Unobserved Clusters of Time-Varying Heterogeneity in Nonlinear Panel Data Models*

Martin Mugnier†

August 21, 2022

(Preliminary draft, please do not circulate)

Abstract

In non-experimental longitudinal studies, researchers often estimate causal effects assuming time-constant unobserved heterogeneity or linear-in-parameters conditional expectations. Violation of these assumptions may lead to poor counterfactuals. I study the identification and estimation of a class of nonlinear grouped fixed effects models where the relationship between observed covariates and cross-sectional unobserved heterogeneity is left unrestricted but the latter only takes a restricted number of paths over time. To identify the corresponding "clusters" and common parameters of the model, I consider a two-step method. First, I transform the NP-hard clustering problem into an individuals-pairing problem and recover the latent clustering under an injectivity condition. Second, I rely on within- and across cluster variation in the observed covariates and a monotonicity property to infer the remaining infinite-dimensional parameters. I propose a practically useful semiparametric maximum likelihood estimator whose implementation is feasible and establish its large sample properties in popular binary and count outcome models (including Probit, Logit, Poisson). Distinctive features of the estimator are that it is higher-order unbiased and it allows the number of periods to grow slowly with the number of cross-sectional units. Monte Carlo simulations suggest good finite sample performance. I apply this new method to revisit Aghion, Bloom, Blundell, Griffith, and Howitt (2005)'s inverted-U relationship between product market competition and innovation.

Keywords: nonparametric identification, semiparametric estimation, fixed effects, nonlinear panel data models, clustering, time-varying heterogeneity, M-estimation.

JEL Codes: C14, C23, C25.

*I am deeply grateful to my Ph.D. supervisor Xavier D'Haultfœuille for his continuous support and being a source of inspiration for the choice of this topic. Special thanks go to Stéphane Bonhomme, Victor-Emmanuel Brunel, and Ao Wang for invaluable advice and guidance. I am also thankful to Arnak Dalalyan, Laurent Davezies, Jim Heckman, Francis Kramarz, Elena Manresa, Isabelle Méjean, Pauline Rossi, Anna Simoni, Anthony Strittmatter, Michael Visser, Andrei Zeleneev, and seminar participants at CREST, the University of Chicago, 2022 NASMES in Miami, and 2021 EWMES in Barcelona for their comments. All remaining errors are my own.

†Ph.D. Candidate in Economics, CREST, ENSAE, Institut Polytechnique de Paris. Email: martin.mugnier@ensae.fr. Website: https://martinmugnier.github.io. This research is supported by several grants of the French National Research Agency (ANR): "Investissements d'Avenir" (LabEx Ecodec/ANR-11-LABX-0047 and EUR DATA EFM/ANR-18-EURE-0005) and Otelo (ANR-17-CE26-0015-041).

1 Introduction

Unobserved heterogeneity is a prevalent feature of most reduced-form and structural work in economics and other social sciences. As "individuals are not inert and interchangeable particles of matter, but discernible and discerning agents",[1] observational outcomes (e.g., consumption choices, mental health, number of patents) and explanatory variables of interest (e.g., income, abortion, product market competition) typically correlate over time with factors unobserved to the researcher (e.g., Bourdieu's habitus, risky behaviors, technological change). This so-called confounding problem renders identification of average partial effects and counterfactuals difficult.[2]

By sampling N individuals at T different points in time, panel data offer opportunities to account for latent structures embedded in low-dimensional manifolds (see, e.g., Bai, 2009; Bonhomme, Lamadon, and Manresa, 2022; Hsiao, 2015; Moon and Weidner, 2019; Wooldridge, 2010).[3] While random effects approaches specify a parametric family for the conditional distribution of the unobserved heterogeneity given the covariates, fixed effects leave this distribution unrestricted at the cost of introducing many additional parameters. A celebrated example is pooled linear regression with additively separable individual and time effects, which has been widely used to model workers, firms, or countries' permanent unobserved heterogeneity and *common trend* in labor and international trade (Abowd, Kramarz, and Margolis, 1999; Helpman, Melitz, and Rubinstein, 2008).

In many cases, however, not only a nonlinear model arises naturally (e.g., discrete choice, point mass in outcome), it is likely that unobserved heterogeneity is time-varying and takes a clustering/grouping structure.[4] Individuals in the population partition into a moderate number of clusters such that members of each cluster share the same path of unobserved heterogeneity over time but the partition is unknown to the researcher (e.g., a few trajectories of social group transitions, risky behaviors, technological change). Missing external information about the clustering, the researcher is faced with the problems of classifying a large number of individuals into clusters and estimating a large number of cluster-specific time effects in large-N, T nonlinear panel models, where N and T diverge jointly to infinity.[5] First, little is known about the nonparametric identification of many

[1] Pierre Bourdieu, *La noblesse d'État. Grandes écoles et esprit de corps*, Paris, Les Éditions de Minuit, 1989.

[2] See, e.g., Abowd, Kramarz, and Margolis (1999); Angrist and Pischke (2009); Imbens and Rubin (2015).

[3] This echoes Occam's razor principle and the "manifold hypothesis" (Goodfellow, Bengio, and Courville, 2016).

[4] Altonji and Matzkin (2005) asserts: *"The linear probability model is biased in almost all circumstances."*; Athey and Imbens (2006) argues *"If an individual gains experience or just age over time, her unobserved skill (...) is likely to change."*. Discreteness assumptions are pervasive in economic modeling: see, among many others, Bonhomme, Lamadon, and Manresa (2019); Bonhomme and Manresa (2015); Deb and Trivedi (1997); Hahn and Moon (2010); Heckman and Singer (1984); Keane and Wolpin (1997); Vogt and Linton (2017).

[5] Such rectangular-array asymptotics have recently become increasingly popular given the growing availability of

popular nonlinear models widely used in empirical research (e.g., random utility binary/ordered choice models) under such clustered patterns of unobserved heterogeneity.[6] Second, available semi-parametric nonlinear fixed effects estimators tend to perform quite poorly, as shown in Monte Carlo experiments (see Section 6). Most are based on partialling-out or joint maximum likelihood estimation and generally fail to provide, as T grows much slower than N, asymptotically normal centered estimates of the typical common slope parameter (resp. time-varying paths of unobserved heterogeneity) at the parametric root-NT (resp. root-N) rate and uniformly (across all pair of individuals) consistent estimates for the cluster memberships (see, e.g., Arellano and Hahn, 2007; Bonhomme, Lamadon, and Manresa, 2022; Chamberlain, 1980; Charbonneau, 2017; Chen, Fernández-Val, and Weidner, 2021; Fernández-Val and Weidner, 2016; Hahn and Moon, 2010; Hahn and Newey, 2004; Rasch, 1960). These limitations are important because common and fixed effects parameters, as well as the distribution of idyosyncratic error terms (e.g., random shocks in taste), are building blocks for estimating counterfactual events and policy relevant parameters (e.g., average causal effects), and T is often moderately large compared to N in practice.

In this paper, I address both issues by introducing a new class of nonlinear grouped fixed effects (NGFE hereafter) static models for discrete outcomes. Three defining features are: (i) individuals with the same unobserved time-invariant cluster membership share the same path of unobserved heterogeneity across time; (ii) the conditional distribution of cluster memberships and cluster-specific time-varying effects given observed covariates is left unrestricted (thus allowing for flexible selection patterns); (iii) and observed covariates and cluster effects enter each individual's conditional choice probability as a single index mapped to the outcome by an unknown link function.

First, I propose a novel identification strategy and prove, under low-level conditions, point identification of all parameters, contrasting with most identification results in the fixed-T setting. The proof is constructive and relies on two steps. I start by transforming the NP-hard clustering problem into countably many individuals-pairing testing problems and rely on an injectivity condition à la Bonhomme, Lamadon, and Manresa (2022) (see their Assumption 2) to build test functions which identify the latent clustering by comparing conditional probability functions identified from the time series dimension of the data. In particular, I show that the injectivity condition holds if,

high-frequency data (e.g., scanner, financial data). See, among others, Arellano and Hahn (2007); Chen, Fernández-Val, and Weidner (2021); Dhaene and Jochmans (2015); Fernández-Val and Weidner (2016); Hahn and Newey (2004).

[6] Obviously, the increasing time dimension should allow to identify many parameters of interest compared to standard fixed-T panels, in whichs few parameters are generally point identified outside specific cases (see, e.g., Chamberlain, 2010; Davezies, D'Haultfoeuille, and Mugnier, 2021).

for instance, there is continuous local variation in a "special" regressor (not necessarily with large support) and the link function is real-analytic (see, e.g., Krantz and Parks, 2002).[7] Given identification of the latent clustering, I alleviate within-cluster variation and apply a well-known result by Ichimura (1993) to obtain identification of the common slope coefficient up to scale. Then, I rely on compensating variations of single-indices within and across clusters together with a monotonicity property to infer the remaining infinite-dimensional parameters. All in all, the identification results pave the way for estimation of the link function (e.g., distribution of random shocks).

Second, I develop simple NGFE semiparametric estimators and establish their asymptotic properties. I first introduce a general M-estimation framework to estimating nonlinear models with clusters of time-varying unobserved heterogeneity. Semiparametric NGFE estimators are obtained by specializing the framework to models with a known link function and a finite number of clusters: they maximize the likelihood of the data conditional on the clustering and time-effects.[8] Importantly, the method does not require any tuning parameter (because the number of clusters is known) but can still be computationally cumbersome in large samples. I propose a heuristic Lloyd (1982)'s algorithm described in Section 4.3, and show that it performs well in various Monte Carlo experiments with moderate sample sizes and number of clusters (see Section 6). From a practical viewpoint, and in contradistinction with popular fixed effects estimators such as Chamberlain (1980) or Charbonneau (2017)'s conditional Logit, NGFE estimators are not confined to the specific and restrictive case of time-variant regressors, nor do they drop individuals without any variation in outcome, thus exploiting the full sample variation. Moreover, compared to Bonhomme, Lamadon, and Manresa (2022)'s 2-step GFE estimator, they have only one optimization step and maintain the discreteness assumption. From a theoretical viewpoint, I show that the latter suffices to restore a rich asymptotic theory alike that of linear GFE estimators developed in Bonhomme and Manresa (2015). As these authors, to study the theoretical properties of NGFE estimators, I focus on the statistical properties of the exact NGFE estimates and abstract from optimization errors stemming from the non-convex and non-smooth objective function and the underlying NP-hard combinatorial problem they require to solve.[9] In a companion paper, I adress the latter problem and develop

[7] Special regressors are widely used in econometrics (e.g., Candelaria, 2020; Honoré and Lewbel, 2002). There is a trade-off between imposing (i) analyticity of the link function which allows to interpolate from bounded variation in the regressors at the cost of a strong functional form assumption and (ii) the existence of a special regressor with unbounded support. Relaxing both conditions at once seems challenging (see, e.g., Gaillac and Gautier, 2021).

[8] In Mugnier (2022), I relax the asumption that the number of groups is known. Results there apply to linear models only but I have been able to extend them for a class of nonparametric directed network nonlinear models.

[9] Investigating the impact of optimization errors on subsequent inference based on NP-hard infeasible exact solutions seems an interesting but difficult avenue for future research.

computationally trivial estimators based on an agglomerative clustering rule (see Mugnier, 2022).

I derive the statistical properties of semiparametric NGFE estimators when the number of clusters, in addition to being known to the researcher, does not grow with the size of the panel (similarly to Bonhomme and Manresa, 2015) and by taking semiparametric binary choice models as a leading example. Under well-separation of clusters and a noncollinearity condition, NGFE estimators of the slope coefficient and cluster-specific effects are consistent as N and T diverge jointly to infinity. The results heavily draw on proof arguments used in Bonhomme and Manresa (2015), and the observation that strong concavity of the log-likelihood function is sufficient to extend some of their results by mean of Taylor expansions. Estimated cluster membership enjoy the "perfect recovery" property: provided T grows at least as some power of N, the misclassification probability tends to zero uniformly across individuals.[10] As in the linear case, this implies that, under additional regularity conditions, NGFE estimators of the slope and cluster-specific effects are asymptotically equivalent to the infeasible oracle MLE based on knowledge of the clustering. When $T = o(N)$, this oracle is asymptotically unbiased so that standard MLE inference yields tests and confidence intervals with correct asymptotic level. When $N/T \to \kappa \in (0, +\infty)$, existing results can be applied to the oracle to derive analytical or jackknife bias correction methods for the slope and average marginal effects estimates.[11]

Third, I investigate the finite sample performance of NGFE estimators, as well as large-N, T estimators of their variance, by mean of Monte Carlo simulations. I compare the results with state-of-the-art competing methods (e.g., nonlinear two-way fixed effects, 2step-GFE). I find that NGFE estimators perform remarkably well in settings they are meant for. In particular, in a static logit model with clustered time-varying correlated unobserved heterogeneity, NGFE estimators have the smallest bias and RMSE compared to linear methods and nonlinear ones such as Bonhomme, Lamadon, and Manresa (2022)'s 2-step GFE, nonlinear two-way fixed effects, or Rasch (1960)'s CMLE. In a DGP without unobserved heterogeneity, the RMSE is of the same order as that of the CMLE, but NGFE estimators have a finite sample bias. The CPU time is similar to that of competing clustering methods. For a 90×7 data set (order of magnitude of the empirical application), it takes 10 seconds to compute on a generic professional laptop. However, NGFE estimators are much less noisy than 2-step GFE as they explicitely rely on the discreteness assumption. I obtain similar results in a dynamic setting including a lagged outcome as explanatory variable. Estimates

[10] A concentration inequality for martingale differences due to Lesigne and Volný (2001) is used to show this result.
[11] See, e.g., Hahn and Newey (2004), Arellano and Hahn (2007), and Chen, Fernández-Val, and Weidner (2021).

become less precise in settings with continuous (even time-invariant) unobserved heterogeneity.

Finally, I illustrate the practical usefulness of NGFE estimators by using this new approach to study whether and how product market competition (measured as one minus the Lerner index) affect innovation (measured as citation-weighted patents) in a panel of UK industries that spans the last part of the twentieth century (1973-1994), revisiting an influential paper by Aghion, Bloom, Blundell, Griffith, and Howitt (2005) published in the top 5 economic journal *The Quarterly Journal of Economics*. Challenging their nonlinear additively separable two-way fixed effects main specification, I find evidence of clustered time-varying unobserved heterogeneity, which results in a mildly inverted-U shape relationship and sheds new light on the unobserved mechanisms driving both market structure and technological change across time. Specifically, the data-driven clustering procedure reveals steady "high/low-innovation" clusters of industries as well as "caching-up" industries.

Overall, the theoretical results broaden the scope of application of GFE estimators and clustering techniques in econometrics, complementing the available toolbox for applied economists interested in assessing robustness of their results to specification choices (in particular when unobserved heterogeneity is plausibly clustered and time-varying). Results from the empirical applications confirm the usefulness of considering flexible specifications such as NGFE for modeling unobserved heterogeneity.

Related Literature This paper contributes to the literature on nonparametric identification of nonseparable panel data models, by providing new identification results in long nonlinear panel models while most previous papers from this literature have either assumed time-homogeneity conditions and fixed-T,[12] continuous outcomes,[13] relied on additive separability of the unobserved heterogeneity,[14] or specified parametrically the link function.[15] In contrast, by alleviating the large-T dimension and the single-index structure, I show that all parameters of NGFE models can be (nonparametrically) point-identified.

[12] See, in particular, Chernozhukov, Fernández-Val, Hahn, and Newey (2013), Evdokimov (2010), Evdokimov (2011), Hoderlein and White (2012), Botosaru and Muris (2017), Manski (1987) and Altonji and Matzkin (2005).

[13] See, e.g., Athey and Imbens (2006) and Freyberger (2018).

[14] See, e.g., Botosaru and Muris (2017) and Mugnier and Wang (2022). Differently from the additively separable case considered in Mugnier and Wang (2022), interactions between individual-specific (i.e., vector of group membership dummies) and time-specific (i.e., vector of cluster effects) effects complicates analysis, which requires new arguments on top of the compensating variation technique already used in that paper and D'Haultfoeuille, Hoderlein, and Sasaki (2021).

[15] Zeleneev (2020).

The second contribution of this paper is to propose a novel and convenient estimation method for semiparametric nonlinear panel data models with time-varying unobserved heterogeneity and derive its asymptotic properties. Most previous research in the large-N, large-T panel data literature has focused on factor-analytic type linear models while nonlinear models with multiple fixed-effects have only recently drawn considerable attention.[16] Fernández-Val and Weidner (2016), Graham (2017), and Charbonneau (2017) provide consistent and asymptotically normal semiparametric estimators of the homogeneous slope coefficient (as well as average partial effects in Fernández-Val and Weidner, 2016) in nonlinear two-way fixed effects models, assuming that unobserved heterogeneity is additively separable into individual-specific and time-specific components. Neither two-way fixed effects nonlinear models nor NGFE models are nested one into another and the two approaches should therefore seen as complementary. However, differently from NGFE estimators, Graham (2017) and Charbonneau (2017)'s conditioning estimators, by partialling out the unobservables, do not provide consistent estimates for them, and Fernández-Val and Weidner (2016) require $N/T \to \kappa \in (0, +\infty)$ to obtain statistical guarantees.

The closest papers to ours are Chen, Fernández-Val, and Weidner (2021), Bonhomme, Lamadon, and Manresa (2022), and a recent working paper by Ando and Bai (2022). Chen, Fernández-Val, and Weidner (2021) extend Fernández-Val and Weidner (2016)'s results to semiparametric nonlinear factor-analytic models under concavity conditions. When the link function is parametrically specified, NGFE models fall into their framework. Yet, Chen, Fernández-Val, and Weidner (2021) do not derive any formal nonparametric identification result and, because of their generality, also require $N/T \to \kappa \in (0, +\infty)$ and need bias correction methods to obtain correctly centered limiting distributions allowing for valid inference on slope coefficient and average marginal effects (but not on the latent factors). The two-step discretization approach of developed in Bonhomme, Lamadon, and Manresa (2022), albeit its remarkable generality, comes at a similar price. When heterogeneity is discrete, it ressembles a Lloyd's algorithm where the first clustering step would not take advantage of improvement on the other parameters (as noted by the author, the choice of moments is important in practice) but, different from the NGFE approach, it does not have yet any inference method. Moreover, Monte Carlo simulations suggest that relying directly on maximum likelihood (NGFE) is better in terms of bias and RMSE when unobserved heterogeneity is time-varying and discrete. Alternatively, NGFE estimators are asymptotically equivalent to the oracle MLE with

[16] For linear factor-type models, see, among many others, Bai (2003), Pesaran (2006), Bai (2009), Bonhomme and Manresa (2015), Moon and Weidner (2015), Moon and Weidner (2017), and Ando and Bai (2017). For nonlinear ones, see, e.g., Chen, Fernández-Val, and Weidner (2021) and Ando and Bai (2022).

known clusters which, itself, is shown asymptotically centered and normal at parametric rates (or for which bias-correction techniques might be available in the same flavour of Chen, Fernández-Val, and Weidner (2021)), and provide a parsimonious approximation if one is willing to assume discrete unobserved heterogeneity. Independently from this paper, Ando and Bai (2022) generalizes Bonhomme and Manresa (2015)'s semiparametric GFE estimator to an exponential family of nonlinear grouped factor models with heterogeneous coefficients (including Probit, Logit, Poisson). As in this paper, they consider the MLE and their results extend our NGFE estimator for semiparametric NGFE models with heterogeneous coefficients. Differently, their general framework imposes stronger restrictions (requires larger T in the asymptotics), delivers slower $\sqrt{T}$-rate for the slope coefficient estimates (v.s. $\sqrt{NT}$ for the NGFE estimate of the common slope), and they do not provide nonparametric identification results.

Some papers assume that clusters are known to the econometrician (see, e.g. Arkhangelsky and Imbens, 2018; Bester and Hansen, 2016). Many papers allow for clustered structure on the unobserved heterogeneity but otherwise impose time-invariant unobserved heterogeneity.[17] For instance, Hahn and Moon (2010) and Bonhomme and Manresa (2015), which focus respectively on discrete but time-invariant unobserved heterogeneity in general models and linear versions of NGFE models with time-varying unobserved heterogeneity, have been extended to some nonlinear models with time-invariant unobserved heterogeneity in Saggio (2012) and Cheng, Schorfheide, and Shao (2021). Yet, accounting for clustered patterns of time-varying unobserved heterogeneity in nonlinear models seems to be a difficult and less investigated problem that I address in this paper. In particular, NGFE estimators are a natural semiparametric extension of Bonhomme and Manresa (2015); Bryant and Williamson (1978); Hahn and Moon (2010); Saggio (2012)'s classification maximum likelihood estimators to cover the class considered in this paper and allow for nonlinearity and time-varying unobserved heterogeneity simultaneously. As the latter, they are based on optimal clustering of individuals given a M-estimation likelihood criterion. However, while the least-squares formulation of Bonhomme and Manresa (2015)'s GFE estimator and linearity allow many useful connections with clustering theory and, in particular, that of the *kmeans* algorithm (the GFE clustering estimate is based on k-means clustering of individuals' profiles of outcome net of the effects of the covariates), the binary outcome $Y_{it} = \mathbf{1}\{Y_{it}^* \geq 0\}$ in, e.g., a Logit or Probit NGFE latent utility model, is not linear in parameters and the latent variable $Y_{it}^* = X_{it}'\theta^0 + \alpha^0_{g_i^0 t} - \varepsilon_{it}$, although linear

[17] See, e.g., Hahn and Moon (2010), Su, Shi, and Phillips (2016), Gu and Volgushev (2019); Yu, Gu, and Volgushev (2022), Saggio (2012), Vogt and Linton (2017), and Cheng, Schorfheide, and Shao (2021).

in parameters, is not observed by the econometrician. Hence, the *kmeans* agorithm is not directly applicable to the within profiles of outcomes $\{Y_{it}^* : t\}$ net of the effect of covariates. Differently from us, a line of research put the grouping assumption on the unknown slope coefficient (heterogeneous models), letting again the unobserved heterogeneity individual-specific.[18]

The third strand of literature this paper contributes to is that of dimension reduction methods applied to nonlinear panel data models. A surge of papers have leveraged state-of-the-art statistical learning tools such as matrix completion devices and extensions of Tibshirani (1996)'s Least Absolute Shrinkage Estimator (LASSO) estimator to tackle the problem of estimating a large number of unobserved effects in parsimonious panel data models.[19] A common idea underlying these methods (as well as grouping/clustering techniques) is to exploit restrictions on the support of the unobserved heterogeneity, which echoes the concept of sparsity in high-dimensional statistics.[20]

Finally, this paper is related to a strand of the statistical literature concerned with the classical NP-hard problem of clustering (see, e.g., Forgy, 1965; Lloyd, 1982; MacQueen, 1967) and the closely related statistical concept of (nonparametric) finite mixtures models (see, e.g., McLachlan and Peel, 2000). In NGFE models, and conversely to classical EM approaches (see, e.g. Dempster, Laird, and Rubin, 1977; Sun, 2005), the probabilities to belong to each cluster are not restricted. In sharp contrast with nonparametric finite mixture approaches, where the underlying heterogeneity is usually continuous, NGFE models have an underlying discrete structure which is the object of interest. For each individual, a unique cluster-membership is estimated instead of a vector of probabilities (e.g., the Bayes predictor) to belong to each cluster (see, e.g. Bryant and Williamson, 1978, for a discussion). Yet, as in EM algorithms, it is important to acknowledge that the popular iterative Lloyd (1982)'s algorithm used to compute the NGFE estimator is subject to the problem of being attracted to local minima.

Organization The remainder of the paper is organized as follows. In Section 2, I introduce the class of NGFE models. The main identification result is presented in Section 3. In Section 4, I propose a general M-estimation framework, develop semiparametric NGFE estimators, and discuss

[18] See, Boneva, Linton, and Vogt (2015), Su, Shi, and Phillips (2016), Su, Wang, and Jin (2019), Gao, Xia, and Zhu (2020), Zhang, Wang, and Zhu (2019), Liu, Shang, Zhang, and Zhou (2020), and Wang and Su (2021).

[19] See, among others, Kock (2016), Moon and Weidner (2019), Zeleneev (2020), and Athey, Bayati, Doudchenko, Imbens, and Khosravi (2021).

[20] See, e.g., the monograph by Giraud (2014) for a thorough introduction to the topic. Note that "sparsity" of the unobserved heterogeneity is different from "sparsity" of common parameters, which distinguishes this literature from that focused on the use of the LASSO in panel data models with high-dimensional covariates or instruments (see, e.g., Belloni, Chen, Chernozhukov, and Hansen, 2012; Belloni, Chernozhukov, Hansen, and Kozbur, 2016).

their computation. Section 5 provides theoretical properties of NGFE estimators in semiparametric binary choice models. Section 6 presents Monte Carlo results. Section 7 contains the empirical application. Section 8 concludes. All proofs are collected in the appendix.

Notation For any set A, I let $A^* := A \setminus \{0\}$ and $|A|$ denote the cardinal of A. For any $(a,b) \in \mathbb{R}^2$, I let $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$. λ denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ collects the Borel sets on $\mathbb{R}$. The abbreviation "a.e." stands for "almost everywhere" (with respect to an appropriate measure). Let $\xrightarrow{d}$ and $\xrightarrow{p}$ denote convergence in distribution and convergence in probability respectively. For any sequence of random variables $\{U_n : n \in \mathbb{N}\}$ such that $U_n \xrightarrow{p} U$, let $\text{plim}_{n\to\infty} U_n := U$. $U_n = O_p(1)$ (resp. $o_p(1)$) means U_n is bounded in probability (resp. converges in probability to zero). $U_n = O_p(R_n)$ means that $U_n = R_n \times V_n$ with $V_n = O_p(1)$; $U_n = o_p(R_n)$ means that $U_n = R_n \times V_n$ with $V_n = o_p(1)$. Henceforth, I denote by $\text{Supp}(U)$ the support of any random variable U.

2 Nonlinear Discrete Outcome Models With Unobserved Clusters of Time-Varying Heterogeneity

Suppose to observe a balanced random sample of panel data $\{(Y_{it}, X_{it}')' : (i,t) \in \mathcal{N} \times \mathcal{T}\}$ with dimensions $N := |\mathcal{N}|$ and $T := |\mathcal{T}|$.[21] In many applications, $\mathcal{N}$ is an index for individuals or "units", and $\mathcal{T}$ indexes time periods or "unit members". I consider the problem of modeling, for individual $i \in \mathcal{N}$, the T-vector of discrete outcomes $Y_i = (Y_{it})_{t\in\mathcal{T}}'$ in relation with its $T \times p$ matrix of weakly exogeneous covariates $X_i = (X_{it}')_{t\in\mathcal{T}}'$. The dependent variable Y_{it} represent agents' (choice) decisions and X_{it} represent agents' attributes over time and it is often plausible that time-varying unobservables (to the econometrician) confound the "effect" of X_{it} on Y_{it}.[22]. For instance, in the empirical application, $Y_{it} \in \mathbb{N}$ denotes the number of patents produced by industry i at time t and X_{it} collects industry i's characteristics at time t such as the level of product market competition.

With this purpose, I introduce below a class of nonlinear clustered or "grouped" fixed effects

[21] Unbalanced panels can be accomodated easily under exogeneous attrition (i.e., missing-at-random). Endogeneous attrition is beyond the scope of this paper. Throughout the paper, I rule out undirected graph (or network or "pseudo-panel") data for which there is no proper $\mathcal{T}$ and observations are indexed by pairs of indices $(i,t) \in \mathcal{N}^2$ such that $(Y_{it}, X_{it}')' = (Y_{ti}, X_{ti}')'$ for all $(i,t) \in \mathcal{N}^2$. There is a vast literature on models of link formations and networks (see, e.g., de Paula, 2020, for a recent review).

[22] E.g., agents choose X_{it} depending on time-varying unobservables that also affect Y_{it} before idiosyncratic shocks are realized. One might also want to distinguish between state dependence and unobserved (time-varying) heterogeneity (see, e.g. Heckman, 1981).

