Limited Memory, Time-varying Expectations and Asset Pricing*

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Abstract

We propose a theory of asset pricing based on limited memory and time-varying expectations. The former guarantees a tendency to revert to fundamentals. The latter induces 'momentum' in asset prices and it is motivated by a novel empirical observation about a time-varying mapping from price-dividend ratio to return expectations in survey data. The simulated method of moments shows that the model quantitatively replicates a host of asset-pricing features, including equity premium, excessive volatility, persistence of price-dividend ratio, predictability of excess returns and the consumption correlation puzzle. The model also generates empirical plausible subjective expectations.

Keywords: Asset pricing, Expectations, Limited memory, Equity premium.

JEL classification: G0, G12, G40.

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1 Introduction

Expectations play a central role in asset pricing. Asset prices are essentially forward-looking, implying that expectations about the asset's future payouts and future prices would determine the equilibrium today. The predominant modelling approach assumes rational expectations (RE). Under RE, investors' expectations are tied down by the true underlying law of motion that generates future asset payouts (Sargent, 2008). This paper proposes an asset pricing model built on relaxing two main features of the RE approach.

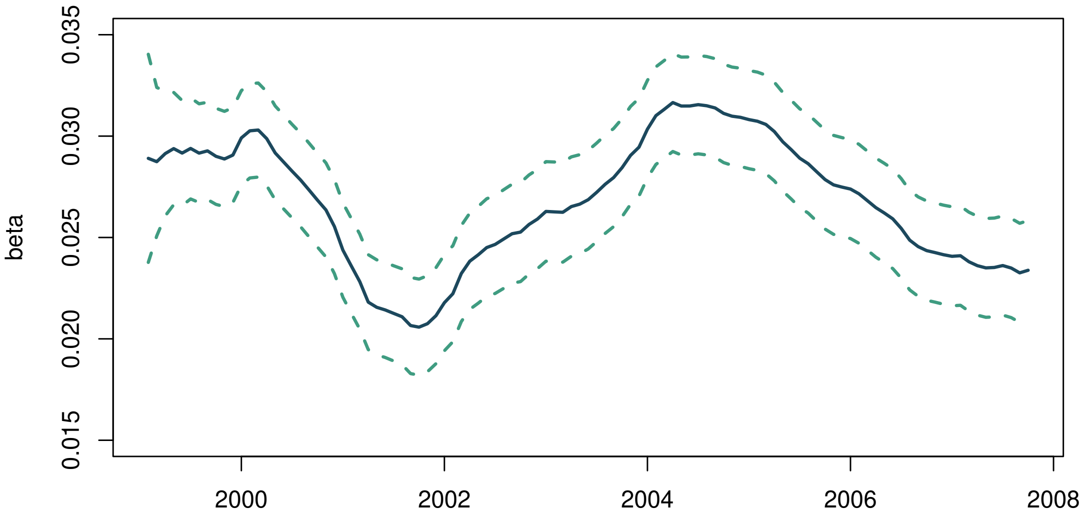
First, recent empirical evidence suggests investor expectations deviate from the RE paradigm. Among others, investor expectations tend to be extrapolative, in the sense that their expectations are positively correlated with current or past prices (see e.g., Haruvy et al., 2007; Greenwood and Shleifer, 2014). This is at odds with RE models with a unique forward-looking equilibrium where history does not matter for equilibrium prices and hence for expectations. In this respect, this paper starts by uncovering a novel fact: this extrapolative behaviour varies over time. Figure 1 plots the path of the coefficient obtained by a time-varying parameter regression of survey (excess) return expectations on the price-dividend (PD) ratio. A visual inspection suggests a time-varying mapping from observed PD ratio to return expectations, that is, the way investors map observations to expectations seems to be time-varying.[1]

Second, RE assumes that agents retain full memory of all past events. However, this standard assumption is not only conceptually implausible but also rejected by both empirical and experimental evidence (see e.g., Jonides et al., 2008). Indeed, the limited memory assumption has been embedded in the recent finance and macroeconomics literature (See Section 1.1 for a brief review).

These considerations motivate the construction of an asset-pricing model in which agents feature: (i) a time-varying expectation formation process and (ii) limited memory. We show that a mechanism based on these two features generates endogenous stochastic volatility, which existing asset pricing models usually need to assume exogenously to match the empirical asset pricing behaviour. Embedding this mechanism in the standard

[1] This is confirmed by the Hansen (1992) stability test. Section 6.4 presents a detailed discussion of this TVP estimation.

Figure 1: Time-varying parameter estimations



The Figure plots the coefficient in a time-varying parameter regression of survey expected excess return on log PD ratio, see Section 6.4 for details. Dashed lines show the 95% probability intervals standard error bands for the coefficient.

Bansal and Yaron (2004) long-run risks asset pricing model without stochastic volatility, the resulting model is able to not only quantitatively reproduce a variety of stylized asset pricing facts, but also produce plausible expectations in light of survey data on investor expectations, such that this mechanism is empirically very encouraging.

We explore two forms of memory constraint: (i) decay memory, whereby historical data's influence on expectations gradually fades over time as it recedes into the past; (ii) finite memory, whereby a past data point does not influence agents' expectations after a given period of time.[2]

Despite this arguably minor deviation from RE from a theoretical point of view, the limited memory assumption opens up a very different world from the point of view of the admissible dynamics. As a consequence of limited memory, backward-looking solutions of an explosive system are no longer explosive; hence the typical saddle-point dynamic system admits infinite possible solutions. In other words, once we perturb the original Muth (1961) RE solution with the limited memory assumption, there is an infinite number of bounded solutions. Hence, Blanchard and Kahn's (1980) stability criterion to select

[2] In both cases, we assume that the agent does not internalize the memory constraints, that is, the agent at t does not anticipate that time $t+1$ will have experienced some loss of memory of the data that she knows at time t.

the unique stable solution of a saddle-point dynamic system is not applicable and Muth's (1961) original problem of the multiplicity of solutions remains. This raises the question of how to select the equilibrium path among all bounded solutions.

As in the literature on indeterminacy of RE equilibria, we appeal to the existence of a sunspot shock to choose one among all these stable paths, modelling our sunspot shock as in Ascari et al. (2019).[3] This sunspot shock has an appealing economic interpretation as it generates a time-varying expectation formation process, i.e., it generates changes over time in the way the agents weight past data to calculate their expectations, as suggested by the survey evidence (Figure 1). As a result, the equilibrium asset price switches randomly among the infinite limited memory equilibria as agents modify how they combine past data to calculate their expectations. Moreover, this approach has two other attractive properties. First, as explained by Ascari et al. (2019), this sunspot is a multiplicative sunspot, rather than an additive one. Hence, it generates both a time-varying parameter solution and endogenous stochastic volatility. Second, this approach encompasses the usual RE equilibrium as a special case.

Introducing limited memory and time-varying expectations into an asset pricing model makes both a time-varying equity premium and stochastic volatility arise endogenously. Specifically, in our model, the representative agent has Epstein and Zin (1989) recursive preferences, and there is a small predictable component in the consumption and dividend growth process - i.e., the long-run risk in Bansal and Yaron (2004). Our mechanism affects the risk premium through two channels. First, the expectation shocks carry an additional source of risk different from the fundamental ones, generating a higher equity premium. Second, as the expectation shock enters non-linearly in the solution, it also affects the risk premium agent demands for the long-run risk in a time-varying way. This second channel also gives rise to stochastic volatility. Hence, in contrast with the Bansal and Yaron (2004) model, we do not need to assume stochastic volatility, which instead arises endogenously through the time-varying expectation formation process. Moreover, it is

[3] Ascari et al. (2019) generalizes RE solutions to accommodate temporarily unstable paths. They build solutions that randomly jump between all the admissible RE paths, both stable and unstable. There is an important distinction between their framework and ours. In their framework, time variation in the solution allows temporary walks on unstable RE trajectories, and they thus need to impose an exogenous force for the system to eventually converge to the unique stable solution. In our framework, instead, the limited memory assumption provides such stabilizing force and the expectation will never explode.

important to stress that the assumed time-varying expectation formation process imposes some theoretical structure on stochastic volatility. Despite this constraint imposed by the theoretical structure, our model quantitatively outperforms the Bansal and Yaron (2004) model that assumes an unconstrained exogenous process for stochastic volatility, implying that the assumed expectation process seems corroborated by the data. Our mechanism also produces realistic price dynamics. The expectation process induces "momentum" on stock prices, while the limited memory assumption entails "mean reversion" over long horizons to stable fundamentals. As such, the model naturally generates boom-and-bust dynamics.

Based on the simulated method of moments (SMM) approach, the model can quantitatively replicate a host of key asset-pricing features, including equity premium, excessive volatility and persistence of PD ratio, without yielding counterfactual high-level correlation between returns and fundamentals. In addition, the model could match the counter-cyclical realized excess returns - i.e., actual excess returns negative correlate with lagged PD ratios. The model passes a formal econometric test for the overall fit.

Furthermore, we compare our theory's predictions with survey evidence on expectations. First, using the estimated parameters from SMM, we show that our model is statistically consistent with the evidence about the extrapolative behaviour in the formation of investors' expectations, that is, expected excess market return is positively correlated with lagged price-dividend ratios. Second, the solution of our model implies that this link between the PD ratio and expected return is time-varying, as suggested by the predictive regression based on the UBS/Gallup survey data (see Figure 1).

Finally, we compare our model with other leading alternatives, the long-run risk model by Bansal and Yaron (2004) and the learning model by Adam et al. (2016).

The structure of the paper is as follows. After the next subsection that briefly reviews the relevant literature, Section 2 features a simple example to explain our approach to modelling limited memory and time-varying expectations. Section 3 describes how we incorporate the time-varying expectation formation process in the Bansal and Yaron (2004) asset-pricing model. Section 4 derives analytical results that explain the model potential to replicate the dynamics of the PD ratio and the equity premium. Section 5 describes the data sources and the estimation method. Section 6 presents the quantitative results. Section 7 discusses the comparison with the models in Bansal and Yaron (2004) and Adam et al. (2016). Finally, Section 8 concludes.

1.1 Literature Review

Many key features of asset markets are puzzling from the perspective of theoretical models. Among others, these include the equity premium puzzles (see, e.g., Mehra and Prescott, 1985) and asset price volatility puzzles (see, e.g., Shiller, 1980). An extensive literature has developed since then, introducing additional dynamics in investors' stochastic discount factor and made progress in better matching stock price behaviour. For example, Campbell and Cochrane (1999) add external habits (past consumption) as an extra state variable in the stochastic discount factor. Bansal and Yaron (2004) introduce fluctuations in the economy's long-run growth prospects and the level of economic uncertainty (or stochastic volatility). Several extensions to the Bansal and Yaron's (2004) long-run risk framework aiming at providing micro-foundations to the time-varying uncertainty include Bansal and Shaliastovich (2010) and Bansal and Shaliastovich (2011). Our paper expands on this literature showing that stochastic volatility can also arise from the time-varying expectations formation process. Specifically, in our model, the variation in the expectation formation process is due to an exogenous sunspot shock, hence, one may argue we do not provide a micro-foundation of time-varying uncertainty. However, this is not left completely unconstrained, because our modelling assumptions regarding both limited memory and Muth's (1961) type of solutions provide an economic interpretation of the reason why stochastic volatility arises, and a theoretical microstructure that imposes certain restrictions on the way stochastic volatility can affect the model variables.

Our model also shares some elements with asset pricing models with time preference shocks. For example, Albuquerque et al. (2016) stress the importance of demand shocks coming from stochastic changes in agents' rate of time preference in resolving the correlation puzzle (see Cochrane, 2009, for the discussion on correlation puzzle). As explained later, despite being very different in terms of assumptions and modelling, our time-varying expectation shock has some similarity regarding economic intuition with the valuation shock in Albuquerque et al. (2016). In particular, both models yields stochastic changes in agents' valuation of assets, which in turn determines the equilibrium distribution of prices. However, Albuquerque et al. (2016)'s valuation risk comes from changes in agents' rate of time preference and it alters the valuation of assets in the absence of any shocks to the fundamentals. In our model, instead, the existence and the magnitude of the change in valuations would depend crucially on the history of fundamental shocks.

Although the variant of consumption-based RE models has been the benchmark of asset pricing models, they are incapable of answering questions such as what drives the periodical boom and bust in the financial market? Why subjective excess returns expectations (from survey data) are virtually unrelated to dividend growth but strongly positively correlate with price levels? These motivate a recent literature deviating from full rationality and developing alternative theoretical frameworks. Adam et al. (2016) argue that past price increases generate optimism about future capital gains and thus a further rise in asset prices. Another strand of literature that deviates from RE involves some form of learning where investors use observed data to form expectations about future payouts or prices. Learning about the underlying process of assets' payouts can contributes to variations in expected payouts and hence to price volatility. Early literature includes Timmermann (1993, 1996). Models in which investors have memory constraints are closely related to our modelling assumption. Nagel and Xu (2021) studied asset price behaviour in an economy where agents learn about the asset dividend growth with fading memory. The fading memory in the Bayesian framework produces perpetual learning, which induces substantial long-run uncertainty. However, in their model, agent holds subjective beliefs only about exogenous objects (i.e., exogenous payouts) that are independent of agent's beliefs, therefore, the dynamic feedback from past price changes to future prices is absent. Other theoretical models and discussions on learning from experience (see, e.g., Malmendier and Nagel, 2016) is generally based on the notion that memory of past data is lost. On top of the behavioural finance literature, limited memory and memory constraints are also been embedded in macroeconomics (see, e.g., Woodford, 2018; Angeletos and Lian, 2021).

