcal{P}$, with greater optimism and pessimism as defined in section 2. The behavioral value functions will then be given, as in section 2, by $u^{\mathrm{B}}_s(x_s) = U^{\mathrm{opt}}_s(x_s) - U^{\mathrm{opt}}_s(y_s)$ for $0 \leq x_s \leq y_s$ and $u^{\mathrm{B}}_s(x_s) = U^{\mathrm{pess}}_s(x_s) - U^{\mathrm{pess}}_s(y_s)$ for $x_s > y_s$. Now if $x \trianglerighteq y$ then $x \succsim^{\Theta} y$ for the Θ that consists of the states s with $x_s \leq y_s$ and hence the representation result implies $\sum_{s\in\Omega} u^{\mathrm{B}}_s(x_s - y_s) \geq 0$. Conversely if $\sum_{s\in\Omega} u^{\mathrm{B}}_s(x_s - y_s) \geq 0$ then the greater optimism of U^{opt}_s and the greater pessimism of U^{pess}_s will imply, as in section 2, that any $\succsim \in \mathcal{P}$ can be represented by a $(U_s)_{s\in\Omega}$ such that $U_s(x_s) - U_s(y_s) \geq U^{\mathrm{opt}}_s(x_s) - U^{\mathrm{opt}}_s(y_s)$ for the s with $0 \leq x_s \leq y_s$ and $U_s(x_s) - U_s(y_s) \geq U^{\mathrm{pess}}_s(x_s) - U^{\mathrm{pess}}_s(y_s)$ for the s with $x_s \geq y_s$. Since therefore $x \succsim y$ for any $\succsim \in \mathcal{P}$, we have $x \trianglerighteq y$.

Theorem 2 carves out an exception for $S = 3$. With three states the representation result need not hold: it may not be possible to represent all of the $\succsim^{\Theta}$'s in Definition 3 using a single family of extreme utilities. The 'if' of the Theorem may then fail to hold. While it remains true that if $x \trianglerighteq y$ and $x' \trianglerighteq y$ then $x \succsim^{\Theta} y$ for $\Theta = \{s : x_s \leq y_s\}$ and $x' \succsim^{\Theta'} y$ for $\Theta' = \{s : x'_s \leq y_s\}$, there may not be a single family of behavioral value functions that satisfies both $\sum_{s\in\Omega} u_s^{\mathrm{B}}(x_s - y_s) \geq 0$ and $\sum_{s\in\Omega} u_s^{\mathrm{B}}(x'_s - y_s) \geq 0$.

For an example, let $S = 3$ and suppose $\mathcal{P}$ consists solely of one $\succsim^{\Theta}$ for each nonempty $\Theta \subsetneq \Omega$ where $\succsim^{\Theta}$ is represented by $\Big((2(x_s - y_s))_{s\in\Theta}, (x_s - y_s)_{s\in\Omega\setminus\Theta}\Big)$. To focus on essentials, I have dropped the curvature requirements of diverse preferences. I have also set the utilities of the status quo to 0. The behavioral value functions that result are, for each s, $u_s^{\mathrm{B}}(x_s) = 2(x_s - y_s)$ for $x_s \leq y_s$ and $u_s^{\mathrm{B}}(x_s) = x_s - y_s$ for $x_s > y_s$. These functions together with the unanimity preference for $\mathcal{P}$ satisfy the conclusion of Theorem 2. Now consider an alternative set of candidate preferences $\mathcal{P}'$ where, for some a near 1, $\succsim^{\{3\}}$ has the representation $(x_1 - y_1, a(x_2 - y_2), 2a(x_3 - y_s))$, $\succsim^{\{2,3\}}$ has the representation $(x_1 - y_1, 2(x_2 - y_2), 2a(x_3 - y_s))$, and the other $\succsim^{\Theta}$'s are unchanged. Diversity then remains satisfied: the only marginal rate of substitution relevant to diversity that $\mathcal{P}'$ changes is the MRS between consumption at states 3 and 1 for $\succsim^{\{3\}}$ and $\succsim^{\{2,3\}}$and this MRS continues to be held in common by these two preferences (as diversity requires). Due to the change in this MRS, the original behavioral value functions and the unanimity preference for $\mathcal{P}'$ will no longer satisfy the 'if' in Theorem 2. But since the $\succsim^{\Theta}$'s besides $\succsim^{\{3\}}$ and $\succsim^{\{2,3\}}$ remain unchanged, the 'if' in Theorem 2 also requires that we retain, up to a common affine transformation, the original value functions.

This counterexample relies on two ingredients: (i) a Θ and Θ' that both contain some state s (3 above) and fail to contain some other state s' (1 above) and (ii) a reweighting of utilities where the only diversity-relevant MRS that changes is between consumption at s and s'. The first ingredient cannot arise with two states since the only two nonempty strict subsets of Ω are $\{1\}$ and $\{2\}$. The second cannot arise with four or more states due to diversity: the MRS between consumption at one of the two states, say s, and some further state s'' would change, in contradiction to the diversity requirement that $\succsim^{\Theta}$ (or $\succsim^{\Theta'}$) is optimism- or pessimism-dominant with respect to consumption at s relative to s''.

A Appendix

Proof of Theorem 1. Suppose $(U_1, U_2) \in \mathcal{U}$ and define, for $i \in \{1, 2\}$, $u_i : [-\bar{x}_i, \infty) \to \mathbb{R}$ by $u_i(\Delta x_i) = U_i(\bar{x}_i + \Delta x_i) - U_i(\bar{x}_i)$. Fix $i \in \{1, 2\}$.