(NGFE) models to flexibly incorporate time-varying patterns of unobserved heterogeneity. I let $\text{Supp}(Y_{it}, X_{it}) = \mathcal{Y} \times \mathcal{X}_i$ and assume that $\mathcal{Y} \subset \mathbb{R}$ is at most countable and $\mathcal{X}_i \subset \mathbb{R}^p$ for some fixed $p \in \mathbb{N}^*$. In its simplest version, individual $i \in \mathcal{N} := \{1, ..., N\}$ at time $t \in \mathcal{T} := \{1, ..., T\}$ chooses $Y_{it} \in \mathcal{Y}$ given her weakly exogeneous covariates $X_i^t := (X_{i1}', ...X_{it}')'$, her unobserved cluster membership variable $g_i^0 \in \mathcal{G}^0 := \{1, ..., G^0\}$, and unobserved time-varying cluster-specific effect $\alpha_{gt}^0 \in \mathcal{A} \subset \mathbb{R}$ such that, for all $y \in \mathcal{Y}$,

$$\Pr\left(Y_{it} = y \mid X_{i1}, ..., X_{it}, g_i^0, \alpha_{g_i^0 t}^0\right) = h^0\left(y, X_{it}'\beta^0 + \alpha_{g_i^0 t}^0\right), \tag{1}$$

where $\beta^0 \in \mathcal{B} \subset \mathbb{R}^p$ in an unknown fixed parameter of interest, $G^0 \in \mathbb{N}^*$ is unknown but "small" relative to N, and $h^0 \in \mathcal{H}$ is an unknown link function from the set

$$\mathcal{H} \subset \left\{ h : \mathcal{Y} \times \mathbb{R} \to (0,1) \text{ measurable}, \sum_{y \in \mathcal{Y}} h(y, \cdot) = 1, \text{ and } \sum_{y \in \mathcal{Y}} |y|\, h(y, \cdot) < \infty \text{ a.e.} \right\}.$$

The common parameter β^0 is often of key interest in applications (e.g., marginal utilities). Unobserved effects $(\alpha_{g_i^0 t}^0)_{t \geq 1}$ account for time-varying unobserved heterogeneity shared by all members of cluster g_i^0, i.e., all individuals $\left\{j : g_j^0 = g_i^0\right\}$, that might confound β^0 (i.e., arbitrarily correlated with X_{it}). The link function h^0 captures the conditional distribution of random idiosyncratic shocks in exogeneous latent variable utility choice models. The contemporaneous covariates X_{it} and the unobserved effect $\alpha_{g_i^0 t}^0$ enter the response function as the combination of a linear single-index $X_{it}'\beta^0 + \alpha_{g_i^0 t}^0$ and an unknown link function h^0.[23] Single index assumptions are common in the nonseparable panel data models literature and serve mainly computational and interpretation purposes (relying on another smooth index would not significantly change our subsequent results, but likely some identification assumptions). Note that (i) neither the clustering nor the number of clusters is observed by the econometrician and (ii) the number of possible assignments of N individuals into G^0 clusters grows exponentially fast with N.

Model (1), although static, complement models that have been routinely employed in the empirical microeconometric, industrial organisation, macroeconometric, innovation, and international trade literature, which, in contrast, assume additively separable (and time-invariant) fixed effects. I provide below some leading examples complementing Mugnier and Wang (2022).

[23] If h^0 were known to the econometrician, model (1) would become a special case of the semiparametric nonlinear factor models considered in Chen, Fernández-Val, and Weidner (2021).

Example 1 (Binary outcome)

$$Y_{it} = \mathbf{1}\left\{X_{it}'\beta^0 + \alpha^0_{g_i^0 t} - \varepsilon_{it} \geq 0\right\},$$

where ε_{it} *is independent from* $\left(X_{i1}', ..., X_{it}', g_i^0, \alpha^0_{g_i^0 t}\right)'$ *and distributed with (unknown) cumulative distribution function (cdf)* Ψ^0*. Then,*

$$h^0\left(y, X_{it}'\beta^0 + \alpha^0_{g_i^0 t}\right) = \mathbf{1}\{y = 1\} \times \Psi^0\left(X_{it}'\beta^0 + \alpha^0_{g_i^0 t}\right) + \mathbf{1}\{y = 0\} \times \left[1 - \Psi^0\left(X_{it}'\beta^0 + \alpha^0_{g_i^0 t}\right)\right].$$

Example 2 (Ordered outcome)

$$Y_{it} = \begin{cases} 0 & \text{if } X_{it}'\beta^0 + \alpha^0_{g_i^0 t} - \varepsilon_{it} < d_1^0. \\ 1 & \text{if } d_1^0 \leq X_{it}'\beta^0 + \alpha^0_{g_i^0 t} - \varepsilon_{it} < d_2^0. \\ 2 & \text{if } X_{it}'\beta^0 + \alpha^0_{g_i^0 t} - \varepsilon_{it} \geq d_2^0, \end{cases} \tag{2}$$

where $d_2^0 > d_1^0$*, and* ε_{it} *is independent from* $\left(X_{i1}', ..., X_{it}', g_i^0, \alpha^0_{g_i^0 t}\right)'$ *and distributed with (unknown) cdf* Ψ^0*. Then,*

$$h^0\left(y, X_{it}'\beta^0 + \alpha^0_{g_i^0 t}\right) = \begin{cases} 1 - \Psi^0\left(X_{it}'\beta^0 + \alpha^0_{g_i^0 t} - d_1^0\right) & \text{if } y = 0. \\ \Psi^0\left(X_{it}'\beta^0 + \alpha^0_{g_i^0 t} - d_1^0\right) - \Psi^0\left(X_{it}'\beta^0 + \alpha^0_{g_i^0 t} - d_2^0\right) & \text{if } y = 1. \\ \Psi^0\left(X_{it}'\beta^0 + \alpha^0_{g_i^0 t} - d_2^0\right) & \text{if } y = 2. \end{cases}$$

Example 3 (Count outcome) $\mathcal{Y} = \{0, 1, 2, ...\}$*. A Poisson parametrization specifies*

$$h^0\left(y, X_{it}'\beta^0 + \alpha^0_{g_i^0 t}\right) = \frac{\left(\lambda_{it}^0\right)^y \exp\left(-\lambda_{it}^0\right)}{y!}, \tag{3}$$

where $\lambda_{it}^0 = \exp\left(X_{it}'\beta^0 + \alpha^0_{g_i^0 t}\right)$*. Alternatively,* h^0 *could encapsulate, e.g., the negative binomial distribution.*

I adopt the so-called "fixed effects" approach, treating the realizations of unobserved time effects and group membership variables as unrestricted parameters to be estimated. I asume that G^0 fixed and exogeneous. Policy parameters of interest such as average marginal effects often write as functionals of β^0, h^0, $\alpha^0 := \left(\alpha^0_{g1}, ..., \alpha^0_{gT}, ..., \alpha^0_{G^0 1}, ..., \alpha^0_{G^0 T}\right)' \in \mathcal{A}^{G^0 T}$, and latent clustering structure $\gamma^0 := (g_1^0, ..., g_N^0)' \in \mathcal{G}^{0N}$. Hereafter, I focus on identification and estimation of the sequence of

parameters $\theta_{NT}^0 := \left(G^0, h^0, \beta^{0\prime}, \gamma^{0\prime}, \alpha^{0\prime}\right)' \in \Theta_{NT}$, where I let

$$\Theta_{NT} = \bigcup_{G=1}^{+\infty} \{G\} \times \mathcal{H} \times \mathcal{B} \times \{1, ..., G\}^N \times \mathcal{A}^{GT}.$$

While $\mathcal{B}$ is a finite-dimensional space, $\mathcal{H}$ is clearly not and the dimensions of both the discrete set $\{1, ..., G\}^N$ and $\mathcal{A}^{GT}$ grow with the sample size. This makes model (1) a high-dimensional combinatorial semi-parametric nonseparable model.

Remark 1 *It is straightforward to adapt the analysis to allow for cluster-specific slope coefficient* $\boldsymbol{\beta}^0 := \left(\beta_1^{0\prime}, ..., \beta_{G^0}^{0\prime}\right)'$ *such that*

$$\Pr\left(Y_{it} = y \mid X_{i1}, ..., X_{it}, g_i^0, \alpha_{g_i^0 t}^0, \beta_{g_i^0}^0\right) = h^0\left(y, X_{it}'\beta_{g_i^0}^0 + \alpha_{g_i^0 t}^0\right), \quad \forall y \in \mathcal{Y}. \tag{4}$$

We discuss this extension, as well as heterogeneous link functions, additional individual- and time-specific effects, and grouped time-periods in Appendices B.1-B.3. Model (1) can also be extended to allow for multimodal outcomes. The notation are more lengthy and would essentially follow the same lines as in Mugnier and Wang (2022).

Remark 2 *Model (1) extends Bonhomme and Manresa (2015) to nonparametric discrete choice modeling. In contrast to Bonhomme, Lamadon, and Manresa (2022), the link function h^0 is unknown, the true underlying unobserved heterogeneity is discrete, and all parameters of the models are considered as target parameters.*

3 Nonparametric Identification and Estimation

In this section, I investigate the identification of θ_{NT}^0 in model (1) and provide guideline for fully nonparametric estimation. Note that model (1) is related to nonseparable panel data models with latent factors as it implies the following semiparametric regression equations:

$$\mathbf{1}\{Y_{it} = y\} = h^0\left(y, X_{it}'\beta^0 + \alpha_{g_i^0 t}^0\right) + \varepsilon_{it}(y), \quad \forall (i, t, y) \in \times\mathcal{N} \times \mathcal{T} \times \mathcal{Y}, \tag{5}$$

where $\mathbb{E}\left[\varepsilon_{it}(y) \mid X_i, g_i^0, \alpha_{g_i^0 t}^0\right] = 0$, and

$$Y_{it} = \sum_{y \in \mathcal{Y}} y h^0\left(y, X_{it}'\beta^0 + \alpha_{g_i^0 t}^0\right) + v_{it}, \quad \forall (i, t) \in \times\mathcal{N} \times \mathcal{T}, \tag{6}$$

where $v_{it} = \sum_{y \in \mathcal{Y}} y \varepsilon_{it}(y)$ and, by linearity, $\mathbb{E}\left[v_{it} \mid X_i, g_i^0, \alpha^0_{g_i^0 t}\right] = 0$. The representation given by (5) is useful to identify the clustering structure, while the representation given by (6) allows to apply results in Ichimura (1993) under appropriate dependence conditions that I now introduce.

3.1 Large-N, Large-T Nonparametric Identification

In this section, I prove the nonparametric identification of θ^0_{NT} in model (1) as N and T diverge jointly to infinity. Since both g_i^0 and $\alpha^0_{g_i^0 t}$ are unobserved, identification holds up to clusters relabeling only.[24] It is also necessary to impose location and scale normalizations, which I specify as $\|\beta^0\| = 1$ and $\alpha^0_{11} = 0$, where $\|\cdot\|$ denotes the Euclidean norm.[25] Identification is based on Assumptions 1-5 below.

Assumption 1 (Random sampling) *There exist sequences of random vectors of fixed dimensions* $\lambda^0 := \left\{\lambda^0_{gt} : (g,t)\right\}$, $\mu^0 := \left\{\mu^0_g : g\right\}$, $\xi^0 := \{\xi^0_i : i\}$, *such that:*

(a) $(Y_i', X_i', g_i^0)'$ *is i.i.d. across* $i \in \mathcal{N}$ *conditional on* α^0, λ^0, μ^0.

(b) For all $i \in \mathcal{N}$: $\left\{\left(Y_{it}, X_{it}', \alpha^0_{g_i^0 t}\right)'\right\}_{t \geq 2}$ *is a strictly stationary strong mixing process with mixing coefficients* $\alpha_i(\cdot)$ *conditional on* $g_i^0, \mu^0_{g_i^0}, \xi_i^0$. *Let* $\alpha(\cdot) = \sup_i \alpha_i(\cdot)$ *satisfy* $\alpha(l) \leq c_\alpha \rho^l$ *with* $c_\alpha > 0$, *and* $\rho \in (0,1)$.

(c) For all $t \in \mathcal{T}$: $Y_{1t} \mid X_{1t}, g_1^0, \alpha^0, \lambda^0, \mu^0, \xi^0 \overset{d}{=} Y_{1t} \mid X_{1t}, g_1^0, \alpha^0_{g_1^0 t}$.

Assumptions 1(a)-1(b) restrict cross-sectional and time dependence in the data. Contrasting with many papers, they allow for flexible patterns of unconditional spatial and time-series correlations as captured by the clustering latent structure $\alpha^0, \lambda^0, \mu^0$ and individual-specific effects ξ^0. Assumption 1(c) requires that λ^0, μ^0, ξ^0 have no effect on the outcome after conditioning for the covariates, cluster membership and the cluster-specific effects α^0. This assumption is mostly for a matter of simplicity in exposition. In Appendix B.1, I discuss several extensions such as cluster-specific slopes, individual-fixed and time-fixed effects which possibly affect all observed variables.[26]

Assumption 2 (Latent clustering) $\mathcal{X} := \bigcap_{i=1}^{\infty} \mathcal{X}_i$ *is not empty and:*

[24] This mirrors standard rotational invariance normalizations in interactive fixed effects models (see, e.g., Bai, 2009).

[25] These choices are, of course, arbitrary but normalizing $\left\|\beta^0\right\| = 1$ is standard in nonparametric single-index models (see, e.g. Botosaru and Muris, 2017; Ichimura, 1993).

[26] In some application, it could be useful to allow for a non-scalar α^0_{gt}. Estimation in semiparametric nonlinear grouped factor models with many factors has recently been considered in Ando and Bai (2022).

(a) *There exist known* $\mathcal{X}^0 \subset \mathcal{X}$, $y \in \mathcal{Y}$, *and functional* ϕ *such that, for all fixed* $(i,j) \in \mathcal{N}^2$, *letting* $\rho_i(x) : \mathcal{X}^0 \ni x \mapsto \Pr\left(Y_{i2} = y \mid X_{i2} = x, g_i^0, \mu_i^0, \xi_i^0\right)$, $\phi(\rho_i, \rho_j) = \mathbf{1}\left\{g_i^0 = g_j^0\right\}$.

(b) *For all* $g \in \mathcal{G}^0$, *almost surely* $\Pr\left(g_1^0 = g \mid \alpha^0, \lambda^0, \mu^0, \xi^0\right) > 0$.

Assumption 2(a) requires clusters to be sufficiently well-separated in terms of individual-level conditional probability functions. It is a low-level injectivity or "completeness"-type assumption à la Bonhomme, Lamadon, and Manresa (2022) which ensures that latent variables are recoverable from observed moments and leaves flexibility to the researcher for defining clusters of unobserved heterogeneity. In Appendix A.2, I provide sufficient conditions for Assumption 2(a) to hold, including smoothness and the existence of a special regressor à la Honoré and Lewbel (2002) but without large support. For such a mapping to exist, the intuition is that whenever $g_i^0 \neq g_j^0$, the conditional distributions $\alpha_{g2}^0 \mid X_{i2} = x, g_i^0, \mu_i^0, \xi_i^0$ and $\alpha_{g2}^0 \mid X_{j2} = x, g_j^0, \mu_j^0, \xi_j^0$ across $x \in \mathcal{X}^0$ should differ sufficiently (and the link function h^0 should be sufficiently smooth to convey such a difference) so as to trigger a difference in the integrated-out conditional outcome probabilities captured by ϕ. In many application, $\phi(f,g) = \mathbf{1}\{f = g\}$ makes sense (see, e.g., Vogt and Linton, 2017). Yet, the setting is kept slightly more general as other clustering structures might be plausible. Assumption 2(b) rules out asymptotically negligible clusters. Notice that allowing for an increasing number of clusters or negligible clusters would need substantial changes to Assumption 1 (e.g., as the cross-sectional identical distribution would not hold anymore).

Assumption 3 (Regularity and smoothness)

(a) *Conditional on* $g_i^0, \mu_{g_i^0}^0, \xi_i^0$, X_{i2} *admits a uniformly continuous density function* $f_{X_{i2}|g_i^0,\mu_{g_i^0}^0,\xi_i^0}$ *such that* $0 < \underline{\delta} \leq \inf_{x \in \mathcal{X}^0} f_{X_{i2}|g_i^0,\mu_{g_i^0}^0,\xi_i^0}(x) \leq \sup_{x \in \mathcal{X}^0} f_{X_{i2}|g_i^0,\mu_{g_i^0}^0,\xi_i^0}(x) \leq \overline{\delta} < \infty$.

(b) *Almost surely,* $\mathbb{E}\left(\left\|X_{12}\right\|^2 \mid g_1^0, \alpha^0, \lambda^0, \mu^0\right)$ *is finite and* $\mathbb{E}\left(X_{12}X_{12}' \mid g_1^0, \alpha^0, \lambda^0, \mu^0\right)$ *is nonsingular.*

(c) $\sum_{y \in \mathcal{Y}} y h^0(y, \cdot)$ *is differentiable on* $\mathbb{R}$ *and not constant on the support of* $X_{it}'\beta^0 + \alpha_{g_i^0 t}^0$.

Assumption 3 collects sufficient technical conditions that are useful to achieve point identification of β^0, α^0 given that h^0 is unknown, by relying on existing results in Ichimura (1993) for nonparametric i.i.d. single index models. In particular, it requires continuous covariates (which could be relaxed at the expense of heavier conditions) and invertibility of conditional Gram matrices.

Assumption 4 (Monotonicity) *There exists* $y \in \mathcal{Y}$ *such that* $h^0(y, v)$ *is strictly monotonic in* v.

Assumption 4 is a shape restriction which may be expected to hold at boundary points of $\mathcal{Y}$ (e.g., outside option in random utility models, absence of trade, absence of patenting in acount outcome model). Shape restrictions such as monotonicity have been routinely used to obtain point-identification in nonseparable panel data models.[27]

Assumption 5 (Compensating variations) *For all fixed $(g,\widetilde{g},t)$, there exist $x_1,x_2 \in \mathcal{X}$ such that*

$$\alpha^0_{\widetilde{g}t} + x_1'\beta^0 = \alpha^0_{gt} + x_2'\beta^0. \tag{7}$$

Similarly, for all $(g,t,\widetilde{t})$, there exist $x_3,x_4 \in \mathcal{X}$ such that

$$\alpha^0_{g\widetilde{t}} + x_3'\beta^0 = \alpha^0_{gt} + x_4'\beta^0. \tag{8}$$

Assumption 5 requires sufficient variation in the covariates and has the same flavor as the *compensating variations* used in D'Haultfoeuille, Hoderlein, and Sasaki (2021) and Mugnier and Wang (2022). As in the latter paper, it does not necessarily require a covariate with large support (it depends on the support of unobserved group-specific effects) and ensures that there is overlap in the single index across unobserved clusters (not individuals) and periods. Let $W^0_N = \left(\mathbf{1}\left\{g^0_i = g^0_j\right\}\right)_{(i,j)\in\{1,\dots,N\}^2}$. Theorem 1 below is the main identification result of the paper.

Theorem 1 *Let Assumptions 1-3(a) hold, and let N and T diverge jointly to infinity. Then,*

1. *$(W^0_N)_{N\in\mathbb{N}^*}$ and G^0 are point identified.*
2. *If Assumptions 3(b)-5 further hold, then h^0, β^0, and $(\alpha^0_{gt})_{(g,t)\in\mathcal{G}^0\times\mathbb{N}^*}$ are point identified.*

For the proof see Appendix A.1.

Remark 3 *A key argument of the proof of Theorem 1 is to frame the identification of the clustering γ^0 up to cluster relabeling as the equivalent problem of recovering the lower (or uppper)-triangular submatrix of the adjency matrix W^0_N of the undirected graph $\mathcal{G}_N = \{V,E\}$ whose set of vertices V contains units $i \in \mathcal{N}$ and whose edges E contains all $(i,j) \in \mathcal{N}^2$ such that $g^0_i = g^0_j$. Given the clustering structure of the model, note that W^0_N has rank $R_N \leq G^0$ which is also its number of distinct rows as clusters forms disconnected cliques in $\mathcal{G}_N$.*[28] *In other words, it is easily seen*

[27] See, among many others, Athey and Imbens (2006); Evdokimov (2011); Klein and Spady (1993); Mugnier and Wang (2022), and Altonji and Matzkin (2005).

[28] The related problem of "community detection" in networks has been widely studied in the statistical learning literature, and in particular in the compressed sensing literature. I do not pursue adaptation of spectral clustering techniques or recent development in Graph-cut problems for which very few asymptotic results in statistical settings with complex structure of dependencies are known. See von Luxburg (2007); Wang and Su (2021).

that identification of γ^0 up to cluster relabeling is equivalent to identification of all sets $\mathcal{C}^0(i) := \left\{j \in \mathcal{N} : g_j^0 = g_i^0\right\}$ for $i \in \mathcal{N}$. Such a characterization has two advantages: (i) it is invariant to clusters relabeling and (ii) it reduces the NP-hard G^0-mean clustering problem to that of solving $N(N-1)/2$ binary classification problems.[29] *Once the clustering γ^0 has been identified for all N, identification of G^0 follows easily by letting $N \to \infty$. Identification of β^0 can be obtained relying on intra-cluster cross-sectional variation for a single cluster and a result by Ichimura (1993) for a large class of cross-sectional nonparametric single-index models. Identification of cluster-specific effects relies on the compensating variations and monotonicity of $h^0(y,\cdot)$.*

3.2 Nonparametric Estimation

A nonparametric estimator can be build following exactly the identification strategy. I provide below the roadmap and main steps:

1. For all $i \in \mathcal{N}$: get an estimate $\widehat{\rho}_i$ of ρ_i using nonparametric density estimation (including machine learning) methods.
2. Compute $\widehat{W}_{ij} = \phi(\widehat{\rho}_i, \widehat{\rho}_j)$ or an approximate regularized version.
3. Set $\widehat{G} = \left|\left\{\widehat{W}_{1,\cdot}, ..., \widehat{W}_{N,\cdot}\right\}\right|$ and pick $(\widehat{g}_1, ..., \widehat{g}_N)' \in \left\{1, ..., \widehat{G}\right\}^N$ satisfying, for all $(i,j) \in \{1, ..., N\}^2$,
$$\left[\widehat{g}_i = \widehat{g}_j \iff \widehat{W}_{i,\cdot} = \widehat{W}_{j,\cdot}\right].$$
4. Estimate $\widehat{\beta}_{gt}$ within each group at each period using Ichimura (1993)'s SLS method, and let $\widehat{\beta} = \frac{1}{T\widehat{G}} \sum_{g=1}^{\widehat{G}} \sum_{t=1}^{T} \widehat{\beta}_{gt}$.
5. Estimate $(\widehat{\alpha}_{gt})_{g,t}$ by the compensating variation.
6. Estimate $\widehat{h}(y,\cdot)$ by non-parametric regression of $\mathbf{1}\{Y_{it} = y\}$ on $\widehat{Z}_{it} := X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t}$.

This approach has the drawback of requiring a lot of nonparametric density estimation, i.e., a lot of tuning parameters as it requires combining nonparametric estimates of conditionals probabilities. This is similar to Gao, Li, and Xu (2022)'s approach in a pure network. I do not pursue the theoretical analysis of an estimator of this type, because I aim at developing a simple method for

[29] Building on this insight, Mugnier (2022) proposes computationally straightforward pairwise differencing estimators for linear grouped fixed effects models. A similar-in-philosophy though different trick to break NP hardness is the binary segmentation approach of Wang and Su (2021).

which inference tools are available. An open question is how the pairwise approach compares to the bruteforce fully nonparametric maximum likelihood approach. I note that, for a class of directed network models, the pairwise differencing approach developed in Mugnier (2022) yields a convenient estimation procedure under conditional moment restrictions, without requiring any nonparametric estimation, which reconciles computational simplicity and powerful inference.

4 Semiparametric Estimation

In the first part of this section, I propose a general M-estimation framework accomodating nonlinear models when the number of clusters, $G^0 \in \mathbb{N}^*$, is known to the researcher.[30] In the second part, I specialize the framework to cases where $h^0 \in \mathcal{H}$ is further assumed to be known (e.g., Probit, Logit, Poisson) to define semiparametric NGFE estimators. In the third part, I discuss computation.

4.1 A Generic M-Estimation Framework

Assume from now that $G^0 \in \mathbb{N}^*$ is known to the researcher, and suppose there exists a known function $\rho : \mathcal{Y} \times \mathcal{X} \times \mathcal{B} \times \mathcal{H} \times \mathcal{G}^{0N} \times \mathcal{A}^{G^0 T} \longrightarrow \mathbb{R}$ such that $\theta^0_{NT} := (\beta^{0\prime}, h^0, \gamma^{0\prime}, \alpha^{0\prime})'$ satisfies

$$\theta^0_{NT} = \underset{\theta \in \mathcal{B} \times \mathcal{H} \times \mathcal{G}^{0N} \times \mathcal{A}^{G^0 T}}{\arg\max} \mathbb{E}\left(\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \rho(Y_{it}, X_{it}; \theta) \mid \gamma, \alpha \right), \tag{9}$$

where $\mathcal{G}^{0N} = \{1, ..., G^0\}^N$ is the set of all partitions of $\{1, ..., N\}$ into at most G^0 clusters. Provided it exists, the M-NGFE nonparametric estimator $\widehat{\theta}^{\mathrm{M}}_{\rho} := (\widehat{\beta}^{\mathrm{M}\prime}, \widehat{h}^{\mathrm{M}}, \widehat{\gamma}^{\mathrm{M}\prime}, \widehat{\alpha}^{\mathrm{M}\prime})'$ of θ^0_{NT} solves

$$\widehat{\theta}^{\mathrm{M}}_{\rho} \in \underset{\theta \in \mathcal{B} \times \mathcal{H} \times \mathcal{G}^{0N} \times \mathcal{A}^{G^0 T}}{\arg\max} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \rho(Y_{it}, X_{it}; \theta). \tag{10}$$

Finding a suitable ρ-function, proving identification of θ^0_{NT} (i.e., that eq. (9) holds), and deriving the asymptotic properties of the sequence of $\widehat{\theta}^{\mathrm{M}}_{\rho}$ are certainly difficult problems beyond the scope of the paper, each of them would require further assumptions. Moreover, computation of $\widehat{\theta}^{\mathrm{M}}_{\rho}$ is generally infeasible because maximization problem (10) is a non-smooth non-concave optimization

[30] Estimating G^0 in nonlinear models with time-varying unobserved heterogeneity is a difficult problem that is beyond the scope of the paper. See Chen, Fernández-Val, and Weidner (2021) for a discussion in some concave nonlinear factor type models. An AIC or BIC-type criterion à la Bai and Ng (2002); Bonhomme and Manresa (2015) could be employed but would require to know at least an upper bound on G^0. Letting G^0 grow slowly with N, T could also be of interest but would require a different analysis that is beyond the scope of the paper. Note that Bonhomme, Lamadon, and Manresa (2022) *need* the number of clusters to increase as they assume a (possibly) continuous underlying unobserved heterogeneity.

problem with combinatorial optimization (due to the clustering part) over an infinite-dimensional space (due to $\mathcal{H}$). A practical solution to make the problem finite-dimensional is sieve-estimation of h^0 but this is beyond the scope of this paper. Instead, I focus on semiparametric versions where h^0 is assumed to be known and that are of practical interest in many empirical applications.