Our paper is also related to the empirical findings from survey data which show that investors' belief tends to be extrapolative. Haruvy et al. (2007) shows that extrapolation shows up in data on expectations of participants in experimental bubbles, where subjects can be explicitly asked about their expectations of returns. Greenwood and Shleifer (2014) analyzes time series of investor expectations of future stock market returns from six data sources between 1963 and 2011 and find strong evidence of extrapolation - i.e., investors' expected returns are positively correlated with the PD ratio. Adam et al. (2017) show that this is robust to a range of surveys (see also Nagel and Xu, 2021). Based on the survey evidence, many works investigate theoretical models based on extrapolation (see, e.g., Hirshleifer, 2015; Barberis et al., 2018; Cassella and Gulen, 2018).

2 Time-Varying Expectations

This section presents our approach to model expectations. It modifies the standard RE assumption by introducing two features: 1) limited memory; 2) time-variation in expectation formation as in Ascari et al. (2019).

We use a simple forward-looking equation to explain the basic intuition

$$y_t = \theta \mathbb{E}_t y_{t+1} + \varepsilon_t, \tag{1}$$

where ε_t is an i.i.d shock $\sim N(0, \sigma_\varepsilon^2)$ and $\mathbb{E}_t\, y_{t+1} = \mathbb{E}(y_{t+1}|\mathcal{I}_t)$ is the expected value of y_{t+1} conditional on the information set at time t. Here ε_t can be interpreted as the asset's dividend and y_t as its price.

Any forward-looking equation as (1) implies a fundamental degree of freedom, and an infinite number of solutions, because one can find an infinite number of pairs $(y_t, \mathbb{E}_t\, y_{t+1})$ that satisfy it. To see this, recall that Muth's (1961) RE seminal idea assumes agents form their expectations so that the expected forecast error cannot be systematic or predictable, i.e., $\mathbb{E}_{t-1}(\eta_t) = 0$, where $\eta_t = y_t - \mathbb{E}_{t-1}\, y_t$ is the forecast error. The RE requirement, however, is generally not enough to pin down a unique solution, as it is evident by rewriting (1) using conditional expectations $\xi_t = \mathbb{E}_t(y_{t+1})$ as

$$\xi_t = \theta^{-1}\left(\xi_{t-1} - \varepsilon_t + \eta_t\right). \tag{2}$$

Any process η_t such that $E_{t-1}(\eta_t) = 0$ defines a different solution to (2).[4] Any forecast error of the form

$$\eta_t = b\varepsilon_t + \zeta_t, \tag{3}$$

then yields a RE solution, where ζ_t is a mean zero non-fundamental/sunspot disturbance, uncorrelated with the fundamental one. Equation (3) shows that there are two main degrees of freedom in the admissible solutions: the parameter b and the disturbance ζ_t.

The point was evident in Muth's (1961) original formulation that looks for solutions for equation (1) expressed as a weighted sum of past, current and expected future values of the structural shocks (hence, abstracting from sunspot disturbances)

$$y_t = \sum_{j=1}^{\infty} u_j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \, \mathbb{E}_t \, \varepsilon_{t+j}, \tag{4}$$

where u_j, b and c_j are coefficients to be determined. Plug (4) back into (1), and use the undetermined coefficient method to derive the set of admissible solutions as

$$y_t = (b-1)\sum_{j=1}^{\infty} \frac{1}{\theta^j}\varepsilon_{t-j} + b\varepsilon_t + b\sum_{j=1}^{\infty} \theta^j \, \mathbb{E}_t \, \varepsilon_{t+j} = (b-1)\sum_{j=1}^{\infty} \frac{1}{\theta^j}\varepsilon_{t-j} + b\varepsilon_t, \tag{5}$$

given that $\mathbb{E}_t \, \varepsilon_{t+j} = 0, \forall j > 0$.[5] Equation (5) shows that all the infinite solutions of equation (1) that are a function only of the history of the structural shocks can be parameterized by a free parameter $b \in (-\infty, +\infty)$. A particular value of b defines a particular solution. Following the terminology used by Blanchard (1979), two important solutions often considered in the literature are: (i) the pure forward looking solution corresponding to $b = 1 : y_t^F = \varepsilon_t$; (ii) the pure backward looking solution, corresponding to $b = 0 : y_t^B = -\sum_{j=1}^{\infty} \theta^{-j} \varepsilon_{t-j} = \theta^{-1}(y_{t-1}^B - \varepsilon_{t-1})$. Moreover, Muth (1961) stressed that b has a natural interpretation: it defines the way agents form their expectations. This is

[4] One could interpret the error of expectations η_t as a martingale difference process, and the requirement of a zero expected error simply implies that the solution is characterized up to an arbitrary martingale.

[5] Without loss of generality we assume that expected future shock are zero. Appendix A contains all the derivations for the equations in this Section.

easy to see by writing the implied expected value of y_{t+1} at t as (assuming $b \neq 0$)

$$\mathbb{E}_t\, y_{t+1} = (b-1) \sum_{i=1}^{\infty} \left(\frac{1}{b\theta}\right)^i y_{t+1-i}, \tag{6}$$

which shows that agents combined past value of the observable variable, $\{y_t, y_{t-1}, y_{t-2}, \ldots\}$, to form their expectation about its future value, $\mathbb{E}_t\, y_{t+1}$.[6] The weights on past values are determined by b. First, b measures the extent to which past observations matter for expectations in absolute terms. If $b = 1$, for example, past values have no effect; this is the forward-looking solution. Second, b determines the relative weight $(\frac{1}{b\theta})$ put on the past data when agents form their expectations. Hence, any given value of b pins down a particular way that agents combine past data to form their expectations, thus leading to one particular RE solution.

In other words, there is a multiplicity of solutions satisfying the rationality condition, meaning that additional conditions must be placed in order to pick a unique equilibrium. Blanchard and Kahn (1980) famously proposed the stability (i.e., boundedness) of the solution as such a condition. In the case where $\theta < 1$ and the agent has full information and retains full memory of the past history of the shocks (i.e., the agent knows $h^t = \{\varepsilon_t, \varepsilon_{t-1}, \ldots\}$), the backward-looking component in the solution (5) is explosive. Therefore, the stability condition pins down the pure forward-looking solution (corresponding to $b = 1$, $y_t = \varepsilon_t$, $\mathbb{E}_t\, y_{t+1} = 0$), which is indeed the unique bounded one.

1. Limited Memory. We twist this framework and deviate from the usual RE in one fundamental way. As in Muth's (1961) original formulation, we look for solutions for equation (1) expressed as a weighted sum of past, current and expected future values of the structural shocks, but we assume limited memory. We investigate two different specifications of limited memory: (i) finite memory; (ii) decay memory. In both cases, we assume that the agent does not internalize the memory constraints, that is, the agent at t does not anticipate that at time $t+1$ she will experience some loss of memory of the

[6] One of the purpose of Muth (1961) original paper is to write the expectation at time t as an exponentially weighted average of past observations - as in the adaptive expectations or constant gain learning framework - because he showed in a previous paper - Muth (1960) - that, under some assumptions, this is the optimal estimator.

data known at time t.

(i) Finite Memory. We assume the agent remembers the past structural shocks only up to T period ago. In other words, a past data point does not influence agent expectations after an extended period of time. Under this specification, the expectation can only condition on a subset of structural shocks in the past: $\mathbb{I}_t = \{\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-T+1}, \varepsilon_{t-T}\}$. Then, denoting the expectations under limited memory as $\bar{\mathbb{E}}$, we guess the solution has the following formulation

$$y_t = \sum_{j=1}^{T} u_j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \bar{\mathbb{E}}_t \varepsilon_{t+j}. \tag{7}$$

The above equation (7) is similar to equation (4) but with an additional memory constraint. Following the same procedure to determine u_j, b and c_j yields the following solution

$$y_t = (b-1)\sum_{j=1}^{T} \frac{1}{\theta^j}\varepsilon_{t-j} + b\varepsilon_t + b\sum_{j=1}^{\infty} \theta^j \bar{\mathbb{E}}_t \varepsilon_{t+j} = (b-1)\sum_{j=1}^{T} \frac{1}{\theta^j}\varepsilon_{t-j} + b\varepsilon_t, \tag{8}$$

that mirrors Muth's equation (5), and again admits infinite solutions parameterized by b.

(ii) Decay Memory. We assume the agent progressively loses memory of past structural shocks at a rate λ, that is, historical data's influence on expectations gradually fades over time as it recedes into the past. The solution conditions on the decayed memory of structural shocks in the past: $\mathbb{I}_t = \{\varepsilon_t, \lambda\varepsilon_{t-1}, \lambda^2\varepsilon_{t-2}, \ldots\}$. Following Muth (1961)'s formulation, we are going to look for a solution where (4) becomes

$$y_t = \sum_{j=1}^{\infty} u_j \lambda^j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \bar{\mathbb{E}}_t \varepsilon_{t+j}, \tag{9}$$

The solution has a similar form as (5) but with a constant decay rate on past shocks.

$$y_t = (b-1)\sum_{j=1}^{\infty} \left(\frac{\lambda}{\theta}\right)^j \varepsilon_{t-j} + b\varepsilon_t + b\sum_{j=1}^{\infty} \theta^j \bar{\mathbb{E}}_t \varepsilon_{t+j} = (b-1)\sum_{j=1}^{\infty} \left(\frac{\lambda}{\theta}\right)^j \varepsilon_{t-j} + b\varepsilon_t, \tag{10}$$

Despite these apparently minor differences with respect to the original RE formulation, the assumption of limited memory has a major implication for the stability property of the differential equation (1). Under limited memory, the backward-looking solution might not be explosive anymore even in the case where $\theta < 1$. Hence, any linear combination (i.e., any given b) of the backward and forward-looking solutions is an admissible solution according to the stability criterion of Blanchard and Kahn (1980) . The original pure backward looking solution, corresponding to $b = 0$, is now equal to $y_t^B = -\sum_{j=1}^{T} \theta^{-j} \varepsilon_{t-j}$ from (8) in the finite memory case or to $y_t^B = -\sum_{j=1}^{\infty} \left(\frac{\lambda}{\theta}\right)^j \varepsilon_{t-j}$ from (10) in the decay memory case. In the former case, y_t^B is always bounded. In the latter case, where $\lambda > \theta$, y_t^B is explosive so that the stability criterion pins down the unique stable solution, which is the forward-looking one, i.e., $b = 1$. In the case where $\lambda < \theta$, however, the Blanchard and Kahn's (1980) stability condition can no longer select a unique solution, because (10) is always bounded for any value of b. It follows that, in the finite memory case or whenever the degree of memory decay is sufficiently high in the decay memory case, the original problem of the multiplicity of solutions remains, even when the RE solution would be saddle point stable, i.e., $\theta < 1$. As a result, b is no longer constrained by the stability condition, and thus can take any value.

In the limited memory case, b has the same interpretation as in RE: it pins down a particular way that agents combine past data to form their expectations. In the decay memory case, for example, future expectation obeys

$$\bar{\mathbb{E}}_t y_{t+1} = (b-1) \sum_{i=1}^{\infty} \left(\frac{1}{\theta b}\right)^i \lambda^{i-1} y_{t+1-i}, \tag{11}$$

which carries the same intuitive interpretation as equation (6), though now agents form their expectations conditional on a fraction of past observations, i.e., $\mathbb{I}_t = \{y_t, \lambda y_{t-1}, \lambda^2 y_{t-2}, \ldots\}$. As b is not constrained to be equal to one by the stability condition, past observations matter for expectations even in the case pure forward-looking equation as (1). The value of b would determine how agent maps discounted past observations to form their expectations. Given that b is now a free parameter, this raises the following question: how to deal with the limited memory assumption since there are many (or infinite) admissible bounded solutions?

2. Time-varying Expectation Formation Process. As in the macroeconomic literature on indeterminacy, we appeal to the existence of a sunspot shock to choose one among all these stable paths. Following the approach in Ascari et al. (2019), we assume that b_t is time-varying, so that b_t follows a random walk, $b_t = b_{t-1} + \sigma_b \xi_t$, with $\xi \sim$ i.i.d $N(0,1)$, being the sunspot shock. This assumption is convenient and has a natural interpretation. As said above, there is an infinite number of possible ways agents could combine (the memory of) past observations mimicking the RE. All of them are admissible and they are parameterized by b_t. The sunspot shock hence captures the fact that our agent can change over time how to combine past data to form her expectations. Any given value of b_t picks up a particular solution. Two brief comments follows. First, as explained by Ascari et al. (2019), this sunspot is a multiplicative sunspots, rather than an additive one. Hence, it generates both a time-varying parameter solution and endogenous stochastic volatility. Second, this setup encompasses the standard RE equilibrium as a special case, corresponding to $b_t = 1, \forall t$.

Therefore, the solution is randomising among different admissible stable equilibria. Appendix A shows that under these assumptions, in the finite memory case the solution with time-varying expectations is

$$y_t = \sum_{j=1}^{T} \left(\frac{1}{\theta}\right)^j (b_t - 1)\varepsilon_{t-j} + b_t \varepsilon_t \tag{12}$$

and in the decay memory case is

$$y_t = \sum_{j=1}^{\infty} \left(\frac{\lambda}{\theta}\right)^j (b_t - 1)\varepsilon_{t-j} + b_t \varepsilon_t. \tag{13}$$

2.1 How Price Expectations are Updated

This section derives the implication of our approach for the updating of expectations - for brevity we just focus on the decay memory case. Corresponding to (6) or (11), the expectation in the decay memory case with time-varying b_t is

$$\bar{\mathbb{E}}_t y_{t+1} = (b_t - 1) \sum_{i=1}^{\infty} \left(\frac{\lambda^{i-1}}{\theta^i \prod_{j=0}^{i-1} b_{t-j}} \right) y_{t+1-i}, \tag{14}$$

which can be written recursively as

$$\bar{\mathbb{E}}_t y_{t+1} = \frac{1}{\theta}\left[\frac{\nu_t}{\nu_{t-1}}\lambda\bar{\mathbb{E}}_{t-1}y_t + \nu_t\left(y_t - \lambda\bar{\mathbb{E}}_{t-1}y_t\right)\right], \tag{15}$$

where $\nu_t = \frac{b_t - 1}{b_t}$. This expression reminds the updating implied by constant gain learning, employed by, e.g., Adam et al. (2016) and Nagel and Xu (2021), that is

$$\bar{\mathbb{E}}_t y_{t+1} = \bar{\mathbb{E}}_{t-1}y_t + \nu\left(y_t - \bar{\mathbb{E}}_{t-1}y_t\right), \tag{16}$$

where ν is the gain parameter. The gain parameter measures how much agents will update their expectation accordingly to the previous period expectation error, as well as the weight on the last observed data point.