I will first show that $u_i^{\text{opt}}(\Delta x_i) \leq u_i(\Delta x_i)$ for all $\Delta x_i \leq 0$. Since $u_i^{\text{opt}}(0) = u_i(0)$, it is sufficient to show that $u_i^{\text{opt}} - u_i$ is a nondecreasing function. As there exists a concave f such that $u_i = f \circ u_i^{\text{opt}}$, I show that the function $h : [-\bar{x}_i, \infty) \to \mathbb{R}$ defined by $h(\Delta x_i) = u_i^{\text{opt}}(\Delta x_i) - f(u_i^{\text{opt}}(\Delta x_i))$ is nondecreasing. Since u_i^{opt} is convex and f is concave, each is right differentiable and therefore h is right differentiable. Since a continuous and right differentiable function whose right derivatives are everywhere nonnegative is nondecreasing (see, e.g., Miller and Výborný (1986)), I will show that $D_+h(\Delta x_i) \geq 0$ for all Δx_i.

Observe first that since u_i^{pess} is concave and strictly increasing, $D_+u_i^{\text{pess}}(-\bar{x}_i) > 0$ and hence $D_+u_i^{\text{opt}}(-\bar{x}_i) > 0$. Since moreover u_i^{opt} is convex, $D_+u_i^{\text{opt}}(-\bar{x}_i)$ is finite. Since in addition $D_+u_i^{\text{opt}}(-\bar{x}_i) \geq D_+u_i(-\bar{x}_i)$ by assumption, $D_+u_i^{\text{opt}}(-\bar{x}_i) \geq D_+f(u_i^{\text{opt}}(-\bar{x}_i))D_+u_i^{\text{opt}}(-\bar{x}_i)$. Hence $D_+f(u_i^{\text{opt}}(-\bar{x}_i)) \leq 1$ and, given that f is concave, $D_+f(u_i^{\text{opt}}(\Delta x_i)) \leq 1$ for all Δx_i. Due to the right differentiability of u_i^{opt} and f and since in addition u_i^{opt} is increasing, we may apply the chain rule to h. Thus $D_+h(\Delta x_i) = D_+u_i^{\text{opt}}(\Delta x_i) - D_+f(u_i^{\text{opt}}(\Delta x_i))D_+u_i^{\text{opt}}(\Delta x_i) = D_+u_i^{\text{opt}}(\Delta x_i)(1 - D_+f(u_i^{\text{opt}}(\Delta x_i))) \geq 0$.

A similar argument shows that $u_i^{\text{pess}}(\Delta x_i) \leq u_i(\Delta x_i)$ for all $\Delta x_i \geq 0$.

Now suppose that $\bar{x} + \Delta x \trianglerighteq \bar{x}$ or equivalently $u_1(\Delta x_1) + u_2(\Delta x_2) \geq 0$ for all $(u_1, u_2) \in \cdot\mathcal{U}$. If $\Delta x \geq 0$ then $u_i^{\text{B}}(\Delta x_i) + u_{-i}^{\text{B}}(\Delta x_{-i}) = u_i^{\text{pess}}(\Delta x_i) + u_{-i}^{\text{pess}}(\Delta x_{-i}) \geq 0$ while if $\Delta x_i \geq 0$ and $\Delta x_{-i} < 0$ then $u_i^{\text{B}}(\Delta x_i) + u_{-i}^{\text{B}}(\Delta x_{-i}) = u_i^{\text{pess}}(\Delta x_i) + u_{-i}^{\text{opt}}(\Delta x_{-i}) \geq 0$.

Conversely, suppose that $u_i^{\text{B}}(\Delta x_i) + u_{-i}^{\text{B}}(\Delta x_{-i}) \geq 0$. Fix some $(u_1, u_2) \in \Delta\mathcal{U}$. If $\Delta x \geq 0$ then $u_j^{\text{B}}(\Delta x_j) = u_j^{\text{pess}}(\Delta x_j) \leq u_j(\Delta x_j)$ for $j = i$ and $j = -i$ and hence $u_i(\Delta x_i) + u_{-i}(\Delta x_{-i}) \geq 0$. If $\Delta x_i \geq 0$ and $\Delta x_{-i} < 0$ then $u_i^{\text{B}}(\Delta x_i) = u_i^{\text{pess}}(\Delta x_i) \leq u_i(\Delta x_i)$ and $u_{-i}^{\text{B}}(\Delta x_{-i}) = u_{-i}^{\text{opt}}(\Delta x_{-i}) \leq u_{-i}(\Delta x_{-i})$ and again $u_i(\Delta x_i) + u_{-i}(\Delta x_{-i}) \geq 0$. ■

