A Solution to the Global Identification Problem in DSGE Models

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Outline

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Motivation

- Identification problem in linear, Gaussian state-space models is well understood
- Identification problem in linearized DSGE models is still unresolved
  - DSGE model has a state-space form <u>after</u> it is solved
  - Link between deep parameters in DSGE models and its state-space form is not analytical
- Since lack of identification is a faulty design of any underlying model, it creates problems for
  - Estimation
  - Drawing conclusions from structural models

Literature

- Local identification:
  - Iskrev (2010), Komunjer & Ng (2011), Qu & Tkachenko (2012)
- Global identification - special cases:
  - Fukac, Waggoner & Zha (2007), Morris (2013)
- Global identification - general approaches:
  - Qu & Tkachenko (2017, 2022), Kocięcki & Kolasa (2018)
  - No sufficient conditions for global identification
  - Only numerical algorithms to search for observationally equivalent parameter sets

This paper

- Still no sufficient conditions for global identification
- **A constructive proof whether the model is globally identified or not**
- Using methods from algebraic geometry (Gröbner basis)
- No need to use numerical algorithms to search for observationally equivalent sets of parameters
- Can handle non-square systems and indeterminacy

Outline

Model setup

Linearized DSGE model

$$\Gamma_0(\theta)\begin{bmatrix} s_t \\ p_t \end{bmatrix} = \Gamma_1(\theta)E_t\begin{bmatrix} s_{t+1} \\ p_{t+1} \end{bmatrix} + \Gamma_2(\theta)s_{t-1} + \Gamma_3(\theta)\varepsilon_t$$

where:

- s_t is vector of states
- p_t is vector of policy variables
- $\Gamma_i(\theta)$'s are matrices that **explicitly** depend on deep model parameters $\theta \in \Theta$
- $\varepsilon_t \sim N(0, \Sigma(\theta))$ are exogenous variables (innovations to structural shocks, measurement errors, sunspot shocks)

Model solution

- A stable model solution can be written as

$$s_t = A(\theta)s_{t-1} + B(\theta)\varepsilon_t$$

$$p_t = F(\theta)s_{t-1} + G(\theta)\varepsilon_t$$

where $A(\theta)$, $B(\theta)$, $F(\theta)$, $G(\theta)$ are matrices that **implicitly** depend on deep model parameters θ

- From now on: $A(\theta) := A$, $A(\bar{\theta}) := \bar{A}$, etc.

ABCD-representation

- Using measurement equation for observables y_t

$$y_t = H \begin{bmatrix} s_t \\ p_t \end{bmatrix} + J\varepsilon_t$$

- We arrive at the ABCD-representation of the linearized DSGE model

$$s_t = As_{t-1} + B\varepsilon_t$$

$$y_t = Cs_{t-1} + D\varepsilon_t$$

Assumptions

Assumption 1

(Stability) A is a stable matrix for all $\theta \in \Theta$.

- Define $N = APC' + B\Sigma D'$, where $P := E(s_t s_t')$

Assumption 2

(Stochastic minimality) For all $\theta \in \Theta$, $O = [C' \vdots A'C' \vdots A'^2 C' \vdots \dots \vdots A'^{n-1} C']'$ has full column rank and $K = [N \vdots AN \vdots A^2 N \vdots \dots \vdots A^{n-1} N]$ has full row rank n (number of states).

- Generalization and unification of the framework used by Komunjer & Ng (2011) and Kocięcki & Kolasa (2018)

Observational equivalence

- We deal with a stationary Gaussian environment
- We leave out intercept in the measurement equation (no loss of generality)
- This allows us to define observational equivalence by using only second moments
- Define the z-spectrum of the ABCD-representation as $\Phi(z) = H(z)\Sigma H'(z^{-1})$, where $H(z) = D + C(z\mathrm{I}_n - A)^{-1}B$

Definition 1

θ and $\bar{\theta}$ are observationally equivalent ($\theta \sim \bar{\theta}$) if $\Phi(z) = \bar{\Phi}(z)$ for all $z \in \mathbb{C}$ in an open annulus containing the unit circle.