4.2 Semiparametric NGFE Estimators

From now on, I assume that $h^0 \in \mathcal{H}$ is known (e.g., Logit, Probit, Poisson, etc.) and consider the problem of estimating $\theta^0_{NT} := (\beta^{0\prime}, \gamma^{0\prime}, \alpha^{0\prime})'$ in the semiparametric model (1) with known G^0. The semiparametric NGFE estimator of θ^0_{NT}, denoted $\widehat{\theta}^{\text{NGFE}} := (\widehat{\theta}', \widehat{\gamma}', \widehat{\alpha}')'$, is the M-NGFE estimator $\widehat{\theta}^{\text{M}}_{\rho}$ (once suppressing dependence on h) with $\rho(Y_{it}, X_{it}; \theta) = \ln h^0\left(Y_{it}, X_{it}'\beta + \alpha_{g_i t}\right)$. In other words, $\widehat{\theta}^{\text{NGFE}}$ is solution to the following minimization problem:

$$\widehat{\theta}^{\text{NGFE}} \in \underset{\theta \in \mathcal{B} \times \mathcal{G}^{0N} \times \mathcal{A}^{G^0 T}}{\arg\min} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} - \ln h^0\left(Y_{it}, X_{it}'\beta + \alpha_{g_i t}\right), \tag{11}$$

where the minimum is taken over all possible common parameters β, partitions $\gamma = (g_1, ..., g_N)'$ of the N individuals into G^0 clusters, and cluster-specific time effects $\{\alpha_{gt} : (g,t)\}$. Note that the NGFE estimator is a "classification likelihood" estimator. For given values of β and α, the optimal cluster assignment for individual i is

$$\widehat{g}_i(\beta, \alpha) = \underset{g \in \mathcal{G}^0}{\arg\min} \frac{1}{NT} \sum_{t=1}^{N} \sum_{t=1}^{T} - \ln h^0\left(Y_{it}, X_{it}'\beta + \alpha_{gt}\right), \tag{12}$$

where the minimum g is taken in case of a non-unique solution. The NGFE estimator of $(\beta^{0\prime}, \alpha^{0\prime})'$ in (11) can then be written as

$$(\widehat{\beta}, \widehat{\alpha}) = \underset{(\beta,\alpha) \in \mathcal{B} \times \mathcal{A}^{G^0 T}}{\arg\min} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} - \ln h^0\left(Y_{it}, X_{it}'\beta + \alpha_{\widehat{g}_i(\beta,\alpha) t}\right), \tag{13}$$

where $\widehat{g}_i(\beta, \alpha)$ is given by (12).

4.3 Computation

The minimization problem (11) is not differentiable nor convex in ϕ. In particular, it may be subject to the existence of local minima. Note that the number of partitions of N individuals

into G^0 clusters increases steeply with N, making exhaustive search impossible.[31] I propose the following simple algorithm which is an extension of the popular Lloyd (1982)'s algorithm for k-means, a "greedy" algorithm providing a converging sequence of heuristic solutions in polynomial time.

Iterative Algorithm:

1. Let $(\beta^{(0)}, \alpha^{(0)}) \in \mathcal{B} \times \mathcal{A}^{G^0 T}$ be some starting value. Set $s = 0$.

2. Compute for all $i \in \{1, ..., N\}$:

$$g_i^{(s+1)} = \underset{g \in \mathcal{G}^0}{\arg\min} \sum_{t=1}^{T} -\ln h^0 \left(Y_{it}, X_{it}' \beta^{(s)} + \alpha_{gt}^{(s)} \right). \tag{14}$$

3. Compute:

$$\left(\beta^{(s+1)}, \alpha^{(s+1)} \right) = \underset{(\beta, \alpha) \in \mathcal{B} \times \mathcal{A}^{G^0 T}}{\arg\min} \sum_{i=1}^{N} \sum_{t=1}^{T} -\ln h^0 \left(Y_{it}, X_{it}' \beta + \alpha_{g_i^{(s+1)} t} \right). \tag{15}$$

4. Set $s = s + 1$ and go to Step 2 (until numerical convergence).

Algorithm 1 alternates between two steps. In the "assignment" step, each individual i is assigned to cluster g_i whose vector of time effects minimizes individual's i time-averaged log-likelihood given the slope parameter. In the "update step", β and α are computed using maximum likelihood and controlling for interactions of cluster and time dummies. A potential issue is that the solution depends on the chosen starting values. Drawing starting values at random and selecting the solution that yields the lowest objective is a practical solution in low-dimensional problems. Finding a fast approximation of NGFE for larger-scale problems and controlling its optimization error is left for further research.[32]

5 Asymptotic Properties of Semiparametric NGFE Estimators

In this section, I assume that $\theta_{NT}^0 := (\beta^{0\prime}, \alpha^{0\prime}, \gamma^{0\prime})'$ is identified (e.g., by Theorem 1) and derive the asymptotic properties of semiparametric NGFE estimators. I consider an asymptotic framework

[31] The number of partitions of N objects into G^0 disjoint and non-empty subsets is $\frac{1}{N!} \sum_{i=1}^{N} (-1)^{N-i} \binom{N}{i} N^{G^0} \propto \frac{G^{0N}}{G^{0!}}$. In fact the G^0-means problem without regressors in a cross-sectional setting is NP-hard (see, e.g., Aloise, Deshpande, Hansen, and Popat, 2009).

[32] Note that an algorithm similar to Algorithm 2 in Bonhomme and Manresa (2015) can be employed to improve the trade-off between exploration and exploitation during the optimization process.

where N and T tend jointly to infinity but G^0 does not grow with N and T. I focus on binary choice models with grouped fixed effects as the leading case. Similar results can be obtained for other strictly concave models (see Appendix B.4), but I stick to binary choice models to keep the exposition simple. I abstract from optimization errors and study the asymptotic behaviour of the exact sequence of estimates defined in eq. (11).

5.1 Binary Choice Model With Grouped Fixed Effects

Consider the following data generating process:

$$Y_{it} = \mathbf{1}\left\{X_{it}'\beta^0 + \alpha^0_{g_i^0 t} - \varepsilon_{it} \geq 0\right\}, \quad i = 1,...,N, t = 1,...,T. \tag{16}$$

For any $\mathbf{Z} = (Z_{11},...,Z_{1T},...,Z_{N1},...,Z_{NT})'$, let $\mathbf{Z}_-^{(t)} = \{Z_{is} : 1 \leq i \leq N, 1 \leq s \leq t\}$, $\mathbf{Z}_+^{(t)} = \{Z_{is} : 1 \leq i \leq N, t \leq s \leq T\}$, and $\boldsymbol{\varepsilon} := \{\varepsilon_{it} : (i,t)\}$.

Assumption 6

Eq. (16) holds and:

(a) For all t: $\left(\mathbf{X}_-^{(t)}, \gamma^0, \alpha^0, \boldsymbol{\varepsilon}_-^{(t-1)}\right)$ and $\boldsymbol{\varepsilon}_+^{(t)}$ are independent.[33]

(b) The $\{\varepsilon_{it} : (i,t)\}$ are identically distributed with known cumulative distribution function Ψ that is fully supported on $\mathbb{R}$, twicely continuously differentiable, strictly increasing, and such that $(\ln \Psi)'' < 0$. Moreover, Ψ' is symmetric around 0.

Assumption 6(a) is a weak exogeneity assumption, standard in the panel data literature, which allows X_{it} to contain predetermined variables with respect to Y_{it}. In particular, X_{it} can include lags of Y_{it} to accommodate dynamic models. Such assumption does not restrict the correlation between (γ^0, α^0) and $\{\mathbf{X}_i : i\}$. Assumption 6(b) is standard in semiparametric panel discrete choice models and yields strict concavity of the log-likelihood function under minimal amount of cluster-specific and time-specific variation in the covariates (as assumed, e.g., in Bonhomme, Lamadon, and Manresa, 2022; Chen, Fernández-Val, and Weidner, 2021; Fernández-Val and Weidner, 2016).[34] The second part of Assumption 6(b) is weak and is satisfied by the Probit ($\Psi(u) = \int_{-\infty}^{u}(1/\sqrt{2\pi})e^{-t^2/2}dt$) and Logit ($\Psi(u) = 1/(1+e^{-u})$) distributions. Symmetry of Ψ is not necessary but it conveniently

[33] If one lag Y_{it-1} is included as regressor, I assume that Y_{i0} is observed and contained in $\mathbf{X}_-^{(t)}$. Higher-order dependence can be accommodated similarly.

[34] See also, Pratt (1981).

simplifies notation in the proofs. Under Assumption 6, note that eq. (16) is a semiparametric NGFE model (1) with known link function $h^0(y,z) = \Psi(z)^{\mathbf{1}\{y=1\}}(1-\Psi(z))^{\mathbf{1}\{y=0\}}$. The corresponding NGFE estimator writes

$$(\widehat{\beta},\widehat{\gamma},\widehat{\alpha}) \in \underset{(\beta,\gamma,\alpha)\in\mathcal{B}\times\mathcal{G}^{0N}\times\mathcal{A}^{G^0T}}{\arg\min} \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} -\ln\Psi\left(Q_{it}\left(X_{it}'\beta+\alpha_{g_it}\right)\right), \tag{17}$$

where $Q_{it} = 2Y_{it} - 1$.

5.2 Consistency

Consider the following assumption.

Assumption 7

(a) $\mathcal{B}$ *and* $\mathcal{A}$ *are compact convex subsets of* $\mathbb{R}^p$ *and* $\mathbb{R}$*, respectively.*

(b) There exists a constant $M > 0$ *such that* $\|X_{it}\| \leq M$ *almost surely.*

(c) Let $\overline{X}_{g\wedge\widetilde{g},t}$ *denotes the mean of* X_{it} *in the intersection of clusters* $g_i^0 = g$*, and* $g_i = \widetilde{g}$*. For all partitions* $\gamma = \{g_1,...,g_N\} \in \Gamma_{\mathcal{G}^{0N}}$*, let* $\widehat{\rho}(\gamma)$ *denote the minimum eigenvalue of the following matrix:*

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(X_{it}-\overline{X}_{g_i^0\wedge g_i,t})(X_{it}-\overline{X}_{g_i^0\wedge g_i,t})'.$$

Then, $\operatorname{plim}_{N,T\to\infty}\min_{\gamma\in\Gamma_{G^0}}\widehat{\rho}(\gamma) = \rho > 0$.

Assumption 7(a) and 7(c) are the same as Assumption 1(a) and 1(g) in Bonhomme and Manresa (2015). Assumption 7(b) strengthens Assumption 1(b) in Bonhomme and Manresa (2015). It ensures (together with Assumption 7(a)) strong concavity of the log-likelihood function and rules non-stationary covariates.[35] Assumption 7(c) requires that X_{it} shows sufficient within-cluster variation over time and across individuals, and is related to standard noncolinearity assumptions encountered in the large-N, large-T panel data literature (see, e.g., Ando and Bai, 2022; Bai, 2009; Chen, Fernández-Val, and Weidner, 2021; Vogt and Linton, 2017). It allows for time-invariant covariates provided that they have a sufficiently rich support. As a special case highlighted in Bonhomme and Manresa (2015), Assumption 7(c) is satisfied if X_{it} are discrete and, for all g, the conditional distribution of X_i given $g_i^0 = g$ has strictly more than G^0 points of supports.

[35] One could relax this assumption by allowing covariates to have sub-gaussian tails (see, e.g., Vershynin, 2019, for a definition). I do not pursue this avenue in order to keep the exposition light. Moment conditions in Bonhomme and Manresa (2015) also rule out covariates.

Theorem 2 (Consistency) *Let Assumptions 6 and 7 hold. Then, as N and T tend to infinity:*

1. $\widehat{\beta} \xrightarrow{p} \beta^0$.
2. $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\widehat{g}_i t} - \alpha^0_{g_i^0 t}\right)^2 \xrightarrow{p} 0$.

For the proof see Appendix A.3.

Theorem 2 shows that NGFE estimators of the common slope coefficient and cluster-specific effects in NGFE binary choice models are both consistent.

5.3 Asymptotic Distribution

Consider the following assumption.

Assumption 8

(a) For all $g \in \mathcal{G}^0$: $\text{plim}_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i^0 = g\} = \pi_g > 0$.

(b) For all $(g,\widetilde{g}) \in \mathcal{G}^{02}$ such that $g \neq \widetilde{g}$: $\text{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}(\alpha^0_{gt} - \alpha^0_{\widetilde{g}t})^2 = c_{g,\widetilde{g}} > 0$.

(c) There exist constants $a > 0$ and $d > 0$ and a sequence $\alpha(t) \leq \exp(-at^d)$ such that, for all $i \in \{1,...,N\}$ and $(g,\widetilde{g}) \in \mathcal{G}^{02}$ such that $g \neq \widetilde{g}$, $\left\{\alpha^0_{gt} - \alpha^0_{\widetilde{g}t} : t\right\}$ are strongly mixing processes with mixing coefficients $\alpha(t)$.

Assumptions 8(a)-(c) are identical to Assumptions 2(a)-(c) in Bonhomme and Manresa (2015), respectively. Assumption 8(a) ensures that no cluster is asymptotically negligible relative to the others and that each cluster has a large number of observations in the population. This is equivalent to the "strong factor" condition in approximate factor models (see, e.g., Assumption 1.(v) in Chen, Fernández-Val, and Weidner, 2021). Assumption 8(b) imposes that the G^0 clusters are well separated in the population. As discussed in a recent work by Chetverikov and Manresa (2021), departing from such an assumption seems quite difficult. Assumption 8(c) restricts the dependence and tail properties of the processes $(\alpha^0_{gt} - \alpha^0_{\widetilde{g}t})$, which are assumed to be strongly mixing. Assumption 8(d) is standard and requires a sufficient amount of variation in the covariates.

Assumption 8 allows me to rely on exponential inequalities for dependent processes (e.g., Rio, 2000) in order to bound misclassification probabilities, almost the same way as in the proof of Theorem 2 in Bonhomme and Manresa (2015). The novelty here is that their assumption that the idiosyncratic shock in the linear model is a strong mixing process is hidden in the parametric and independence restrictions formulated in Assumption 6, the latter allowing to rely on exponential

inequalities for martingale differences (see, e.g., Lesigne and Volný, 2001) to control remainder terms in the proofs (essentially the score).

Let $(\widetilde{\beta},\widetilde{\alpha})$ be such an infeasible version of the NGFE estimator where cluster membership g_i, instead of being estimated, is fixed to its population counterpart g_i^0:

$$(\widetilde{\beta},\widetilde{\alpha}) = \underset{(\beta,\alpha)\in\mathcal{B}\times\mathcal{A}^{G^0T}}{\operatorname{argmin}} \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} -\ln\Psi\left(Q_{it}\left(X_{it}'\beta+\alpha_{g_i^0 t}\right)\right). \tag{18}$$

This is the maximum likelihood estimator in the pooled regression of Y_{it} on X_{it} and the interactions of population cluster dummies and time dummies.

Assumptions 6, 7, and 8 provide conditions under which estimated cluster memberships converge to their population counterparts, and the NGFE estimator defined in (17) is asymptotically equivalent to the infeasible maximum likelihood estimator $(\widetilde{\beta},\widetilde{\alpha})$, when N and T tend to infinity and $N/T^{\nu}\to 0$ for some $\nu>0$ (see Lemma 7 in Appendix A.4.1). In particular, this allows T to grow considerably more slowly than N. Because of invariance to relabeling of the clusters, the results for cluster membership and cluster-specific effects are understood to hold given a suitable choice of the labels (see the proof for details). Theorem 2 and eq. (52) crucially hinge on the restrictive assumption that the number of well-separated clusters G^0 is known and fixed, but it could be that consistent estimation of $\widehat{\beta}$ remains possible under weaker assumptions that would nonetheless prevent consistent estimation of cluster memberships.[36]

Given Lemma 7, showing asymptotic normality of the NGFE estimator then reduces to the simpler problem of showing asymptotic normality of the infeasible (oracle) MLE $(\widetilde{\beta},\widetilde{\alpha})$. Let $Z_{it}^0 = X_{it}'\beta^0+\alpha^0_{g_i^0 t}$. For all $g\in\mathcal{G}$, all $t\in\{1,...,T\}$, let $\widetilde{X}_{gt}$ denote the projection of X_{it} on the space spanned by the cluster membership variable under a metric weighted by $(-\ln\Psi)''(Q_{it}Z_{it}^0)$:

$$\widetilde{X}_{gt} = \left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left\{g_i^0=g\right\}(\ln\Psi)''(Q_{it}Z_{it}^0)\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left\{g_i^0=g\right\}(\ln\Psi)''(Q_{it}Z_{it}^0)X_{it}\right),$$

i.e., the weighted average of X_{it} for individuals $\{i: g_i^0=g\}$. Also, let $\widehat{\pi}_{gt}$ denote the following weighted average:

$$\widehat{\pi}_{gt} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left\{g_i^0=g\right\}(-\ln\Psi)''(Q_{it}Z_{it}^0).$$

Assumption 9 below allows to characterize the asymptotic distribution of the infeasible MLE $(\widetilde{\beta},\widetilde{\alpha})$.

[36] I thank Martin Weidner for pointing out this to me, something also discussed in Dzemski and Okui (2021).

Assumption 9

(a) $\{Y_{it} : (i,t)\}$ *are independent conditional on* $(\mathbf{X}, \gamma^0, \alpha^0)$.

(b) There exists a positive definite matrix Σ_β *such that*

$$\Sigma_\beta = \text{plim}_{N,T\to\infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (-\ln \Psi)''(Q_{it} Z_{it}^0) \left[X_{it} - \widetilde{X}_{g_i^0 t} \right] \left[X_{it} - \widetilde{X}_{g_i^0 t} \right]'.$$

(c) As N *and* T *tend to infinity,*

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} \left\{ (-\ln \Psi)''(Q_{it} Z_{it}^0) \left(X_{it} - \widetilde{X}_{g_i^0 t} \right) \right\} \left\{ Q_{it} (-\ln \Psi)'(Q_{it} Z_{it}^0) \right\} \xrightarrow{d} \mathcal{N}(0, \Sigma_\beta).$$

(d) For all (g,t)*:* $\text{plim}_{N\to\infty} \widehat{\pi}_{gt} = \widetilde{\pi}_{gt} > 0$.

(e) For all (g,t)*:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} E \left(\mathbf{1} \left\{ g_i^0 = g \right\} \mathbf{1} \left\{ g_j^0 = g \right\} Q_{it} Q_{jt} (\ln \Psi)'(Q_{it} Z_{it}^0) (\ln \Psi)'(Q_{jt} Z_{jt}^0) \right) = \omega_{gt} > 0.$$

(f) For all (g,t)*, and as* N *and* T *tend to infinity:*

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathbf{1} \left\{ g_i^0 = g \right\} Q_{it} (\ln \Psi)'(Q_{it} Z_{it}^0) \xrightarrow{d} \mathcal{N}(0, \omega_{gt}).$$

(g) The true value of β*,* β^0*, is in the interior of* $\mathcal{B}$*. For all* T*, the true value of* α*,* α^0*, is in the interior of* $\mathcal{A}^{G^0 T}$.

Assumption 9(a) rules out dynamic or feedbacks.

Theorem 3 (Asymptotic Distribution) *Let Assumptions 6-9 hold and let* N *and* T *tend to infinity such that* $N/T \to \infty$ *and, for some* $\nu > 1$*,* $N/T^\nu \to 0$*. Then:*

$$\sqrt{NT} (\widehat{\beta} - \beta^0) \xrightarrow{d} \mathcal{N} \left(0, \Sigma_\beta^{-1} \right), \tag{19}$$

and, for all (g,t)*,*

$$\sqrt{N} \left(\widehat{\alpha}_{gt} - \alpha_{gt}^0 \right) \xrightarrow{d} \mathcal{N} \left(0, \frac{\omega_{gt}}{\widetilde{\pi}_{gt}^2} \right), \tag{20}$$

where Σ_β*,* ω_{gt}*, and* $\widetilde{\pi}_{gt}$ *are defined in Assumption 9.*

For the proof see Appendix A.4.2.

Theorem 3 demonstrates that NGFE estimators in NGFE binary choice models achieve the parametric root-NT and root-N rates of convergence and are free of Neyman and Scott (1948)'s incidental parameters problem. These rates are in contrast with standard interactive fixed-effects models (see, e.g. Ando and Bai, 2022; Bai, 2003, 2009) for which root-N consistency of the time-varying factors requires $N/T^2 \to 0$ or more generally $N/T \to \kappa$, $0 < \kappa < \infty$, as it is assumed for instance in Chen, Fernández-Val, and Weidner (2021); Fernández-Val and Weidner (2016). The intuition behind is that the extreme sparsity of the factor loading structure in model (16) allows NGFE estimators to achieve fast accurate classification of individuals which, reduces the problem to that of a multidimensional fixed effect in the time-series dimension in the limit.[37] Consistent estimators of the asymptotic variances are given in Appendix C.

5.4 Average Partial Effects (APEs)

Under Assumption 6, if $X_{it,k}$, the kth element of X_{it} is binary, its partial effect on the conditional probability of Y_{it} is

$$\Delta(X_{it},\beta^0,\alpha^0_{g_i^0 t}) = \Psi(\beta^0_k + X'_{it,-k}\beta^0_{-k} + \alpha^0_{g_i^0 t}) - \Psi(X'_{it,-k}\beta^0_{-k} + \alpha^0_{g_i^0 t}),$$

where β^0_k is the kth element of β^0, and $X_{it,-k}$ and β^0_{-k} include all elements of X_{it} and β^0 except the kth element. If $X_{it,k}$ is continuous, the partial effect of $X_{it,k}$ on the conditional probability of Y_{it} is

$$\Delta(X_{it},\beta^0,\alpha^0_{g_i^0 t}) = \beta^0_k \Psi'(X'_{it}\beta^0 + \alpha_{g_i^0 t}),$$

where Ψ' is the derivative of Ψ. As discussed in Fernández-Val and Weidner (2016), if $(X_{it}, g_i^0, (\alpha^0_{gt})_{g\in\mathcal{G}^0})$ is identically distributed over i but can be heterogeneously distributed over t, then $\mathbb{E}(\Delta_{it}) = \delta^0_t$ and $\delta^0_{NT} = \frac{1}{T}\sum_{t=1}^T \delta^0_t$ changes only with T. If $(X_{it}, g_i^0, (\alpha^0_{gt})_{g\in\mathcal{G}^0})$ is identically distributed over i and stationary over t, then $\mathbb{E}(\Delta_{it}) = \delta^0_{NT}$, and $\delta^0_{NT} = \delta^0$ does not change with N and T.

Deriving the asymptotic properties of plug-in estimators of average partial effects of the type $\widehat{\delta}_{NT} = \Delta(\widehat{\beta},\widehat{\alpha},\widehat{\gamma})$ is left for further research.

[37] To see the factor-loading structure, note that model (16) can be written as $Y_{it} = \mathbf{1}\left\{X'_{it}\beta + \lambda'_i f_t - \varepsilon_{it} \geq 0\right\}$, where $\lambda'_i = (\mathbf{1}\left\{g_i^0 = 1\right\}, ..., \mathbf{1}\left\{g_i^0 = G^0\right\}) \in \left\{b \in \{0,1\}^{G^0} : \sum_{g=1}^{G^0} b_g = 1\right\}$ and $f_t = (\alpha^0_{gt})'_{g=1,...,G^0} \in \mathcal{A}^{G^0\times 1}$. If $N/T \to \kappa \in (0,+\infty)$, similar arguments than Chen, Fernández-Val, and Weidner (2021) apply and bias-correction methods are needed.

6 Monte Carlo Simulations

In this section, I conduct Monte Carlo experiments to assess the numerical performance of NGFE estimators in finite samples, in terms of bias, root mean squared errors (RMSE), classification (Precision, Recall, Rand Index), execution (CPU) time, and inference accuracy (standard errors, standard deviation and coverage). I also report the results for currently available competitors. I consider Chamberlain (1980); Rasch (1960)'s conditional logit (CMLE), nonlinear two-way fixed effects (NLTWFE, see, e.g. Fernández-Val and Weidner, 2016; Mugnier and Wang, 2022), Bonhomme, Lamadon, and Manresa (2022)'s 2-step grouped fixed effects (2GFE), pooled OLS regression, linear two-way fixed effects (LTWFE), and Bonhomme and Manresa (2015)'s GFE estimators.[38]

As in Bonhomme and Manresa (2015), I focus on settings of moderate size ($N = 90$, $T = 7$) to highlight the performance of NGFE with small datasets as often encountered in macro/meso-economics (e.g., in my empirical application). Having large N is not computationally demanding. When T is very large, computation of the NGFE estimate might be demanding and results in Mugnier (2022) could probably be adapted. I consider static and dynamic logit models, and four DGPs for the time-varying covariates (more or less correlated with the unobserved heterogeneity, UH hereafter), where the number of groups G^0 each time varies across $\{2,3,5\}$. Variation across time periods in the covariates is not necessary for NGFE but allows for comparisons (e.g., with CMLE).

Overall, I find that NGFE estimators perform best uniformly across competitors in the design they are meant to adress: correlated time-varying unobserved heterogeneity (DGP 1). In other DGPs, where the unobserved heterogeneity does not vary with time, they might be slightly more noisy than well-suited estimators (e.g., CMLE or NLTWFE) and have a larger finite sample bias.

6.1 Static Logit Model

The data generating process is

$$Y_{it} = \mathbf{1}\{X_{it}\beta + \alpha_{g_i t} > \varepsilon_{it}\},\ i = 1,...,N,\ t = 1,...,T, \tag{21}$$

where $\beta = 1$ and $\varepsilon_{it} \sim$Logit$(0,\pi^2/3)$, $g_i \sim$Unif$\{1,...,G^0\}$ for $G^0 \in \{2,3,5\}$, and, letting with $\mu = (-1,1)'$ if $G^0 = 2$, $\mu = (-\pi/\sqrt{3},0,\pi/\sqrt{3})'$ if $G^0 = 3$, and $\mu = (-2\pi/\sqrt{3},-\pi/\sqrt{3},0,\pi/\sqrt{3},2\pi/\sqrt{3})'$

[38] I leave comparison with Charbonneau (2017)'s conditional logit and Chen, Fernández-Val, and Weidner (2021)'s nonlinear factor models for further research. A definition of the metrics and more details are given in Appendix D.

if $G^0 = 5$, V_i such that $\Pr(V_i = -2) = 1/12, \Pr(V_i = -1) = 1/4, \Pr(V_i = 0) = 1/3, \Pr(V_i = 1) = 1/4, \Pr(V_i = 2) = 1/12$, and $W_{it} \sim \mathcal{N}(0,1)$:

- DGP1 (grouped patterns of time-varying UH): $\alpha_{g0} = \mu_g$, for $t \geq 1$, $\alpha_{gt} = 0.1\alpha_{gt-1} + (-1)^{g-1}U_{gt}$, $U_{gt} \sim$Unif$[0,1]$, $X_{it} = 0.5V_i + 0.8U_{g_i^0 t}$.
- DGP2 (grouped patterns of time-invariant UH): $\alpha_{gt} = \mu_g$, $X_{it} = 0.3\mu_{g_i} + V_i + 0.8W_{it}$.
- DGP3 (continuous time-invariant UH): $\alpha_i \sim \mathcal{N}(0,1)$, $X_{it} = \alpha_i + 0.5V_i + 0.8W_{it}$.
- DGP4 (No UH): $\alpha_{gt} = 0$, $X_{it} = 0.5V_i + 0.8W_{it}$.

The variables U_{gt}, V_i, W_{it}, g_i and ε_{it} are independent and i.i.d. across individuals and time periods. All the results are based on 50 Monte-Carlo replications and computed using Algorithm 1 with 200 randomized initialization points (results improve by increasing this number).

Table 1 reports the bias and RMSE of NGFE and five competing estimators. It shows that NGFE estimates minimize both metrics across all estimators in DGP 1 (e.g., one order of magnitude less than CMLE or 2STEPGFE, the best competitors). If there is no UH (DGP 4), NGFE keeps a reasonable RMSE compared to CMLE but has small bias (e.g. RMSE of .151 v.s. .152 if $G^0 = 2$ and .178 v.s. .118 if $G^0 = 5$, Bias of 0.040 v.s. -0.002 and 0.114 v.s. 0.018 respectively). All linear estimators perform very poorly. The 2-step GFE is more noisy in general.

Table 2 shows that any measure of the clustering accuracy remains at a high level because of the high level of UH. For instance, the misclassification rate only falls below 50% when $G^0 = 2$. In unreported simulations, we show that it actually drops to 5% if one has $G^0 = 2$ and cluster-specific effects are not correlated with the covariates. There is a continum between the two that should merit further investigation. The CPU time of the method is comparable to that of other clustering methods such as Bonhomme, Lamadon, and Manresa (2022)'s 2-step GFE.