Proof of Proposition 2. Observe first that if y_s is a s-midpoint between x_s and z_s for $\succsim$ then $u_s(z_s) - u_s(y_s) = u_s(y_s) - u_s(x_s)$. Now suppose u_s is concave. Then $u_s(\frac{1}{2}x_s + \frac{1}{2}z_s) \geq \frac{1}{2}u_s(x_s) + \frac{1}{2}u_s(z_s)$ and hence $2u_s(\frac{1}{2}x_s + \frac{1}{2}z_s) \geq u_s(z_s) + u_s(x_s) = 2u_s(y_s)$. Since u_s is increasing, $\frac{1}{2}x_s + \frac{1}{2}z_s \geq y_s$. If u_s is strictly concave then, since u_s is strictly increasing, each of these inequalities is strict. Conversely, suppose that $\frac{1}{2}x_s + \frac{1}{2}z_s \geq y_s$ for all x_s, y_s, and z_s such that y_s is a s-midpoint between x_s and z_s for $\succsim$ and $x_s \neq z_s$. Since u_s is continuous and any $0 \leq \alpha \leq 1$ equals the limit of a sequence of dyadic fractions, it suffices to show that $u_s(\frac{1}{2}x_s + \frac{1}{2}z_s) \geq \frac{1}{2}u_s(x_s) + \frac{1}{2}u_s(z_s)$ for any $x_s, z_s \in \mathbb{R}_+$ with $x_s < z_s$. Due to the continuity of u_s, there exists a $y_s \in [x_s, z_s]$ such that $u_s(y_s) = \frac{1}{2}u_s(x_s) + \frac{1}{2}u_s(z_s)$. As y_s is a s-midpoint between x_s and z_s for $\succsim$, $\frac{1}{2}x_s + \frac{1}{2}z_s \geq y_s$ and, since u_s is increasing, $u_s(\frac{1}{2}x_s + \frac{1}{2}z_s) \geq u_s(y_s) = \frac{1}{2}u_s(x_s) + \frac{1}{2}u_s(z_s)$. Now suppose that $\frac{1}{2}x_s + \frac{1}{2}z_s > y_s$ at the same triples (x_s, y_s, z_s). Given the concavity of u_s, a failure of strict concavity implies that there is a (x_s, y_s, z_s) and $0 < \alpha < 1$ such that $u_s(\alpha x_s + (1-\alpha)z_s) = \alpha u_s(x_s) + (1-\alpha)u_s(z_s)$. But then u_s must be linear on the interval between x_s and z_s, which contradicts $\frac{1}{2}x_s + \frac{1}{2}z_s > y_s$. The convexity conclusions are proved similarly. ■

Proof of Proposition 3. Suppose u_s is more concave than u'_s. There is then an increasing concave $f : u_s(\mathbb{R}_+) \to \mathbb{R}$ such that $u'_s = f \circ u_s$. For $x_s, z_s \in \mathbb{R}_+$, let y_s be the s-midpoint between x_s and z_s for $\succsim$ and let w_s be the s-midpoint between x_s and z_s for $\succsim'$. Then $u_s(y_s) = \frac{1}{2}u_s(x_s) + \frac{1}{2}u_s(z_s)$ and $f(u_s(w_s)) = \frac{1}{2}f(u_s(x_s)) + \frac{1}{2}f(u_s(z_s))$. Since f is concave, $f(\frac{1}{2}u_s(x_s) + \frac{1}{2}u_s(z_s)) \geq \frac{1}{2}f(u_s(x_s)) + \frac{1}{2}f(u_s(z_s)) = f(u_s(w_s))$. Since f is increasing, $\frac{1}{2}u_s(x_s) + \frac{1}{2}u_s(z_s) \geq u_s(w_s)$ and hence $u_s(y_s) \geq u_s(w_s)$. Since u_s is increasing, $y_s \geq w_s$.

Conversely suppose that, for each $s \in \Omega$, $\succsim$ has lesser s-midpoints than $\succsim'$. Since both u_s and u'_s are strictly increasing, there is an increasing $f : u_s(\mathbb{R}_+) \to \mathbb{R}$ such that $u'_s = f \circ u_s$. Since f is increasing and maps onto an interval, it is continuous. Hence, as in the proof of Proposition 2, to conclude that f is concave it is sufficient to show that $f(\frac{1}{2}u_s(x_s)+\frac{1}{2}u_s(z_s)) \geq \frac{1}{2}f(u_s(x_s)) + \frac{1}{2}f(u_s(z_s))$ for any $x_s, z_s \in \mathbb{R}_+$ with $x_s < z_s$. By assumption, $y_s \geq w_s$ where $u_s(y_s) = \frac{1}{2}u_s(x_s) + \frac{1}{2}u_s(z_s)$ and $f(u_s(w_s)) = \frac{1}{2}f(u_s(x_s)) + \frac{1}{2}f(u_s(z_s))$. Since u_s and f are increasing, $u_s(y_s) \geq u_s(w_s)$ and $f(u_s(y_s)) \geq f(u_s(w_s))$. Hence

$$f(\tfrac{1}{2}u_s(x_s) + \tfrac{1}{2}u_s(z_s)) = f(u_s(y_s)) \geq f(u_s(w_s)) = \tfrac{1}{2}f(u_s(x_s)) + \tfrac{1}{2}f(u_s(z_s)).$$

■

Proof of Theorem 2. For each nonempty $\Theta \subset \Omega$ with $\Theta \neq \Omega$, henceforth a *partition*, let the preference $\succsim^{\Theta}$ given by Definition 3.1 be represented by $\left(\left(\hat{U}^{\text{opt}}_s[\Theta]\right)_{s\in\Theta}, \left(\hat{U}^{\text{pess}}_s[\Theta]\right)_{s\in\Omega\setminus\Theta}\right)$. Define $\hat{u}^{\text{opt}}_s[\Theta]$ for $s \in \Theta$ and $\hat{u}^{\text{pess}}_s[\Theta]$ for $s \in \Omega \setminus \Theta$ by $\hat{u}^{\text{opt}}_s[\Theta](x_s) = \hat{U}^{\text{opt}}_s[\Theta](x_s) - \hat{U}^{\text{opt}}_s[\Theta](y_s)$ and $\hat{u}^{\text{pess}}_s[\Theta](x_s) = \hat{U}^{\text{pess}}_s[\Theta](x_s) - \hat{U}^{\text{pess}}_s[\Theta](y_s)$. By adding constants to each $\hat{u}^{\text{opt}}_s[\Theta]$ and

$\hat{u}_s^{\text{pess}}[\Theta]$, we may assume that $\hat{u}_s^{\text{opt}}[\Theta](y_s) = 0$ for each $s \in \Theta$ and $\hat{u}_s^{\text{pess}}[\Theta](y_s) = 0$ for each $s \in \Omega \setminus \Theta$. More generally, the use of lowercase in the $(u_s)_{s\in\Omega}$ below will indicate a representation that satisfies $u_s(y_s) = 0$ for each $s \in \Omega$.