The comparison between (15) and (16) highlights the three differences between our approach and constant gain learning. These three differences come first from starting from the RE assumption, and then from twisting it with both limited memory and time-varying expectation. First, the updating rule (15) is multiplied by $1/\theta$, because under RE agents take the model into account in formulating their forecasts. In other words, the agent knows the objective underlying law of motion. Combining (2) and (3) yields the updating rule under RE as [7]

$$\mathbb{E}_t y_{t+1} = \frac{1}{\theta}\left[\mathbb{E}_{t-1}y_t + \nu\left(y_t - \mathbb{E}_{t-1}y_t\right)\right] \qquad \text{where} \qquad \nu = \frac{b-1}{b}. \tag{17}$$

The updating rule under RE is similar to constant gain learning, but the updating is multiplied by $1/\theta$ and the constant gain is given by $\nu = \frac{b-1}{b}$. Second, the term λ appears in front of the past expectation in (15) as a result of the decay memory assumption. Agents discount the previous period expectation by a factor λ, because they partially lose memory of past expectations. Third, the time-varying expectation assumption makes not only the gain parameter to be time-varying, ν_t, but it also changes the way past expectation affects current expectation, i.e., the term $\frac{\nu_t}{\nu_{t-1}}$. As clear from (14), a change in b_t reshuffles the weights agents use for past data in forming their expectations - thus

[7] From (3) $y_t - \mathbb{E}_{t-1}y_t = \eta_t = b\varepsilon_t$, abstracting for simplicity from the additive sunspot shock ξ_t.

agents also change the weight of the last observation, i.e., the gain parameter.[8]

3 An Asset Pricing Model with Time-varying Expectations

This Section presents an asset-pricing model that embeds the assumption on time-varying expectation formation and limited memory described above in a structure based on the seminal model of Bansal and Yaron (2004).[9]

The representative agent has Epstein and Zin (1989) recursive preference, so the utility function satisfies

$$V_t = [(1-\delta)C_t^{1-1/\psi} + \delta(\mathbb{E}_t V_{t+1}^{1-\gamma})^{\frac{1-1/\psi}{1-\gamma}}]^{\frac{1}{1-1/\psi}},$$

where C_t is consumption, γ is the coefficient of risk aversion, ψ is the intertemporal elasticity of substitution (IES), and δ is the discount factor. From the Euler equation, the asset pricing equation for gross return from any asset i ($R_{i,t+1}$) is

$$\mathbb{E}_t[\delta^\theta G_{c,t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}] = 1, \tag{18}$$

where $\theta = \frac{1-\gamma}{1-1/\psi}$, $G_{c,t+1} = (C_{t+1}/C_t)$, and $R_{a,t+1}$ is the unobservable return on an asset that delivers aggregate consumption as its dividends each period, or the so-called 'return on the wealth portfolio'. This is the usual asset pricing equation $\mathbb{E}_t[M_{t+1}R_{i,t+1}] = 1$, where $M_{t+1} = \delta^\theta G_{c,t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)}$ is the stochastic discount factor (SDF) for Epstein and Zin (1989) preferences. In logs[10]

$$m_{t+1} = \theta \log\delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta-1) r_{a,t+1}. \tag{19}$$

Process for consumption and dividends. As in Bansal and Yaron (2004), the consumption growth, i.e., $g_{c,t}$, and the dividend growth $g_{d,t}$, processes contain a small predictable component x_t - the so-called long run risk in Bansal and Yaron (2004) - with

[8] Note that with a constant b and no decay memory, $\lambda = 1$, then (14) becomes (17). Moreover, when $\theta < 1$ and $b = 1$ we get the usual RE forward-looking solution $\mathbb{E}_t y_{t+1} = 0, \forall t$.

[9] Appendices B.1 and C.1 contains all the derivations for the equations in this Section.

[10] Lower case variables indicates logs. So for example, $g_{c,t+1} = \log G_{c,t+1} = \log(C_{t+1}/C_t)$.

x_t evolving according to a AR(1) process, so that

$$x_{t+1} = \rho x_t + \varphi_e \sigma e_{t+1}, \tag{20}$$

$$g_{c,t+1} = \mu + x_t + \sigma \eta_{t+1}, \tag{21}$$

$$g_{d,t+1} = \mu_d + \phi x_t + \varphi_d \sigma u_{t+1}, \tag{22}$$

The exogenous shocks e_{t+1},u_{t+1} and η_{t+1} are white noise and mutually independent, ρ is the persistence of the expected growth rate process. $\varphi_d > 1$ captures the fact that the evolution of dividend observed is much more volatile than that of consumption, while ϕ implies that the persistent component x_t induces correlation between consumption and dividend growth. However, in contrast to Bansal and Yaron's (2004) model, we do not assume stochastic volatility a priori[11], because it arises endogenously through our assumptions on expectations, as we will show later. The stochastic volatility in our model is not unconstrained as in Bansal and Yaron (2004), but it obeys the restrictions imposed by the time-varying expectation formation process.

The solution method involves two steps. First, as Bansal and Yaron (2004) and Albuquerque et al. (2016) among others, we solve the model using the approximation proposed by Campbell and Shiller (1988), which involves linearizing the expressions for the returns and exploiting the properties of the log-normal distribution. Second, we assume relevant state variable for deriving the solution and then apply the undetermined coefficient method using the assumptions explained in the previous section - decay memory and time-varying expectations.

Solution for the return on the wealth portfolio. We first solve for the log return on the wealth portfolio $r_{a,t+1}$, as it determines the SDF and therefore the market portfolio return $r_{i,t+1}$ given (18). Applying the log-linear approximations in Campbell and Shiller (1988), the (approximated) log return on the wealth portfolio can by written as

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1} \tag{23}$$

[11] In Bansal and Yaron's (2004) model, they model the variance follows an exogenous AR(1) process.

where $z_t = \log(P_{C,t}/C_t)$ is the log of the price-consumption (PC) ratio of this asset that delivers aggregate consumption as its dividends each period, $P_{C,t} =$ is the price of this asset, and the κ's are the following approximation coefficients

$$\kappa_0 = \log(1 + \exp(\bar{z})) - \kappa_1 \bar{z}; \qquad \kappa_1 = \frac{\exp(\bar{z})}{1 + \exp(\bar{z})}. \tag{24}$$

To find the solution for z_t, we use the method of undetermined coefficient, as described in Section 2. Guess that the solution for z is a linear function of the past, present and expected future values of the endogenous state variable x, but subject to memory constraints and a time-varying parameter, indicated by b_t. In the case of finite memory

$$z_t = A_{0,t} + \left(1 - \frac{1}{\psi}\right) \left[\sum_{j=1}^{T} u_{j,t} x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t} \, \mathbb{E}_t \, x_{t+j} \right], \tag{25}$$

while in the case of decay memory

$$z_t = A_{0,t} + \left(1 - \frac{1}{\psi}\right) \left[\sum_{j=1}^{\infty} u_{j,t} \lambda^j x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t} \, \mathbb{E}_t \, x_{t+j} \right]. \tag{26}$$

The time-varying expectation parameter follows the random walk process $b_t = b_{t-1} + \sigma_b \xi_t$, where $\xi_t \sim$ i.i.d N(0, 1) is assumed to be uncorrelated to all other fundamental shocks in the model. The parameters $A_{0,t}$, $u_{j,t}$ and $c_{j,t}$ are coefficients to be determined. Substituting the above equation (25) or (26) into the Euler equation (18) yields, respectively

$$z_t = A_{0,t} + \left(1 - \frac{1}{\psi}\right) \left[\sum_{j=1}^{T} \left(\frac{1}{\kappa_1}\right)^j (b_t - 1) x_{t-j} + b_t x_t + b_t \sum_{j=1}^{\infty} (\kappa_1 \rho)^j x_t \right], \tag{27}$$

or

$$z_t = A_{0,t} + \left(1 - \frac{1}{\psi}\right) \left[\sum_{j=1}^{\infty} \left(\frac{\lambda}{\kappa_1}\right)^j (b_t - 1) x_{t-j} + \frac{b_t}{1 - \kappa_1 \rho} x_t \right]. \tag{28}$$

Given the solution for z_t, equation (23) yields the log wealth return $r_{a,t+1}$, which in turn yields the log of the SDF from (19).[12]

Solution for the market return. The same procedure solves for the market return,

[12] To avoid repetitions, we present only the decay memory case in what follows. Similar expressions holds for the finite memory case, see Appendix C.1 and C.2 for this and next Section respectively.

which has the analogous expression as equation (23), that is

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + g_{d,t+1}, \tag{29}$$

where κ_m's are coefficients as in (24) consistent with the average PD ratio $\bar{z}_m$. Similarly to (26), the guess for $z_{m,t}$ is

$$z_{m,t} = A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{\infty} u_{j,t}\lambda^j x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\,\mathbb{E}_t\, x_{t+j}\right], \tag{30}$$

where the parameters $A_{0,m,t}$, $u_{j,t}$, and $c_{j,t}$ are coefficients to be determined by substituting the conjectured equation (30) into the Euler equation (18). The solution for $z_{m,t}$ is

$$z_{m,t} = A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{\infty}\left(\frac{\lambda}{\kappa_{1,m}}\right)^j (b_t - 1)x_{t-j} + \frac{b_t}{1-\kappa_{1,m}\rho} x_t\right]. \tag{31}$$

The solution (31) has the property that the PD ratios are constant, absent uncertainty on the consumption process, i.e., $\sigma = 0$ and thus $x_i = 0$, $\forall i$. This property follows that the expectation parameter b_t enters into the solution in a multiplicative way.

Appendix B.1 presents the expressions for $A_{0,t}$, $A_{0,m,t}$ and the solution for the risk-free rate $r_{f,t+1}$. Finally, (28) and (31) show that z_t and $z_{m,t}$ are bounded, for any given bounded level of b_t, whenever, $\lambda < \kappa_1$ and $\lambda < \kappa_{1,m}$, respectively. Hence, as explained in the previous Section, the decay memory assumption implies a multiplicity of bounded solutions, and a given value of b_t pins down a particular solution among the infinite admissible ones.[13]

4 Analytical Results

This section derives analytical results that provide the intuition why the asset pricing model with time-varying expectation formation process and limited memory can be potentially consistent with many asset pricing puzzles. In particular, subsection 4.1 shows that the PD ratio can persistently deviate from the stable fundamentals. Moreover, these

[13]The same applies to the finite memory case because the backward-looking summations are finite.

persistent deviations are often associated with high price volatility. Subsection 4.2 shows that the model could generate a higher and time-varying equity premium as a result of the expectation formation process. Again we focus mainly on the decay memory case to avoid repetitions, but similar arguments apply to the finite memory case.[14]

4.1 Price-Dividend Ratio

The fundamental RE solution in standard asset-pricing models has difficulty matching the high price volatility in the data. It is immediate to obtain the RE solution for the log of the PD ratio in our model by imposing $b_t = 1$ in (31)

$$z_{m,t}^{RE} = A_{0,m} + \frac{\phi - \frac{1}{\psi}}{1 - \kappa_1 \rho} x_t. \tag{32}$$

The volatility of the PD ratio equals roughly the volatility of dividend growth, inherited by the long-risk process through the term, ϕx_t. For the PD ratio to feature stochastic price volatility - an important empirical observation - Bansal and Yaron (2004) add an exogenous stochastic component in the volatility of the growth rate of dividends (again common to the long-run risk).

To highlight how our model solution for $z_{m,t}$ deviates from the standard RE solution, we can write (31) as a combination of usual RE solution and a bounded backward-looking component[15]

$$z_{m,t} = b_t \underbrace{\left(A_{0,m} + \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m}\rho} x_t \right)}_{\text{fundamental eq., } z_{m,t}^{RE}} + (1 - b_t) \underbrace{\left(A_{0,m} - \sum_{j=1}^{\infty} \left(\frac{\lambda}{\kappa_{1,m}} \right)^j \left(\phi - \frac{1}{\psi} \right) x_{t-j} \right)}_{\text{bounded backward-looking eq.}} \tag{33}$$

The solution encompasses the usual RE result as a special case (corresponding to $b_t = 1$). When $b_t \neq 1$, the asset prices deviate from the usual RE values and we can distinguish the 'fundamental regime' and the 'trend-following regime', following the Boswijk et al.'s

[14] See Appendix B.2 for the derivations in this Section and Appendix C.2 for the corresponding derivations in the finite memory case.

[15] The time-varying components in the A_0 and $A_{0,m}$ were abbreviated when deriving the analytical solutions of this Section as they do not affect the main argument, but their effects were obviously considered when doing the quantitative analysis.