Table 3 suggests that estimates of the standard errors based on the large-T clustered variance formula match on average the effective finite sample dispersion of the NGFE estimates. The resulting confidence intervals have an almost correct coverage though showing a small finite-sample under-coverage.[39] In particular, Table 3 suggests good coverage rates around the prescribed theoretical level of 95% (e.g., .86, .80, .84 in DGP 1 and .92, .92, .88 in DGP 4), which fall with the number of groups and, more generally, with the degree of continuity of the UH (e.g., below .5 in DGP 3 but still .82 in DGP 2 with $G^0 = 2$).

[39] A similar finite-sample undercoverage phenomenon is also reported in Bonhomme and Manresa (2015), who suggest the use of a bootstrap estimator instead.

6.2 Dynamic Logit Model

The data generating process is

$$Y_{it} = \mathbf{1}\{Y_{it-1}\beta_1 + X_{it}\beta_2 + \alpha_{g_it} > \varepsilon_{it}\},\ i =,...,N,\ t = 1,...,T,$$
$$Y_{i0} = \mathbf{1}\{X_{i0}\beta_2 + \alpha_{g_i0} > \varepsilon_{i0}\},\ i = 1,...,N. \tag{22}$$

Tables 4-6 report the same statistics as Tables 1-3 but for the dynamic model. Results for β_2 are very similar to that for β. On the other hand, the precision of NGFE estimates of β_1 is more mixed (the conditional independence assumption 9(a) does not hold here). Previous comparison still apply there.

7 Empirical Application: Revisiting the Inverted-U Relationship Between Innovation and Competition

Does more competition lead to more innovation? This question of fundamental importance, e.g., for Antitrust and Competition policy, has been the subject of a longstanding debate in the fields of industrial organization and macroeconomics of endogeneous growth theory (e.g., Gilbert, 2006; Griffith and Van Reenen, 2021). On the one hand, more competition reduces profit and postinnovation rents, and therefore disincentivizes innovation: this is the so-called Schumpetarian effect. On the other hand, competition may reduce a firm's preinnovation rents by more than it reduces its postinnovation rent so that an escape-competition effect may dominate and foster innovation and growth.

In an influential paper, Aghion, Bloom, Blundell, Griffith, and Howitt (2005) reconcile these two contradictory views by documenting an inverted-U relationship between the number of citation-weighted patents and product market competition within a panel data set of UK industries over the period 1973-1994. The inverted-U is predicted by a model of endogeneous growth, and estimated after controlling for additively separable industry and year fixed effects controlling for permanent unobserved technological levels and common trends in a conditional FE Poisson model. The authors assume in their preferred specification: for $p \in \{0,1,...\}$,

$$\Pr(p_{it} = p|c_{it},\nu_i,\xi_t) = \frac{\exp(p(g(c_{it}) + \nu_i + \xi_t)\exp(-\exp(g(c_{it}) + \nu_i + \xi_t)))}{p!}, \tag{23}$$

where p_{it} represents the number of citation-weighted patents in industry i in year t, c_{it} is 1−Lerner index$_{it}$, ν_i is a permanent unobserved level of innovation, ξ_t captures macroeconomic trend, and $g(\cdot)$ is a second-degree polynomial.[40]

Figure 1 replicates their Figure II, which is a scatterplot of an innovation measure (citation-weighted patents) on an competition measure (1 minus the Lerner index) with exponential and nonparametric spline fits predicted by their preferred specification:[41]

While model (23) is in line with a large body of the previous literature (see, e.g., Gourieroux, Monfort, and Trognon, 1984; Hausman, Hall, and Griliches, 1984), it imposes strong assumptions on the data generating process: conditional Poisson distribution and additive separability of unobserved effects. In particular, the inverted-U relationship seems fragile as recent empirical research has reported both increasing and decreasing monotonic relationships depending on the controls included (Aghion, Van Reenen, and Zingales, 2013), whether accounting for structural breaks or not (Correa, 2012), or the country data used (Askenazy, Cahn, and Irac, 2013; Hashmi, 2013), spurring a variety of explanations and theoretical models. Yet, to the best of our knowledge, no paper has assessed the robustness of the inverted-U relationship to modeling choices regarding unobserved heterogeneity. A natural question is then:

Are all industries subject to the same economic trend (time-effect) during the 1973-1994 period where, e.g., the development of I.T. has been exponential and plausibly shaped market structures?

As Aghion, Bloom, Blundell, Griffith, and Howitt (2005) and Correa (2012) argue, innovation is a dynamic process and the potential endogeneity might comes from unobserved forces that drive both innovation and the market structure in a dynamic way. Moreover, while industry might be a good level to control for permanent scaling, it is likely that among the 311 firms of Aghion, Bloom, Blundell, Griffith, and Howitt (2005)'s UK panel, a few time-varying path emerge.

In this section, I illustrate how the class of NGFE models together with semiparametric NGFE estimators introduced in this paper can be used to adress this question, challenging the fact that firms are all subjects to the same macroeconomic trends and that the unobserved propensity to innovate and compete is industry-specific and fixed across time.

[40] The fact that the number of patents is weighted and averaged at the industry level makes it a "continuous" variable with a mass point at 0. This is probably a reason why the authors apply a discrete model. See the summary statistics in Table 7. See Aghion, Bloom, Blundell, Griffith, and Howitt (2005) for details on the construction of each variable.

[41] I note that the scale of the y-axis in ABBGH's Figure II is incorrect, as well as the legend of their Figure I since the graph in fact corresponds to specification (1) in their Table I (and not (2) as claimed).

Data. I use Aghion, Bloom, Blundell, Griffith, and Howitt (2005)'s data available at N. Bloom's website.[42] This is an unbalanced industry-level panel based on 311 firms listed on the London Stock Exchange and grouped in 17 two-digit SIC code industries, which received patent grants from the United States Patent and Trademark Office (USPTO). The period covered by the dataset is from 1973 until 1994 and there are 354 observations. Table 7 reports summary statistics borrowed from Hashmi (2013). In particular, one can see that some industries are never granted patents.[43] Table 8 lists industries of the sample.

Evidence of Time-Varying Unobserved Heterogeneity. Before estimating a NGFE model, I investigate the existence of a latent clustering structure by applying the tetrad pairwise differencing estimator developed in Mugnier (2022) to ABBGH's residuals $p_{it} - \widehat{\mathbb{E}}[p_{it}|c_{it}, \widehat{\nu}_i, \widehat{\xi}_t] - \exp(\widehat{g}(c_{it}) + \widehat{\nu}_i + \widehat{\xi}_t)$, plotted in Figure 2. This smooth exploration method allows for an unconstrained number of clusters, run in polynomial time, provides a regularization path for the number of groups and estimate time-varying effects without relying, e.g., on k-means and local minima.[44]. Figure 3 and Figure 4 plot the regularization path corresponding to the largest plateau, i.e., for a regularization parameter such that $\widehat{G} = 3$, and time effects respectively. Figures 4 reveals one cluster with residuals centered around zero and low variance (red), one cluster with higher volatility and statistically from zero at several periods and statistically different from the first cluster at least at one period (blue), and a very high volatility cluster (green) that consists of industries with missing values. There is evidence of time-varying unobserved heterogeneity.

A Mildly Inverted-U Relationship. I now estimate the following NGFE model:

$$\Pr(p_{it} = p \mid c_{it}, \alpha_{g_i t}, g_i) = \frac{\exp(p(c_{it}\beta_1 + c_{it}^2\beta_2 + \alpha_{g_i t}) \exp(-\exp(c_{it}\beta_1 + c_{it}^2\beta_2 + \alpha_{g_i t})))}{p!}, \tag{24}$$

where $g_i \in \{1, ..., G^0\}$ is industry i's unknown cluster membership and $(\alpha_{1t}, ..., \alpha_{G^0 t})' \in \mathbb{R}^{G^0}$ are time-specific unobserved effects accounting for unobserved confounding variations in the propensity to patent and product market competition in each of the G^0 clusters. Given the small number of industries, I report results for $G^0 \in \{2, 3, 4\}$. Models (23) and (24) are non-nested as $G^0 << N$.

Table 9 and Figure 5 replicate ABBGH's Table I and Figure I, and additionally show results

[42] https://nbloom.people.stanford.edu/sites/g/files/sbiybj4746/f/abbgh.zip.

[43] This does not mean that such industries never innovate. Patenting is an imperfect measure of innovation in several aspects (Boldrin and Levine, 2013). Many studies perform robustnes checks by using R&D expenses as an alternative measure (Aghion, Bloom, Blundell, Griffith, and Howitt, 2005).

[44] Yet, its statistical guarantees are currently not known in the Poisson model.

of NGFE estimation for the choices $G^0 \in \{2,3,4\}$, using 2,000 random initializers around 0^{2+G^0T}. Two results are striking. When $G^0 = 2$, the in-sample relationship (no extrapolation) is a significant increasing relationship. This can be explained by the structure of the clustered effects discussed in the next paragraph: when $G^0 = 2$, I only estimate two clusters that do not exhibit a lot of variation over time. Estimation then acts as a constrained classical fixed effect estimator (where industry-specific effects only have two points of support). When G^0 increases, I find strong evidence of a mildly inverted-U relationship. Estimates of the competition parameters are still significantly different from zero but the inverted-U relationship is dramatically less pronounced (the curve is flatter) when unobserved heterogeneity is allowed to be time-varying.

Clustered Unobserved Innovation Dynamics. The 70-90's are characterized by the extremely rapid development of electronics, networks and the Internet. It is likely that economies of scale, shocks and unobsered innovation trends are not the same for each industry. Figure 6 confirms this intuition by plotting the estimated cluster-specific effects obtained in specifications (3)-(5) from Table 9, where predicted clusters of industries are given in Figure 7.

The NGFE estimates of the unobserved determinants of innovation reveal heterogeneous, time-varying patterns, in particular for $G^0 \geq 3$.

Allowing for two clusters delivers two clusters that experience stable innovation paths over time, albeit at very different levels. Cluster 1, which I refer to as the "high-innovation" cluster, mostly contains highly-patenting, highly-competitive industries. It includes Manufacture of office machinery and data processing equipment, Electrical and electronic engineering, Manufacture of motor vehicles and parts therof, and Manufacture of other transport equipment, but also Chemical industry. Cluster 2, which I refer to as "low-innovation" mostly includes low-patenting, low-competition: metal manufacturing, textile industry, and processing of rubber and plastics, among others. This clustering structure of unobserved heterogeneity is broadly consistent with an additive fixed-effects representation, as the cluster effects $\widehat{\alpha}_{1t}$ and $\widehat{\alpha}_{2t}$ are approximately parallel over time. In contrast, when allowing for more than two clusters, newly estimated clusters are not consistent with a fixed effects model. For $G^0 = 3$, Cluster 2 does not change significantly but the vast majority of industry from Cluster 1 now belongs to Cluster 3 ("steady-catchers") as they experience a steadily increase during the all period towards the unobserved innovation level of Cluster 1. Only the car, food and tobacco, and chemical industries remain in the stable "high-innovation" cluster 1 whereas cluster 3 now includes electrical and electronical engineering, office machinery and data processing equip-

ment. Finally, when $G^0 = 4$, Cluster 3 further split into two neck-to-neck catching-up clusters of industries. The new Cluster 4 ("Noisy-catchers"), which is more volatile in the race, contains other manufacturing industries and transport equipment. Steadily increasing industries now include, among others: Manufacture of office machinery and data processing equipment, and Electrical and electronic engineering.

Figure 8 plots estimated cluster effects, competition and innovation by estimated cluster memberships. It suggests that the relationship between observables and unobservables is complex and hardly predictable from observables only.

Endogeneity. Because competition is likely to be an endogeneous variable, ABBGH use a control function approach by including as an additional regressor in their main specification the residual of a first-stage where the lerner index is predicted by a set of policy instruments such as the Thatcher era privatizations, the EU Single Market Programme, and the Monopoly and Merger Commission investigations ate the industry level (see Table II in ABBGH). The first and fourth columns of Table 10 show that coefficient estimates are similar to Table 9 in the case of NGFE models.

Testing for Structural Break. Finally, I revisit Correa (2012) who tests for the existence of a structural break in 1981, and finds a decreasing relationships before and no effects of competition afterwards. This would spuriously explains ABBGH's inverted-U relationship. In contrast to Correa (2012)'s results, I find no evidence of any relationships in both spells when using a NGFE specification (see Table 10).

8 Conclusion

In this paper, I study the nonparametric identification and estimation of a new class of nonlinear panel data model that accomodates clustered patterns of time-varying unobserved heterogeneity. Sufficient low-level conditions delivering identification of all parameters of the models are provided. Because nonparametric estimation might be overwhelmingly cumbersome in panel data with moderate length, I propose semiparametric NGFE estimators that enjoy nice statistical properties (even when $T << N$) and are free of the incidental parameters problem when $T = o(N)$, which sharply contrasts with many competing approaches. Individual are uniformly classified in the limit as T grows at least as some power of N and cluster-specific and slope coefficient estimates are asymptotically normal (and centered at the true value). A simple Lloyd's algorithm is shown to perform

well in Monte-Carlo simulation. By applying this new estimator to revisit Aghion, Bloom, Blundell, Griffith, and Howitt (2005), I demonstrate that their so-called inverted-U relationship between innovation and product market competition is sensitive to the researcher's choice of whether including time-varying grouped effects in the model or not, and document a data-driven clustering of industries. In particular, once controlling for two groups, the relationships becomes increasing. Once controlling for $3 \leq G \leq 4$ clusters, the relationship becomes a mildly inverted-U.

Interesting research avenues include quantifying the random uncertainty from picking the best output from multiple runs of Lloyd's algorithm with random initializers while basing inference instead on the true NGFE estimate; bridging the gap between the nonparametric identification result and the estimation method, and deriving a complete asymptotic theory accounting for uncertainty in the clustering. I leave such extensions for future work.

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Appendix

A Proof of the Results

A.1 Proof of Theorem 1

Part 1.

Identification of $W_N^0 \in \{0,1\}^{N\times N}$ *for all* $N \in \mathbb{N}^*$. Let $N \in \mathbb{N}^*$. By Assumption 2, there exist $\mathcal{X}^0 \subset \mathcal{X}$, $\overline{y} \in \mathcal{Y}$, and a known functional ϕ such that, for all $(i,j) \in \mathcal{N}^2$, the (i,j)-th entry of W_N^0 satisfies $W_{ijN}^0 := \mathbf{1}\left\{g_i^0 = g_j^0\right\} = \phi(\rho_i, \rho_j)$ with $\rho_i(x) : \mathcal{X}^0 \ni x \mapsto \Pr\left(Y_{i2} = \overline{y} \mid X_{i2} = x, g_i^0, \mu_i^0, \xi_i^0\right)$. It is then sufficient to show that, for all $i \in \mathcal{N}$, ρ_i is identified. Let $(i,x) \in \mathcal{N} \times \mathcal{X}^0$. Under Assumptions 1(b) and 3(a), and conditional on the σ-algebra generated by $\left(g_i^0, \mu_{g_i^0}^{0\prime}, \xi_i^{0\prime}\right)'$, the time-series process $\left\{(Y_{it}, X_{it}')' : t \geq 2\right\}$ is strictly stationary strong mixing and satisfies regularity conditions given in Hansen (2008) to obtain consistency of the Nadaraya-Watson estimator of $\mathbb{E}\left[\mathbf{1}\left\{Y_{it} = \overline{y}\right\} \mid i, X_{it} = x\right]$. Hence, point identification of $\mathbb{E}\left[\mathbf{1}\left\{Y_{i2} = \overline{y}\right\} \mid X_{i2} = x, g_i^0, \mu_{g_i^0}^0, \xi_i^0\right] = \rho_i(x)$ follows by pooling unit i's choices when $(Y_{it}, X_{it}')' \in \{\overline{y}\} \times \mathcal{B}_T(x)$, where $\mathcal{B}_T(x)$ is a well-chosen shrinking neighborhood of x as $T \to \infty$ (e.g., using any well-chosen kernel $\mathcal{K}$ and bandwidth h_T).

Identification of G^0. For any fixed $N \in \mathbb{N}^*$, R_N^0 the number of distinct rows in W_N^0, is identified. But R_N^0, which is also the rank of W_N^0, is exactly the number of clusters represented in the finite sample of size N. Under Assumptions 1(a) and 2(b), $G^0 = \limsup_{N\to\infty} R_N^0$ is thus identified.[45]

Part 2.

Identification of β^0. Let $(i,t) \in \mathbb{N}^{*2}$. By Part 1, $\mathcal{C}^0(i) := \left\{j \in \{1,...,N\} : g_j^0 = g_i^0\right\}$ is identified for all $N \in \mathbb{N}^*$. Under Assumption 1(a) and 2(b), conditional on $(\gamma^{0\prime}, \alpha^{0\prime}, \lambda^{0\prime}, \mu^{0\prime})'$, $\{Y_{jt}, X_{jt} : j \in \mathcal{C}^0(i) \setminus \{i\}\}$ is an identified infinite sequence of i.i.d. random variables. By applying Theorem 4.1 in Ichimura (1993) with $\varphi(\cdot) = \sum_{y\in\mathcal{Y}} y h^0\left(y, \cdot + \alpha_{g_i^0 t}^0\right)$, whose conditions 4.1 and 4.2(1-3) hold under Assumptions 1(c) and 3, β^0 is identified up-to-scale. Because $\|\beta^0\| = 1$, β^0 is identified.

Identification of cluster-specific time effects α_{gt}^0 *for all* $(g,t) \in \mathcal{G}^0 \times \mathbb{N}^*$, *up to cluster relabeling.* Given identification of W_N^0 for all $N \in \mathbb{N}^*$, I build the G^0 groups sequentially starting from $N = 2$, $N = 3$,... and regrouping at each step units with same rows in W_N^0. Without loss of generality, I assume that the resulting labeling matches the true labeling. Let $t \in \mathbb{N}^*$, $x \in \mathcal{X}$, and $\underline{y} \in \mathcal{Y}$ verifying

[45] From an estimation perpective, one would need conditions on the joint rate of convergence of (N,T) to ensure adequate controls on tails of the error terms (ρ_i should typically be estimated in sup-norm on $\mathcal{X}^0$ at some polynomial rate in T).

Assumptions 4. By pooling choices of individuals in cluster g and $\widetilde{g}$ at time t for which $Y_{it} = \underline{y}$ and $X_{it} = x$, and applying a standard LLN using Assumptions 1(a) and 1(c), the following probabilities are identified:

$$\Pr\left(Y_{1t} = \underline{y} \mid X_{1t} = x, g_1^0 = g, \alpha_{gt}^0\right) = h^0\left(\underline{y}, x_1'\beta^0 + \alpha_{gt}^0\right),$$
$$\Pr\left(Y_{1t} = \underline{y} \mid X_{1t} = x, g_1^0 = \widetilde{g}, \alpha_{\widetilde{g}t}^0\right) = h^0\left(\underline{y}, x_1'\beta^0 + \alpha_{\widetilde{g}t}^0\right).$$

By Assumption 5 (eq. (7)), I can find $x_1, x_2 \in \mathcal{X}$ such that

$$\Pr\left(Y_{1t} = \underline{y} \mid X_{1t} = x_2, g_1^0 = g, \alpha_{gt}^0\right) = \Pr\left(Y_{1t} = \underline{y} \mid X_{1t} = x_1, g_1^0 = \widetilde{g}, \alpha_{\widetilde{g}t}^0\right),$$

or, equivalently,

$$h^0\left(\underline{y}, x_1'\beta^0 + \alpha_{\widetilde{g}t}^0\right) = h^0\left(\underline{y}, x_2'\beta^0 + \alpha_{gt}^0\right). \tag{25}$$

By strict monotonicity of $h^0\left(\underline{y}, \cdot\right)$, I can invert (25) and identify $\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0 = (x_2 - x_1)'\beta^0$. As β^0 is already identified, it follows that $\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0$ is identified. Because the result holds for all $(g, \widetilde{g}, t)$, it holds for $g = t = 1$ (for which $\alpha_{gt}^0 = 0$ by the normalization assumption), thus $(\alpha_{g1}^0)_{g \in \mathcal{G}^0}$ is identified. A similar reasoning but now identifying $x_1, x_2 \in \mathcal{X}$ such that eq. (8) holds in place of eq. (7) yields identification of $\alpha_{g\widetilde{t}}^0 - \alpha_{gt}^0$ for all $(g, t, \widetilde{t})$, and, in turn, that of $(\alpha_{1t}^0)_{t \in \mathbb{N}^*}$. Identification of α_{gt}^0 for all (g, t) then follows because, for all (g, t) with $g \neq 1$ and $t \neq 1$, α_{gt}^0 can be decomposed as

$$\alpha_{gt}^0 = \underbrace{\alpha_{gt}^0 - \alpha_{1t}^0}_{:=a} + \underbrace{\alpha_{1t}^0}_{:=b},$$

where a and b are identified. Finally, $h^0(y, z)$ is identified as a function of $y \in \mathcal{Y}$ and index $z = X_{it}'\beta^0 + \alpha_{g_i^0 t}^0$.

The proof of Theorem 1 is complete.

A.2 Sufficient Condition for Assumption 2(a)

Lemma 1 below shows that Assumption 10 is sufficient for Assumption 2(a) to hold.

Assumption 10

(a) There exists an open set $\mathcal{X}^1 \subset \mathcal{X}$ such that, for all $(i, j, g, \tilde{g}, x) \in \mathbb{N}^{*2} \times \mathcal{G}^{02} \times \mathcal{X}^1$, the conditional distribution $\alpha_{g2}^0 \mid X_{i2} = x, g_i^0 = g, \mu_{g_i^0}^0, \xi_i^0$ admits a fully supported density $f_{\alpha_{g2}^0 \mid X_{i2} = x, g_i^0 = g, \mu_g^0, \xi_i^0}(\alpha)$

with respect to the Lebesgue measure such that

$$f_{\alpha^0_{g2}|X_{i2}=x,g^0_i=g,\mu^0_g,\xi^0_i}(\alpha) = f_{\underset{\sim}{\alpha^0_{g2}}|X_{j2}=x,g^0_j=\widetilde{g},\underset{\sim}{\mu^0_g},\xi^0_j}(\alpha), \quad \lambda(\alpha)\text{-}a.e.$$

if and only if $g = \widetilde{g}$.

(b) There exists $k \in \{1,...,p\}$ *such that* $\beta^0_k \neq 0$ *and* $X_{i2,k} \perp\!\!\!\perp \alpha^0_{g^0_i 2} \mid X_{i2,(-k)}, g^0_i, \mu^0_{g^0_i}, \xi^0_i$. *Moreover, almost surely,* $\text{Supp}\left(X_{i2,k} \mid X_{i2,(-k)}, g^0_i, \mu^0_{g^0_i}, \xi^0_i\right)$ *is open.*

(c) There exists $y \in \mathcal{Y}$ *such that* $\psi_y : v \mapsto h^0(y,v)$ *is stricly monotonic, real analytic with bounded first derivative* ψ'_y *such that* $\int \left|\psi'_y\right| d\lambda < \infty$.[46] *Moreover, the characteristic function of* ζ *with density* $f_\zeta(z) = \frac{|\psi'_y(z)|}{\int |\psi'_y| d\lambda}$ *does not vanish and is infinitely often differentiable in* $\mathbb{R}\backslash A$ *for some set* A *such that* $\lambda(A) = 0$.

Real analyticity can be relaxed to continuous differentiability by strenghtenning the support in Assumption 10(b) to be the full real line, which is equivalent to having a special regressor with large support (see, e.g., Honoré and Lewbel, 2002).

Lemma 1 *If Assumptions 1(c) and 10 hold, then Assumption 2(a) holds.*

Proof of Lemma 1 W.l.o.g. I assume that $k = 1$ and denote $x_{(-1)} = (x_j)_{j\in\{2,...,p\}}$. Let $x = (x_1, x'_{(-1)})' \in \mathcal{X}^1$, and $y \in \mathcal{Y}$ verifying Assumption 10(c). I proceed in two steps. In the first step, I construct $\mathcal{X}^0 \subset \mathcal{X}^1$. In the second step, I construct ϕ that fulfills Assumption 2.

Step 1: Let $(i,x) \in \mathcal{N} \times \mathcal{X}^1$ and $\rho_i(x) := \Pr\left(Y_{i2} = y \mid X_{i2} = x, g^0_i, \mu^0_{g^0_i}, \xi^0_i\right)$. By the law of total expectations, Assumption 1(c), using equation (1), and Assumption 10(a), I obtain

$$\begin{aligned}
\rho_i(x) &= \mathbb{E}\left[\Pr\left(Y_{i2} = y \mid X_{i2} = x, g^0_i, \alpha^0, \lambda^0, \mu^0, \xi^0\right) \mid X_{i2} = x, g^0_i, \mu^0_{g^0_i}, \xi^0_i\right] \\
&= \mathbb{E}\left[\Pr\left(Y_{i2} = y \mid X_{i2} = x, g^0_i, \alpha^0_{g^0_i 2}\right) \mid X_{i2} = x, g^0_i, \mu^0_{g^0_i}, \xi^0_i\right] \\
&= \mathbb{E}\left[\psi_y\left(x'\beta^0 + \alpha^0_{g^0_i 2}\right) \mid X_{i2} = x, g^0_i, \mu^0_{g^0_i}, \xi^0_i\right] \\
&= \int_{\mathbb{R}} \psi_y\left(x'\beta^0 + \alpha\right) f_{\alpha^0_{g^0_i 2}|X_{i2}=x,g^0_i,\mu^0_{g^0_i},\xi^0_i}(\alpha) d\lambda(\alpha).
\end{aligned} \tag{26}$$

[46] Let $I \subset \mathbb{R}$ be an open set. A function $f : I \to \mathbb{R}$ is called "analytic" if for any $x_0 \in I$ there is a neighborhood J of x_0 and a power series $\sum a_n (x - x_0)^n$ such that $f(x) = \sum_n a_n (x - x_0)^n \quad \forall x \in J$ (see, e.g., Krantz and Parks, 2002).

By Assumption 10(b), there exists $\epsilon > 0$ and an open set $\mathcal{X}^0 = \{x + (v, 0')' : v \in (-\epsilon, \epsilon)\} \subset \mathcal{X}^1$ with $\Pr(X_{i2} \in \mathcal{X}^0) > 0$ such that, for all $w \in \mathcal{X}^0$, almost everywhere $f_{\alpha^0_{g_i^0 2}|X_{i2}=w, g_i^0, \mu^0_{g_i^0}, \xi_i^0}(\alpha) = f_{\alpha^0_{g_i^0 2}|X_{i2}=x, g_i^0, \mu^0_{g_i^0}, \xi_i^0}(\alpha)$. Since $\mathcal{X}^0 \subset \mathcal{X}^1$, eq. (26) yields, for all $w \in \mathcal{X}^0$,

$$\rho_i(w) = \int_{\mathbb{R}} \psi_y\left(w'\beta^0 + \alpha\right) f_{\alpha^0_{g_i^0 2}|X_{i2}=x, g_i^0, \mu^0_{g_i^0}, \xi_i^0}(\alpha) d\lambda(\alpha).$$

By Assumption 10(c), $w \mapsto \rho_i(w)$ is differentiable on $\mathcal{X}^0$ and, for all $w \in \mathcal{X}^0$,

$$\begin{aligned}\frac{\partial \rho_i(z_1, ..., z_p)}{\partial z_1}\Big|_{z=w} &= \beta_1^0 \int_{\mathbb{R}} \psi_y'\left(w'\beta^0 + \alpha\right) f_{\alpha^0_{g_i^0 2}|X_{i2}=x, g_i^0, \mu^0_{g_i^0}, \xi_i^0}(\alpha) d\lambda(\alpha) \\ &= \beta_1^0 \left(1 - 2\mathbf{1}\left\{\psi_y'(0) < 0\right\}\right) \int_{\mathbb{R}} \left|\psi_y'\left(w'\beta^0 + \alpha\right)\right| f_{\alpha^0_{g_i^0 2}|X_{i2}=x, g_i^0, \mu^0_{g_i^0}, \xi_i^0}(\alpha) d\lambda(\alpha),\end{aligned} \tag{27}$$

where the second equality follows from strict monotonicity of ψ_y.

