Information Flows and Memory in Games

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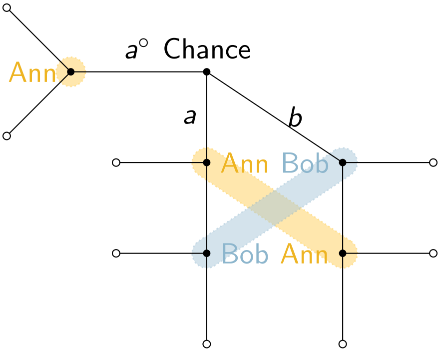

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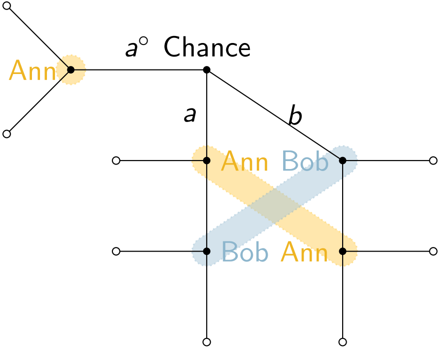

- But impossible to disentangle aspects of the **rules of the game** from **cognitive features**.

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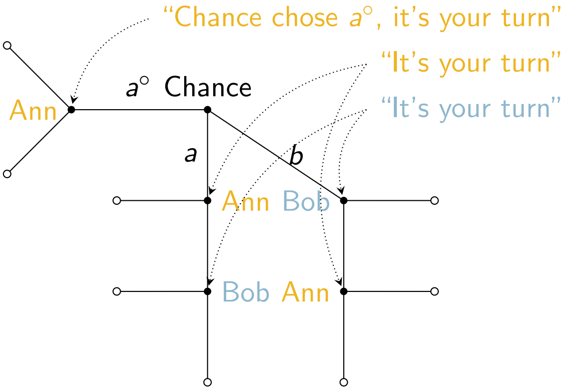

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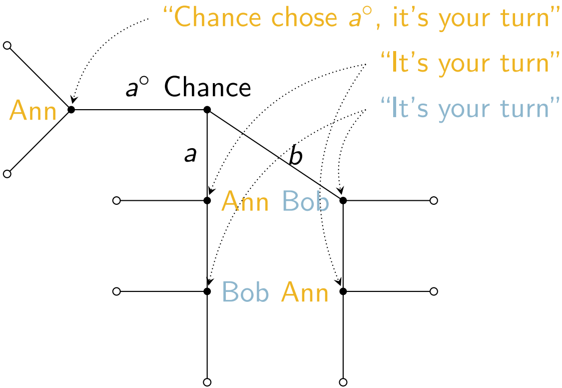

- Note: this is determined by the **rules of the game**.

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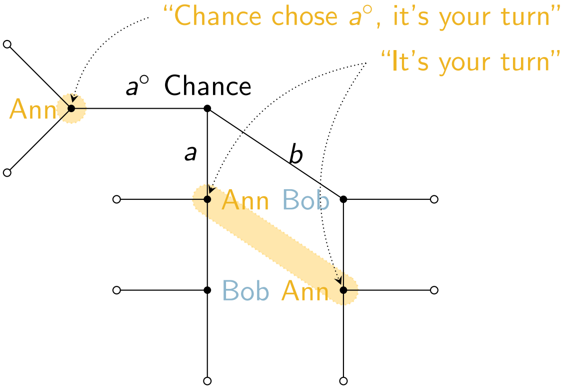
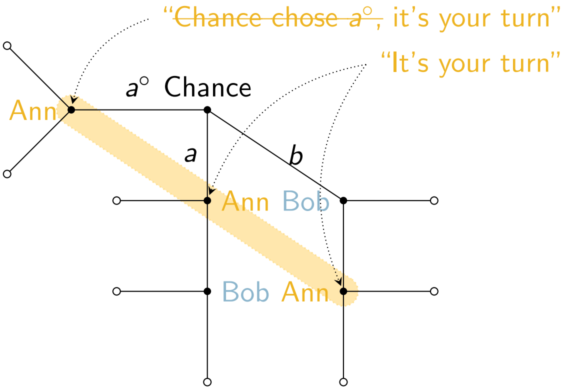
- For example, if Ann has "good" memory:



- But Ann may be forgetful:



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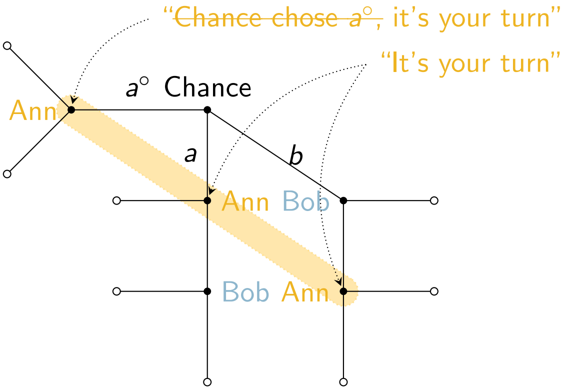

- Modeling cognitive limitations explicitly is important: a designer has control over the rules of interaction only!

Roadmap

1. Introduction
2. Flows of information
3. Memory
4. Conclusion
5. Bonus: An illustration

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- For $J \subseteq I$, $A_J := \times_{j \in J} A_j$ and $M_J := \times_{j \in J} M_j$.

  $A := \bigcup_{J \in 2^I \setminus \{\emptyset\}} A_J$ and $M := \bigcup_{J \in 2^I \setminus \{\emptyset\}} M_J$ $\rightarrow$ sets of **action** and **message profiles**.

  - ⋆ Note: we allow for situations where not every player moves or receives messages.