(2007) terminology. As the value of the expectation parameter b_t varies, the model yields different ways agent maps past observations to form their expectations and thus the equilibrium price distribution. When the value of b wanders around one, the model solution is close to the RE one and asset prices to fundamentals, so we are close to a 'fundamental regime'. When b differs from 1, the model enters a 'trend-following regime', where persistent under-and over-valuations of asset prices appear but are compatible with stable economic fundamentals. To see this, define $\hat{z}_{m,t}$ as the deviation from the usual RE solution, i.e., $\hat{z}_{m,t} = z_{m,t} - z_{m,t}^{RE}$, then, for $b_t \neq 1$,[16]

$$\begin{aligned}\hat{z}_{m,t+1} =& \frac{\lambda}{\kappa_{1,m}} \frac{b_{t+1}-1}{b_t - 1} \hat{z}_{m,t} + (b_{t+1}-1)\left(\phi - \frac{1}{\psi}\right)\frac{1}{1-\kappa_{1,m}\rho}\varphi_e \sigma e_{t+1} \\ &+ \underbrace{(b_{t+1}-1)\left(\phi - \frac{1}{\psi}\right)\frac{1}{1-\kappa_{1,m}\rho}(1-\lambda)\,\rho x_t}_{\text{Due to memory loss, approach to zero as } \lambda \to 1}. \end{aligned} \tag{34}$$

$\hat{z}_{m,t+1}$ positively depends on the deviation in the last period when b_t and b_{t+1} on the same side relative to 1. Given that b_t is a persistent process, it follows that persistent under- and over-valuations of asset prices arise. Moreover, even if expectations about stock prices are very high at a given point in time, the PD ratio has the tendency to return to fundamentals in absence of fundamental shocks as $\lambda/\kappa_{1,m} < 1$. In other words, the expectation formation process affects the response to the fundamental shocks, through b_t, and its persistence induces "momentum" on stock prices, while the decay memory assumption, through λ, entails "mean-reversion" over long horizons to stable fundamentals. Together, the model is able to explain the periodical boom and bust in the financial market.

Equation (35) also shows that our model provides a micro-structure for stochastic volatility, that arises from the time-variation in b_t. Stochastic volatility is one desirable feature in the asset pricing literature. However, such stochastic volatility is often disentangled from the fundamentals. For example, Bansal and Yaron (2004) model stochastic volatility by adding fluctuations in economic uncertainty, which are completely free in the

[16] In the finite memory case (34) is given by

$$\hat{z}_{m,t+1} = \frac{(b_{t+1}-1)}{\kappa_{1,m}(b_t - 1)}\hat{z}_{m,t} + (b_{t+1}-1)\left(\phi - \frac{1}{\psi}\right)\left(\frac{1}{1-\kappa_{1,m}\rho}\varphi_e \sigma e_{t+1} - \frac{1}{\kappa_{1,m}^T} x_{t-T}\right).$$

sense that they follow an independent process which is unrelated to any of the fundamentals. In contrast, in our model stochastic volatility arises as a by-product of time-variation in b_t and thus it is tight to the particular assumptions about the expectation formation process. This implies certain restrictions on the stochastic volatility due to the structure of the assumed expectation process - one stochastic expectation formation parameter b_t governs both the level and volatility of the PD ratio at once. Formally, both in the decay and in the finite memory case, the conditional variance of $z_{m,t+1}$ (assume $\rho = 0$ for simplicity, the full derivation for $\rho \neq 0$ can be found in the Appendix B.1) is equal to

$$\mathbb{V}ar_t(z_{m,t+1}) = \left(\frac{z_{m,t} - z_{m,t}^{RE}}{b_t - 1}\right)^2 \sigma_b^2 + \left(\phi - \frac{1}{\psi}\right)^2 (b_t \varphi_e \sigma)^2, \qquad \text{for } b_t \neq 1, \tag{35a}$$

$$\mathbb{V}ar_t(z_{m,t+1}) = \left(\phi - \frac{1}{\psi}\right)^2 \varphi_e^2 \sigma^2, \qquad \text{for } b_t = 1. \tag{35b}$$

Our model implies that the time variation in expectations affects not only the level of the conditional variance of the PD ratio directly through σ_b^2 as a new source of risk, but it also induces time-variation in how the variance of structural shocks feed into PD volatility because the coefficients that multiply σ^2 depend on b_t. The reason why it is evident from (33) that shows that b_t affects both the level (i.e., intercept) of the PD ratio - i.e., it changes the weight given to the discounted sum of past x_{t-j}'s - and the slope of the PD ratio - i.e., it changes the way the PD ratio reacts to a given new realisation x_t.[17] Moreover, from equation 35, the volatility of the PD ratio increases when the price dividend ratio is deviating from the fundamental solution, or in other words, the price volatility increases in a bubbly market, a phenomenon often observed in the data.

4.2 Equity Premium

In our framework, the model can generate a higher and time-varying equity premium as there are now two sources of systemic risk: the first relates to fluctuations in expected consumption growth; the second relates to fluctuations in the expectation formation process.

[17] Note that this nothing else that the application to the asset pricing model of the implication of the solution (13) in the simple example.

Appendix B.2 shows that the innovation to the pricing kernel, i.e., the SDF, is

$$m_{t+1} - \mathbb{E}_t m_{t+1} = -\lambda_{m,\eta}\sigma\eta_{t+1} - \lambda_{m,e,t+1}\sigma e_{t+1} - \lambda_{m,\xi,t+1}\sigma_b\xi_{t+1}, \tag{36}$$

where $\lambda_{m,\eta}$, $\lambda_{m,e,t+1}$ and $\lambda_{m,\xi,t+1}$ capture the pricing kernel's exposure to the consumption growth shocks, η_{t+1}, to the long-run risk shock, e_{t+1}, and to the time-varying expectation shock, ξ_{t+1}, respectively, and they are equal to

$$\lambda_{m,\eta} = -\left(-\frac{\theta}{\psi} + \theta - 1\right) = \gamma, \tag{37a}$$

$$\lambda_{m,e,t+1} = (1-\theta)\kappa_1\left(1 - \frac{1}{\psi}\right) b_{t+1}\frac{1}{1-\kappa_1\rho}\varphi_e, \tag{37b}$$

$$\lambda_{m,\xi,t+1} = (1-\theta)\kappa_1\left(1 - \frac{1}{\psi}\right)\left[\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_1})^j x_{t+1-j} + \frac{1}{1-\kappa_1\rho}\rho x_t\right]. \tag{37c}$$

The pricing kernel's exposure to long run risk $\lambda_{m,e,t+1}$ rises with the persistence parameter ρ or as agents are more forward-looking (higher b_{t+1}). The pricing kernel's exposure to the expectation risk $\lambda_{m,\xi,t+1}$ rises with the difference between the expected pure forward looking solution - when $b_{t+1} = 1$ - and the pure backward looking solution - $b_{t+1} = 0$, as can be shown by rewriting (37c) as (see (28))

$$\lambda_{m,\xi,t+1} = (1-\theta)\kappa_1(1 - \frac{1}{\psi})\Bigg(\underbrace{\mathbb{E}_t\left(z_{t+1}|b_{t+1}=1\right)}_{\mathbb{E}_t z^f_{t+1}} - \underbrace{\mathbb{E}_t\left(z_{t+1}|b_{t+1}=0\right)}_{\mathbb{E}_t z^b_{t+1}}\Bigg). \tag{38}$$

It is instructive to look at the comparison with the standard RE case (i.e., $b_{t+1} = 1$ and $\sigma_b = 0$), where (36) would correspond to the Bansal and Yaron's (2004) model without stochastic volatility (see equation (6) at p. 1486 therein). The Bansal and Yaron's (2004) model with stochastic volatility delivers an expression that has three terms, as in (36), where the last term would capture the fact that the innovation to the pricing kernel responds to the assumed shock to the volatility of consumption (see equation (10) at p. 1487) rather than the shock to expectation formation as in our model.

Similarly, the innovation in the market return is given by

$$r_{m,t+1} - \mathbb{E}_t\, r_{m,t+1} = \beta_{m,u}\sigma u_{t+1} + \beta_{m,e,t+1}\sigma e_{t+1} + \beta_{m,\xi,t+1}\sigma_b\xi_{t+1}, \tag{39}$$

where $\beta_{m,u}$, $\beta_{m,e,t+1}$ and $\beta_{m,\xi,t+1}$ are time-dependent convolutions given in Appendix B.2. The conditional equity premium for the market portfolio $r_{m,t+1}$ is equal to $\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = -cov(m_{t+1} - \mathbb{E}_t\, m_{t+1}, r_{m,t+1} - \mathbb{E}_t\, r_{m,t+1}) - 0.5\mathbb{V}ar_t(r_{m,t+1})$, which yields

$$\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = \vartheta_{e,t}\sigma^2 + \vartheta_{\xi,t}\sigma_b^2 - 0.5\mathbb{V}ar_t(r_{m,t+1}), \tag{40}$$

with

$$\vartheta_{e,t} = (1-\theta)\left(1 - \frac{1}{\psi}\right)\frac{\kappa_1 b_t}{1-\kappa_1\rho}\left(\phi - \frac{1}{\psi}\right)\frac{\kappa_{1,m} b_t}{1-\kappa_{m,1}\rho}\varphi_e^2\sigma^2, \tag{41a}$$

$$\vartheta_{\xi,t} = \frac{\vartheta_{e,t}}{b_t^2\varphi_e^2\sigma^2}\left[(1-\kappa_1\rho)\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_1})^j x_{t-j} + x_t\right]\left[(1-\kappa_{m,1})\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j x_{t-j} + x_t\right] \tag{41b}$$

$$\mathbb{V}ar_t(r_{m,t+1}) = (\beta_{m,u} + \beta_{m,e,t+1})^2\sigma^2 + \beta_{m,\xi,t+1}^2\sigma_b^2 \tag{41c}$$

The equity premium is determined by two sources of risk: the fluctuations in consumption growth and the fluctuations in expectations. The second term in equation (40) says that the equity premium must compensate for the risk due to time-varying expectation formation process (σ_b^2). Moreover, $\vartheta_{e,t}$ illustrates that the realization of b_t also determines how equity premium compensate for consumption growth volatility (σ^2). Therefore, as for the PD ratio, the expectation formation process affects both the level of the equity premium, i.e., the intercept of the equation for the equity premium (40), and its slope, i.e., how it reacts to consumption growth volatility.

5 Data and Methodology

We estimate our asset pricing model with a time-varying expectation formation process using the simulated method of moments (SMM). This Section describes first the data sources (subsection 5.1) and then the methodology employed (subsection 5.2). Subsection 5.3 formally examines which moments should be included in the SMM, as including all moments of interest may result in the violation of certain regularity conditions.

5.1 Data Sources

The seasonally adjusted per-capital annual consumption growth used in this paper is from Barro and Ursúa (2012) and was extended by the authors to 2019 using data from the Bureau of Economic Analysis (BEA) (National Income and Product Accounts: Table 7.1 Selected Per Capita Product and Income Series). The real S&P 500 return, PD ratio and dividend growth are from Robert Shiller's website. The data on the annualized nominal return to one-month Treasury bills, deflated by the CPI, is taken from Robert Shiller's website. We use the data from 1929 to 2018 which contains several crisis episodes in financial markets.

5.2 Simulated Method of Moments

This paper applies the simulated method of moments (SMM) estimation to evaluate the ability of the model to match salient features of data. The SMM approach aims to find model parameter values that make model simulated moments match the data moments as closely as possible.

In this application, the moments of interest include: the mean, standard deviation and the first-order autocorrelation of consumption growth; the mean, standard deviation and the first-order autocorrelation of dividend growth; the mean, standard deviation and the first-order autocorrelation of PD ratio; the mean and standard deviation of real stock return; the contemporaneous correlation between consumption growth and dividend growth; the contemporaneous correlation between stock return and consumption growth; the correlation between stock returns and one-period lagged consumption growth; the mean and standard deviation of the risk-free rate. Moreover, we include also excess return predictability, that is, the coefficient c^2 and the R^2 in the following regression:

$$r_{s,t+n} - r_{f,t} = c_n^1 + c_n^2 \log(PD_t) + u_{t,n} \tag{42}$$

where the dependent variable $(r_{s,t+n} - r_{f,t})$ is the observed real excess return of stocks over bonds from t to $t+n$ years (here we consider five-year horizon, $n = 5$), and $u_{n,t}$ is the regression residual.

There are 11 parameters we are trying to estimate, namely: the coefficient of relative risk aversion γ; the elasticity of intertemporal substitution ψ; the rate of time preference δ; the drift in the log consumption and in the dividend growth processes μ; the persistence of the expected growth rate process ρ; the volatility of innovation σ; the volatility of the persistent component of the growth process (i.e., the LRR) φ_e; the elasticity of dividend growth to the persistent component of the growth process ϕ; the volatility of dividend growth process φ_d; the volatility of the innovation in expectation formation process σ_b; the decay rate of memory λ (or the limit period of memory T in the case of finte memory). We summarise these parameters in the vector $\theta = \{\gamma, \psi, \delta, \mu, \rho, \sigma, \varphi_e, \phi, \varphi_d, \sigma_b, \lambda \text{ (or } T)\}$.