Step 2: Let $\Delta(a, b) := a - b$ and ∂_1 be the partial differencing operator with respect to the first argument (for multivalued functions). I prove below that $\phi(f, g) := \mathbf{1}\{\Delta(\partial_1 f, \partial_1 g) = 0\}$ verifies Assumption 2(a). I have to show that, for all $(i, j) \in \mathcal{N}^2$,

$$\frac{\partial \rho_i(z_1, ..., z_p)}{\partial z_1}\Big|_{z=w} = \frac{\partial \rho_j(z_1, ..., z_p)}{\partial z_1}\Big|_{z=w} \quad \forall w \in \mathcal{X}^0 \iff g_i^0 = g_j^0. \tag{28}$$

Let $(i, j) \in \mathcal{N}^2$.

$\underline{\Longleftarrow:}$ Suppose that $g_j^0 = g_i^0$ and let $w \in \mathcal{X}^0$. By Assumption 10(c), I have

$$f_{\alpha^0_{g_i^0 2}|X_{i2}=x, g_i^0, \mu^0_{g_i^0}, \xi_i^0}(\alpha) = f_{\alpha^0_{g_j^0 2}|X_{j2}=x, g_j^0, \mu^0_{g_j^0}, \xi_j^0}(\alpha), \quad \lambda(\alpha) - \text{a.e.}.$$

Equation (27) then implies $\frac{\partial \rho_i(z_1, ..., z_p)}{\partial z_1}\Big|_{z=w} = \frac{\partial \rho_j(z_1, ..., z_p)}{\partial z_1}\Big|_{z=w}$.

$\underline{\Longrightarrow:}$ Suppose that, for all $w \in \mathcal{X}^0$,

$$\frac{\partial \rho_i(z_1, ..., z_p)}{\partial z_1}\Big|_{z=w} = \frac{\partial \rho_j(z_1, ..., z_p)}{\partial z_1}\Big|_{z=w}.$$

Dividing each side of this equation by $\int \left|\psi_y'\right| d\lambda > 0$, using (27) and the fact that

$$\left|\left(1 - 2\mathbf{1}\left\{\psi_y'(0) < 0\right\}\right)\beta_1^0\right| = \left|\beta_1^0\right| > 0,$$

I obtain, denoting $f_{\alpha^0_{g_i^0}}(\alpha) := f_{\alpha^0_{g_i^0 2}|X_{i2},g_i^0,\mu^0_{g_i^0},\xi_i^0}(\alpha)$, for all $w \in \mathcal{X}^0$,

$$\int_{\mathbb{R}} f_\zeta\left(w'\beta^0+\alpha\right) f_{\alpha^0_{g_i^0}}(\alpha)d\lambda(\alpha) = \int_{\mathbb{R}} f_\zeta\left(w'\beta^0+\alpha\right) f_{\alpha^0_{g_j^0}}(\alpha)d\lambda(\alpha).$$

I show below that this constraint is equivalent to $f_{\alpha^0_{g_j^0}} = f_{\alpha^0_{g_i^0}}$ a.e., which, by Assumption 10(a), in turn implies $g_i^0 = g_j^0$. Specifically, I show that the solution set $\mathcal{S}^* \subset L^1(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$ to the integral inverse problem: $f_\alpha \in \mathcal{S}^*$ if and only if

$$\int_{\mathbb{R}} f_\zeta\left(w'\beta^0+\alpha\right) f_{\alpha^0_{g_i^0}}(\alpha)d\lambda(\alpha) = \int_{\mathbb{R}} f_\zeta\left(w'\beta^0+\alpha\right) f_\alpha(\alpha)d\lambda(\alpha) \quad \forall w \in \mathcal{X}^0, \tag{29}$$

verifies $\mathcal{S}^* = \left\{ f \in L^1(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda) : f_\alpha = f_{\alpha^0_{g_i^0}} \text{ a.e.} \right\}$. Suppose $f^*_\alpha \in \mathcal{S}^*$ and consider the change of variable $z = w'\beta^0+\alpha$ in (29). Then, for all $\delta \in (x'\beta^0 - \beta_1^0\epsilon, x'\beta^0+\beta_1^0\epsilon) \subset \mathbb{R}$,

$$\int_{\mathbb{R}} f_\zeta(z) f_{-\alpha^0_{g_i^0}}(\delta - z)d\lambda(z) = \int_{\mathbb{R}} f_\zeta(z) f^*_{-\alpha}(\delta - z)d\lambda(z). \tag{30}$$

Note that both sides of eq. (30) are convolutions of f_ζ with $df_{-\alpha^0_{g_i^0}}$ or $df^*_{-\alpha}$. By letting

$$\mathcal{W} : \delta \mapsto \int_{\mathbb{R}} f_\zeta(\delta - z)\left[f_{-\alpha^0_{g_i^0}}(z) - f^*_{-\alpha}(z)\right]d\lambda(z),$$

and using commutativity of the convolution product, eq. (30) implies that there exists an open set $U \subset \mathbb{R}$ such that

$$\mathcal{W}(\delta) = 0, \quad \forall \delta \in U. \tag{31}$$

Given Assumption 10(c), it can be shown that $\mathcal{W} : \mathbb{R} \to \mathbb{R}$ is real-analytic (see footnote 46). A continuation theorem for real analytic functions (see e.g. Corollary 1.2.5 in Krantz and Parks, 2002) implies that eq. (31) holds for all $\delta \in \mathbb{R}$, i.e.:

$$\int_{\mathbb{R}} f_\zeta(\delta - z)\left[f_{-\alpha^0_{g_i^0}}(z) - f^*_{-\alpha}(z)\right]d\lambda(z) = 0, \quad \forall \delta \in \mathbb{R}. \tag{32}$$

Since the functions $f_\zeta, f_{-\alpha^0_{g_i^0}}$, and $f^*_{-\alpha}$ belong to $L^1(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, I can apply Fourrier transformation

on both sides of eq. (32) to obtain

$$\varphi_{f_\zeta}(v) \times \left[\varphi_{f_{-\alpha^0_{g_i^0}}}(v) - \varphi_{f^*_{-\alpha}}(v) \right] = 0, \quad \forall v \in \mathbb{R}, \tag{33}$$

where φ_f is the Fourrier transform of f. By Assumption 10(c) again, the set

$$\{v \in \mathbb{R} : \varphi_\zeta(v) = 0\}$$

is of zero Lebesgue measure. Equation (33) therefore implies $\varphi_{f_{-\alpha^0_{g_i^0}}} = \psi_{f^*_{-\alpha}}$ a.e.. Since Fourrier transforms are continuous, I obtain $\varphi_{f_{-\alpha^0_{g_i^0}}} = \varphi_{f^*_{-\alpha}}$ everywhere and thus $f_{\alpha^0_{g_i^0}} = f^\star_\alpha$ everywhere. The proof of Lemma 1 is complete.

A.3 Proof of Theorem 2

The key argument is to linearize problem (17) by mean of a second-order Taylor expansion, bounding the log-likelihood function by below by a quadratic function similar to that appearing in Lemma A.2 in Bonhomme and Manresa (2015). For all $\theta = (\beta', \alpha', \gamma')' \in \mathcal{B} \times \mathcal{G}^{0N} \times \mathcal{A}^{G^0 T}$, define

$$\widehat{Q}(\theta) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} -\ln\left(\Psi\left(Q_{it} Z_{it}\right)\right),$$

where $Z_{it} = X'_{it}\beta + \alpha_{g_i t}$ and $Q_{it} = 2Y_{it} - 1$. Note that Z_{it} is an implicit function of θ but I drop this conditioning for the sake of clarity and let $Z^0_{it} = X'_{it}\beta^0 + \alpha^0_{g_i^0 t}$ denote Z_{it} evaluated at the true parameter value θ^0. Note that the NGFE estimator $\widehat{\theta}$ minimizes $\widehat{Q}(\cdot)$ over all $\theta \in \mathcal{B} \times \mathcal{G}^{0N} \times \mathcal{A}^{G^0 T}$. Define the auxiliary quadratic function:

$$\check{Q}(\theta) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(X'_{it}\left(\beta - \beta^0\right) + \alpha_{g_i t} - \alpha^0_{g_i^0 t} \right)^2,$$

and let $\overline{z} := \sup_{(\beta', \alpha', g, x)' \in \mathcal{B} \times \mathcal{A}^{\mathcal{G}^0 T} \times \mathcal{G}^0 \times \cup_{t=1,\ldots,i=1,\ldots} \mathrm{Supp}(X_{it})} |Z_{it}|$ and $\mathcal{Z} = [-\overline{z}, \overline{z}]$. Note that $\mathcal{Z}$ is a well-defined segment of $\mathbb{R}$ by Assumptions 7(a) and 7(b). By second-order Taylor expansion, for any z_1, z_2 in $\mathcal{Z}$,

$$-\ln\Psi(z_1) = -\ln\Psi(z_2) - (\ln\Psi)'(z_2)(z_1 - z_2) - \frac{1}{2}(\ln\Psi)''(z^*)(z_1 - z_2)^2,$$

for some $z^* \in]z_1 \wedge z_2, z_1 \vee z_2[$. By continuity of $z \mapsto -(\ln \Psi)''(z)$ and because $-(\ln \Psi)''(z) > 0$ by Assumption 6(b), there exists a constant $b_{\min} > 0$ such that, for all $z \in \mathcal{Z}$,

$$b_{\min} \leq -(\ln \Psi)''(z).$$

Hence, for all $z_1, z_2 \in \mathcal{Z}$

$$-\ln \Psi(z_1) \geq -\ln \Psi(z_2) + s(z_2)(z_1 - z_2) + \frac{b_{\min}}{2}(z_1 - z_2)^2, \tag{34}$$

where $s(z) = -(\ln \Psi)'(z)$. Now substitute $Q_{it}Z_{it}$ for z_1 and $Q_{it}Z_{it}^0$ for z_2, and averaging (34) over i,t, I have, for all $\theta \in \mathcal{B} \times \mathcal{G}^{0N} \times \mathcal{A}^{G^0 T}$,

$$\widehat{Q}(\theta) - \widehat{Q}(\theta^0) \geq \frac{b_{\min}}{2}\check{Q}(\theta) + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} E_{it}\left(Q_{it}\left(Z_{it} - Z_{it}^0\right)\right), \tag{35}$$

where $E_{it} = s\left(Q_{it}Z_{it}^0\right)$. Since the estimated parameter $\widehat{\theta}$ minimizes $\widehat{Q}(\cdot)$, deduce

$$0 \geq \widehat{Q}(\widehat{\theta}) - \widehat{Q}(\phi^0) \geq \frac{b_{\min}}{2}\check{Q}(\widehat{\theta}) + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} E_{it}\left(Q_{it}\left(\widehat{Z}_{it} - Z_{it}^0\right)\right), \tag{36}$$

where $\widehat{Z}_{it} = X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_{it}t}$. I start by showing the following uniform convergence result, which is reminiscent of Lemma A.1 in Bonhomme and Manresa (2015).

Lemma 2 *Let Assumption 6 and Assumptions 7(a)-(b) hold. Then,*

$$\sup_{\theta \in \mathcal{B} \times \mathcal{G}^{0N} \times \mathcal{A}^{G^0 T}} \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} E_{it}\left(Q_{it}\left(Z_{it} - Z_{it}^0\right)\right) = o_p(1).$$

Proof of Lemma 2: The proof closely follows that of Lemma A.1 in Bonhomme and Manresa (2015), up to a few adjustments.

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} E_{it}\left(Q_{it}\left(Z_{it} - Z_{it}^0\right)\right) \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} Q_{it}E_{it}\left(X_{it}'\left(\beta - \beta^0\right) + \alpha_{g_i t} - \alpha_{g_i^0 t}^0\right) \\
&= \left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} Q_{it}E_{it}X_{it}\right)'\left(\beta - \beta^0\right) + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} E_{it}Q_{it}\alpha_{g_i t} - \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} E_{it}Q_{it}\alpha_{g_i^0 t}^0.
\end{aligned}$$

Let $\mathcal{F}_t = \sigma\left(\left\{\gamma^0, \alpha^0, \mathbf{X}_-^{(t)}, \boldsymbol{\varepsilon}_-^{(t-1)}\right\}\right)$ denote the σ-field generated by $\gamma^0, \alpha^0, \mathbf{X}_-^{(t)}$, and $\boldsymbol{\varepsilon}_-^{(t-1)}$. Under Assumptions 6(a) and 6(b), for all $s < t$, I have

$$\begin{aligned}
\mathbb{E}\left(Q_{it}Q_{is}E_{it}E_{is}X_{it}'X_{is}\right) &= \mathbb{E}\left(\mathbb{E}(Q_{it}Q_{is}E_{it}E_{is}X_{it}'X_{is} \mid \mathcal{F}_t)\right) \\
&= \mathbb{E}\left(X_{it}'X_{is}Q_{is}E_{is}\mathbb{E}(Q_{it}E_{it} \mid \mathcal{F}_t)\right) \\
&= \mathbb{E}\left(X_{it}'X_{is}Q_{is}E_{is}\mathbb{E}\left(\frac{Y_{it} - \Psi(Z_{it}^0)}{\Psi(Z_{it}^0)(1-\Psi(Z_{it}^0))}\Psi'(Z_{it}^0) \mid \mathcal{F}_t\right)\right) \\
&= \mathbb{E}\left(X_{it}'X_{is}Q_{is}E_{is}\underbrace{\frac{\mathbb{E}(Y_{it} - \Psi(Z_{it}^0) \mid \mathcal{F}_t)}{\Psi(Z_{it}^0)(1-\Psi(Z_{it}^0))}\Psi'(Z_{it}^0)}_{=0}\right) \\
&= 0,
\end{aligned}$$

where the penultimate equality follows because $\Psi'(Z_{it}^0)$ is $\mathcal{F}_t$-measurable, and the last equality follows from $\mathbb{E}(Y_{it} \mid \mathcal{F}_t) = \Psi(Z_{it}^0)$. By Cauchy-Schwarz (CS) inequality, and using Assumption 6(b), 7(b), and $Q_{it}^2 = 1$, there exists a constant $M' > 0$ such that, for $s = t$,

$$\mathbb{E}\left(Q_{it}Q_{is}E_{it}E_{is}X_{it}'X_{is}\right) = \mathbb{E}\left(E_{it}^2\|X_{it}\|^2\right) \leq \sqrt{\mathbb{E}\left(E_{it}^4\right)\mathbb{E}\left(\|X_{it}\|^4\right)} \leq M' < \infty.$$

Hence, I have

$$\left|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}\left(Q_{it}Q_{is}E_{it}E_{is}X_{it}'X_{is}\right)\right| \leq M'. \tag{37}$$

By (37), I have

$$\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}Q_{it}E_{it}X_{it}\right\|^2\right) \leq \frac{M'}{T},$$

so it follows from the Markov inequality that

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}Q_{it}E_{it}X_{it} = o_p(1).$$

In addition, $\|\beta - \beta^0\|$ is bounded under Assumption 7(a), hence

$$\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}Q_{it}E_{it}X_{it}\right)'(\beta - \beta^0) = o_p(1).$$

I next show that $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} Q_{it}E_{it}\alpha_{g_it}$ is $o_p(1)$, uniformly on the parameter space. This will imply that $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} Q_{it}E_{it}\alpha^0_{g_i^0 t} = o_p(1)$. I have

$$\begin{aligned}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} Q_{it}E_{it}\alpha_{g_it} &= \sum_{g\in\mathcal{G}^0}\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{1}\{g_i=g\}Q_{it}E_{it}\alpha_{gt}\right] \\ &= \sum_{g\in\mathcal{G}^0}\left[\frac{1}{T}\sum_{t=1}^{T}\alpha_{gt}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i=g\}Q_{it}E_{it}\right)\right].\end{aligned}$$

Moreover, by the CS inequality and for all $g\in\mathcal{G}^0$:

$$\left(\frac{1}{T}\sum_{t=1}^{T}\alpha_{gt}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i=g\}Q_{it}E_{it}\right)\right)^2 \le \left(\frac{1}{T}\sum_{t=1}^{T}\alpha_{gt}^2\right)\times\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i=g\}Q_{it}E_{it}\right)^2\right),$$

where, by Assumption 7(a), $\frac{1}{T}\sum_{t=1}^{T}\alpha_{gt}^2$ is uniformly bounded. Now, note that

$$\begin{aligned}\frac{1}{T}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i=g\}Q_{it}E_{it}\right)^2 &= \frac{1}{TN^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{1}\{g_i=g\}\mathbf{1}\{g_j=g\}\sum_{t=1}^{T}Q_{it}Q_{jt}E_{it}E_{jt} \\ &\le \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}Q_{it}Q_{jt}E_{it}E_{jt}\right| \\ &\le \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}(Q_{it}Q_{jt}E_{it}E_{jt})\right| \\ &\quad + \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}(Q_{it}Q_{jt}E_{it}E_{jt}-\mathbb{E}(Q_{it}Q_{jt}E_{it}E_{jt}))\right|.\end{aligned}$$

Since $\mathbb{E}(Q_{it}Q_{jt}E_{it}E_{jt})=0$ for $i\neq j$, there exists a constant $M''>0$ such that

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}(Q_{it}Q_{jt}E_{it}E_{jt})\right| \le M'' < \infty,$$

and, therefore, $\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}(Q_{it}Q_{jt}E_{it}E_{jt})\right| \le \frac{M''}{N}$. Moreover, by the CS inequality,

$$\begin{aligned}&\left(\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}(Q_{it}Q_{jt}E_{it}E_{jt}-\mathbb{E}(Q_{it}Q_{jt}E_{it}E_{jt}))\right|\right)^2 \\ &\quad\le \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}(Q_{it}Q_{jt}E_{it}E_{jt}-\mathbb{E}(Q_{it}Q_{jt}E_{it}E_{jt}))\right)^2. \qquad (38)\end{aligned}$$

Similarly again, I can show that there exists a constant $M''' > 0$ such that

$$\left| \frac{1}{N^2 T} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \text{Cov}(Q_{it} Q_{jt} E_{it} E_{js}, Q_{is} q_{js} E_{is} E_{js}) \right| \leq M''' < \infty.$$

Hence, the term in the right-hand side of (38) is bounded in expectation by M'''/T. This shows that $\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} Q_{it} E_{it} \alpha_{g_i t}$ is uniformly $o_p(1)$, and ends the proof of Lemma 2. □

Next, by Lemma A.2 in Bonhomme and Manresa (2015), it follows that

$$\check{Q}(\widehat{\theta}) \geq \widehat{\rho} \|\widehat{\beta} - \beta^0\|^2, \tag{39}$$

where $\text{plim}_{N,T \to \infty} \widehat{\rho} = \rho > 0$. Hence, combining (36), Lemma 2, and (39) I obtain

$$0 \geq \frac{b_{\min} \rho}{2} \|\widehat{\beta} - \beta^0\|^2 + o_p(1),$$

from which it is concluded that $\widehat{\beta} = \beta^0 + o_p(1)$.

Lastly, to show convergence in quadratic mean of the estimated unit-specific effects, note that

$$\begin{aligned}
&\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\widehat{\alpha}_{\widehat{g}_i t} - \alpha^0_{g_i^0 t} \right)^2 \\
&\quad = \check{Q}(\theta) - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} X_{it}' \left(\beta^0 - \widehat{\beta} \right) X_{it}' \left(\beta^0 - \widehat{\beta} \right) - \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} X_{it}' \left(\beta^0 - \widehat{\beta} \right) \left(\alpha^0_{g_i^0 t} - \widehat{\alpha}_{\widehat{g}_i t} \right) \\
&\quad \leq \check{Q}(\theta) - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \|X_{it}\|^2 \times \|\beta^0 - \widehat{\beta}\|^2 \\
&\qquad + \left(4 \sup_{\alpha \in \mathcal{A}} |\alpha| \right) \times \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \|X_{it}\| \times \|\beta^0 - \widehat{\beta}\|,
\end{aligned}$$

which is $o_p(1)$ by Assumptions 7(a)-7(b), by consistency of $\widehat{\beta}$, and because Lemma 2 and (36) together imply $\check{Q}(\widehat{\theta}) = o_p(1)$.

This completes the proof of Theorem 2.

A.4 Proof of Theorem 3

A.4.1 Step 1: A Useful Asymptotic Equivalence

Lemma 7 below provides an asymptotic equivalence result which is key to prove Theorem 3. I first prove three lemmas (3, 4, and 5) that help in showing that NGFE estimators achieve uniformly

consistent classification of individuals (Lemma 6). This, in turn, allows me to prove Lemma 7.

First, consistency of $\widehat{\alpha}$ for α^0 can be established as in Bonhomme and Manresa (2015). Because the objective function is invariant to relabeling of the cluster labels, the consistency result holds with respect to the Hausdorff distance d_H in $\mathbb{R}^{G^0T}$, defined by

$$d_H(a,b)^2 = \max\left\{\max_{g\in\mathcal{G}^0}\left(\min_{\widetilde{g}\in\mathcal{G}^0}\frac{1}{T}\sum_{t=1}^{T}\left(a_{\widetilde{g}t}-b_{gt}\right)^2\right), \max_{\widetilde{g}\in\mathcal{G}^0}\left(\min_{g\in\mathcal{G}^0}\frac{1}{T}\sum_{t=1}^{T}\left(a_{\widetilde{g}t}-b_{gt}\right)^2\right)\right\}.$$

Lemma 3 *Let Assumptions 6-7, and 8(a)-8(b) hold. Then, as N and T tend to infinity,*

$$d_H\left(\widehat{\alpha},\alpha^0\right) \xrightarrow{p} 0.$$

Proof of Lemma 3: Given Theorem 2, the proof is identical to that of Lemma B.3 in Bonhomme and Manresa (2015). □

Second, I rely on the use of exponential inequalities for dependent processes. Lemma 4 and Lemma 5 are direct consequences of Theorem 6.2 in Rio (2000) (see also Merlevède, Peligrad, and Rio, 2011) and Theorem 3.2 in Lesigne and Volný (2001), respectively.

Lemma 4 (Bonhomme and Manresa (2015), Lemma B.5) *Let z_t be a strongly mixing process with zero mean, with strong mixing coefficient $\alpha[t] \leq \exp(-at^{d_1})$, and tail probabilities $\Pr(|z_t| < z) \leq \exp\left(1-\left(\frac{z}{b}\right)^{d_2}\right)$, where a, b, d_1, and d_2 are positive constants. Then, for all $z>0$, for all $\delta>0$,*

$$T^{\delta}\Pr\left(\left|\frac{1}{T}\sum_{t=1}^{T}z_t\right| \geq z\right) \to 0, \text{ as } T\to\infty.$$

Lemma 5 [47] *Let $\{z_t,\mathcal{F}_t\}_{t=1}^{T}$ be a martingale difference sequence and assume that there exists $\delta, M > 0$ such that $E(\exp(\delta|z_t|)) \leq M$ for all $t=1,...,T$. Then, for $a>0$, there exist positive constants A and B such that for all $z \geq a/\sqrt{T}$*

$$\Pr\left(\left|\frac{1}{T}\sum_{t=1}^{T}z_t\right| \geq z\right) \leq A\exp\left(-B(z^2T)^{1/3}\right).$$

[47] I found this result in a 2013 unpublished manuscript by A.-B. Kock entitled "Oracle inequalities and variable selection in high-dimensional panel data models" (Lemma 2). For completeness, I report the original proof of the author here.

Proof of Lemma 5: In the proof of their Theorem 3.2 Lesigne and Volný (2001) show that if $E(\exp(|z_t|) \leq M$ for all $t = 1,...,T$, then for any $x > 0$ and $t \in (0,1)$, I have

$$\begin{aligned}
&\Pr\left(\left|\sum_{t=1}^{T} z_t\right| > Tz\right) \\
&< \left(2 + \frac{M}{(1-t)^2}\left[\frac{1}{4}t^{4/3}(z^{-2}T^{-1})^{1/3} + t^{2/3}(z^{-2}T^{-1})^{2/3} + 2z^{-2}T^{-1}\right]\right) \\
&\quad \times \exp\left(-\frac{1}{2}t^{2/3}(z^2T)^{1/3}\right). \qquad (40)
\end{aligned}$$

Note that $\Pr\left(\left|\sum_{t=1}^{T} z_t\right| > Tz\right) = \Pr\left(\left|\sum_{t=1}^{T}(\delta z_t)\right| > T(\delta z)\right)$ where $\{\delta z_t\}_{1\leq t\leq T}$, by assumption now satisfy the conditions of Theorem 3.2 in Lesigne and Volný (2001) and so replacing z by δz in (40) yields

$$\begin{aligned}
&\Pr\left(\left|\sum_{t=1}^{T} z_t\right| > Tz\right) \\
&< \left(2 + \frac{M}{(1-t)^2}\left[\frac{1}{4}t^{4/3}\delta^{-2/3}(z^{-2}T^{-1})^{1/3} + t^{2/3}\delta^{-4/3}(z^{-2}T^{-1})^{2/3} + 2\delta^{-2}z^{-2}T^{-1}\right]\right) \\
&\quad \times \exp\left(-\frac{1}{2}t^{2/3}\delta^{2/3}(z^2T)^{1/3}\right).
\end{aligned}$$

Restricting z to be greater than $a/\sqrt{T}$, implying that $z^{-2}T^{-1} \leq 1/a^2$, and using that M, t and δ are constants the conclusion of the lemma follows. □

I am now in position to prove Lemma 6 which extends Lemma B.4 in Bonhomme and Manresa (2015) and shows that $\widehat{g}_i(\beta,\alpha)$ achieves uniformly consistent classification of individuals over a neighbourhood of the true parameter values (β^0,α^0). Note that by the same arguments as in the proof of Lemma B.3 in Bonhomme and Manresa (2015), there exists a permutation $\sigma : \mathcal{G}^0 \to \mathcal{G}^0$ such that

$$\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\sigma(g)t} - \alpha^0_{gt}\right)^2 \xrightarrow{p} 0. \qquad (41)$$

By simple relabeling of the elements of $\widehat{\alpha}$, I may take $\sigma(g) = g$. I adopt this convention in the rest of the proof. For any $\eta > 0$, I let $\mathcal{N}_\eta$ denote the set of parameters $(\beta,\alpha) \in \mathcal{B} \times \mathcal{A}^{G^0T}$ that satisfy $\|\beta - \beta^0\|^2 < \eta$ and $\frac{1}{T}\sum_{t=1}^{T}\left(\alpha_{gt} - \alpha^0_{gt}\right)^2 < \eta$ for all $g \in \mathcal{G}^0$.