We begin with two lemmas whose proofs are given at the end.

Lemma 1. For each $s \in \Omega$, $\hat{u}_s^{\text{opt}}[\{s\}]$ *and the* $\hat{u}_{s'}^{\text{opt}}[\Omega \setminus \{s\}]$ *with* $s' \neq s$ *are convex and hence left and right differentiable and* $\hat{u}_s^{\text{pess}}[\Omega \setminus \{s\}]$ *and the* $\hat{u}_{s'}^{\text{pess}}[\{s\}]$ *with* $s' \neq s$ *are concave and hence left and right differentiable.*

Lemma 2. For each $\succsim \in \mathcal{P}$ *and representation* $(u_{s'})_{s'\in\Omega}$ *of* $\succsim$ *and each* $s \in \Omega$, u_s *is differentiable at* 0, $Du_s(0)$ *is finite,* $\lim_{x_s\to 0+} \frac{z_{s'} - x_{s'}(x_s,\succsim)}{x_s}$ *(the marginal rate of substitution between* s *and* $s' \neq s$ *when consumption at* s' *is* $z_{s'}$*) equals* $\frac{Du_s(0)}{D_- u_{s'}(z_{s'})}$ *for* $z_{s'} > 0$, and $\lim_{z_{s'}\to 0+} \frac{Du_s(0)}{D_- u_{s'}(z_{s'})} = \frac{Du_s(0)}{Du_{s'}(0)}$.

For $S \neq 3$, we show that there exists, for each $s \in \Omega$, a pair of functions u_s^{opt} and u_s^{pess} with $u_s^{\text{opt}}(y_s) = u_s^{\text{pess}}(y_s) = 0$ such that, for each partition Θ, $\left((u_s^{\text{opt}})_{s\in\Theta}, (u_s^{\text{pess}})_{s\in\Omega\setminus\Theta}\right)$ represents $\succsim^{\Theta}$. If $S = 2$, the functions will be given by $u_1^{\text{opt}} = \hat{u}_1^{\text{opt}}[\{1\}])$, $u_2^{\text{pess}} = \hat{u}_2^{\text{pess}}[\{1\}])$, $u_1^{\text{pess}} = \hat{u}_1^{\text{pess}}[\{2\}])$, and $u_2^{\text{opt}} = \hat{u}_2^{\text{opt}}[\{2\}])$.

Let $S > 3$ and define, for each $n \in \{1, ..., S-1\}$, the partition $\Theta_n = \{1, ..., n\}$. Set $u_1^{\text{opt}} = \hat{u}_1^{\text{opt}}[\Theta_1]$ and $u_s^{\text{pess}} = \hat{u}_s^{\text{pess}}[\Theta_1]$ for $s \in \Omega \setminus \Theta_1$. Assume for purposes of induction that, for some $1 \leq n < S$, there is a u_s^{opt} with $u_s^{\text{opt}}(y_s) = 0$ for each $s \in \Theta_n$ and a u_s^{pess} with $u_s^{\text{pess}}(y_s) = 0$ for each $s \in \Omega \setminus \Theta_n$ such that, for each partition $\Theta \subset \Theta_n$, $\left((u_s^{\text{opt}})_{s\in\Theta}, (u_s^{\text{pess}})_{s\in\Omega\setminus\Theta}\right)$ represents $\succsim^{\Theta}$.

Diversity and Definition 3.1 imply that $\succsim^{\{n\}} R_{n,s} \succsim^{\{n,n+1\}}$ and $\succsim^{\{n,n+1\}} R_{n,s} \succsim^{\{n\}}$ where $s \notin \{n, n+1\}$. Given that $\left(u_n^{\text{opt}}, (u_s^{\text{pess}})_{s\in\Omega\setminus\{n\}}\right)$ represents $\succsim^{\{n\}}$, we conclude that u_n^{opt} has lesser n-midpoints than $\hat{u}_n^{\text{opt}}[\{n, n+1\}]$ and that $\hat{u}_n^{\text{opt}}[\{n, n+1\}]$ has lesser n-midpoints than u_n^{opt}. Proposition 3 therefore implies that u_n^{opt} is more concave than $\hat{u}_n^{\text{opt}}[\{n, n+1\}]$ and $\hat{u}_n^{\text{opt}}[\{n, n+1\}]$ is more concave than u_n^{opt}. Hence $\hat{u}_n^{\text{opt}}[\{n, n+1\}]$ and u_n^{opt} differ by an increasing affine transformation and, since $u_n^{\text{opt}}(y_n) = \hat{u}_n^{\text{opt}}[\{n, n+1\}](y_n) = 0$, there exists a