Formally, the SMM entails the following. Let $(y_1, ..., y_N)$ be the observed data sample with size N. The sample moments is defined as $\hat{M}_N \equiv \frac{1}{N}\sum_{t=1}^{N} h(y_t)$ for a given moment function h. Some of the statistics of interests we considered here are functions of moments so that $\hat{S}_N \equiv S(\hat{M}_N)$. We base our SMM estimates and test on matching the statistics $\hat{S}_N$. Let $\hat{S}_N \in R^s$ denote a vector of statistics that will be matched in the estimation given the N observations in the data. $\tilde{S}(\theta)$ is a vector of moments implied by the model for some parameter value θ. The SMM parameter estimate θ_N is formally defined as

$$\hat{\theta}_N \equiv \arg\min_{\theta} [\hat{S}_N - \tilde{S}(\theta)]' \hat{\Sigma}_{S,N}^{-1} [\hat{S}_N - \tilde{S}(\theta)]. \tag{43}$$

The SMM estimates $\hat{\theta}_N$ then choose the model parameter such that the model moments $\tilde{S}(\theta)$ fit the observed moments $\hat{S}_N$ as closely as possible in terms of a quadratic form with weighting matrix $\hat{\Sigma}_{S,N}^{-1}$. In our paper, we create 1000 synthetic time series using the Monte Carlo procedure, each length equal to our sample size.[18] $\tilde{S}(\theta)$ that enters the criterion function is the mean value of the sample moments across the synthetic time series for a given parameter vector of $\theta \in \Theta$.[19] Let ν^i denote a realization of shocks drawn randomly from their known distributions, and let $(y_1(\theta, \nu^i), \ldots, y_N(\theta, \nu^i))$ denote the random variables corresponding to a history of length N generated by the model for

[18] We assume the agent makes decisions on a monthly basis and we compute model moments at an annual frequency.

[19] An unconstrained minimization of the objective function over the parameter space Θ can be numerically unstable and computationally costly. Therefore, additional restrictions have been imposed on Θ. These restrictions can only deteriorate the model's empirical performance so that the goodness of fit results presented in the next section represent a lower bound on what the model can achieve.

shock realization ν^i and parameter value θ. Then, the model statistics $\tilde{S}(\theta)$ are computed as

$$\tilde{S}(\theta) \equiv \frac{1}{K}\sum_{i=1}^{K} S(M_N(\theta,\nu^i)) = \frac{1}{K}\sum_{i=1}^{K} S\left(N\sum_{t=1}^{N} h(y_t(\theta,\nu^i))\right) \tag{44}$$

where we use $K = 1000$. In other words, $\tilde{S}(\theta)$ is an average across a large number of simulations of length N of the statistics $S(M_N(\theta,\nu^i))$ implied by each simulation.

The weighting matrix in our estimation $\hat{\Sigma}_{S,N}^{-1}$ is the inverse of variance-covariance matrix of the empirical moment conditions $\hat{S}_N$, as required for efficient SMM estimation. The weighting matrix is estimated using the Newey-West estimator - which is heteroskedasticity and autocorrelation consistent - with a lag of 3.[20]

The SMM approach also allows a formal econometric test to evaluate individual and overall goodness of fit based on asymptotic distribution. Under the null hypothesis of the test that the model is correct, we have

$$\hat{W}_N \equiv N[\hat{S}_N - \tilde{S}(\hat{\theta}_N)]'\hat{\Sigma}_{S,N}^{-1}[\hat{S}_N - \tilde{S}(\hat{\theta}_N)] \rightarrow \chi^2(s-n) \text{ as } N \rightarrow \infty, \tag{45}$$

where convergence is in distribution. In addition, t-statistics can be conducted based on the asymptotic distribution for each element of the deviations $\hat{S}_N - \tilde{S}(\hat{\theta}_N)$ to evaluate how close each individual moment is to the data moment.

5.3 Which Moments to Match?

The validity of the SMM requires certain regularity conditions, as documented by Adda and Cooper (2003) and Davidson et al. (2004). As some components of the moment functions listed above are not sample moments but nonlinear functions of sample moments, this paper is concerned with violating one of the regularity conditions of standard SMM, that is, the non-singularity of the covariance matrix of the moment vector. The violation of the singularity requirement would result in the test $\hat{W}_N$ varying greatly with small model changes or testing procedures. Moreover, the maximization algorithm as the non-singularity condition is violated would be nearly unstableas evident in equation

[20] We follow a common practice that specifies the lags as the smallest integer greater than or equal to $(T^{1/4})$.

(45) - the formula for $\hat{W}_N$ nearly divides zero by zero, the objective function is nearly undefined, and the asymptotic distribution may not be a good approximation to the true distribution of the test statistic. Therefore, to make sure that $\hat{\Sigma}_{S,N}$ is invertible, we need to carefully select statistics so they do not give rise to multicollinearity. To decide which statistics to use, following Adam et al. (2016) and Adam et al. (2017), we exclude some moments from the estimation that are nearly redundant. The idea is to compute the variability of each statistic that cannot be explained by a linear combination of the remaining statistics, similarly to the R^2 coefficient of a regression of each statistic on all the other statistics. The regression coefficients and the ensuing R^2 are computed from $\hat{\Sigma}_{S,N}$ in a standard manner. We exclude those statistics with $R^2 < 0.04$ as they are nearly redundant. Specifically, they are the R^2 of the excess return predictability regression (42) and the first-order autocorrelation of consumption growth, $\rho_{\Delta c/c,-1}$ (for which $R^2 = 0.0283$ and $R^2 = 0.0140$, respectively). After we drop these two statistics, the weighting matrix is non-singular. Moreover, even though we drop these two statistics, the model is able to match them.[21]

6 Quantitative Results

This Section presents the results for both the finite memory model and the decay memory model. Table 1 reports our parameter estimates along with standard errors. The estimated parameters are very close between the two different specifications of our models. First, the coefficient of risk aversion, γ, is estimated to be 3.67 for the finite memory model and 3.9 for the decay memory one. These values are smaller than the value of 10 used in Bansal and Yaron (2004) to generate the observed equity premium but larger than the value of 1.5 estimated by Albuquerque et al. (2016) in their benchmark model. Second, the IES, ψ, is precisely estimated to be 1.115 in both models. This value is larger than one, and close to the value of 1.5 calibrated by Bansal and Yaron (2004) and estimated (1.46) in Albuquerque et al. (2016). Third, the parameter ρ, which governs the persistence of the long-run risk that affects both consumption and dividend growth, is very high at

[21] This applies to both the decay memory and finite memory case.

Table 1: Parameter Estimates

Parameter	Finite Memory	Decay Memory
γ	3.6721	3.9015
	(0.0026)	(0.1551)
δ	0.9965	0.9961
	(0.0003)	(0.0003)
ψ	1.1150	1.1148
	(0.001)	(0.0085)
ρ	0.9941	0.9915
	(0.0013)	(0.0020)
φ_e	0.1097	0.0788
	(0.0012)	(0.0121)
σ	0.003	0.004
	(0.0003)	(0.0002)
μ	0.0017	0.0016
	(0.0001)	(0.0001)
ϕ	2.5317	2.5344
	(0.1212)	(0.1851)
φ_d	9.7109	6.2188
	(1.4423)	(0.5413)
σ_b	0.0123	0.0245
	(0.0014)	(0.0051)
	$T = 6.8750$	$\lambda = 0.9419$
	(2.0583)	(0.0016)

The table presents estimated parameter values for the finite memory and decay memory models (with standard errors reported in parentheses). The model simulates on a monthly basis and appropriately compounds to the annual frequency. The Simulated Method of Moment method is used to obtain the estimates.

0.99, but consistent with the value used by Bansal and Yaron (2004) of 0.979. Fourth, the memory limit is similar between the two specifications. The finite memory is estimated to be 6.87 years (equal to 83 months) and the parameter of memory decay is 0.94 per month, implying that the memory weight after 83 months is around 0.006. Finally, the parameter ϕ that measures the effect of the long-run risk on dividend growth is close to the calibrated value in Bansal and Yaron (2004) of 3. The main difference between the finite and decay specifications is the estimated variance of the sunspot shock, which is larger for the latter case.

The estimated model closely matches the data moments, and the model's performance is robust across the two specifications for limited memory. Table 2 reports data moments (with standard errors reported in parentheses) and model moments (with t-statistics for each moment reported in parentheses) under the two specifications. Taking sampling uncertainty into account, the model moments are all statistically indistinguishable from the US data moments at 5% significance level.[22] In particular, both models closely match the observed main asset pricing moments: level and volatility of the stock returns, the PD ratio, the risk-free rate, and the dividend yield, without yielding unrealistic strong correlations between stock returns and measurable fundamentals. The model also succeeds in replicating the return predictability and the autocorrelation of the consumption process even though these two statistics are not included in the set of moments to be matched. The p-value of the Wald test, a measure of the model's overall performance, indicates that both models cannot be rejected at 5% significance level, even though the finite memory model is only marginally so.

6.1 The Equity Premium

Both the finite and the decay memory models are capable of producing a large equity premium (around 5%) with a relatively moderate estimated degree of risk aversion ($\gamma =$ 3.67 or 3.90). The intuition behind this is that the model requires additional compensation for expectation risks on top of fundamental risks. This compensation for expectation risks - estimated by setting the variance of the time preference shock in the model to zero and $b_t = 1$ for $\forall t$ - accounts for around 1.49% of the total 5.25% equity premium in the decay memory case and to 0.48% of the total 4.95% in the finite memory case.

Notably, in our model, the compensation for expectation risks increases as the gap between the expected forward-looking solution and the expected backward-looking solution widens. As the gap increases, the change in the relative weights (expectation shock) on those two solutions can induce large price fluctuations, which increases the agent's desire to hedge the risk. Moreover, the estimated values of risk aversion and of the IES imply $\theta < 1$, so that $\gamma > 1/\phi$ which Epstein et al. (2014) shows to be the condition for a preference for early resolution of uncertainty. Hence, as in standard long-run risk models, long-run risks are penalized more heavily than current risks in our model, because they

[22]The t-statistics based on formal asymptotic distribution are all at or below two in absolute value, with the only exceptions of the mean of the PD ratio and the standard deviation of bond return for the finite memory model.

Table 2: Quantitative Model Performance

	U.S. Data	Finite Memory	Decay Memory
Mean stock return E_{r^s}	7.79	6.00	6.27
	(1.83)	(0.99)	(0.83)
Mean bond return E_{r^b}	0.45	1.05	1.05
	(0.49)	(-1.20)	(-1.23)
Mean PD ratio E_{PD}	32.05	35.84	34.80
	(1.43)	(-2.47)	(-1.91)
Mean dividend growth $E_{\Delta D/D}$	1.74	2.77	2.56
	(1.12)	(-0.97)	(-0.73)
Std. dev. stock return σ_{r^s}	18.71	19.05	19.26
	(0.94)	(-0.36)	(-0.56)
Std. dev. PD ratio σ_{PD}	16.40	17.91	17.68
	(2.05)	(-0.69)	(-0.62)
Std. dev. Dividend Growth $\sigma_{\Delta D/D}$	10.67	11.29	11.08
	(1.60)	(-0.40)	(-0.25)
Std. dev. bond return σ_{r^b}	3.91	2.70	3.28
	(0.43)	(2.48)	(1.46)
Autocorrel. PD ratio $\rho_{PD,-1}$	0.90	0.85	0.80
	(0.12)	(0.41)	(0.74)
Mean consumption growth $E_{\Delta C/C}$	2.01	2.15	2.00
	(0.32)	(-0.46)	(0.03)
Std. dev. consumption growth $\sigma_{\Delta C/C}$	2.96	2.79	2.93
	(0.32)	(0.56)	(0.08)
Autocorrel. consumption growth $\rho_{\Delta C/C,-1}$	0.61	0.80	0.62
	(0.12)	(-0.07)	(-0.00)
Autocorrel. dividend growth $\rho_{\Delta D/D,-1}$	0.24	0.31	0.79
	(0.37)	(-0.09)	(-0.08)
Corr. $corr_{\Delta C/C,\Delta D/D}$	0.47	0.55	0.50
	(0.13)	(-0.58)	(-0.25)
Predictability β_{PD}	-0.0110	-0.0098	-0.0090
	(0.003)	(-0.38)	(-0.70)
Predictability R^2	0.1327	0.0804	0.0756
	(0.086)	(0.60)	(0.66)
Contemporaneous correlation between	0.03	0.07	0.24
stock return and consumption growth	(0.11)	(-0.31)	(-1.88)
Correlation between stock return	-0.13	0.52	0.10
and one-period lag consumption growth	(0.27)	(-0.53)	(-0.45)
Test statistic $\hat{W}_N$		10.5926	7.7713
p-value of $\hat{W}_N$		6.01%	16.93%

The table compares the asset-pricing moments from the US data (column 2, with standard errors reported in parentheses) with the one implied by the finite memory and decay memory models (column 3 and 4, respectively, the t-ratios for each moment are reported in parentheses). The t-ratios are calculated as (data moment - model moment)/(estimated standard deviation of the model moment). The measure of the overall goodness of fit $\hat{W}_N$, defined as (45), and the corresponding p-value are reported in the last two rows of the table.

are resolved in the distant future. In the CRRA case ($\theta = 1$), the equity premium is not affected by the two sources of risks, both the consumption and expectation risk, i.e., $\vartheta_{e,t}, \vartheta_{\xi,t} = 0$ in equation (40).

From an a analytical point of view, equations (28) and (31) help to explain why the model embodies a compensation for expectation risk and that is particularly pronounced for stocks relative to bonds. Equations (28) and (31) show that the solution for the price-consumption ratio and the price-dividend ratio, respectively. First, from Table 2, the parameter $\phi > 1$, which implies that dividends are more sensitive both to changes in the long-run risks, $x's$, and in the expectation formation, b_t, (i.e., see (28) and (31)). Second, under our parameter estimates, $\kappa_1 < \kappa_{1,m} \approx 1$, which implies that the price-dividend ratio is more sensitive to changes in expected future price-dividend ratio, and thus any changes in expectation formation process would have a leveraged impact on the current price-dividend ratio.