Lemma 6 *For $\eta > 0$ small enough, I have, for all $\delta > 0$ and as N and T tend to infinity,*

$$\sup_{(\beta,\alpha)\in\mathcal{N}_\eta} \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i(\beta,\alpha) \neq g_i^0\right\} = o_p(T^{-\delta}).$$

Proof of Lemma 6: Note that, from the definition of $\widehat{g}_i(\cdot)$, for all $g \in \mathcal{G}^0$,

$$\mathbf{1}\{\widehat{g}_i(\beta,\alpha) = g\} \leq \mathbf{1}\left\{\sum_{t=1}^{T} \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + \alpha_{g_i^0 t}\right)\right)\right) \leq \sum_{t=1}^{T} \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + \alpha_{gt}\right)\right)\right)\right\},$$

so

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i(\beta,\alpha) \neq g_i^0\right\} &= \sum_{g \in \mathcal{G}^0} \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\left\{g_i^0 \neq g\right\} \mathbf{1}\left\{\widehat{g}_i(\beta,\alpha) = g\right\} \\ &\leq \sum_{g \in \mathcal{G}^0} \frac{1}{N}\sum_{i=1}^{N} W_{ig}(\beta,\alpha),\end{aligned}$$

where

$$W_{ig}(\beta,\alpha) = \mathbf{1}\left\{g_i^0 \neq g\right\} \times \mathbf{1}\left\{\sum_{t=1}^{T} \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + \alpha_{g_i^0 t}\right)\right)\right) \leq \sum_{t=1}^{T} \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + \alpha_{gt}\right)\right)\right)\right\}.$$

I start bounding $W_{ig}(\beta,\alpha)$, for all $(\beta,\alpha) \in \mathcal{N}_\eta$, by a quantity that does not depend on (β,α). To proceed first note that, by Assumption 6(b), and 7(a)-7(b), $v \mapsto \ln\left(\Psi\left(Q_{it}\left(X_{it}'v + \alpha_{gt}\right)\right)\right)$ is uniformly Lipschitz over $(i,t,\alpha,g) \in \{1,...,N\} \times \{1,...,T\} \times A^{G^0 T} \times \mathcal{G}^0$, i.e., there exists a constant $L_\beta > 0$ such that, for all $(i,t,\alpha,g) \in \{1,...,N\} \times \{1,...,T\} \times A^{G^0 T} \times \mathcal{G}^0$, all $\beta_1, \beta_2 \in \mathcal{B}$, almost surely

$$\left|\ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta_1 + \alpha_{gt}\right)\right)\right) - \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta_2 + \alpha_{gt}\right)\right)\right)\right| \leq L_\beta \|\beta_1 - \beta_2\|. \tag{42}$$

Similarly, $a \mapsto \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + a\right)\right)\right)$ is uniformly Lipschitz over $(i,t,\beta) \in \{1,...,N\} \times \{1,...,T\} \times \mathcal{B}$, i.e., there exists a constant $L_\alpha > 0$ such that, for all $(i,t,\beta) \in \{1,...,N\} \times \{1,...,T\} \times \mathcal{B}$, all $a, b \in \mathcal{A}$, almost surely

$$\left|\ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + a\right)\right)\right) - \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + b\right)\right)\right)\right| \leq L_\alpha |a - b|. \tag{43}$$

Then, by choosing $g = g_i^0, \beta_1 = \beta^0$ and $\beta_2 = \beta$ in (42), I have, for all (β,α) and all i,

$$\begin{aligned}W_{ig}(\beta,\alpha) \leq\ & \mathbf{1}\left\{g_i^0 \neq g\right\} \\ & \times \mathbf{1}\left\{\sum_{t=1}^{T} \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta^0 + \alpha_{g_i^0 t}\right)\right)\right) \leq \sum_{t=1}^{T} \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + \alpha_{gt}\right)\right)\right) + TL_\beta\|\beta - \beta^0\|\right\}.\end{aligned}$$

By choosing $a=\alpha_{g_i^0 t}$, $b=\alpha^0_{g_i^0}$, and $\beta=\beta^0$ in (43), I have, for all (β,α) and all i,

$$\begin{aligned}W_{ig}(\beta,\alpha) \leq\ & \mathbf{1}\left\{g_i^0 \neq g\right\} \\ & \times \mathbf{1}\left\{\sum_{t=1}^{T} \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta^0+\alpha^0_{g_i^0 t}\right)\right)\right) \leq \sum_{t=1}^{T} \ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta+\alpha_{gt}\right)\right)\right)\right. \\ & \left. +TL_\beta\|\beta-\beta^0\|+TL_\alpha\|\alpha^0_{g_i^0}-\alpha_{g_i^0}\|\right\},\end{aligned}$$

where I used the norm inequality $\|u\|_1 \leq \sqrt{T}\|u\| \leq T\|u\|$ for all $u\in\mathbb{R}^T$, $T\in\mathbb{N}^*$, where $\|\cdot\|_1$ is the ℓ^1-norm. Next, a second-order Taylor expansion of $z\mapsto \ln\Psi(z)$ at $Q_{it}Z_{it}$ around $Q_{it}Z^0_{it}$ combined with (A.3), yields

$$\begin{aligned}W_{ig}(\beta,\alpha) \leq\ & \mathbf{1}\left\{g_i^0 \neq g\right\} \\ & \times \mathbf{1}\left\{0 \leq \sum_{t=1}^{T} \frac{Y_{it}-\Psi\left(Z^0_{it}\right)}{\Psi\left(Z^0_{it}\right)\left(1-\Psi\left(Z^0_{it}\right)\right)}\Psi'\left(Z^0_{it}\right)\left(X_{it}'\left(\beta-\beta^0\right)+\alpha_{gt}-\alpha^0_{g_i^0 t}\right)\right. \\ & \left. -\frac{b_{\min}}{2}\left(X_{it}'\left(\beta-\beta^0\right)+\alpha_{gt}-\alpha^0_{g_i^0 t}\right)^2+TL_\beta\|\beta-\beta^0\|+TL_\alpha\|\alpha^0_{g_i^0}-\alpha_{g_i^0}\|\right\} \\ \leq\ & \max_{\widetilde{g}\neq g} \mathbf{1}\left\{0 \leq \sum_{t=1}^{T}\left[\frac{Y_{it}-\Psi\left(Z^0_{it}\right)}{\Psi\left(Z^0_{it}\right)\left(1-\Psi\left(Z^0_{it}\right)\right)}\Psi'\left(Z^0_{it}\right)\left(X_{it}'\left(\beta-\beta^0\right)+\alpha_{gt}-\alpha^0_{\widetilde{g}t}\right)\right.\right. \\ & \left.\left. -\frac{b_{\min}}{2}\left(X_{it}'\left(\beta-\beta^0\right)+\alpha_{gt}-\alpha^0_{\widetilde{g}t}\right)^2\right]+TL_\beta\|\beta-\beta^0\|+TL_\alpha\|\alpha^0_{\widetilde{g}}-\alpha_{\widetilde{g}}\|\right\},\end{aligned}$$

Now, let define $V_{it}=\frac{Y_{it}-\Psi\left(Z^0_{it}\right)}{\Psi\left(Z^0_{it}\right)\left(1-\Psi\left(Z^0_{it}\right)\right)}\Psi'\left(Z^0_{it}\right)$, and

$$\begin{aligned}A_T=\Bigg|&\sum_{t=1}^{T}\left[V_{it}\left(X_{it}'\left(\beta-\beta^0\right)+\alpha_{gt}-\alpha^0_{\widetilde{g}t}\right)-\frac{b_{\min}}{2}\left(X_{it}'\left(\beta-\beta^0\right)+\alpha_{gt}-\alpha^0_{\widetilde{g}t}\right)^2\right]+TL_\beta\|\beta-\beta^0\| \\ &+TL_\alpha\|\alpha^0_{\widetilde{g}}-\alpha_{\widetilde{g}}\|-\sum_{t=1}^{T}V_{it}\left(\alpha^0_{gt}-\alpha^0_{\widetilde{g}t}\right)+\frac{b_{\min}}{2}\left(\alpha^0_{gt}-\alpha^0_{\widetilde{g}t}\right)^2\Bigg|.\end{aligned}$$

As I have

$$\begin{aligned}A_T \leq\ & \left|\sum_{t=1}^{T}V_{it}X_{it}'\left(\beta-\beta^0\right)\right|+\left|\sum_{t=1}^{T}V_{it}\left(\alpha_{gt}-\alpha^0_{\widetilde{g}t}\right)-\sum_{t=1}^{T}V_{it}\left(\alpha^0_{gt}-\alpha^0_{\widetilde{g}t}\right)\right|+\frac{b_{\min}}{2}\left|\sum_{t=1}^{T}X_{it}'\left(\beta-\beta^0\right)\right| \\ & +b_{\min}\left|\sum_{t=1}^{T}X_{it}'\left(\beta-\beta^0\right)\left(\alpha_{gt}-\alpha^0_{\widetilde{g}t}\right)\right|+\frac{b_{\min}}{2}\left|\sum_{t=1}^{T}\left(\alpha^0_{gt}-\alpha_{gt}\right)\left(\alpha^0_{gt}-2\alpha^0_{\widetilde{g}t}\right)\right| \\ & +TL_\beta\|\beta-\beta^0\|+TL_\alpha\|\alpha^0_{\widetilde{g}}-\alpha_{\widetilde{g}}\|,\end{aligned}$$

it is easy to show using the CS inequality that, for $(\beta,\alpha)\in\mathcal{N}_\eta$,

$$\begin{aligned}A_T &\leq T\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T}V_{it}^2\right)^{1/2}\left(\frac{1}{T}\sum_{t=1}^{T}\|X_{it}\|^2\right)^{1/2}+TC_1\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T}V_{it}^2\right)^{1/2}\\ &\quad+b_{\min}\left(\frac{1}{2}+2\sup_{\alpha\in\mathcal{A}}|\alpha|\right)\sqrt{\eta}\sum_{t=1}^{T}\|X_{it}\|\\ &\quad+T\sqrt{\eta}\frac{3b_{\min}}{2}\sup_{\alpha\in\mathcal{A}}\|\alpha\|+T\sqrt{\eta}\,(L_\beta+L_\alpha)\\ &\leq T\sqrt{\eta}\left[(c_1\vee c_2)\times(M+C_1)+b_{\min}C_2M+C_3+L_\beta+L_\alpha\right],\end{aligned}$$

where C_1,C_2,C_3,

$$\begin{aligned}c_1 &:= \sup_{(\beta,\alpha,g,x)\in\mathcal{B}\times\mathcal{A}^{\mathcal{G}^0T}\times\mathcal{G}^0\times\cup_{t=1,\ldots,i=1,\ldots}\mathrm{Supp}(X_{it})}\Psi'(Z_{it})/\Psi(Z_{it}),\\ c_2 &:= \sup_{(\beta,\alpha,g,x)\in\mathcal{B}\times\mathcal{A}^{\mathcal{G}^0T}\times\mathcal{G}^0\times\cup_{t=1,\ldots,i=1,\ldots}\mathrm{Supp}(X_{it})}\Psi'(Z_{it})/(1-\Psi(Z_{it})),\end{aligned}$$

are positive constants, independent of η and T. I thus obtain that

$$\begin{aligned}W_{ig}(\beta,\alpha)\leq\max_{\widetilde{g}\neq g}\mathbf{1}\Bigg\{&\sum_{t=1}^{T}V_{it}\left(\alpha^0_{\widetilde{g}t}-\alpha^0_{gt}\right)\leq-\frac{b_{\min}}{2}\sum_{t=1}^{T}\left(\alpha^0_{gt}-\alpha^0_{\widetilde{g}t}\right)^2\\ &+T\sqrt{\eta}\left[(c_1\vee c_2)\times(M+C_1)+b_{\min}C_2M+C_3+L_\beta+L_\alpha\right]\Bigg\}.\end{aligned}$$

Noting that the right-hand side of this inequality does not depend on (β,α), it follows that $\sup_{(\beta,\alpha)\in\mathcal{N}_\eta}W_{ig}(\beta,\alpha)\leq\overline{W}_{ig}$, where

$$\overline{W}_{ig}=\max_{\widetilde{g}\neq g}\mathbf{1}\left\{\sum_{t=1}^{T}V_{it}\left(\alpha^0_{\widetilde{g}t}-\alpha^0_{gt}\right)\leq-\frac{b_{\min}}{2}\sum_{t=1}^{T}\left(\alpha^0_{gt}-\alpha^0_{\widetilde{g}t}\right)^2\right. \tag{44}$$

$$\left.+=T\sqrt{\eta}\left[(c_1\vee c_2)\times(M+C_1)+b_{\min}C_2M+C_3+L_\beta+L_\alpha\right]\right\}. \tag{45}$$

As a result,

$$\sup_{(\beta,\alpha)\in\mathcal{N}_\eta}\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left\{\widehat{g}_i(\beta,\alpha)\neq g_i^0\right\}\leq\frac{1}{N}\sum_{i=1}^{N}\sum_{g\in\mathcal{G}^0}\overline{W}_{ig}. \tag{46}$$

I have, using standard probability algebra and for all $g \in \mathcal{G}^0$,

$$\begin{aligned}
\Pr\left(\overline{W}_{ig}=1\right) &\leq \sum_{\widetilde{g}\neq g} \Pr\Bigg(\sum_{t=1}^{T} V_{it}\left(\alpha^0_{\widetilde{g}t}-\alpha^0_{gt}\right) \leq -\frac{b_{\min}}{2}\sum_{t=1}^{T}\left(\alpha^0_{gt}-\alpha^0_{\widetilde{g}t}\right)^2 \\
&\qquad + T\sqrt{\eta}\left[(c_1 \vee c_2)\times(M+C_1)+b_{\min}C_2M+C_3+L_\beta+L_\alpha\right]\Bigg) \\
&\leq \sum_{\widetilde{g}\neq g}\Bigg\{\Pr\left(\frac{1}{T}\sum_{t=1}^{T}\left(\alpha^0_{gt}-\alpha^0_{\widetilde{g}t}\right)^2 \leq \frac{c_{g,\widetilde{g}}}{2}\right) \\
&\qquad + \Pr\Bigg(\sum_{t=1}^{T} V_{it}\left(\alpha^0_{\widetilde{g}t}-\alpha^0_{gt}\right) \leq -T\frac{c_{g,\widetilde{g}}b_{\min}}{4} \\
&\qquad + T\sqrt{\eta}\left[(c_1 \vee c_2)\times(M+C_1)+b_{\min}C_2M+C_3+L_\beta+L_\alpha\right]\Bigg)\Bigg\}.
\end{aligned} \tag{47}$$

To end the proof, let $\mathcal{F}_t = \sigma\left(\left\{\mathbf{X}_-^{(t)}, \boldsymbol{\varepsilon}_-^{(t)}, \gamma^0, \alpha^0\right\}\right)$ denote the σ-field generated by $\mathbf{X}_-^{(t)}, \boldsymbol{\varepsilon}_-^{(t)}, \gamma^0$, and α^0 and set $S_{it} = \sum_{s=1}^{t} V_{is}\left(\alpha^0_{\widetilde{g}s}-\alpha^0_{gs}\right)$. Then, $\{(S_{it},\mathcal{F}_t), 1 \leq t \leq T\}$ is a martingale under Assumptions 6(a) and 6(b) since

$$\begin{aligned}
&\mathbb{E}\left(\sum_{s=1}^{t} V_{is}\left(\alpha^0_{\widetilde{g}s}-\alpha^0_{gs}\right) \mid \mathcal{F}_{t-1}\right) \\
&= \sum_{s=1}^{t-1} V_{is}\left(\alpha^0_{\widetilde{g}s}-\alpha^0_{gs}\right) + \left(\alpha^0_{\widetilde{g}t}-\alpha^0_{gt}\right)\mathbb{E}\left(\frac{Y_{it}-\Psi\left(Z^0_{it}\right)}{\Psi\left(Z^0_{it}\right)\left(1-\Psi\left(Z^0_{it}\right)\right)}\Psi'\left(Z^0_{it}\right) \mid \mathcal{F}_{t-1}\right) \\
&= \sum_{s=1}^{t-1} V_{is}\left(\alpha^0_{\widetilde{g}s}-\alpha^0_{gs}\right) + \left(\alpha^0_{\widetilde{g}t}-\alpha^0_{gt}\right)\mathbb{E}\left(\mathbb{E}\left(\frac{Y_{it}-\Psi\left(Z^0_{it}\right)}{\Psi\left(Z^0_{it}\right)\left(1-\Psi\left(Z^0_{it}\right)\right)}\Psi'\left(Z^0_{it}\right) \mid \mathcal{F}_{t-1}, \sigma\left(\mathbf{X}_-^{(t)}\right)\right) \mid \mathcal{F}_{t-1}\right) \\
&= \sum_{s=1}^{t-1} V_{is}\left(\alpha^0_{\widetilde{g}s}-\alpha^0_{gs}\right),
\end{aligned}$$

where the last equality follows from independence of ε_t and $(\mathbf{X}_-^{(t)}, \boldsymbol{\varepsilon}_-^{(t-1)}, \gamma^0, \alpha^0)$ and

$$\mathbb{E}\left(Y_{it} \mid X_{i1}, ..., X_{it}, \alpha^0, \gamma^0\right) - \Psi\left(Z^0_{it}\right) = 0.$$

By Assumption 7(b), for all $i \in \{1, ..., N\}$, $\left\{V_{it}\left(\alpha^0_{\widetilde{g}t}-\alpha^0_{gt}\right) : t\right\}$ is such that $\left|V_{it}\left(\alpha^0_{\widetilde{g}t}-\alpha^0_{gt}\right)\right| \leq (\widetilde{c}_1 \vee \widetilde{c}_2) < \infty$, where the positive constants $\widetilde{c}_j = 2c_j \sup_{\alpha\in\mathcal{A}}|\alpha| > 0$, for $j \in \{1,2\}$, do not depend on (i,t). Let $a > 0$. By Lemma 5, there exist positive constants A and B, independent from (i,t), such that

for all $z > a/\sqrt{T}$,

$$\Pr\left(\left|\frac{1}{T}\sum_{t=1}^{T} V_{it}\left(\alpha^0_{\widetilde{g}t} - \alpha^0_{gt}\right)\right| \geq z\right) \leq A\exp\left(-B(z^2T)^{1/3}\right). \tag{48}$$

I now bound the two terms on the right-hand side of (47).

- By applying Lemma 4, and conducting the same reasoning as in the first bullet point page 1176 in Bonhomme and Manresa (2015), under Assumptions 7(a) and 8(b)-(c), for all $\delta > 0$ and as T tends to infinity,

$$\Pr\left(\frac{1}{T}\sum_{t=1}^{T}\left(\alpha^0_{gt} - \alpha^0_{\widetilde{g}t}\right)^2 \leq \frac{c_{g,\widetilde{g}}\underline{b}_{\min}}{2}\right) = o(T^{-\delta}).$$

- Lastly, to bound the second term on the right-hand side of (47), I denote as $\underline{c}$ the minimum of $c_{g,\widetilde{g}}$ over all $g \neq \widetilde{g}$ and I take

$$\eta \leq \left(\frac{\underline{c}}{8[(c_1 \vee c_2) \times (M + C_1) + b_{\min}C_2M + C_3 + L_\beta + L_\alpha]}\right)^2. \tag{49}$$

  Note that this upper bound on η does not depend on T. Taking η satisfying (49) yields, for all $\widetilde{g} \neq g$,

$$\begin{aligned}
&\Pr\left(\sum_{t=1}^{T} V_{it}\left(\alpha^0_{\widetilde{g}t} - \alpha^0_{gt}\right) \leq -T\frac{c_{g,\widetilde{g}}\underline{b}_{\min}}{4} + T\sqrt{\eta}[(c_1 \vee c_2) \times (M + C_1) + b_{\min}C_2M + C_3 + L_\beta + L_\alpha]\right) \\
&\leq \Pr\left(\frac{1}{T}\sum_{t=1}^{T} V_{it}\left(\alpha^0_{\widetilde{g}t} - \alpha^0_{gt}\right) \leq -\frac{c_{g,\widetilde{g}}}{8}\right).
\end{aligned}$$

  Lastly, by applying (48) with $z = \frac{c_{g,\widetilde{g}}}{8}$, for T sufficiently large, I obtain

$$\Pr\left(\frac{1}{T}\sum_{t=1}^{T} V_{it}\left(\alpha^0_{\widetilde{g}t} - \alpha^0_{gt}\right) \leq -\frac{c_{g,\widetilde{g}}}{8}\right) = O(\exp(-C_3T^{1/3})) = o(T^{-\delta}), \tag{50}$$

  for all $\delta > 0$, and for some constant C_3 that does not depend on i, T, and g.

Combining results, I thus obtain, using (47), that for η satisfying (49) and for all $\delta > 0$,

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{g \in \mathcal{G}^0} \Pr\left(\overline{W}_{ig} = 1\right) \leq \left|\mathcal{G}^0\right|\left(\left|\mathcal{G}^0\right| - 1\right)\left[o(T^{-\delta}) + o(T^{-\delta})\right] = o(T^{-\delta}). \tag{51}$$

To complete the proof of Lemma 6, note that, for η that satisfies (49), I have, for all $\delta > 0$ and all $\varepsilon > 0$,

$$\Pr\left(\sup_{(\beta,\alpha)\in\mathcal{N}_\eta} \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i(\beta,\alpha) \neq g_i^0\right\} > \varepsilon T^{-\delta}\right) \leq \Pr\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{g\in\mathcal{G}^0} \overline{W}_{ig} > \varepsilon T^{-\delta}\right)$$
$$\leq \frac{\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{g\in\mathcal{G}^0} \overline{W}_{ig}\right)}{\varepsilon T^{-\delta}} = o(1),$$

where I have used (46), the Markov inequality, and (51), respectively. This ends the proof of Lemma 6. □

I am now in position to prove the three parts of the following asymptotic equivalence result.

Lemma 7 (Asymptotic Equivalence) *Let Assumptions 6, 7, and 8 hold. Then, for all $\delta > 0$ and as N and T tend to infinity*

$$\Pr\left(\sup_{i\in\{1,\ldots,N\}} \left|\widehat{g}_i - g_i^0\right| > 0\right) = o(1) + o(NT^{-\delta}), \tag{52}$$

and

$$\widehat{\beta} = \widetilde{\beta} + o_p(T^{-\delta}), \tag{53}$$

and

$$\widehat{\alpha}_{gt} = \widetilde{\alpha}_{gt} + o_p(T^{-\delta}) \text{ for all } g,t. \tag{54}$$

Proof of Lemma 7: The proof closely follows pages 1178-1180 in Bonhomme and Manresa (2015).

#1. Properties of $\widehat{\beta}$. Define

$$\widehat{Q}(\beta,\alpha) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} -\ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + \alpha_{\widehat{g}_i(\beta,\alpha)t}\right)\right)\right), \tag{55}$$

$$\widetilde{Q}(\beta,\alpha) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} -\ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta + \alpha_{g_i^0 t}\right)\right)\right). \tag{56}$$

Notice that $\widehat{Q}(\cdot)$ is minimized at $(\widehat{\beta},\widehat{\alpha})$ and $\widetilde{Q}(\cdot)$ is minimized at $(\widetilde{\beta},\widetilde{\alpha})$. Let $\eta > 0$ be small enough such that the conclusion of Lemma 6 holds. Using Assumptions 7(a) and 7(b), it is then easy to see that, for all $\delta > 0$,

$$\sup_{(\beta,\alpha)\in\mathcal{N}_\eta} \left|\widehat{Q}(\beta,\alpha) - \widetilde{Q}(\beta,\alpha)\right| = o_p(T^{-\delta}). \tag{57}$$

By consistency of $\widehat{\beta}$ (Theorem 2) and $\widehat{\alpha}$ (Lemma 3), and because $\widetilde{\beta}$ and $\widetilde{\alpha}$ are also consistent under

the conditions of Theorem 2, we have, as N and T tend to infinity,

$$\Pr\left(\left(\widehat{\beta},\widehat{\alpha}\right)\notin\mathcal{N}_\eta\right)\to 0, \tag{58}$$

$$\Pr\left(\left(\widetilde{\beta},\widetilde{\alpha}\right)\notin\mathcal{N}_\eta\right)\to 0. \tag{59}$$

Then, the same arguments as those appearing between (B-14) and (B-17) in page 1179 in Bonhomme and Manresa (2015) can be used to show that eq. (57)-(59) imply

$$\widetilde{Q}(\widehat{\beta},\widehat{\alpha})-\widetilde{Q}(\widetilde{\beta},\widetilde{\alpha})=o_p(T^{-\delta}). \tag{60}$$

Now, using that $(\widetilde{\beta},\widetilde{\alpha})$ minimizes the twicely continuously differentiable function $\widetilde{Q}(\cdot)$, we obtain under Assumption 6(b)

$$\begin{aligned}
\widetilde{Q}(\widehat{\beta},\widehat{\alpha})-\widetilde{Q}(\widetilde{\beta},\widetilde{\alpha}) &\geq \frac{b_{\min}}{2}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(X_{it}'\left(\widetilde{\beta}-\widehat{\beta}\right)+\widetilde{\alpha}_{g_i^0t}-\widehat{\alpha}_{g_i^0t}\right)^2,\\
&\geq \frac{b_{\min}}{2}\left(\widetilde{\beta}-\widehat{\beta}\right)'\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(X_{it}-\overline{X}_{g_i^0t}\right)\left(X_{it}-\overline{X}_{g_i^0t}\right)'\right)\left(\widetilde{\beta}-\widehat{\beta}\right)\\
&\geq \frac{\widehat{\rho}b_{\min}}{2}\|\widetilde{\beta}-\widehat{\beta}\|^2,
\end{aligned}$$

where $\widehat{\rho}\xrightarrow{p}\rho>0$ as a consequence of Assumption 7(c). Hence, $\widetilde{\beta}-\widehat{\beta}=o_p(T^{-\delta})$ for all $\delta>0$. This shows (53).

#2. Properties of $\widehat{\alpha}$. The proof is identical to page 1180 in Bonhomme and Manresa (2015).

#3. Properties of $\widehat{g}_i=\widehat{g}_i(\widehat{\beta},\widehat{\alpha})$. The proof is identical to page 1180 in Bonhomme and Manresa (2015).

The proof of Lemma 7 is complete. □

A.4.2 Step 2: Asympotic Properties of the Oracle MLE

By Lemma 7 and Slutsky's lemma, it is sufficient to analyze the limiting distribution of the unfeasible maximum likelihood estimator, $(\widetilde{\beta},\widetilde{\alpha})$, defined as

$$(\widetilde{\beta},\widetilde{\alpha})=\underset{(\beta,\alpha)\in\mathcal{B}\times\mathcal{A}^{G^0T}}{\arg\min}\ \widetilde{Q}(\beta,\alpha),$$

where

$$\widetilde{Q}(\beta,\alpha)=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{g\in\mathcal{G}^0}\mathbf{1}\left\{g_i^0=g\right\}\times\left[-\ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta+\alpha_{gt}\right)\right)\right)\right].$$

First, I show

$$\sqrt{NT}\left(\widetilde{\beta}-\beta^0\right)\xrightarrow{d}\mathcal{N}\left(0,\Sigma_\beta^{-1}\right). \tag{61}$$

Second, I show for all g,t,

$$\sqrt{N}\left(\widetilde{\alpha}_{gt}-\alpha_{gt}^0\right)\xrightarrow{d}\mathcal{N}\left(0,\frac{\omega_{gt}}{\widetilde{\pi}_{gt}^2}\right), \tag{62}$$

and conclude by Slutsky's lemma.

 # 1. (61) holds. Under Assumption 9, results in Hahn and Newey (2004) (eq. (3)) and Arellano and Hahn (2007) (in case of multi-dimensional fixed effects of size G^0) ensure

$$\sqrt{NT}\left(\widetilde{\beta}-\beta^0\right)=S_{NT}+\sqrt{\frac{T}{N}}B+O_p\left(\sqrt{\frac{T}{N^3}}\right),$$

for some deterministic $B\in\mathbb{R}^{p\times p}$ and $S_{NT}\xrightarrow{d}\mathcal{N}\left(0,\Sigma_\beta^{-1}\right)$. The result then follows from $T=o(N)$.