$\lambda > 0$ such that $u_n^{\text{opt}} = \lambda \hat{u}_n^{\text{opt}}[\{n, n+1\}]$. Then

$$\left((\tilde{u}_s^{\text{opt}}[\{n, n+1\}])_{s\in\{n,n+1\}}, (\tilde{u}_s^{\text{pess}}[\{n, n+1\}])_{s\in\Omega\setminus\{n,n+1\}} \right) \\ = \lambda \left((\hat{u}_s^{\text{opt}}[\{n, n+1\}])_{s\in\{n,n+1\}}, (\hat{u}_s^{\text{pess}}[\{n, n+1\}])_{s\in\Omega\setminus\{n,n+1\}} \right)$$

represents $\succsim^{\{n,n+1\}}$and satisfies $\tilde{u}_n^{\text{opt}}[\{n, n+1\}] = u_n^{\text{opt}}$. Define $u_{n+1}^{\text{opt}} = \tilde{u}_{n+1}^{\text{opt}}[\{n, n+1\}]$. Due again to diversity and Definition 3.1, the marginal rate of substitution (henceforth MRS) between n and $k \in \{1, ..., n-1\}$ is both greater for $\succsim^{\{n,n+1\}}$ than for $\succsim^{\{n\}}$ and greater for $\succsim^{\{n\}}$ than for $\succsim^{\{n,n+1\}}$. Lemma 2 then implies both $\frac{Du_n^{\text{opt}}(0)}{D_-\tilde{u}_k^{\text{pess}}[\{n,n+1\}](z_k)} = \frac{Du_n^{\text{opt}}(0)}{D_-u_k^{\text{pess}}(z_k)}$ for each $z_k > 0$ and, letting $z_k \to 0$, $D\tilde{u}_k^{\text{pess}}[\{n, n+1\}](0) = Du_k^{\text{pess}}(0)$. Due to diversity, $\succsim^{\{n\}} R_{k,n} \succsim^{\{n,n+1\}}$ and $\succsim^{\{n,n+1\}} R_{k,n} \succsim^{\{n\}}$ which, given Proposition 3, implies that each of $\tilde{u}_k^{\text{pess}}[\{n, n+1\}]$ and u_k^{pess} is more concave than the other. The functions therefore differ by an increasing affine transformation. That fact, $D\tilde{u}_k^{\text{pess}}[\{n, n+1\}](0) = Du_k^{\text{pess}}(0)$, and $\tilde{u}_k^{\text{pess}}[\{n, n+1\}](y_k) = u_k^{\text{pess}}(y_k) = 0$ give $\tilde{u}_k^{\text{pess}}[\{n, n+1\}] = u_k^{\text{pess}}$. Thus $\left((u_s^{\text{opt}})_{s\in\{n,n+1\}}, (u_s^{\text{pess}})_{s\in\Omega\setminus\{n,n+1\}} \right)$ represents $\succsim^{\{n,n+1\}}$.

We next show that $\left((u_s^{\text{opt}})_{s\in\{j,n+1\}}, (u_s^{\text{pess}})_{s\in\Omega\setminus\{j,n+1\}} \right)$ represents $\succsim^{\{j,n+1\}}$ for each $j \in \{1, ..., n-1\}$. Fix such a j. Replacing n in the previous paragraph with j, there is a representation $\left((\tilde{u}_s^{\text{opt}}[\{j, n+1\}])_{s\in\{j,n+1\}}, (\tilde{u}_s^{\text{pess}}[\{j, n+1\}])_{s\in\Omega\setminus\{j,n+1\}} \right)$ of $\succsim^{\{j,n+1\}}$ such that $\tilde{u}_j^{\text{opt}}[\{j, n+1\}] = u_j^{\text{opt}}$, $\tilde{u}_s^{\text{opt}}[\{j, n+1\}](y_s) = 0$ for $s \in \{j, n+1\}$, and $\tilde{u}_s^{\text{pess}}[\{j, n+1\}](y_s) = 0$ for $s \in \Omega\setminus\{j, n+1\}$. Furthermore, for each $k \in \Omega\setminus\{j, n+1\}$, we have $\tilde{u}_k^{\text{pess}}[\{j, n+1\}] = u_k^{\text{pess}}$. It remains to show that $\tilde{u}_{n+1}^{\text{opt}}[\{j, n+1\}] = u_{n+1}^{\text{opt}}$. Since $S \geq 4$, there exists a $i \in \Omega\setminus\{j, n, n+1\}$. Setting $k = i$, we have $\tilde{u}_i^{\text{pess}}[\{j, n+1\}] = u_i^{\text{pess}} = \tilde{u}_i^{\text{pess}}[\{n, n+1\}]$. Since $\succsim^{\{n,n+1\}} R_{n+1,i} \succsim^{\{j,n+1\}}$ and $\succsim^{\{j,n+1\}} R_{n+1,i} \succsim^{\{n,n+1\}}$, the functions $\tilde{u}_{n+1}^{\text{opt}}[\{n, n+1\}] = u_{n+1}^{\text{opt}}$ and $\tilde{u}_{n+1}^{\text{opt}}\{j, n+1\}$ differ by an increasing affine transformation. Combining the previous two sentences with the fact that the MRS between $n+1$ and i is the same for $\succsim^{\{n,n+1\}}$ as for $\succsim^{\{j,n+1\}}$ and Lemma 2, we have $D\tilde{u}_{n+1}^{\text{opt}}[\{j, n+1\}](0) = Du_{n+1}^{\text{opt}}(0)$. Since finally $\tilde{u}_{n+1}^{\text{opt}}[\{j, n+1\}](y_{n+1}) = u_{n+1}^{\text{opt}}(y_{n+1}) = 0$, we have $\tilde{u}_{n+1}^{\text{opt}}[\{j, n+1\}] = u_{n+1}^{\text{opt}}$.