Our expectation risk shares a certain similarity with the 'valuation risk' in Albuquerque et al. (2016) in that both risks are due to stochastic changes in agents' valuation of assets in the absence of changes in fundamentals. Albuquerque et al. (2016) introduces a time preference shock that changes agents' relative valuation of present consumption against future consumption. A shock that increases agents' valuation of the present relative to the future would drive down the asset price, as they want to sell stocks and consume more. In our model, expectation sunspot shocks also change agents' valuation of assets. In particular, assume an agent that buys an asset at a certain date and then at a later date changes her expectations such that, say, she expects the future price is lower than she initially expected. Since the shock is common to all agents (who are identical), they sell their assets (both stocks and bonds) and drive down the prices.

However, there is a fundamental difference between our expectation shock and the valuation shock. Our sunspot process creates uncertainty on the future valuations, because it changes the way agents form expectations, that is, the weight between the backward and forward-looking solution, and bonds and stocks are exposed differently to this risk. As such, this expectation shock interacts with the fundamental shocks, increasing the risk and inducing stochastic volatility. Here, we insert it in the Bansal and Yaron (2004) model of long-run risk without stochastic volatility. Albuquerque et al.'s (2016) valuation shock is a state variable of the model and it is a fundamental demand shock to time preference, that exhibits exogenous stochastic volatility.[23] In principle, one could embed the expectation shock in a model with a (demand) valuation shock rather than a (supply) long-run risk shock. The expectation shock would then amplify a different fundamental shock. In other words, our shock interacts with fundamental shocks amplifying the risks connected to fundamentals. As such it creates stochastic volatility, but, contrary to a valuation shock, our model would not imply an equity premium in absence of uncertainty on the consumption process, i.e., $\sigma = 0$.

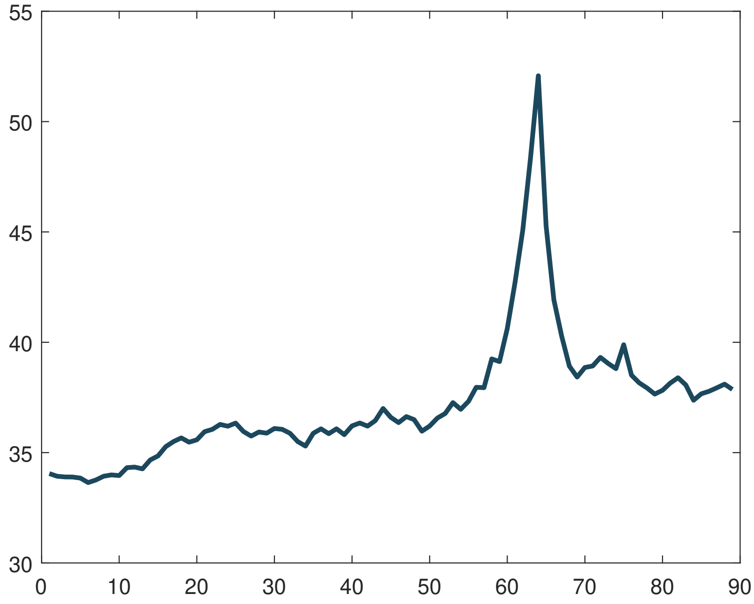
Finally, a problem with some explanations of the equity premium is that they imply counterfactually high levels of volatility for the risk-free rate (for example, Boldrin et al. 2001). Table 2 shows that the volatilities of the risk-free rate and the stock market returns implied by our model are similar to the ones in the data.

6.2 The Correlation Puzzle

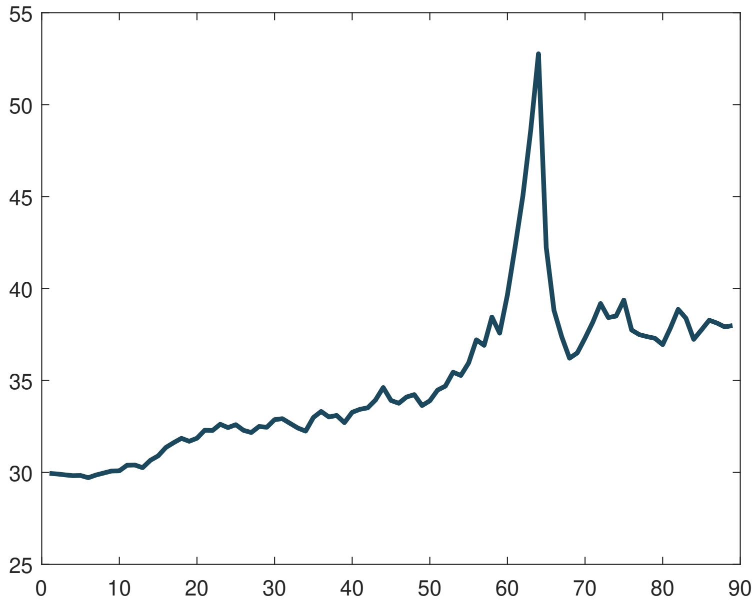
As well documented by Albuquerque et al. (2016), the correlation and covariance between stock returns and measurable fundamentals, especially consumption growth, are weak. To simultaneously account for the equity premium and the correlation puzzle is challenging for models with all uncertainty being loaded to the supply side, such as Bansal and Yaron's (2004) long-run risk models. Albuquerque et al. (2016) consider this problem as one of their main motivation for introducing a valuation risk from the demand side. Our long-run risk model, augmented by the expectations risk, does relatively well at matching the correlation between stock returns and consumption growth in the data (both contemporaneous and with one-period lagged consumption). The intuition for this result is related to the ability of our model to generate boom-and-bust bubbly behaviour, as explained in Section 4 and shown in the next Section.

[23]Moreover, both the growth rate of consumption and dividends are affected by the innovation to the persistent component of the time-preference shock.

Figure 2: Simulated Price-Dividend Ratio



(a) Finite memory



(b) Decay memory

The figure shows the simulated PD ratio using the estimated model from Table 1. The simulated time series is able to generate booms and busts, as observed in the data.

6.3 Price Dividend Ratio

In Table 2, we show that the model matches the mean, volatility, and persistence of the PD ratio very well. Indeed, the model is capable of generating ample boom-and-bust type of movements, one of the main characteristics of the behaviour of the PD ratio in the data. Figure 2 shows a particular realisation of the PD behaviour from simulating both model specifications. Both simulations exhibit a clear boom-bust cycle. This kind of behaviour appears as the PD ratio persistently deviates from the fundamental one, as evident in equation (34). The fact that our sunspot shock is very persistent is key for generating "momentum" on stock prices, while the limited memory assumption entails "mean reversion" over long horizons to stable fundamentals. The fact that the PD ratio drifts away persistently from its fundamental RE value - temporarily delinking stock prices from fundamentals - also explains why the model is able to match the low correlation between the fundamentals, i.e., consumption growth, and stock returns, as we discussed in the previous paragraph.

Moreover, another important feature observed in the financial market is that the PD ratio volatility increases in a bubbly market. Figure 2 suggests that the model is able to replicate this feature. Equation (35a) conveys the analytical explanation behind this, because it shows that the conditional variance of $z_{m,t+1}$ increases in the deviation of the PD ratio, $z_{m,t}$, from its fundamental value, $z_{m,t}^{RE}$.

Finally, another well-known feature of the data is the predictability of excess returns from lagged PD ratios. Following Albuquerque et al. (2016), Table 3 reports the results of regressing real excess returns on equity over holding periods of one, three, and five years on the lagged price-dividend ratio, that is

$$r_{s,t+n} - r_{f,t} = c_n^1 + c_n^2 \log(PD_t) + u_{t,n} \tag{46}$$

where the dependent variable $r_{s,t+n} - r_{f,t}$ is the observed real excess return of stocks over bonds from t to $t+n$ years, and $u_{n,t}$ is the regression residual. The second column in Table 3 reports the slope coefficients c_1^2, c_3^2 and c_5^2, while the sixth column reports the R^2's. The slope coefficients are all negative, signalling that high PD ratios is associated

Table 3: Predictability of excess returns

	Slope coefficient				R^2		
	Data	Finite memory	Decay Memory		Data R^2	Finite memory	Decay Memory
c_1^2	-0.0022	-0.0019	-0.0019	R_1^2	0.0391	0.0208	0.0235
	(0.0010)	(-0.27)	(-0.36)		(0.0375)	(0.49)	(0.42)
c_3^2	-0.0062	-0.0059	-0.0054	R_3^2	0.0890	0.0544	0.0553
	(0.0025)	(-0.14)	(-0.34)		(0.0872)	(0.53)	(0.52)
c_5^2	-0.0110	-0.0098	-0.0090	R_5^2	0.1327	0.0804	0.0756
	(0.0034)	(-0.33)	(-0.58)		(0.0872)	(0.60)	(0.66)

The table reports the results of regression excess return over holding periods of one, three, and five years on the lagged price-dividend ratio based on the parameter estimates in Table 1. $c^2(n)$ and the $R^2(n)$ in the table represents the coefficients and the $R-$squared's, respectively, in the following regression: $X_{t,n} = c_n^1 + c_n^2 PD_t + u_{t,n}$, where $X_{t,n}$ is the observed real excess return of stocks over bonds from t to $t+n$ years. The standard error (for estimated data moments) and t-ratios (for model implied moments) are reported in parentheses.

with lower future excess returns. The other columns in Table 3 show the same results by running this regression over our simulated data. Our model matches both the slope coefficients and the R^2's of the regression, and the $t-$statistics are all well within the significance level.

6.4 Implication for Expectations

Traditional rational expectation models give rise to an important counterfactual prediction for the behaviour of investors' return expectations. Reflecting the data feature we just saw in Table 3, the rationally expected return should correlate negatively with the PD ratio. However, the available survey data on investors' return expectations suggest the opposite. Based on the UBS Survey, the CFO survey and the Shiller individual investor survey, Adam et al. (2017) concludes a positive correlation between the PD ratio and survey expected returns, despite the fact that actual returns is negatively correlated to the PD ratio. Nagel and Xu (2021) shows that the conclusion holds true for both individual investors and professional forecasters. Although the previous literature studied the positive correlation between price-dividend and expected returns, there is much less formal evidence on the stability of this correlation. This is important for us because our model implies time variation in expectation formation, and hence time variation in this correlation, as stressed in the Introduction. Therefore, in this section, we first confirm the well-documented facts that the PD ratio plays a role in investors' future return expectations. Then, we use econometric tests to show that how agent maps observed PD ratios to calculate their return expectation is time-varying.

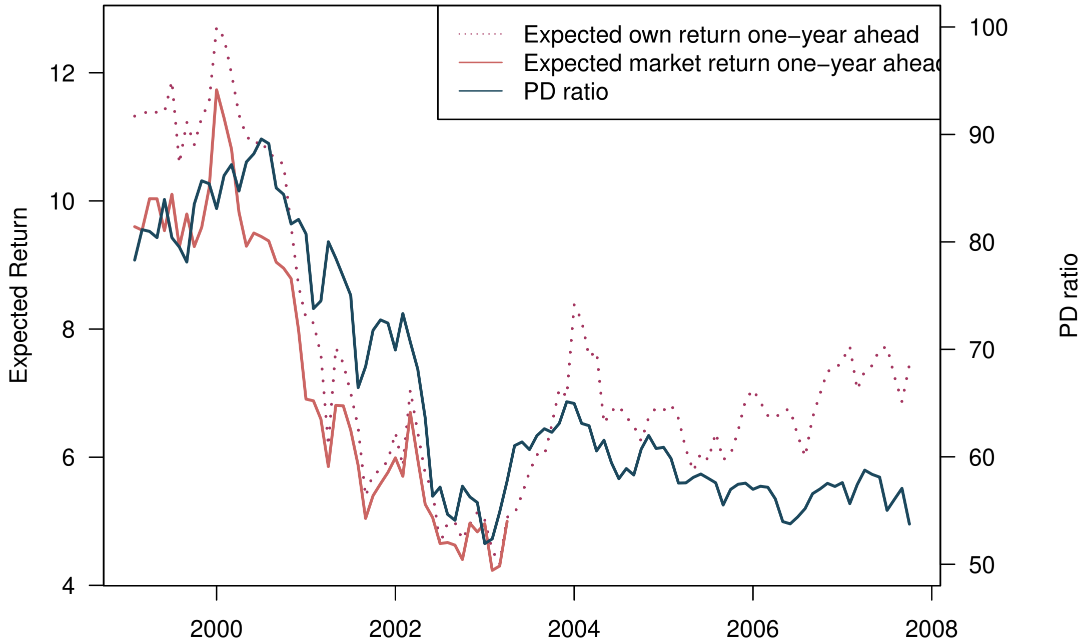
We use the UBS/Gallup Survey data, which is based on a representative sample of approximately 1,000 US investors who own at least US$10,000 in financial wealth. We use the data from February 1999 onwards when the survey was conducted on a regular monthly basis until April 2007. After data cleaning, there are about 600 observations per month. Figure 3 contains the cross-sectional average of investors' one-year ahead expected return on the market portfolio, on respondents' own portfolio and the PD ratio at the time when investors were asked to report their expectations. Notice that: (i) the expected return on own portfolio one year ahead is very close to the expected market return over the same period; (ii) the price-dividend ratio and the expected returns are generally positively correlated. As evident from the Figure, the UBS survey changed its survey questions over time. Before 2003 the respondents reported both the return expectations on their own portfolio and expectations on market return one year ahead, while from 2003 onwards, respondents report only the return they expect on their own portfolio one year ahead.