#2. (62) holds. Let $(g,t)\in\mathcal{G}^0\times\mathbb{N}^*$. For all $\beta\in\mathcal{B}$, define the optimal $\widetilde{\alpha}_{gt}(\beta)$ as

$$\widetilde{\alpha}_{gt}(\beta)=\underset{\alpha\in\mathcal{A}}{\arg\min}\,\frac{1}{N}\sum_{i=1}^{N}-\mathbf{1}\left\{g_i^0=g\right\}\times\ln\left(\Psi\left(Q_{it}\left(X_{it}'\beta+\alpha\right)\right)\right).$$

The first-order optimality condition for $\widetilde{\alpha}_{gt}(\beta)$ writes

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left\{g_i^0=g\right\}Q_{it}\left(\ln\Psi\right)'\left(Q_{it}\left(X_{it}'\beta+\widetilde{\alpha}_{gt}(\beta)\right)\right)=0. \tag{63}$$

Differentiating eq. (63) with respect to β yields

$$\frac{\mathrm{d}\widetilde{\alpha}_{gt}(\beta)}{\mathrm{d}\beta}=-\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left\{g_i^0=g\right\}\left(\ln\Psi_{it}\right)''\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left\{g_i^0=g\right\}\left(\ln\Psi_{it}\right)''X_{jt}\right), \tag{64}$$

where $\left(\ln\Psi_{it}\right)'':=\left(\ln\Psi\right)''\left(Q_{it}\left(X_{it}'\beta+\widetilde{\alpha}_{g_i^0t}(\beta)\right)\right)$. By Taylor's theory, eq. (64) and Assumptions 7(a)-(b) imply that there exists $C>0$ such that, almost surely,

$$\sup_{\beta,\beta'\in\mathcal{B}}\left|\widetilde{\alpha}_{gt}(\beta)-\widetilde{\alpha}_{gt}(\beta')\right|\leq C\left\|\beta-\beta'\right\|. \tag{65}$$

Deduce that

$$\begin{aligned}\sqrt{N}\left(\widetilde{\alpha}_{gt}-\alpha_{gt}^{0}\right) &= \sqrt{N}\left(\widetilde{\alpha}_{gt}(\beta^{0})-\alpha_{gt}^{0}\right)+\sqrt{N}\left(\widetilde{\alpha}_{gt}(\widetilde{\beta})-\widetilde{\alpha}_{gt}(\beta^{0})\right)\\ &= \sqrt{N}\left(\widetilde{\alpha}_{gt}(\beta^{0})-\alpha_{gt}^{0}\right)+O_{p}\left(\sqrt{N}\|\widetilde{\beta}-\beta^{0}\|\right)\\ &= \sqrt{N}\left(\widetilde{\alpha}_{gt}(\beta^{0})-\alpha_{gt}^{0}\right)+O_{p}(1/\sqrt{T})\\ &= \sqrt{N}\left(\widetilde{\alpha}_{gt}(\beta^{0})-\alpha_{gt}^{0}\right)+o_{p}(1), \end{aligned} \tag{66}$$

where the second and third equality use eq. (65) and (61) respectively. Now, by expanding each summand in eq. (63) at $X_{it}'\beta^{0}+\widetilde{\alpha}_{gt}(\beta^{0})$ around Z_{it}^{0}, Taylor's theory ensures again that there exists $Z_{it}^{*}\in\mathcal{Z}$ such that

$$\widetilde{\alpha}_{gt}(\beta^{0})=\alpha_{gt}^{0}-\left(\sum_{i=1}^{N}\mathbf{1}\left\{g_{i}^{0}=g\right\}(-\ln\Psi)''(Q_{it}Z_{it}^{*})\right)^{-1}\left(\sum_{i=1}^{N}\mathbf{1}\left\{g_{i}^{0}=g\right\}Q_{it}(-\ln\Psi)'\left(Q_{it}Z_{it}^{0}\right)\right). \tag{67}$$

Equation (67) yields

$$\begin{aligned}&\sqrt{N}\left(\widetilde{\alpha}_{gt}(\beta^{0})-\alpha_{gt}^{0}\right)\\ &=-\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left\{g_{i}^{0}=g\right\}(-\ln\Psi)''(Q_{it}Z_{it}^{*})\right)^{-1}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{1}\left\{g_{i}^{0}=g\right\}Q_{it}(-\ln\Psi)'\left(Q_{it}Z_{it}^{0}\right)\right)\\ &=\left(\widetilde{\pi}_{gt}^{-1}+o_{p}(1)\right)\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{1}\left\{g_{i}^{0}=g\right\}Q_{it}(\ln\Psi)'\left(Q_{it}Z_{it}^{0}\right)\right)\\ &\xrightarrow{d}\mathcal{N}\left(0,\frac{\omega_{gt}}{\widetilde{\pi}_{gt}^{2}}\right),\end{aligned}$$

where the second equality follows from $\sup_{i=1,\ldots,N}|Z_{it}^{*}-Z_{it}^{0}|=o_{p}(1)$ (it is easy to prove that $\widetilde{\alpha}_{gt}(\beta^{0})-\alpha_{gt}^{0}=o_{p}(1)$ using (67), Assumptions 6(b), 7(a)-(b), and 9(e)) and Assumption 9(c), and the last convergence follows by Assumption 9(e). Given (66), (62) follows by Slutsky's lemma.

#3. Conclusion. Let $\delta>0$. By Lemma 7,

$$\begin{aligned}\sqrt{NT}\left(\widehat{\beta}-\beta^{0}\right) &= \sqrt{NT}\left(\widetilde{\beta}-\beta^{0}\right)+\sqrt{NT}\left(\widehat{\beta}-\widetilde{\beta}\right)\\ &= \sqrt{NT}\left(\widetilde{\beta}-\beta^{0}\right)+o_{p}\left(\sqrt{N}T^{1-\delta}\right),\end{aligned} \tag{68}$$

and, for all $g \in \mathcal{G}^0$, all $t \in \mathbb{N}^*$,

$$\begin{aligned}\sqrt{N}\left(\widehat{\alpha}_{gt} - \alpha^0_{gt}\right) &= \sqrt{N}\left(\widetilde{\alpha}_{gt} - \alpha^0_{gt}\right) + \sqrt{N}\left(\widehat{\alpha}_{gt} - \widetilde{\alpha}_{gt}\right) \\ &= \sqrt{N}\left(\widetilde{\alpha}_{gt} - \alpha^0_{gt}\right) + o_p\left(\sqrt{N}T^{-\delta}\right). \end{aligned} \tag{69}$$

Since (68) and (69) hold for all $\delta > 0$, and there exists $\nu > 0$ such that $N/T^\nu \to 0$, as N and T tend to infinity, I obtain

$$\begin{aligned}\sqrt{NT}\left(\widehat{\beta} - \beta^0\right) &= \sqrt{NT}\left(\widetilde{\beta} - \beta^0\right) + o_p(1), \\ \sqrt{N}\left(\widehat{\alpha}_{gt} - \alpha^0_{gt}\right) &= \sqrt{N}\left(\widetilde{\alpha}_{gt} - \alpha^0_{gt}\right) + o_p(1).\end{aligned}$$

This result, combined with (61), (62), and Slustky's lemma yields (19) and (20).

B Extensions

B.1 Cluster-Specific Slopes and Time-Specific Effects

In this section, I consider the following extension of model (1): for all $(i,t) \in \mathcal{N} \times \mathcal{T}$,

$$\Pr\left(Y_{it} = y \mid X_{i1}, ..., X_{it}, \alpha^0_{g^0_i t}, \beta^0_{g^0_i}, g^0_i, \zeta^0_t\right) = h^0\left(y, X'_{it}\beta^0_{g^0_i} + \alpha^0_{g^0_i t} + \zeta^0_t\right), \tag{70}$$

where $h^0 \in \mathcal{H}$, $\|\beta^0_1\| = 1$ and $\alpha^0_{11} = \zeta^0_1 = 0$ are normalizations. Absent of correlation between the groups and if we knew the groups, we could just run separate analysis of each panel data $\{(i,t) \in \mathcal{N} \times \mathcal{T} : g^0_i = g\}_{g \in \mathcal{G}^0}$. Here, the difficulty arises from the assumption that the group membership variables g^0_i are unknown to the econometrician. Let $\beta^0 := \{\beta^0_g : g\}$. We first adapt Assumption 1:

Assumption 11 (Random sampling)

(a) $(Y'_i, X'_i, g^0_i)'$ *is i.i.d. across* $i \in \mathcal{N}$ *conditional on* α^0, β^0, λ^0, μ^0.

(b) For all $i \in \mathcal{N}$: $\{(Y_{it}, X'_{it}, \alpha^0_{g^0_i t}, \zeta^0_t)'\}_{t \geq 2}$ *is a strictly stationary strong mixing process with mixing coefficients* $\alpha_i(\cdot)$ *conditional on* $g^0_i, \mu^0_{g^0_i}, \xi^0_i, \beta^0_{g^0_i}$. *Let* $\alpha(\cdot) = \sup_i \alpha_i(\cdot)$ *satisfy* $\alpha(l) \leq c_\alpha \rho^l$ *with* $c_\alpha > 0$, *and* $\rho \in (0,1)$.

(c) For all $t \in \mathcal{T}$: $Y_{1t} \mid X_{1t}, g_1, \alpha^0, \beta^0, \lambda^0, \mu^0, \xi^0 \overset{d}{=} Y_{1t} \mid X_{1t}, g^0_1, \alpha^0_{g^0_1 t}, \beta^0_{g^0_i}$.

Assumption 12 (Latent clustering)

(a) *There exist known* $\mathcal{X}^0 \subset \mathcal{X}$, $y \in \mathcal{Y}$, *and functional* ϕ *such that, for all fixed* $(i,j) \in \mathcal{N}^2$, *letting* $\rho_i(x) : \mathcal{X}^0 \ni x \mapsto \Pr\left(Y_{i2} = y \mid X_{i2} = x, \beta^0_{g_i^0}, g_i^0, \mu^0_{g_i^0}, \xi_i^0\right)$, $\phi(\rho_i, \rho_j) = \mathbb{1}\{g_i^0 = g_j^0\}$.

(b) *For all* $g \in \mathcal{G}^0$, *almost surely* $\Pr(g_1^0 = g \mid \alpha^0, \beta^0, \lambda^0, \mu^0, \xi^0) > 0$.

Assumption 13 (Regularity and smoothness)

(a) *Conditional on* $g_i^0, \mu^0_{g_i^0}, \xi_i^0, \beta^0_{g_i^0}$, X_{i2} *admits a uniformly continuous density function* $f_{X_{i2}|g_i^0, \mu^0_{g_i^0}, \xi_i^0, \beta^0_{g_i^0}}$ *such that* $\inf_{x \in \mathcal{X}^0} f_{X_{i2}|g_i^0, \mu^0_{g_i^0}, \xi_i^0, \beta^0_{g_i^0}}(x) \geq \delta > 0$ *and* $\sup_{x \in \mathcal{X}^0} f_{X_{i2}|g_i^0, \mu^0_{g_i^0}, \xi_i^0, \beta^0_{g_i^0}}(x) < \infty$.

(b) *Almost surely,* $\mathbb{E}(\|X_{12}\|^2 \mid g_1^0, \alpha^0, \beta^0, \lambda^0, \mu^0)$ *is finite and* $\mathbb{E}(X_{12}X_{12}' \mid g_1^0, \alpha^0, \beta^0, \lambda^0, \mu^0)$ *is non-singular.*

(c) *For all* $g \in \mathcal{G}^0$: $\sum_{y \in \mathcal{Y}} y h^0(y, \cdot)$ *is differentiable on* $\mathbb{R}$ *and not constant on the support of* $X_{it}'\beta^0_{g_i^0} + \alpha^0_{g_i^0 t}$.

Assumption 14 (Monotonicity) *There exists* $y \in \mathcal{Y}$ *such that* $h^0(y, v)$ *is strictly monotonic in* v.

Assumption 15 (Compensating variations)

(a) *For all fixed* $(g, t, \widetilde{t})$, *all* $x_1 \in \mathcal{X}$, *there exists* $x_2 \in \mathcal{X}$ *such that*

$$\alpha^0_{gt} + x_1'\beta^0_g + \zeta^0_t = \alpha^0_{g\widetilde{t}} + x_2'\beta^0_g + \zeta^0_{\widetilde{t}}. \tag{71}$$

(b) *For all fixed* $(g, \widetilde{g}, t)$, *all* $x_3 \in \mathcal{X}$, *there exists* $x_4 \in \mathcal{X}$ *such that*

$$\alpha^0_{gt} + x_3'\beta^0_g + \zeta^0_t = \alpha^0_{\widetilde{g}t} + x_4'\beta^0_{\widetilde{g}} + \zeta^0_t. \tag{72}$$

Theorem 4 (Identification) *Let Assumptions 11, 12 and 13(a) hold, and let* N *and* T *diverge jointly to infinity.*

1. $\{\mathbf{W}^0_N : N \in \mathbb{N}^*\}$ *and* G^0 *are identified.*
2. *If Assumptions 13(b)-15 further hold, then*
   - β^0 *is identified.*
   - $\zeta^0_t + \alpha^0_{gt}$ *is identified for all* $(g, t) \in \mathcal{G}^0 \times \mathbb{N}^*$.

Proof of Theorem 4: The proofs of Part 1 and identification of β^0 are identical to the corresponding parts of the proof of Theorem 1, up to running nonparametric regressions for all $g \in \mathcal{G}^0$ to identify β^0_g. Next, Assumption 15(b) ensures that, for all $(g, \widetilde{g}, t)$, we can identify $(x_1, x_2) \in \mathcal{X}^2$, such that for some $y \in \mathcal{Y}$,

$$h^0\left(y, x_1'\beta^0_g + \alpha^0_{gt} + \zeta^0_t\right) = h^0\left(y, x_2'\beta^0_{\widetilde{g}} + \alpha^0_{\widetilde{g}t} + \zeta^0_t\right).$$

By inverting $h^0(y,\cdot)$, we obtain $\alpha^0_{gt} - \alpha^0_{\widetilde{g}t} = x_1'\beta^0_g - x_2'\beta^0_{\widetilde{g}}$. Since the right-hand side is identified, $\alpha^0_{gt} - \alpha^0_{\widetilde{g}t}$ is identified for all $(g, \widetilde{g}, t)$. In particular, $(\alpha^0_{g1})_{g \in \mathcal{G}^0}$ is identified. Now, suppose that $G^0 \geq 2$. By Assumption 15(a), for all $(g, t, \widetilde{t})$, we can identify $(x_3, x_4) \in \mathcal{X}^2$ such that, for some $y \in \mathcal{Y}$,

$$h^0\left(y, x_3'\beta^0_g + \alpha^0_{gt} + \zeta^0_t\right) = h^0\left(y, x_4'\beta^0_g + \alpha^0_{g\widetilde{t}} + \zeta^0_{\widetilde{t}}\right). \tag{73}$$

By inverting $h^0(y,\cdot)$ again, eq. (73) yields

$$\zeta^0_t - \zeta^0_{\widetilde{t}} = \alpha^0_{g\widetilde{t}} - \alpha^0_{gt} + (x_4 - x_3)'\beta^0_g. \tag{74}$$

Because $\zeta^0_1 = \alpha^0_{11} = 0$, $\zeta^0_t + \alpha^0_{1t}$ and $\zeta^0_t + \alpha^0_{gt} = \zeta^0_t + \alpha^0_{1t} + \alpha^0_{gt} - \alpha^0_{1t}$ are identified for all (g, t).

B.2 Individual-Specific Slopes, Effects, Group-Specific Link Function, Time-Varying Slope

TBA

B.3 Grouping Time Periods

Consider a model in which time effects are also grouped: there exists $(g^0_i, k^0_t) \in \{1, ..., G^0\} \times \{1, ..., K^0\}$ such that

$$\Pr\left(Y_{it} = y \mid X_i^t, \alpha^0_{g^0_i k^0_t}, g^0_i, k^0_t\right) = h^0\left(y, X_{it}'\theta^0 + \alpha^0_{g^0_i k^0_t}\right), \quad i = 1, ..., N, t = 1, ..., T \tag{75}$$

When $\mathcal{N} = \mathcal{T}$, this gives rise to a so-called Holland, Laskey, and Leinhardt (1983)'s stochastic block model on latent variables. Methods developed in the present paper and in Mugnier (2022) can be used to obtain identification results for nonlinear multiplicative models in cases where $G^0 = K^0$ and under symmetry ($\alpha^0_{gg} = \alpha^0_{\widetilde{g}g}$ almost surely).

B.4 NGFE Large Sample Theory for Poisson Count Models

Theorem 2 can be generalized to NGFE models satisfying certain moment and concavity/regularity conditions on the series of partial derivatives of $(\beta,\pi) \mapsto \ln h^0(Y_{it}, X_{it}'\beta + \pi) \equiv \ell_{it}(\beta,\pi)$.

Assumption 16

(a) Smoothness and moments: $(\beta,\pi) \mapsto \ell_{it}(\beta,\pi)$ *is three times continuously differentiable almost surely. The partial derivatives of* $\ell_{it}(\beta,\pi)$ *with respect to the elements of* (β,π) *up to the second order are bounded in absolute value uniformly over* $(\beta,\pi) \in \mathcal{B}\times\mathcal{A}$ *by a function* $M(Y_{it}, X_{it}) > 0$ *almost surely, and*

$$\max_{i,t} E\left[M(Y_{it}, X_{it})^4 \mid \mathbf{X}^{(t)}, \alpha^0_{g_i^0 t}\right]$$

is almost surely uniformly bounded over N, T.

(b) Strict concavity: for all N, T, $\frac{\partial^2 \ell_{it}(\beta,\pi)}{\partial \pi^2} < 0$ *almost surely for all* $(\beta,\pi) \in \mathbb{R}^{p+1}$.

In particular, Assumption 16(b) is verified by the Poisson count model (3).

Theorem 5 (Consistency in General Nonlinear Models) *Let Assumptions 7 and 16 hold. Then, as* N *and* T *tend to infinity:*

1. $\widehat{\beta} \xrightarrow{p} \beta^0$, *and*
2. $\frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T \left(\widehat{\alpha}_{\widehat{g}_i t} - \alpha^0_{g_i^0 t}\right)^2 \xrightarrow{p} 0$.

The proof is available upon request.

Under the existence of a moment generating function for the score on a small interval around zero, the concentration inequalities and most of the arguments in the proof of Theorem 3 could still be applied to obtain asymptotic normality. A technical difficulty here is that Y_{it} is not bounded anymore so that uniform Lipschitz continuity in eq. (43) and (42) does not hold anymore. I only state the result without proof for the Poisson count model. I denote as $\widetilde{X}_{gt}$ the projection of X_{it} on the space spanned by the cluster membership variable under a metric weighted by $\exp(Z_{it}^0)$,

$$\widetilde{X}_{gt} = \left(\frac{1}{N}\sum_{i=1}^N \mathbf{1}\left\{g_i^0 = g\right\}\exp(Z_{it}^0)\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N \mathbf{1}\left\{g_i^0 = g\right\}\exp(Z_{it}^0)X_{it}\right),$$

i.e., the weighted mean of X_{it} in cluster $g_i^0 = g$. Also, let define the weighted average

$$\widehat{\pi}_{gt} = \frac{1}{N}\sum_{i=1}^N \mathbf{1}\left\{g_i^0 = g\right\}\exp(Z_{it}^0).$$

Consider the following assumption.

Assumption 17

(a) $\{(Y_{it}, X_{it}')' : (i,t)\}$ *are independent conditional on the fixed effects.*

(b) There exists a positive definite matrix Σ_β *such that*

$$\Sigma_\beta = \text{plim}_{N,T\to\infty} \frac{1}{NT} \sum_{i=1}^{N}\sum_{t=1}^{T} \exp(Z_{it}^0) \left[X_{it} - \widetilde{X}_{g_i^0 t}\right] \left[X_{it} - \widetilde{X}_{g_i^0 t}\right]'.$$

(c) As N *and* T *tend to infinity,*

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{t=1}^{T} \left\{\exp(Z_{it}^0)\left(X_{it} - \widetilde{X}_{g_i^0 t}\right)\right\} \left\{Y_{it} - \exp(Z_{it}^0)\right\} \xrightarrow{d} \mathcal{N}(0, \Sigma_\beta).$$

(d) For all (g,t)*:* $\text{plim}_{N\to\infty} \widehat{\pi}_{gt} = \widetilde{\pi}_{gt} > 0$.

(e) For all (g,t)*:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} E\left(\mathbf{1}\left\{g_i^0 = g\right\} \mathbf{1}\left\{g_j^0 = g\right\} (Y_{it} - \exp(Z_{it}^0))(Y_{jt} - \exp(Z_{jt}^0)\right) = \omega_{gt} > 0.$$

(f) For all (g,t)*, and as* N *and* T *tend to infinity:*

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathbf{1}\left\{g_i^0 = g\right\} (Y_{it} - \exp(Z_{it}^0)) \xrightarrow{d} \mathcal{N}(0, \omega_{gt}).$$

(g) The true value of β*,* β^0*, is in the interior of* $\mathcal{B}$*. For all* T*, the true value of* $\boldsymbol{\alpha}$*,* $\boldsymbol{\alpha}^0$*, is in the interior of* $\mathcal{A}^{G^0 T}$*.*

Theorem 6 (Asymptotic Distribution in the Poisson Count Model – Conjectured) *Let eq. (3), Assumptions 7, 8, and 17 hold, and let* N *and* T *tend to infinity such that* $N/T \to \infty$ *and, for some* $\nu > 0$*,* $N/T^\nu \to 0$*. Then:*

$$\sqrt{NT}\left(\widehat{\beta} - \beta^0\right) \xrightarrow{d} \mathcal{N}\left(0, \Sigma_\beta^{-1}\right), \tag{76}$$

and, for all (g,t)*,*

$$\sqrt{N}\left(\widehat{\alpha}_{gt} - \alpha_{gt}^0\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\omega_{gt}}{\widetilde{\pi}_{gt}^2}\right), \tag{77}$$

where Σ_β, ω_{gt}, and $\widetilde{\pi}_g$ are defined in Assumption 17.

C Large-N, Large-T Inference

C.1 Binary Choice Model

Assuming independent observations across individual units, the asymptotic variance of $\widehat{\alpha}_{gt}$ for all g,t can be estimated as

$$\text{Var}\left(\widehat{\alpha}_{gt}\right) = \frac{\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i = g\right\} \left((\ln \Psi)' \left(Q_{it}\left(X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t}\right)\right)\right)^2}{\left(\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i = g\right\} (-\ln \Psi)'' \left(Q_{it}\left(X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t}\right)\right)\right)^2}. \tag{78}$$

Given Theorem 3, an estimate of the asymptotic variance of $\widehat{\beta}$ is

$$\text{Var}\left(\widehat{\beta}\right) = \left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} (-\ln \Psi)'' \left(Q_{it}\left(X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t}\right)\right) \left[X_{it} - \widehat{\widehat{X}}_{\widehat{g}_i,t}\right] \left[X_{it} - \widehat{\widehat{X}}_{\widehat{g}_i,t}\right]'\right)^{-1}, \tag{79}$$

where

$$\begin{aligned}\widehat{\widehat{X}}_{gt} = &\left(\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i = g\right\} (\ln \Psi)'' \left(Q_{it}\left(X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t}\right)\right)\right)^{-1} \\ &\times \left(\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i = g\right\} (\ln \Psi)'' \left(Q_{it}\left(X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t}\right)\right) X_{it}\right).\end{aligned}$$

C.2 Poisson Count Model

Assuming independent observations across individual units, the asymptotic variance of $\widehat{\alpha}_{gt}$ for all g,t can be estimated as

$$\text{Var}\left(\widehat{\alpha}_{gt}\right) = \frac{\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i = g\right\} \left(Y_{it} - \exp\left(X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t}\right)\right)^2}{\left(\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i = g\right\} \exp\left(X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t}\right)\right)^2}. \tag{80}$$

Given Theorem 6, an estimate of the asymptotic variance of $\widehat{\beta}$ is

$$\text{Var}\left(\widehat{\beta}\right) = \left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \exp\left(X_{it}'\widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t}\right) \left[X_{it} - \widehat{\widehat{X}}_{\widehat{g}_i,t}\right] \left[X_{it} - \widehat{\widehat{X}}_{\widehat{g}_i,t}\right]'\right)^{-1}, \tag{81}$$

where

$$\widehat{\widehat{X}}_{gt} = \left(\frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\left\{ \widehat{g}_i = g \right\} \exp\left(X_{it}' \widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t} \right) \right)^{-1} \times \left(\frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\left\{ \widehat{g}_i = g \right\} \exp\left(X_{it}' \widehat{\beta} + \widehat{\alpha}_{\widehat{g}_i t} \right) X_{it} \right).$$

D More Details on Monte Carlo Experiments

To measure classification accuracy, I focus on three metrics inspired from the binary classification and clustering statistical literature, which are invariant to cluster relabeling.[48] The three metrics write

$$\begin{aligned} \mathrm{R} \equiv \text{Recall rate} &:= \frac{TP}{TP+FN}, \\ \mathrm{P} \equiv \text{Precision rate} &:= \frac{TP}{TP+FP}, \\ \mathrm{RI} \equiv \text{Rand Index} &:= \frac{TP+TN}{TP+TN+FP+FN}, \end{aligned}$$

where

$$\begin{aligned} FP \equiv \text{False Positives} &:= \sum_{i<j} \mathbf{1}\left\{ \widehat{g}_i = \widehat{g}_j \right\} \mathbf{1}\left\{ g_i^0 \neq g_j^0 \right\}, \\ TP \equiv \text{True Positives} &:= \sum_{i<j} \mathbf{1}\left\{ \widehat{g}_i = \widehat{g}_j \right\} \mathbf{1}\left\{ g_i^0 = g_j^0 \right\}, \\ FN \equiv \text{False Negatives} &:= \sum_{i<j} \mathbf{1}\left\{ \widehat{g}_i \neq \widehat{g}_j \right\} \mathbf{1}\left\{ g_i^0 = g_j^0 \right\}, \\ TN \equiv \text{True Negatives} &:= \sum_{i<j} \mathbf{1}\left\{ \widehat{g}_i \neq \widehat{g}_j \right\} \mathbf{1}\left\{ g_i^0 \neq g_j^0 \right\}. \end{aligned}$$

The Recall rate (R) measures the ability of the NGFE estimator to predict the same group for pairs of individual who truly belong to the same group. The Precision rate (P) measures how precise the pairing prediction is: among all the predicted pairs of individual sharing the same group, what is the proportion of correct ones? The Rand Index (RI) is the proportion of correctly predicted pair (true or false) made by the algorithm.

[48] Bonhomme and Manresa (2015) report a "Misclassification Rate" (M) defined as the minimum of $\sum_{i=1}^{N} \left| \widehat{g}_i - g_i^0 \right| / N$ over all possible cluster relabelings for the $\widehat{g}_i$. Beyond the fact that computing MR can be very demanding for large G^0, it is not totally fair since the final labeling of $\widehat{g}_i$ requires knowledge of g_i^0 to be determined.

Initialization of NGFE I use $1,000$ initialization random points $(\theta'_{\text{init}}, \alpha_{11\text{init}}, ..., \alpha_{G^0T\text{init}})'$ such that $\theta_{\text{init}} = v$ where $v \overset{iid}{\sim} \mathcal{N}(0, (1/4)^2)$ and $\alpha_{gt,\text{init}} = \mu_{g,\text{init}} + w$ where $\mu_{g,\text{init}} \overset{iid}{\sim} \text{Unif}[-4, 4]$ and $w \overset{iid}{\sim} \mathcal{N}(0, (1/4)^2)$.

Computation Having large N is not computationally demanding. When T is very large, computation of the NGFE estimate might be demanding. Mugnier (2022) can be adapted. The statistical asymptotic results are confirmed by increasing (N, T) in unreported simulations.

E Additional Tables & Figures

E.1 Monte Carlo Simulations

Table 1: Bias and Root Mean Squared Error of $\widehat{\beta}$ (Static Model)

		NGFE		CMLE		NLTWFE		2STEPGFE		Pooled OLS		LTWFE		GFE	
DGP	G^0	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
1	2	-0.072	0.268	-0.104	0.551	0.217	0.950	-0.252	1.516	-0.407	0.411	-0.790	0.812	-0.798	0.814
	3	-0.089	0.294	0.294	0.637	0.669	1.000	0.355	0.893	-0.363	0.366	-0.724	0.734	-0.853	0.874
	5	-0.022	0.264	0.167	0.538	0.359	0.824	0.104	0.779	-0.369	0.373	-0.766	0.776	-0.784	0.839
2	2	0.106	0.171	0.010	0.161	0.223	0.302	-0.278	0.309	-0.779	0.780	-0.831	0.831	-0.816	0.818
	3	0.236	0.289	0.014	0.160	0.238	0.309	-0.300	0.345	-0.768	0.769	-0.867	0.867	-0.837	0.841
	5	0.601	0.637	-0.004	0.169	0.250	0.332	-0.324	0.358	-0.747	0.747	-0.916	0.916	-0.853	0.860
3	2	0.352	0.385	-0.001	0.169	0.221	0.313	-0.110	0.211	-0.776	0.777	-0.857	0.857	-0.826	0.827
	3	0.432	0.486	-0.002	0.170	0.219	0.308	-0.066	0.192	-0.788	0.789	-0.859	0.859	-0.845	0.846
	5	0.471	0.499	0.011	0.156	0.235	0.309	-0.057	0.186	-0.787	0.788	-0.858	0.858	-0.833	0.836
4	2	0.040	0.151	-0.002	0.152	0.195	0.269	0.085	0.221	-0.789	0.789	-0.783	0.784	-0.788	0.789
	3	0.095	0.159	0.016	0.124	0.223	0.269	0.109	0.213	-0.776	0.776	-0.778	0.779	-0.790	0.792
	5	0.114	0.178	0.018	0.118	0.222	0.266	0.094	0.204	-0.775	0.775	-0.778	0.779	-0.803	0.809

Notes: Static logit model with $\beta = 1$, $N = 90$, and $T = 7$. G^0 = true number of groups. NGFE (resp. 2STEPGFE and GFE) estimates are based on 1,000 (resp. 100 and 100) initialization points. Results are averaged across 50 Monte Carlo replications.