To conclude the induction, we show, for each partition $\Theta \subset \Theta_{n+1}\setminus\Theta_n$ not yet considered, where $n+1 \in \Theta$ and $|\Theta| \neq 2$, that $\left((u_s^{\text{opt}})_{s\in\Theta}, (u_s^{\text{pess}})_{s\in\Omega\setminus\Theta} \right)$ represents $\succsim^{\Theta}$. Fix some $j \in \Theta$. Applying the argument used to conclude that $\tilde{u}_n^{\text{opt}}[\{n, n+1\}] = u_n^{\text{opt}}$, there is a representation $\left((\tilde{u}_s^{\text{opt}}[\Theta])_{s\in\Theta}, (\tilde{u}_s^{\text{pess}}[\Theta])_{s\in\Omega\setminus\Theta}\right)$ of $\succsim^\Theta$ such that $\tilde{u}_j^{\text{opt}}[\Theta] = u_j^{\text{opt}}$, $\tilde{u}_s^{\text{opt}}[\Theta](y_s) = 0$ for $s \in \Theta$, and $\tilde{u}_s^{\text{pess}}[\Theta](y_s) = 0$ for $s \in \Omega\setminus\Theta$. For $k \in \Omega\setminus\Theta$ and $m \in \Omega\setminus\{k, n+1\}$, $\succsim^{\{m,n+1\}} R_{k,n+1} \succsim^\Theta$ and $\succsim^\Theta R_{k,n+1} \succsim^{\{m,n+1\}}$ and therefore $\tilde{u}_k^{\text{pess}}[\Theta]$ and $\tilde{u}_k^{\text{pess}}[\{m, n+1\}] = u_k^{\text{pess}}$ differ by an increasing affine transformation (with the equality due to the previous paragraph), $D\tilde{u}_k^{\text{pess}}[\Theta](0) = Du_k^{\text{pess}}(0)$ due to the MRS between k and $n+1$ being the same for $\succsim^\Theta$ as for $\succsim^{\{m,n+1\}}$, and $\tilde{u}_k^{\text{pess}}[\Theta](y_k) = u_k^{\text{pess}}(y_k) = 0$. Hence $\tilde{u}_k^{\text{pess}}[\Theta] = u_k^{\text{pess}}$. For any $i \in \Theta\setminus\{n+1\}$ and $l \in \Omega\setminus\Theta$, $\succsim^{\{i\}} R_{i,l} \succsim^\Theta$ and $\succsim^\Theta R_{i,l} \succsim^{\{i\}}$ and so, since $\left(u_i^{\text{opt}}, (u_s^{\text{pess}})_{s\in\Omega\setminus\{i\}}\right)$ represents $\succsim^{\{i\}}$ by the induction assumption, $\tilde{u}_i^{\text{opt}}[\Theta]$ and u_i^{opt} differ by an increasing affine transformation. Fixing some $k \in \Omega\setminus\Theta$, we have $D\tilde{u}_i^{\text{opt}}[\Theta](0) = Du_i^{\text{opt}}(0)$ due to the MRS between i and k being the same for $\succsim^\Theta$ as for $\succsim^{\{i,n+1\}}$. Since in addition $\tilde{u}_i^{\text{opt}}[\Theta](y_i) = u_i^{\text{opt}}(y_i) = 0$, $\tilde{u}_i^{\text{opt}}[\Theta] = u_i^{\text{opt}}$.

We will let each u_s^{B} have the domain $\mathbb{R}_+$ rather than $\mathbb{R}_+ - \{y_s\}$: to arrive at the desired value functions, subtract y_s from each element of the domain of u_s^{B}. For each $s \in \Omega$, define $u_s^{\text{B}}(x_s) = u_s^{\text{opt}}(x_s)$ if $0 \le x_s \le y_s$ and $u_s^{\text{B}}(x_s) = u_s^{\text{pess}}(x_s)$ if $x_s > y_s$.

Establishing the 'if and only if' claim when $x \ge y$ or $y \ge x$ is immediate given A4. We therefore consider only a $x \in \mathbb{R}_+^\Omega$ such that, for some partition $\Phi \subset \Omega$, $x_s \ge y_s$ for $s \in \Phi$ and $x_s < y_s$ for $s \in \Omega\setminus\Phi$.

For 'only if', suppose that $x \trianglerighteq y$. Since $\succsim^{\Omega\setminus\Phi} \in \mathcal{P}$ can be represented by $((u_s^{\text{pess}})_{s\in\Phi}, (u_s^{\text{opt}})_{s\in\Omega\setminus\Phi})$, we have the inequality,

$$\sum\nolimits_{s\in\Omega} u_s^{\text{B}}(x_s) = \sum\nolimits_{s\in\Phi} u_s^{\text{pess}}(x_s) + \sum\nolimits_{s\in\Omega\setminus\Phi} u_s^{\text{opt}}(x_s) \ge \sum\nolimits_{s\in\Phi} u_s^{\text{pess}}(y_s) + \sum\nolimits_{s\in\Omega\setminus\Phi} u_s^{\text{opt}}(y_s) = 0.$$

For 'if', suppose that $\sum_{s\in\Omega} u_s^{\text{B}}(x_s) \ge 0$. Then $u_s^{\text{B}}(x_s) = u_s^{\text{pess}}(x_s)$ for $s \in \Phi$, $u_s^{\text{B}}(x_s) = u_s^{\text{opt}}(x_s)$ for $s \in \Omega\setminus\Phi$, and hence $\sum_{s\in\Phi} u_s^{\text{pess}}(x_s) + \sum_{s\in\Omega\setminus\Phi} u_s^{\text{opt}}(x_s) \ge 0$. Let $\succsim \in \mathcal{P}$. We show below that there is a representation $(u_s)_{s\in\Omega}$ of $\succsim$ such that $u_s(x_s) \ge u_s^{\text{pess}}(x_s)$ for $s \in \Phi$ and $u_s(x_s) \ge u_s^{\text{opt}}(x_s)$ for $s \in \Omega\setminus\Phi$. Given that result, $\sum_{s\in\Omega} u_s(x_s) \ge 0 = \sum_{s\in\Omega} u_s(y_s)$. Hence $x \trianglerighteq y$.