Identification of ABCD-representation

Theorem 1

$\theta \sim \bar{\theta}$ iff
1) $\bar{A} = TAT^{-1}$,
2) $\bar{C} = CT^{-1}$,
3) $AQA' - Q = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} - B\Sigma B'$,
4) $CQC' = \bar{D}\bar{\Sigma}\bar{D}' - D\Sigma D'$,
5) $AQC' = T^{-1}\bar{B}\bar{\Sigma}\bar{D}' - B\Sigma D'$,
for some nonsingular matrix T and symmetric matrix Q.

From ABCD-representation to DSGE (from ABCD to θ)

- So far we have stated results for ABCD-representation, which is not analytically linked to deep parameters θ
- Relying only on Theorem 1 would imply the need to solve for all candidate $\bar{\theta} \in \Theta$, and check if $\theta \sim \bar{\theta}$ is a singleton or not
- We follow Kocięcki & Kolasa (2018):
    - Develop the identification condition by combining Theorem 1 with restrictions imposed on the DSGE model in its original form
    - This results in a system of nonlinear equations in $\bar{\theta}$ (and auxiliary "unknowns"), which after some rearrangement becomes the **set of polynomials**

Global identification

Definition 2

A linearized DSGE is globally identified at θ iff all admissible solutions to the system below are such that $\bar{\theta} = \theta$

$$C = \bar{H}^s T A + \bar{H}^p \bar{F}$$

$$\bar{D} = \bar{H}^s T \bar{B} + \bar{H}^p \bar{G} + \bar{J}$$

$$\bar{\Gamma}_0^s T A + \bar{\Gamma}_0^p \bar{F} - \bar{\Gamma}_1^s T A^2 - \bar{\Gamma}_1^p \bar{F} A = \bar{\Gamma}_2 T$$

$$\bar{\Gamma}_1^s T A \bar{B} + \bar{\Gamma}_1^p \bar{F} \bar{B} - \bar{\Gamma}_0^s T \bar{B} + \bar{\Gamma}_3 = \bar{\Gamma}_0^p \bar{G}$$

$$A Q A' - Q = \bar{B} \bar{\Sigma} \bar{B}' - B \Sigma B'$$

$$A Q C' = \bar{B} \bar{\Sigma} \bar{D}' - B \Sigma D'$$

$$C Q C' = \bar{D} \bar{\Sigma} \bar{D}' - D \Sigma D'$$

- Dependence on $\bar{\theta}$: explicitly: $\bar{\Gamma}_0^s$, $\bar{\Gamma}_0^p$, $\bar{\Gamma}_1^s$, $\bar{\Gamma}_1^p$, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$, $\bar{\Sigma}$, $\bar{H}^s$, $\bar{H}^p$, $\bar{J}$, implicitly: $\bar{B}$, $\bar{D}$, $\bar{F}$, $\bar{G}$, plus T, Q. Those in black are fixed numbers (for given θ).

Outline

Gröbner basis in a nutshell

- Economics literature: Kubler & Schmedders (2010a, 2010b), Datta (2010), Kubler et al. (2014), Foerster et al. (2016)
- Calculating a Gröbner basis is analogous to Gaussian elimination in systems of linear equations
- The Gröbner basis makes the polynomial system "triangular"
- The solution set of the initial polynomials is the same as its Gröbner basis
- Exact arithmetics to get the Gröbner basis (floating point approximation used only to get observationally equivalent points in numerical terms)
- Conveys a lot of information: Is there a finite number of solutions? How many?
- If infinity of solutions (i.e. local nonidentification), it allows for analytical insight what parameters should be fixed to achieve global identification
- Specialized, highly efficient software available - we use SINGULAR

Gröbner basis in toy examples

Example

$$\begin{cases} x^5 + y^2 + z^2 = 4 \\ x^2 + 2y^2 = 5 \\ xz = 1 \end{cases} \Rightarrow \text{Gröbner basis: } \begin{cases} z^7 - \frac{3}{2}z^5 - \frac{1}{2}z^3 + 1 = 0 \\ {\color{red}y^2} - \frac{1}{2}z^5 + \frac{3}{4}z^3 + \frac{1}{4}z - \frac{5}{2} = 0 \\ {\color{blue}x} + z^6 - \frac{3}{2}z^4 - \frac{1}{2}z^2 = 0 \end{cases}$$

Example

$$\begin{cases} x^2 - y^2 + x + y - z = 0 \\ x^2 + 2y^2 - 2x + y - z = 0 \\ x^3 - x^2z - xy^2 - 2y^2z + x^2 + xy + xz - yz + z^2 = 0 \end{cases} \Rightarrow$$