Table 2: Classification Accuracy and CPU Time (Static Model)

		NGFE					2STEPGFE						GFE				
DGP	G^0	P	R	RI	M	CPU	P	R	RI	M	CPU	$\widehat{G}$	P	R	RI	M	CPU
1	2	0.51	0.87	0.51	0.44	10.62	0.54	0.24	0.51	0.77	10.19	5.38	0.54	0.55	0.54	0.38	29.27
	3	0.35	0.81	0.42	0.57	11.42	0.37	0.24	0.60	0.75	11.34	5.48	0.36	0.38	0.57	0.55	29.63
	5	0.21	0.80	0.35	0.70	14.75	0.24	0.25	0.69	0.71	11.73	5.88	0.24	0.25	0.69	0.63	83.18
2	2	0.56	0.86	0.57	0.36	8.02	0.64	0.45	0.60	0.53	3.57	3.06	0.61	0.61	0.61	0.29	21.95
	3	0.40	0.85	0.49	0.51	8.52	0.57	0.49	0.70	0.44	4.70	3.64	0.46	0.49	0.64	0.42	22.00
	5	0.22	0.87	0.34	0.69	10.15	0.44	0.53	0.77	0.44	5.78	4.44	0.35	0.40	0.74	0.54	20.93

Notes: Static logit model with $\beta = 1$, $N = 90$, and $T = 7$. G^0 = true number of groups, P = Precision rate, R = Recall rate, RI = Rand Index, M = Misclassification Rate = minimum of $\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i \neq g_i^0\right\}/N$ over all possible cluster relabelings, CPU = CPU time in seconds computed with Python's `time` command `time.perf_counter()`, $\widehat{G}$ = number of groups estimated by 2STEPGFE. NGFE (resp. 2STEPGFE and GFE) estimates are based on 1,000 (resp. 100 and 100) initialization points. Results are averaged across 50 Monte Carlo replications.

Table 3: Inference for β (Static Model)

		NGFE			CMLE		
DGP	G^0	SE	SD	.95	SE	SD	.95
1	2	0.16	0.26	0.86	0.15	0.54	0.38
	3	0.17	0.28	0.80	0.16	0.56	0.40
	5	0.17	0.26	0.84	0.15	0.51	0.42
2	2	0.12	0.13	0.82	0.06	0.16	0.52
	3	0.12	0.17	0.46	0.07	0.16	0.62
	5	0.14	0.21	0.08	0.08	0.17	0.66
3	2	0.12	0.16	0.22	0.06	0.17	0.52
	3	0.12	0.22	0.18	0.06	0.17	0.52
	5	0.12	0.16	0.04	0.06	0.16	0.56
4	2	0.12	0.15	0.92	0.05	0.15	0.38
	3	0.13	0.13	0.92	0.05	0.12	0.56
	5	0.13	0.14	0.88	0.05	0.12	0.56

Notes: Static logit model with $\beta_1 = 1$, $N = 90$, and $T = 7$. SE reports the median of the estimates of the analytical standard errors based on the large-N, T analytical variance formula (81) across simulations; SD reports the median of the actual standard deviation across simulations; .95 reports the empirical nonrejection probabilities (nominal size 5%) based on the analytical standard errors estimates. Results are averaged across 50 Monte Carlo replications.

Table 4: Bias and Root Mean Squared Error (Dynamic Model)

		NGFE				CMLE				NLTWFE				2STEPGFE			
		Bias		RMSE		Bias		RMSE		Bias		RMSE		Bias		RMSE	
DGP	G^0	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1$	$\widehat{\beta}_2$
1	2	-0.026	-0.128	0.229	0.328	-0.663	-0.174	0.689	0.526	-0.702	0.242	0.737	0.965	-0.032	-0.456	0.309	0.666
	3	0.073	-0.144	0.323	0.447	-0.651	0.238	0.676	0.634	-0.684	0.663	0.716	0.995	-0.142	-0.282	0.254	0.745
	5	0.156	-0.279	0.365	0.448	-0.592	0.090	0.629	0.524	-0.606	0.318	0.659	0.826	-0.051	0.158	0.277	0.492
2	2	0.486	0.043	0.630	0.141	-0.786	0.026	0.825	0.184	-0.839	0.248	0.893	0.337	0.695	-0.036	0.731	0.163
	3	1.007	0.111	1.182	0.184	-0.780	0.017	0.820	0.156	-0.842	0.247	0.902	0.316	0.360	-0.109	0.757	0.165
	5	2.144	0.297	2.272	0.358	-0.845	0.022	0.915	0.204	-0.912	0.295	1.015	0.394	0.682	0.077	1.159	0.254
3	2	0.298	0.300	0.507	0.339	-0.767	0.011	0.796	0.161	-0.821	0.242	0.859	0.325	-0.090	0.092	0.377	0.181
	3	0.319	0.319	0.481	0.353	-0.797	0.016	0.842	0.166	-0.868	0.247	0.932	0.329	0.108	0.050	0.506	0.077
	5	0.514	0.370	0.636	0.418	-0.734	0.030	0.770	0.161	-0.771	0.269	0.815	0.337	0.147	0.183	0.363	0.277
4	2	-0.114	0.052	0.267	0.159	-0.658	-0.003	0.676	0.143	-0.687	0.196	0.711	0.263	-0.045	0.071	0.126	0.105
	3	-0.060	0.078	0.230	0.152	-0.677	0.023	0.694	0.128	-0.712	0.234	0.736	0.283	-0.084	0.114	0.242	0.187
	5	-0.077	0.105	0.268	0.181	-0.685	0.018	0.713	0.118	-0.721	0.228	0.761	0.270	0.116	0.090	0.200	0.142

Notes: Dynamic logit model with $\beta_1 = 1$, $\beta_2 = 0.5$, $N = 90$, and $T = 7$. Results are averaged across 50 Monte Carlo replications. See Table 1 for details.

Table 5: Classification Accuracy and CPU Time (Dynamic Model)

		NGFE					2STEPGFE							GFE				
DGP	G^0	P	R	RI	MR	CPU	P	R	RI	MR	CPU	$\widehat{G}$	Failures	P	R	RI	MR	CPU
1	2	0.50	1.0	0.50	0.46	11.06	0.51	0.91	0.51	0.90	0.49	2.33	0.82	0.53	0.55	0.54	0.38	29.60
	3	0.33	1.0	0.33	0.62	12.98	0.34	0.94	0.36	0.93	0.38	2.14	0.86	0.36	0.39	0.57	0.55	29.62
	5	0.20	1.0	0.20	0.74	16.48	0.20	0.97	0.23	0.97	0.18	2.00	0.92	0.24	0.26	0.69	0.64	29.53
2	2	0.50	1.0	0.50	0.46	8.80	0.50	0.95	0.50	0.91	0.25	2.00	0.86	0.60	0.62	0.60	0.30	21.68
	3	0.33	1.0	0.33	0.61	9.69	0.34	0.99	0.35	0.97	0.10	2.50	0.96	0.45	0.47	0.63	0.43	22.91
	5	0.20	1.0	0.20	0.74	10.05	0.23	0.97	0.28	0.92	0.37	2.33	0.82	0.36	0.46	0.74	0.54	21.09

Notes: Dynamic logit model with $\beta_1 = 1$, $\beta_2 = 0.5$, $N = 90$, and $T = 7$. Failures is the number of failures of the first step of 2STEPGFE. Results are averaged across 50 Monte Carlo replications. See Table 2 for details.

Table 6: Inference for β_1 and β_2 (Dynamic Model)

		NGFE						CMLE					
		SE		SD		.95		SE		SD		.95	
DGP	G^0	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2
1	2	0.20	0.18	0.23	0.30	0.94	0.72	0.08	0.17	0.19	0.50	0.00	0.44
	3	0.20	0.19	0.31	0.42	0.82	0.64	0.09	0.17	0.18	0.59	0.00	0.34
	5	0.20	0.19	0.33	0.35	0.66	0.56	0.09	0.17	0.21	0.52	0.00	0.44
2	2	0.20	0.12	0.40	0.13	0.28	0.90	0.10	0.06	0.25	0.18	0.00	0.52
	3	0.23	0.13	0.62	0.15	0.30	0.72	0.12	0.07	0.25	0.16	0.00	0.60
	5	0.32	0.17	0.75	0.20	0.04	0.14	0.16	0.09	0.35	0.20	0.04	0.62
3	2	0.23	0.13	0.41	0.16	0.54	0.38	0.12	0.07	0.21	0.16	0.00	0.66
	3	0.23	0.13	0.36	0.15	0.48	0.28	0.12	0.07	0.27	0.17	0.02	0.62
	5	0.24	0.13	0.38	0.19	0.22	0.16	0.11	0.07	0.23	0.16	0.00	0.58
4	2	0.18	0.13	0.24	0.15	0.84	0.92	0.08	0.05	0.16	0.14	0.00	0.52
	3	0.18	0.13	0.22	0.13	0.88	0.92	0.08	0.05	0.15	0.13	0.00	0.68
	5	0.19	0.13	0.26	0.15	0.82	0.82	0.08	0.05	0.20	0.12	0.00	0.64

Notes: Dynamic logit model with $\beta_1 = 1$, $N = 90$, and $T = 7$. See Table 3 for more details.

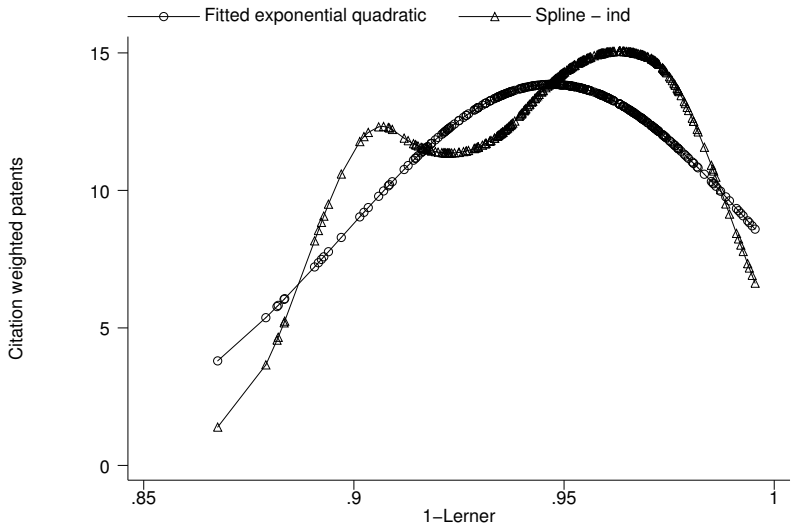
E.2 Empirical Application

Table 7: Summary Statistics

	1-Lerner index	Citation-weighted patents	Technology gap
Mean	0.95	6.66	0.49
SD	0.02	8.43	0.16
p_{10}	0.92	0	0.28
Median	0.95	3.35	0.51
p_{90}	0.98	20.19	0.69

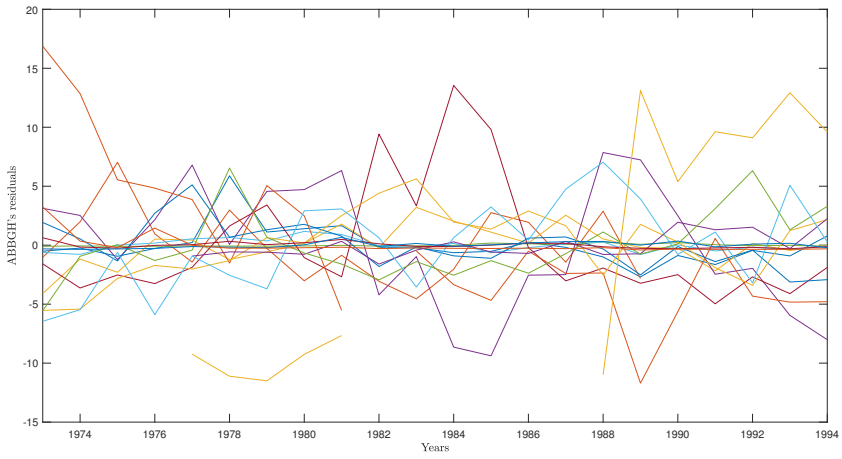
Notes: There are 17 industries, 354 observations and the time period covers 1973-94. See Aghion, Bloom, Blundell, Griffith, and Howitt (2005) for the exact definition of each variable.

Figure 1: The Inverted-U Relationship Between Innovation and Competition



Notes: This figure replicates Aghion, Bloom, Blundell, Griffith, and Howitt (2005)'s Figure II. Data include 17 industries of 311 firms listed on the London Stock Exchange observed between $1973-1994$. For each industry i at year t, the prediction replaces $\widehat{\nu}_i+\widehat{\xi}_t$ with an estimated constant $\widehat{\alpha}$ (one industry and time dummies are dropped).

Figure 2: FE Poisson Residuals



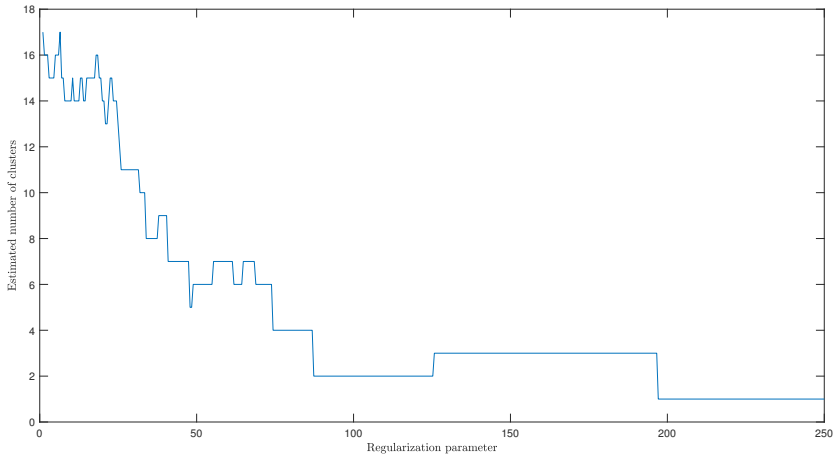
Notes: Each color represents an industry in Aghion, Bloom, Blundell, Griffith, and Howitt (2005)'s dataset. There are 17 industries, period covers 1973-1994.

Table 8: Industries in Aghion, Bloom, Blundell, Griffith, and Howitt (2005) Data Set at the 2-Digit Level

SIC 2	Name
22	Metal manufacturing
23	Extraction of minerals not elsewhere specified
24	Manufacture of non-metallic mineral products
25	Chemical industry
31	Manufacture of metal goods not elsewhere specified
32	Mechanical engineering
33	Manufacture of office machinery and data processing equipment
34	Electrical and electronic engineering
35	Manufacture of motor vehicles and parts therof
36	Manufacture of other transport equipment
37	Instrument engineering
41	Food industry
42	Food, drink and tobacco manufacturing industries
43	Textile industry
47	Manufacture of paper and paper products; printing and publishing
48	Processing of rubber and plastics
49	Other manufacturing industries

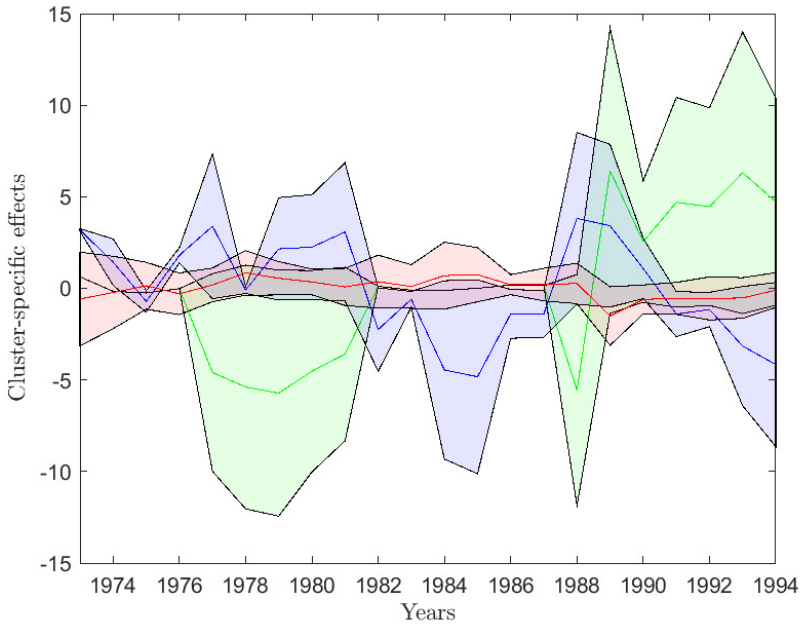
Source: 1980 Notebook of the UK Office of National Statistics available here: https://www.ons.gov.uk /methodology/classificationsandstandards/ ukstandardindustrialclassificationofeconomicactivities/uksicarchive.

Figure 3: TPWD Regularization Path



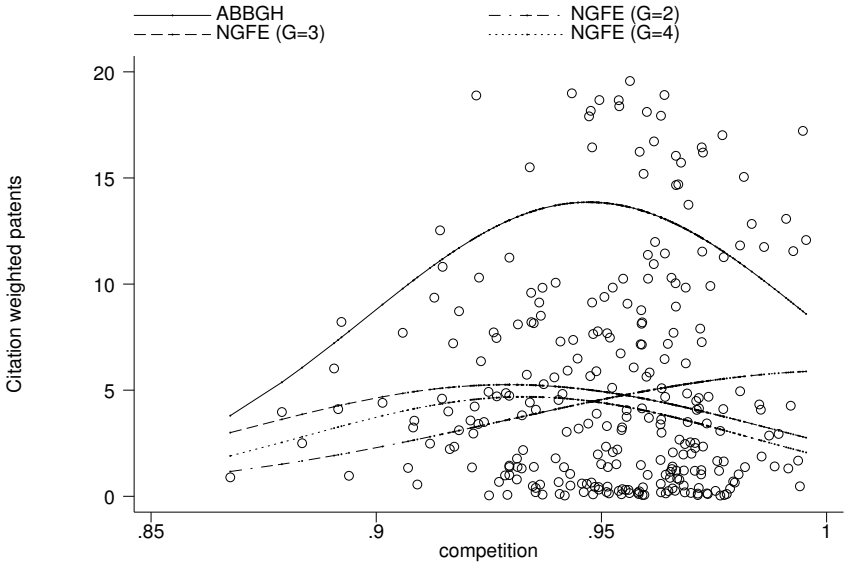
Notes: Number of clusters estimated by the TPWD estimator as a function of the regularization parameter. There are 17 industries, period covers 1973-1994.

Figure 4: TPWD Cluster Estimates (Three Clusters)



Notes: Each color represents an estimated cluster. There are 17 industries, period covers 1973-1994.

Figure 5: Innovation and Competition Revisited: A Mildly Inverted-U Relationship



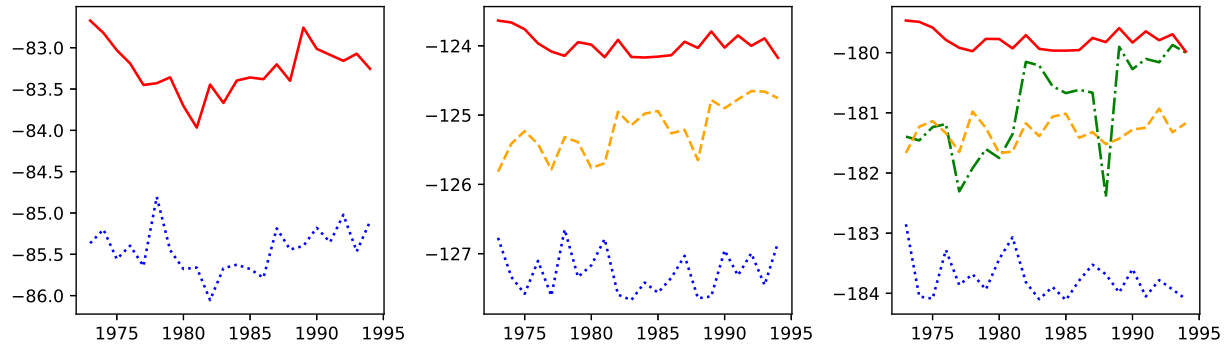
Notes: Aghion, Bloom, Blundell, Griffith, and Howitt (2005) (spe. (2) in Table 9) includes a constant and drop a time and an industry dummy (not included in the fit). NGFE (spe. (3), (4), and (5) in Table 9) does not specify a constant and uses the average of unobserved effects as the intercept in the fit.

Table 9: The Effect of Competition on Innovation

	FE Poisson		NGFE Poisson		
Dependent variable: Citation-weighted patents$_{it}$	(1)	(2)	(3)	(4)	(5)
Competition$_{it}$	152.80***	387.46***	171.28***	273.62***	392.23***
	(55.74)	(67.74)	(71.51)	(70.21)	(70.35)
Competition squared$_{it}$	-80.99***	-204.55***	-85.15***	-147.21***	-210.19***
	(29.61)	(36.17)	(38.18)	(37.62)	(37.73)
Year effects	Yes	Yes			
Industry effects		Yes			
Time-varying clustered effects			Yes	Yes	Yes
Number of clusters			2	3	4

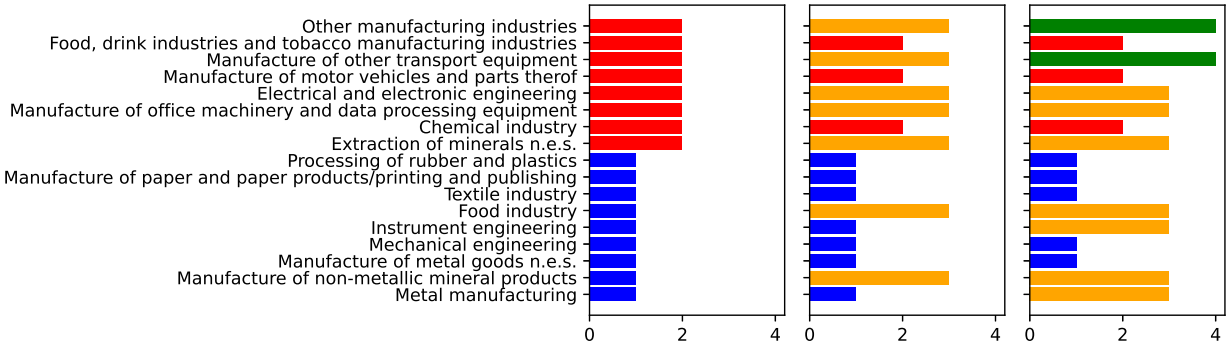
Notes: Analytical standard errors are under parentheses. The sample includes 354 observations from an unbalanced panel of 17 industries over the period 1973-1994. Competition$_{it}$ is measured by (1-Lerner index)$_{it}$ in the industry-year. NGFE estimates are computed using Lloyd's algorithm with 2,000 random initializers. ***, **, * denote statistical significance at 1, 5, and 10% respectively.

Figure 6: Estimated Cluster-Specific Time Effects



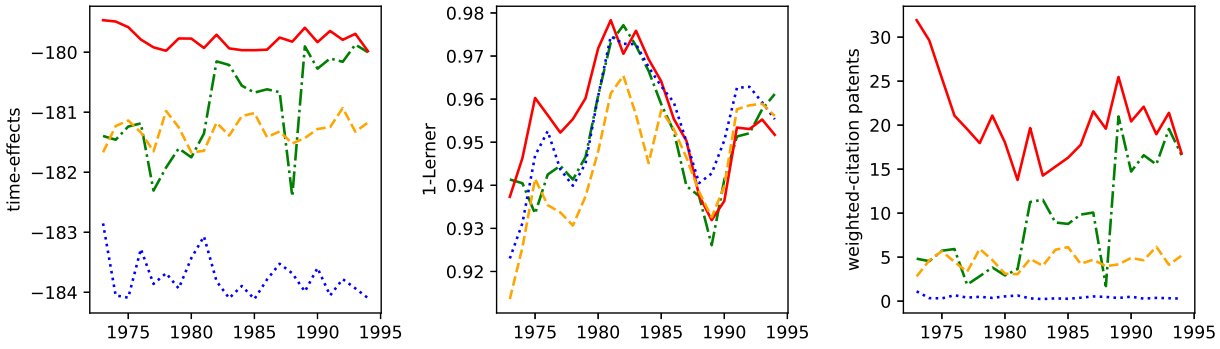
Notes: Solid line=High-Innovation, dotted line=Low-Innovation, dashed line=Steady-Catchers, dashdotted line=Noisy-Catchers. See Table 9 for more details.

Figure 7: Estimated Clusters



Notes: Low-Innovation (1), High-Innovation (2), Steady-Catchers (3), Noisy-Catchers (4). From left to right: NGFE estimates with $G = 2, 3, 4$.

Figure 8: Unobserved Heterogeneity, Competition, and Innovation Vary Across Time and Estimated Clusters



Notes: Solid line=High-Innovation, dotted line=Low Innovation, dashed line=Steady-Catchers, dashdotted line=Noisy-Catchers. From left to right: cluster-specific time-effects estimates ($G = 4$), average of c_{it} by estimated clusters, average of p_{it} by estimated clusters.

Table 10: The Effect of Competition on Innovation (Control Function Approach)

	FE Poisson			NGFE Poisson		
Dependent Variable: Citation-weighted patents$_{it}$	Annual	Before 1983	After 1983	Annual	Before 1983	After 1983
Competition$_{it}$	386.59***	229.18*	113.42	394.23***	265.86***	9.69
	(67.61)	(122.68)	(100.73)	(77.10)	(128.18)	(124.73)
Competition squared$_{it}$	-205.32***	-114.89*	-60.85	-212.35***	-144.18***	-9.41
	(36.11)	(66.49)	(53.37)	(41.14)	(67.95)	(67.46)
Relationship	steep inv-U	increasing		mildly inv-U	mildly inv-U	
Significance of: Competition$_{it}$, Competition squared$_{it}$	33.20	14.66	1.38			
	(0.000)	(0.001)	(0.5022)			
Significance of policy instruments	3.70	1.67	1.77	3.70	1.67	1.77
in reduced form	(0.001)	(0.192)	(0.064)	(0.001)	(0.192)	(0.064)
Significant of other instruments	5.60	3.43	2.11	5.60	3.43	2.11
in reduced form	(0.000)	(0.000)	(0.004)	(0.000)	(0.000)	(0.004)
Control functions in regression	4.38	-.61	-3.56	1.54	16.14	-2.05
	(3.51)	(6.99)	(6.13)	(2.89)	(7.05)	(3.71)
R^2 of reduced form	0.820	0.920	0.822	0.820	0.920	0.822
Year effects	Yes	Yes	Yes			
Industry effects	Yes	Yes	Yes			
Time-varying clustered effects				Yes	Yes	Yes
Number of clusters				4	4	4

Notes: Competition$_{it}$ is measured by (1-Lerner index)$_{it}$ in the industry-year. The sample includes 354 observations from an unbalanced panel of 17 industries over the period 1973 to 1994 (Annual), 1973-1982 (Before 1983), or 1983-1994 (After 1983). Estimates are from a Poisson regression with industry and year fixed effects (FE) or assuming unobserved clusters of time-varying heterogeneity (NGFE) with $G^0 = 4$ clusters of industries. Numbers in brackets are standard errors (not adjusted for the control functions). NGFE estimates are computed using Lloyd's algorithm with 2,000 random initializers. ***, **, * denote statistical significance at 1, 5, and 10% respectively.