Suppose $\hat{s} \in \Omega\setminus\Phi$ and $s' \in \Phi$. Let $(\bar{u}_s)_{s\in\Omega}$ be a representation of $\succsim$ such that $\bar{u}_s(y_s) = 0$ for each $s \in \Omega$. Due to diversity, Definition 3.1, and Lemma 2, $\frac{Du_{\hat{s}}^{\text{opt}}(0)}{D_-u_{s'}^{\text{pess}}(z_{s'})} \ge \frac{D\bar{u}_{\hat{s}}(0)}{D_-\bar{u}_{s'}(z_{s'})}$ for any $z_{s'} > 0$ and hence, by the final part of Lemma 2, $\frac{Du_{\hat{s}}^{\text{opt}}(0)}{Du_{s'}^{\text{pess}}(0)} \ge \frac{D\bar{u}_{\hat{s}}(0)}{D\bar{u}_{s'}(0)}$. Thus $\frac{Du_{s'}^{\text{pess}}(0)}{D\bar{u}_{s'}(0)}$

is well-defined. Letting $\tilde{s} = \text{argmax}_{s\in\Phi} \frac{Du_s^{\text{pess}}(0)}{D\bar{u}_s(0)}$, define $(u_s)_{s\in\Omega} = \frac{Du_{\tilde{s}}^{\text{pess}}(0)}{D\bar{u}_{\tilde{s}}(0)}(\bar{u}_s)_{s\in\Omega}$. Then, for each $s \in \Phi$, $1 = \frac{Du_{\tilde{s}}^{\text{pess}}(0)}{Du_{\tilde{s}}(0)} \geq \frac{Du_s^{\text{pess}}(0)}{Du_s(0)}$ and hence $Du_s^{\text{pess}}(0) \leq Du_s(0)$. Since, for $s \in \Phi$, $\succsim^{\Omega\setminus\{s\}}$ has lesser s-midpoints than $\succsim$, u_s^{pess} is more concave than u_s by Proposition 3. Hence $D_+u_s^{\text{pess}}(x'_s) \leq D_+u_s(x'_s)$ for all $x'_s \geq 0$. Since in addition $u_s^{\text{pess}}(y_s) = u_s(y_s)$, the proof of Theorem 1 implies that $u_s^{\text{pess}}(x'_s) \leq u_s(x'_s)$ for all $x'_s \geq y_s$. Now let $s \in \Omega\setminus\Phi$. Since we have shown that $\frac{Du_s^{\text{opt}}(0)}{Du_{\tilde{s}}^{\text{pess}}(0)} \geq \frac{D\bar{u}_s(0)}{D\bar{u}_{\tilde{s}}(0)}$, $Du_s^{\text{opt}}(0) \geq \frac{Du_{\tilde{s}}^{\text{pess}}(0)}{D\bar{u}_{\tilde{s}}(0)} D\bar{u}_s(0) = Du_s(0)$. Since $\succsim$ has lesser s-midpoints than $\succsim^{\{s\}}$ and hence u_s is more concave than u_s^{opt}, $D_+u_s^{\text{opt}}(x'_s) \geq D_+u_s(x'_s)$ for all $x'_s \geq 0$. Since $u_s^{\text{opt}}(y_s) = u_s(y_s)$, the proof of Theorem 1 now implies that $u_s^{\text{opt}}(x'_s) \leq u_s(x'_s)$ for all $0 \leq x'_s \leq y_s$.

To finish, we confirm that the u_s^{B} are behavioral value functions. For the $\succsim_s^{\text{sconcave}}$ and $\succsim_s^{\text{sconvex}}$ given by diversity, let u_s^{sconcave} and u_s^{sconvex} respectively be the utilities for s given by some pair of representations of $\succsim_s^{\text{sconcave}}$ and $\succsim_s^{\text{sconvex}}$. Due to Proposition 2, u_s^{sconvex} is strictly convex and u_s^{sconcave} is strictly concave. Since for any $(u_s)_{s\in\Omega}$ that represents some $\succsim \in \mathcal{P}$, u_s is more concave than u_s^{opt}, u_s^{sconvex} is more concave than u_s^{opt} and hence u_s^{opt} is strictly convex. Similarly u_s^{pess} is more concave than u_s^{sconcave} and hence u_s^{pess} is strictly concave.