To test our two hypothesis - (i) PD ratio plays a role in investors' future returns expectations; (ii) the mapping from PD ratio to expectations is time-varying - we consider the following regression as in Nagel and Xu (2021) and Adam and Nagel (2022)

$$\hat{\mathbb{E}}_t r_{s,t+1} - r_{f,t} = \beta_0 + \beta_t \log(PD_t) + \varepsilon_t, \qquad t = 1, \ldots, T \tag{47}$$

where $\hat{\mathbb{E}}_t r_{s,t+1}$ is the survey expected market return of one-year ahead at time t. Following the Nagel and Xu's (2021) approach, we imputed the market return expectations by regressing expected market returns on own portfolio expectations using the part of the sample where both are available and using the fitted value from this regression when the market return expectation is not reported.

Figure 3: PD ratio and expected return one year ahead



The figure contains the cross-sectional average of investors' one-year ahead expected return on the market portfolio and on respondents' own portfolio as well as the actual PD ratio at the time when investors were asked to report their expectations.

The first hypothesis can be tested by assuming $\beta_t = \beta$ for all t. Table 4 indicates that the estimated coefficient of log PD ratio (i.e., β) is around 0.0269. We estimate the same regression on model-generated data and the result is remarkably close. As Table 4 shows, the model implied regression coefficient is 0.0240.[24]

Table 4: Survey Return Expectations and PD ratio

	Data Moment		Model		
	Estimate	(SE)	Mean	5%	95%
log PD	0.0269	(0.009)	0.0240	0.0112	0.0368
R^2	0.08		0.22		

The table

To test the parameter stability, we follow the methodology proposed by Nyblom (1989) and Hansen (1992). We extend the standard regression model and allow the regression

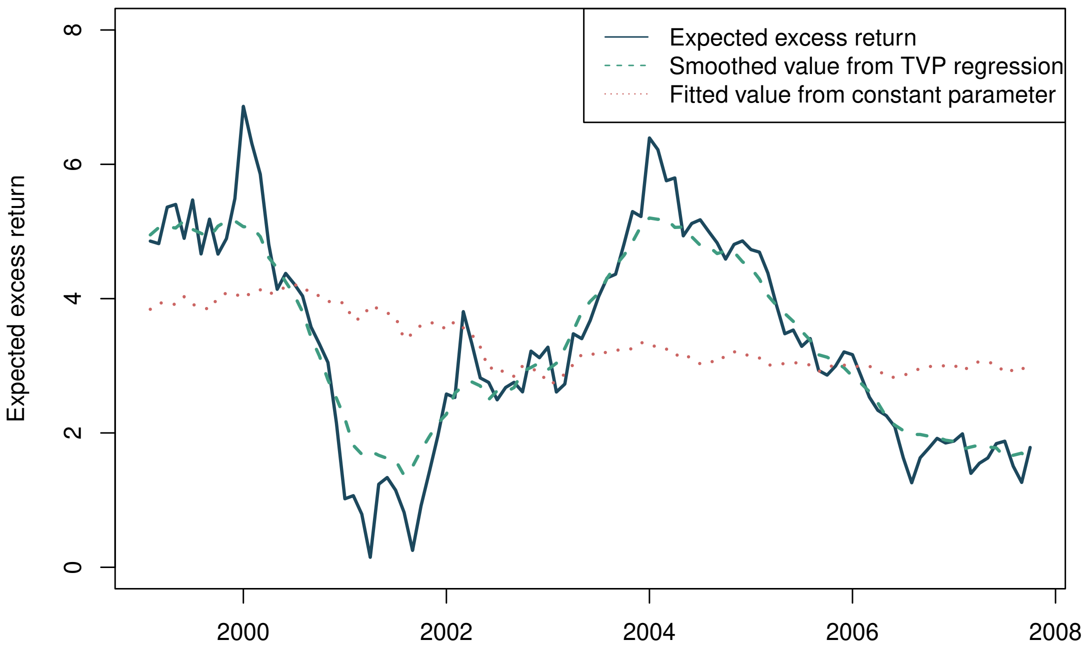
[24]The regression coefficient for finite memory model is 0.0148. Both are calculated as the mean of 1000 simulations.

coefficients to evolve randomly over time, specifically,

$$\beta_t = \beta_{t-1} + \nu_t \tag{48}$$

where ε_t and ν_t are uncorrelated. ν_t is i.i.d $N(0, \tau^2 G)$ (where G is assumed to be known), so that the coefficient β_t follows a random walk and thus evolves smoothly but randomly over the sample period. When ε_t is i.i.d $N(0, \sigma_\varepsilon^2)$, this is refereed as the "time-varying parameter" model (see e.g., Cooley and Prescott, 1976). Under the null hypothesis $\beta_t = \beta$ for all t. A rejection of the null hypothesis implies the parameter is unstable, and thus investors' way of forming expectations based on past price-dividend ratio is time-varying. The Kalman filter is then applied, where β_t is the unobserved state vector, (48) the state equation, and (47) the measurement equation. Figure 1 plots the estimation of β_t. Figure 4 plots the actual value of expected excess return versus the fitted value from constant parameter regression and the smoothed value from TVP regression. It is evident that the smoothed expected excess return from the TVP regression fit much better to the actual data, which further suggests the rejection of a constant β.

Figure 4: Actual and fitted expected excess return



We apply a formal test for time-varying coefficient $\beta_t = \beta$ following Hansen's (1992) approach, the Lagrange Multiplier is estimated to be $L_i = 0.985 > cv_{0.05} = 0.470$ and is greater than the critical value, therefore, we reject the null hypothesis that β_t is constant at 5 percent confidence level.

7 Comparison with Alternative Models

This section compares our model results with two other leading alternatives: the long-run risk model in Bansal and Yaron (2004) (see subsection 7.1) and the learning model in Adam et al. (2016) (see subsection 7.2).

7.1 Comparison with the Long-run Risk Model in Bansal and Yaron (2004)

This subsection discusses the relation between our model and the long-run risk model pioneered by Bansal and Yaron (2004). Both models feature low-frequency fluctuations in consumption growth and stochastic volatility, which induce changes in the agent's SDF. In contrast to our model, Bansal and Yaron (2004) assumes RE and an exogenous AR(1) process for stochastic volatility, $\sigma_{t+1}^2 = \sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}$, where σ^2 is the unconditional variances of consumption and ν_1 is the persistence of the volatility process. Consequently, the pricing kernel is affected by the volatility shock w_{t+1}

$$m_{t+1} - \mathbb{E}_t\, m_{t+1} = \lambda'_{m,\eta}\sigma_t\eta_{t+1} - \lambda'_{m,e}\sigma_t e_{t+1} - \lambda'_{m,w}\sigma_w w_{t+1} \tag{49}$$

and the equity premium in the presence of time-varying volatility becomes

$$\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = \beta'_{m,e}\lambda'_{m,e}\sigma_t^2 + \beta'_{m,w}\lambda'_{m,w}\sigma_w^2 - 0.5\mathbb{V}ar_t(r_{m,t+1}) \tag{50}$$

A few observations follow. While Bansal and Yaron (2004) introduces time-varying consumption volatility, we introduce a time-varying expectation formation process. Comparing the equity premium equations (50) and (40), it is clear that both specifications can induce a time-varying compensation for the long-run risk. Despite this similarity, the two models are not observationally equivalent. First, they have different implications

Table 5: LRR Parameter Estimates

Parameter	Bansal and Yaron (2004)'s original calibration	Estimated value
γ	10	12.6277 (6.0513)
δ	0.9989	0.9999 (1.0119)
ψ	1.5	1.0890 (0.9666)
ρ	0.975	0.8763 (0.7481)
φ_e	0.0373	0.1085 (0.8499)
σ	0.0072	0.0065 (0.9980)
μ	0.0015	0.0020 (1.0034)
μ_d	0.0008	0.0020 (-)
ϕ	2.5	3.0579 (1.1576)
φ_d	5.96	5.1702 (1.2144)
ν_1	0.999	0.9236 (0.9616)
σ_w	2.4e-06	2.4569e-06 (1.0042)

The table presents Bansal and Yaron's (2004) original parameter values in column 2 (however, as their parameters were calibrated based on period 1929 to 1998, we slightly adjusted their original calibration to better match the data moments in this paper) and the estimated parameter values in column 3 (with standard errors reported in parentheses). The SMM is used to obtain the estimates. The model simulates on a monthly basis and appropriately compounds to the annual frequency.

for the correlation between observed consumption growth and asset returns. In our case, the changes in the compensation to expected consumption growth are driven by the expectation formation parameter b_t, so that the correlation between consumption growth and asset returns is relatively weak, as observed in the data. Second, in Bansal and Yaron's (2004) setting, the compensation for the volatility shock is constant. In contrast, the compensation for the expectation formation shock varies over time and positively correlates with the deviation of current prices from the fundamental one. Moreover, our time-varying component arises endogenously from the time-varying expectation formation process, whereas the time-varying volatility in Bansal and Yaron (2004) is exogenous. Despite this, the next subsection **??** shows that our model have better quantitatively performance than the Bansal and Yaron's (2004) one.

Table 6: LRR model performance

	U.S. Data	Calibrated Model	Estimated Model
Mean stock return E_{r^s}	7.79	5.26	5.31
	(1.83)		(1.43)
Mean bond return E_{r^b}	0.45	0.45	-1.16
	(0.49)		(2.11)
Mean PD ratio E_{PD}	32.05	33.13	35.60
	(1.43)		(-0.71)
Mean dividend growth $E_{\Delta D/D}$	1.74	2.16	3.20
	(1.12)		(-1.82)
Std. dev. stock return σ_{r^s}	18.71	18.26	14.64
	(0.94)		(2.23)
Std. dev. PD ratio σ_{PD}	16.40	5.47	3.24
	(2.05)		(3.42)
Std. dev. Dividend Growth $\sigma_{\Delta D/D}$	10.67	11.29	12.76
	(1.60)		(-0.79)
Std. dev. bond return σ_{r^b}	3.91	0.94	1.25
	(0.43)		(3.59)
Autocorrel. PD ratio $\rho_{PD,-1}$	0.90	0.56	0.17
	(0.12)		(7.66)
Mean consumption growth $E_{\Delta C/C}$	2.01	1.85	2.50
	(0.32)		(-1.44)
Std. dev. consumption growth $\sigma_{\Delta C/C}$	2.96	2.79	2.70
	(0.32)		(0.42)
Autocorrel. consump. growth $\rho_{\Delta C/C,-1}$	0.61	0.80	0.14
	(0.12)		(0.31)
Autocorrel. dividend growth $\rho_{\Delta D/D,-1}$	0.24	0.31	0.03
	(0.37)		(0.45)
Corr. $corr_{\Delta C/C,\Delta D/D}$	0.47	0.55	0.18
	(0.13)		(1.96)
Predictability β_{PD}	-0.0110	-0.0098	-0.0145
	(0.003)		(0.9524)
Predictability R^2	0.1327	0.0804	0.0348
	(0.086)		(1.0622)
Contemporaneous correlation between stock return and consumption growth	0.03	0.07	0.14
	(0.11)		(-0.95)
Correlation between stock return and one-period lag consumption growth	-0.13	0.52	0.03
	(0.27)		(-0.59)
Test statistic $\hat{W}_N$			558
p-value of $\hat{W}_N$			0

The table compares the asset-pricing moments from the US data (column 2, with standard errors reported in parentheses) with the one implied by the calibrated and estimated Bansal and Yaron's (2004) model (column 3 and 4, respectively, the t-ratios for each moment are reported in parentheses). The t-ratios are calculated as (data moment - model moment)/(estimated standard deviation of the model moment). The measure of the overall goodness of fit $\hat{W}_N$, defined as (45), and the corresponding p-value are reported in the last two rows of the table.

Estimation Result. In evaluating the quantitative performance of the LRR model, we first focus on Bansal and Yaron's (2004) original calibration.[25] Then, for a fair comparison, we also utilize the SMM method described in the previous section to estimate the model. The parameter values in both cases are reported in Table 5. In the estimation, we assume that the agent's decision interval is monthly and then appropriately compounded to match the annual data. Notice, however, that the structural parameters in this case are estimated less precisely, especially the risk-aversion factor and the IES. The larger standard errors point out difficulties in the estimation, as well documented in the literature (Bansal et al., 2016).

The model moments are reported in Table 6. Columns 3 and 4 display the model moments under calibration and the ones obtained by estimating the model using the SMM, respectively. It turns out that our model outperforms the LRR in terms of both individual moments and overall fitness. In particular, despite the LLR model matching relatively well the levels of equity-premium and price-dividend ratio, the model falls short of the data on some dimensions - the high volatility of price-dividend ratio and market returns as well as the high persistence in the price-dividend ratio (of which t-ratios greater than 2). This might be explained by the fact that the LLR model loads all the uncertainty onto the supply side and thus the price fluctuations only come from fluctuations in the fundamentals, which is considerably small in the data.

7.2 Comparison with the Learning Model in Adam et al. (2016)

This section shows that our key results hold even in a simple version of the Lucas (1978) model. Adam et al. (2016) use this model to show that a departure from RE - in the form of a learning model - enables even such a simple asset pricing model to reproduce a variety of stylized asset pricing facts quantitatively. We then embed our deviation from RE in such a model. We show that our expectation shock in this paper improves the replication of both individual moments and the overall goodness of fit with respect to the learning mechanism in Adam et al. (2016).

[25] In Bansal and Yaron (2004)'s original calibration, their parameters were originally calibrated based on the period 1929 to 1998; therefore, we slightly adjusted their original calibration to better match the data moments in this paper.