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- $\mathcal{A}_i : M_i \rightrightarrows A_i$ **action feasibility** correspondence of $i \in I$.
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- $f : A^{\leq L} \to M$ **feedback function**.
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- A **game structure** is $\Gamma = \langle I, f, (A_i, \mathcal{A}_i, M_i)_{i \in I} \rangle$.
  - ⋆ Note: this describes the **rules of the game** only.

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- We derive the trees of feasible plays and histories (P and H) from the rules Γ.
  - ⋆ Feasible = action profiles are allowed, and message profiles are generated according to the feedback function.

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- **Context-dependent memory**: remember pieces of information observed in contexts similar to the one being experienced.

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- $F_i(p)$ = personal history of i induced by play p = relevant stream of information for i when p has realized.

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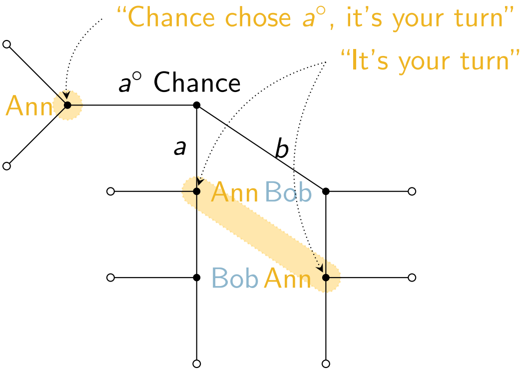
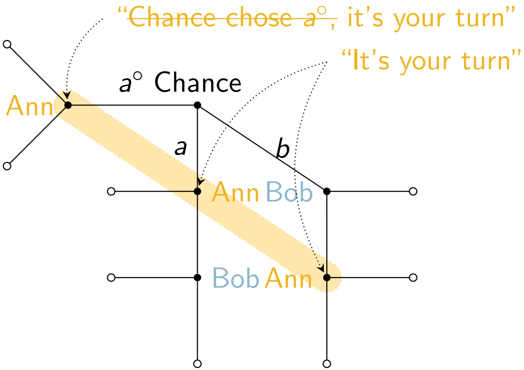
Indistinguishability ($\mathcal{F}_{Ann}$):



Possibility ($\mathcal{P}_{Ann}$):



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   - ⋆ Note: our approach is more expressive $\rightarrow$ it allows for cognitive limitations that do not induce information partitions (e.g., ruling out the actual play).

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Next steps:

- A theory of imperfect memory (or bounded rationality) and **strategic thinking**.
- A **more structured theory of memory**.

Thank you

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An illustration / The absentminded driver

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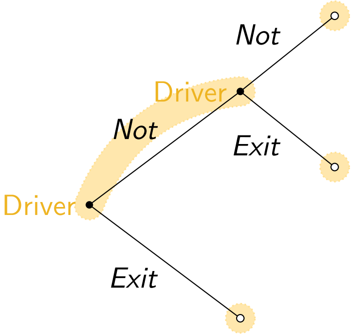

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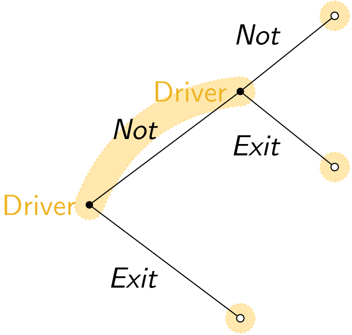

- The information set $\{\varnothing, (\mathit{Not})\}$ is often thought to be problematic (e.g., Alos-Ferrer & Ritzberger 2016).

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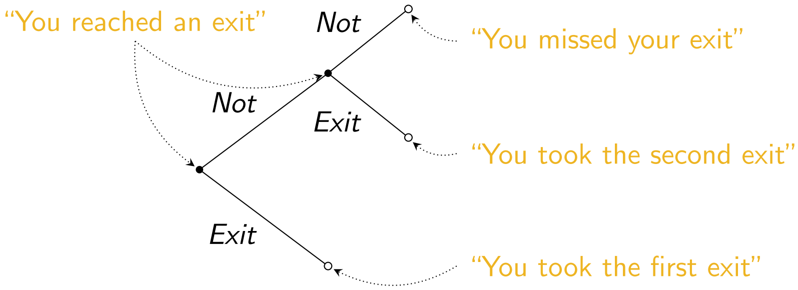

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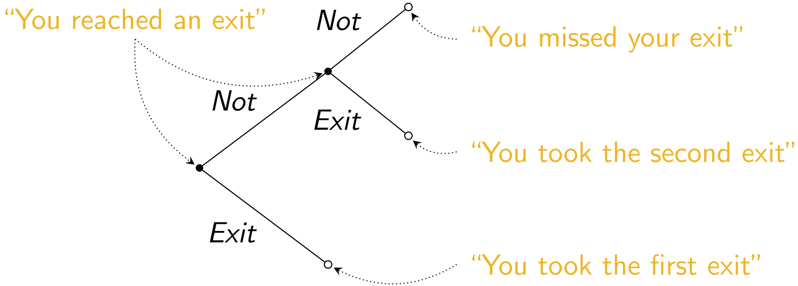

- Interpretation: the driver is just told (or, she can observe while driving) that she encountered an exit, but she is not reminded of whether it is the first or the second one.

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  - So, they can be confused.
- Absentmindedness emerges as a natural cognitive limitation: the agent remembers only the information materially available in that moment.

References

Alos-Ferrer, C., & Ritzberger, K. (2016). *The Theory of Extensive Form Games*.

Piccione, M., & Rubinstein, A. (1997). On the interpretation of decision problems with imperfect recall. *Games and Economic Behavior*, 20(1), 3-24.