Gröbner basis: $\begin{cases} {\color{red}y^4} + {\color{red}y} - z = 0 \\ x - {\color{red}y^2} = 0 \end{cases}$

Outline

An-Schorfheide (AS) model

- An and Schorfheide (2007) + shock correlation as in Herbst and Schorfheide (2016)

$$x_t = E_t x_{t+1} + g_t - E_t g_{t+1} - \frac{1}{\tau}(R_t - E_t \pi_{t+1} - E_t z_{t+1})$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(x_t - g_t)$$

$$R_t = \rho_m R_{t-1} + (1 - \rho_m)[\psi_1 \pi_t + \psi_2(x_t - g_t)] + \varepsilon_{m,t}$$

$$z_t = \rho_z z_{t-1} + \rho_{zg} g_{t-1} + \varepsilon_{z,t}$$

$$g_t = \rho_g g_{t-1} + \rho_{gz} z_{t-1} + \varepsilon_{g,t}$$

where $s_t = [z_t \; g_t \; R_t]'$, $p_t = [x_t \; \pi_t]'$, $y_t = [R_t \; x_t \; \pi_t]'$

Point at which we check identification

Parameter	Value	Parameter	Value
τ	2	ρ_{zg}	0.1
β	0.9975	ρ_{gz}	-0.075
κ	0.33	ρ_m	0.75
ψ_1	1.5	σ_z	0.3
ψ_2	0.125	σ_g	0.6
ρ_z	0.9	σ_m	0.2
ρ_g	0.95		

Gröbner basis

$$
\begin{aligned}
0 &= u^2 - 1.8697u + 0.8697 \\
&\dots \\
\bar{\rho}_z &= 0.9155 - 0.0155u \\
\bar{\rho}_{zg} &= 0.2415 - 0.1415u \\
\bar{\rho}_g &= 0.9345 + 0.0155u \\
\bar{\rho}_{gz} &= 0.0209 - 0.0959u \\
\bar{\tau} &= 2 \\
\bar{\beta} &= 0.5351 + 0.4624u \\
\bar{\kappa} &= 0.4912 - 0.1612u \\
\bar{\rho}_m &= 0.75 \\
\bar{\psi}_1 &= 1.3131 + 0.1869u \\
\bar{\psi}_2 &= 0.2516 - 0.1266u \\
\bar{v}_z &= 0.1279 - 0.0379u \\
\bar{v}_g &= 0.3128 + 0.6728u \\
\bar{v}_m &= 0.04
\end{aligned}
$$

- The first line implies $u = 1$ or $u = 0.8697$
- For $u = 1$ we recover θ (point at which we check identification)
- For $u = 0.8697$ we get $\bar{\theta} \neq \theta$ (observationally equivalent parameter vector)
- **Conclusion: model is locally, but not globally identified at θ**

Gröbner basis for AS model with $\rho_{gz} = \rho_{zg} = 0$

$$
\begin{aligned}
&\dots \\
\bar{\rho}_z &= 0.9 \\
\bar{\rho}_g &= 0.95 \\
\bar{\tau} &= 2 \\
\bar{\beta} &= 0.9975 \\
\bar{\kappa} &= 0.33 \\
\bar{\rho}_m &= 0.75w \\
\bar{\psi}_1(w - 1.3333) &= 3.7211w - 4.2211 \\
\bar{\psi}_2 &= 2.7302 - 1.7368\bar{\psi}_1 \\
\bar{v}_z &= 0.09 \\
\bar{v}_g &= 0.36 \\
\bar{v}_m &= 0.04w^2
\end{aligned}
$$

- No restrictions on w other than $\bar{\theta} \in \Theta$
- **Conclusion: model is locally unidentified at θ**
- **Additional information: explicitly given set of parameters observationally equivalent to θ**

Outline

Conclusions

- Generalize identification framework in Komunjer & Ng (2011)
- Apply Gröbner basis to find **all** observationally equivalent points (constructive proof of identification)
- Accommodate indeterminacy (see the paper)
- Able to deal with medium-scale DSGE models
- Observational equivalence in medium-sized DSGE models might be not so widespread
- Potential to solve global identification problem in many other models, for which the sufficient condition doesn't exist yet (e.g. Gaussian SVAR with inhomogenous restrictions), or is the "Holy Grail" (e.g. Gaussian Affine Term Structure Model)