To show that $D_-u_s^{\text{B}}(y_s) > D_+u_s^{\text{B}}(y_s)$ for each $s \in \Omega$, fix some $\hat{s} \in \Omega$ and a partition Φ with $\hat{s} \in \Omega\setminus\Phi$. As we have seen, for $\succsim \in \mathcal{P}$ and a representation $(\bar{u}_s)_{s\in\Omega}$ of $\succsim \in \mathcal{P}$ that satisfies $u_s(y_s) = 0$ for each $s \in \Omega$, if we set $\tilde{s} = \text{argmax}_{s\in\Phi} \frac{Du_s^{\text{pess}}(0)}{D\bar{u}_s(0)}$ and define $(u_s)_{s\in\Omega} = \frac{Du_{\tilde{s}}^{\text{pess}}(0)}{D\bar{u}_{\tilde{s}}(0)}(\bar{u}_s)_{s\in\Omega}$ then $Du_{\hat{s}}^{\text{opt}}(0) \geq Du_{\hat{s}}(0)$. Setting $\succsim = \succsim^{\Omega\setminus\{\hat{s}\}}$ we have $D_+u_{\hat{s}}^{\text{opt}}(0) \geq D_+u_{\hat{s}}^{\text{pess}}(0)$. Due to the strict convexity of $u_{\hat{s}}^{\text{opt}}$ or the strict concavity of $u_{\hat{s}}^{\text{pess}}$, $D_+u_{\hat{s}}^{\text{opt}}(y_{\hat{s}}) > D_+u_{\hat{s}}^{\text{pess}}(y_{\hat{s}})$. Similarly, $D_-u_{\hat{s}}^{\text{opt}}(y_{\hat{s}}) > D_-u_{\hat{s}}^{\text{pess}}(y_{\hat{s}})$ and, due to the concavity of $u_{\hat{s}}^{\text{pess}}$, $D_-u_{\hat{s}}^{\text{pess}}(y_{\hat{s}}) \geq D_+u_{\hat{s}}^{\text{pess}}(y_{\hat{s}})$. Thus $D_-u_{\hat{s}}^{\text{opt}}(y_{\hat{s}}) > D_+u_{\hat{s}}^{\text{pess}}(y_{\hat{s}})$.

Proof of Lemma 1. Due to the diversity assumption, Definitions 2 and 3.2 and Proposition 2, there is a $\succsim \in \mathcal{P}$ that is represented by a $(u_{s'})_{s'\in\Omega}$ such that u_s is a convex function. Due to Definition 3.1, u_s has lesser s-midpoints than $\hat{u}_s^{\text{opt}}[\{s\}]$ and lesser s-midpoints that each $\hat{u}_{s'}^{\text{opt}}[\Omega\setminus\{s\}]$ with $s' \neq s$. So, by Proposition 3, u_s is more concave than $\hat{u}_s^{\text{opt}}[\{s\}]$ or $\hat{u}_{s'}^{\text{opt}}[\Omega\setminus\{s\}]$ and hence $\hat{u}_s^{\text{opt}}[\{s\}]$ and $\hat{u}_{s'}^{\text{opt}}[\Omega\setminus\{s\}]$ are convex. The concavity claims are proved similarly.

Proof of Lemma 2. Since, by diversity, the marginal rate of substitution between s and s' is greater for the preference given by Definition 3.1 when $\Theta = \{s\}$ than for $\succsim$, $\lim_{x_s \to 0+} \frac{z_{s'} - x_{s'}(x_s, \succsim)}{x_s}$ exists. By Lemma 1, $\hat{u}_{s'}^{\text{opt}}[\{s'\}]$ is convex, it is increasing by assumption, and diversity implies that there is an increasing concave f such that $u_{s'} = f \circ \hat{u}_{s'}^{\text{opt}}[\{s'\}]$. The functions $\hat{u}_{s'}^{\text{opt}}[\{s'\}]$ and f are therefore left differentiable at $z_{s'}$ and $\hat{u}_{s'}^{\text{opt}}[\{s'\}](z_{s'})$ respectively, and $D_{-}\hat{u}_{s'}^{\text{opt}}[\{s'\}](z_{s'})$ and $D_{-}f(\hat{u}_{s'}^{\text{opt}}[\{s'\}](z_{s'}))$ are strictly positive and finite. Since in addition both f and $\hat{u}_{s'}^{\text{opt}}[\{s'\}]$ are increasing, the chain rule implies that $u_{s'}$ is left differentiable at $z_{s'}$ and that $D_{-}u_{s'}(z_{s'})$ is strictly positive and finite. Since $\lim_{x_s \to 0+} \frac{z_{s'} - x_{s'}(x_s, \succsim)}{x_s}$ exists, $x_{s'}(\cdot, \succsim)$ is differentiable at 0, and, $\lim_{x_s \to 0+} \frac{z_{s'} - x_{s'}(x_s, \succsim)}{x_s} = -Dx_{s'}(0, \succsim)$. Given that $x_{s'}(\cdot, \succsim)$ is decreasing and $u_{s'}$ is left differentiable at $z_{s'}$, the chain rule implies $u_{s'}(x_{s'}(\cdot, \succsim))$ is differentiable at 0. Since $u_s(x_s) = -u_{s'}(x_{s'}(x_s, \succsim)) + u_{s'}(z_{s'}) + u_s(0)$ for x_s near 0, $u_s(\cdot)$ is differentiable at 0, $Du_s(0) = -D_{-}u_{s'}(z_{s'})\, Dx_{s'}(0, \succsim)$, and given that $D_{-}u_{s'}(z_{s'}) > 0$, $-Dx_{s'}(0, \succsim) = \frac{Du_s(0)}{D_{-}u_{s'}(z_{s'})}$. For $\lim_{z_{s'} \to 0+} \frac{Du_s(0)}{D_{-}u_{s'}(z_{s'})} = \frac{Du_s(0)}{Du_{s'}(0)}$, apply Rockafellar (1970), Theorem 24.1, to f and $\hat{u}_{s'}^{\text{opt}}[\{s'\}]$ to conclude that $\lim_{z_{s'} \to 0+} D_{-}u_{s'}(z_{s'}) = Du_{s'}(0)$. ■

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