Model Description. The representative agent has standard time-separable CRRA preference. Hence, the problem of the agent is to choose (C_t, S_t, B_t) to maximize the intertemporal utility function

$$\max_{\{C_t \geq 0, S_t, B_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t u(C_t) = \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t \frac{(C_t)^{1-\gamma}}{1-\gamma} \tag{51}$$

subject to the budget constraint

$$C_t + P_t S_t + B_t \leq (P_t + D_t) S_{t-1} + (1 + r_{t-1}) B_{t-1} + Y_t, \tag{52}$$

where C_t denotes the agent's consumption, P_t the competitive price of stock, S_t the stock hold by the representative agent, B_t the bond holding and Y_t the endowment of income that the agent receives each period. Utility maximisation yields

$$C_t^{-\gamma} P_t = \delta\, \mathbb{E}_t[C_{t+1}^{-\gamma} P_{t+1}] + \delta\, \mathbb{E}_t[C_{t+1}^{-\gamma} D_{t+1}] \tag{53}$$

$$C_t^{-\gamma} = \delta(1 + r_t)\, \mathbb{E}_t[C_{t+1}^{-\gamma}] \tag{54}$$

As in Adam et al. (2016), the dividend is assumed to evolve according to $\frac{D_t}{D_{t-1}} = \alpha \varepsilon_t^d$, where $\log \varepsilon_t^d \sim$ i.i.d $N(0, \sigma_d^2)$ and $\alpha \geq 1$. This implies $\mathbb{E}_t(\varepsilon_t^d) = 1$, $\mathbb{E}_{\frac{\triangle D}{D}} \equiv \mathbb{E}\left(\frac{D_t - D_{t-1}}{D_{t-1}}\right) = \alpha - 1$, and $\sigma^2_{\frac{\triangle D}{D}} \equiv \text{var}\left(\frac{D_t - D_{t-1}}{D_{t-1}}\right) = \alpha^2(e^{s_d^2} - 1)$. The consumption growth process is modelled as $\frac{C_t}{C_{t-1}} = \alpha \varepsilon_t^c$, where $\log \varepsilon_t^c \sim$ i.i.d $N(0, \sigma_c^2)$ and $(\log \varepsilon_t^c, \log \varepsilon_t^d)$ are jointly normal. Moreover, the standard deviation of consumption growth is set to be $s^c = \frac{1}{7} s^d$ to capture the lower volatility observed in consumption growth than in dividend growth; the correlation between $\log \varepsilon_t^c$ and $\log \varepsilon_t^d$ is set $\rho_{c,d} = 0.2$ to capture the correlation between dividend and consumption growth.

Adam et al. (2016) then assume agents to think that the process for risk-adjusted stock price growth contains both a transitory and a persistent time-varying component

$$\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma} \frac{P_t}{P_{t-1}} = b_t + \epsilon_t \qquad \text{and} \qquad b_t = b_{t-1} + \xi_t, \tag{55}$$

where ϵ_t and ξ_t are i.i.d white noise also jointly i.i.d. with ε_t^c and ε_t^d. The agents then face a learning problem that consists in optimal filtering out the two unobserved components from the realizations of risk-adjusted stock price growth.

Instead, we apply our expectation shock and limited memory assumptions to this setup. First, rewrite the Euler Equation (53) as an expectational difference equation linking the current and expected future value of the PD ratio as

$$\frac{P_t}{D_t} = \delta\, \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{D_{t+1}}{D_t} \frac{P_{t+1}}{D_{t+1}} \right] + \delta\, \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{D_{t+1}}{D_t} \right]. \tag{56}$$

Then, applying the approach described in Section 2, we can rewrite the solution for the PD ratio as (details can be found in Appendix D)

$$\frac{P_t}{D_t} = (b_t - 1) \left(\sum_{j=0}^{\infty} \left(\frac{\lambda}{\delta} \right)^j \left(\prod_{i=0}^{j} \eta_{t-i} \right)^{-1} \eta_{t-j} \right) + b_t \frac{\delta \alpha^{1-\gamma} \rho_\varepsilon}{1 - \delta \alpha^{1-\gamma} \rho_\varepsilon}, \tag{57}$$

where

$$\eta_{t+1} \equiv \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{D_{t+1}}{D_t} \right) = \alpha^{1-\gamma} \left(\varepsilon_{t+1}^c \right)^{-\gamma} \varepsilon_{t+1}^d, \tag{58}$$

and

$$\rho_\varepsilon = \mathbb{E}_t[(\varepsilon_{t+1}^c)^{-\gamma} \varepsilon_{t+1}^d] = e^{\gamma(1+\gamma)\frac{s_c^2}{2}} e^{-\gamma \rho_{c,d} s_c s_d}. \tag{59}$$

The standard stable RE solution (corresponding to $b_t = 1$) is $PD_t^F = \frac{\delta \alpha^{1-\gamma} \rho_\varepsilon}{1 - \delta \alpha^{1-\gamma} \rho_\varepsilon}$.

Estimation Results. To facilitate the comparison of the quantitative performance of the two models, we use the data are from Adam et al.'s (2016) database. The data is quarterly US stock market data from 1925Q4 to 2012Q2. The model moments are reported on a quarterly frequency, as in Adam et al. (2016).[26] For comparability, we set the risk aversion coefficient to $\gamma = 5$ as in Adam et al. (2016), then there are 4 free parameters to estimate, namely the growth rate of dividend α, the standard deviation of dividend innovations σ, the standard deviation of the expectation parameter σ_b, and the

[26] The data are available at https://www.klaus-adam.com/published-und-forthcoming/. The original data was downloaded from the Global Financial Database. As they consider the return predictability at the five-year horizon, the effective sample is up to 2007Q1, and due to the seasonal adjustment, the effective starting date is 1927Q2. All data are in real terms. Details can be found in Appendix A of Adam et al. (2016).

memory constraint parameter (T in the finite memory case or λ in the decay memory case). Again for comparability, we do not include all the above moments in the estimation for this application, but only the one used in Adam et al. (2016), that is

$$(\hat{E}_{r^s}, \hat{E}_{PD}, \hat{\sigma}_{r^s}, \hat{\sigma}_{PD}, \hat{\rho}_{PD,-1}, \hat{R}^2_5, \hat{E}_{\triangle D/D}, \hat{\sigma}_{\triangle D/D}). \tag{60}$$

The Adam et al.'s (2016) model does not match the mean bond return $\hat{E}_{r^b}$, so they drop it from the estimation in their favourite specification. Moreover, they also drop $\hat{c}^2_5$ to avoid the near-singularity issue of the $\hat{\Sigma}_{S,N}$ matrix. Nonetheless, they report the model implied mean bond return $\hat{E}_{r^b}$ and $\hat{c}^2_5$ based on the estimated parameters. Hence, we proceed in the same way, again for comparability.

Table 7 reports the results. Column 1 displays the data moments (standard errors in parenthesis), column 2 the corresponding estimated moments in Adam et al. (2016) (see Table III therein). Columns 3 and 4 display the moments from the estimated finite memory model and decay memory model, respectively (t-statistics in parenthesis). Again, there is very little difference between the finite memory and the decay memory model. Both specifications closely match both the mean and the standard deviation of stock return, whereas the learning mechanism in Adam et al. (2016) is not able to match. None of the models is able to match the mean bond return pointing to the limitation of the CRRA setup regarding the risk-free rate puzzle. Leaving aside the mean bond return, all the t-statistics for the estimated moments in our limited memory setup have an absolute value less than one, suggesting that the individual model moments can match the data moments pretty well, with the only marginal exception of R^2_5 for the finite memory case equal to -1.03. Note that R^2_5 is the only statistic for which the t-statistics is lower for the Adam et al.'s (2016) model, while our model matches quite well the standard deviation of dividend growth in contrast to Adam et al. (2016). Indeed, the overall goodness of fit test strongly favours our model that displays p-values of 16.7% and 20.1% for the finite memory and decay memory specifications, respectively, so that both are not rejected well above the 10% significance level.

Table 7: Comparison with Adam et al. (2016)'s learning model

	U.S. Data	Adam et al. (2016)	Finite Model	Decay Model
Quarterly mean stock return E_{r^s}	2.25	1.32	2.24	2.23
	(0.37)	(2.50)	(0.04)	(0.07)
Quarterly mean bond return E_{r^b}	0.15	1.09	1.98	1.98
	(0.19)	(-4.90)	(-9.34)	(-9.52)
Mean PD ratio E_{PD}	123.91	109.66	116.92	119.96
	(21.67)	(0.69)	(0.32)	(0.18)
Mean dividend growth $E_{\Delta D/D}$	0.41	0.22	0.36	0.37
	(0.17)	(1.14)	(0.23)	(0.23)
Std. dev. stock return σ_{r^s}	11.44	5.34	9.01	8.73
	(2.71)	(2.25)	(0.90)	(0.99)
Std. dev. PD ratio σ_{PD}	62.43	40.09	54.78	54.80
	(17.27)	(1.33)	(0.50)	(0.43)
Std.dev. dividend growth $\sigma_{\Delta D/D}$	2.88	1.28	2.29	2.17
	(0.82)	(1.95)	(0.72)	(0.86)
Autocorrel. PD ratio $\rho_{PD,-1}$	0.97	0.96	0.97	0.97
	(0.02)	(0.30)	(-0.03)	(-0.17)
Excess return reg. coefficient $\hat{c}_5$	-0.0041	-0.0050	-0.0047	-0.0044
	(0.0014)	(0.64)	(0.40)	(0.19)
R^2 of excess return regression R_5^2	0.2101	0.2282	0.2951	0.2890
	(0.0824)	(-0.22)	(-1.03)	(-0.95)
Risk aversion coefficientγ		5	5	5
Std. dev. of expect. param σ_b			0.084	0.085
T			6	
λ				0.82
Test statistic $\hat{W}_N$		12.87	9.50	7.25
p-value of $\hat{W}_N$		2.5%	16.7%	20.1%

The table reports the US asset-pricing moments (column 2, with standard errors reported in parentheses), the model moments from Table III in Adam et al. (2016) (column 3, the t-ratios for each moment are reported in parentheses), as well as the moments implied by the finite memory and decay memory models (column 4 and 5, respectively, the t-ratios for each moment are reported in parentheses). Growth rates and returns are expressed in terms of real quarterly rates of increase. The PD ratio is the price over the quarterly dividend. A t-ratio less than 2 indicates that moments are closely matched with the data. The measure of the overall goodness of fit $\hat{W}_N$ and the corresponding p-value are reported in the last two rows of the table. Moreover, as in Adam et al.'s (2016), we also exclude the mean risk free rate $\hat{E}_{r^b}$ and c_5^2 from the estimation and set $\gamma = 5$ in both models for consistency.

8 Conclusions

We propose a novel mechanism for asset pricing models based on two features: (i) limited memory; (ii) time-varying expectations. The first assumption guarantees that the model is stable over long horizons, but opens up the possibility of many different temporary equilibria. The second assumption borrows from Ascari et al. (2019) the idea of modelling a multiplicative sunspot shock to select one equilibrium among all admissible ones. This sunspot shock has an appealing economic interpretation as a time-varying expectation formation process, because it entails a change in the way agents combine past data to calculate their expectations.

These two assumptions allow for the possibility of temporary explosive trajectories and boom-and-bust behaviour typical of asset price dynamics. Intuitively, our persistent sunspot shock can induce "momentum" on stock prices, while the limited memory assumption implies "mean reversion" over long horizons to stable fundamentals. We thus embed this mechanism in the standard Bansal and Yaron (2004) model of long-run risk, featuring Epstein-Zin preferences and a persistent predictable component in long-run risk. However, contrary to Bansal and Yaron (2004), we do not assume stochastic volatility, because our mechanism generates it endogenously.

The resulting model is able to quantitatively reproduce a variety of stylized asset pricing facts, such as the equity premium, excessive volatility and persistence of price-dividend ratio, the relatively weak correlation between returns and fundamentals and the observed predictability of excess returns by lagged price-dividend ratios. Moreover, despite the assumed time-varying expectation formation process imposes some theoretical structure on the stochastic volatility, the quantitative performance of our model outperforms the Bansal and Yaron (2004) model, implying that the expectation process seems corroborated by the data.

Furthermore, the model also generates empirical plausible subjective expectations. We use the UBS/Gallup Survey data to show that there is a positive correlation between the PD ratio and survey expected returns. Our model is able to replicate very closely the regression coefficient of expected one-year ahead survey market return on the PD ratio.

Although the previous literature studied the positive correlation between price-dividend and expected returns, we uncover a novel fact showing that this relationship varies over time, as implied by the time-varying expectation formation process in our model. This empirical fact was one of the main motivation to introduce this process in our model in the first place.

Finally, we show that the improvement in the quantitative performance of our model is quite robust, because most of the results continue to hold even when applying to the simplest version of the Lucas (1978) model with time separable preferences and standard stochastic driving processes as in Adam et al. (2016). In this context, our mechanism outperforms the learning model in Adam et al. (2016).

References

Adam, K., A. Marcet, and J. Beutel (2017): "Stock price booms and expected capital gains," *American Economic Review*, 107, 2352–2408.

Adam, K., A. Marcet, and J. P. Nicolini (2016): "Stock market volatility and learning," *The Journal of Finance*, 71, 33–82.

Adam, K. and S. Nagel (2022): "Expectations Data in Asset Pricing," Crc tr 224 discussion paper series, University of Bonn and University of Mannheim, Germany, forthcoming in The Handbook of Economic Expectations.

Adda, J. and R. W. Cooper (2003): *Dynamic economics: quantitative methods and applications*, MIT press.

Albuquerque, R., M. Eichenbaum, V. X. Luo, and S. Rebelo (2016): "Valuation risk and asset pricing," *The Journal of Finance*, 71, 2861–2904.

Angeletos, G.-M. and C. Lian (2021): "Determinacy without the Taylor principle," NBER working paper 28881, National Bureau of Economic Research.

Ascari, G., P. Bonomolo, and H. F. Lopes (2019): "Walk on the wild side: Temporarily unstable paths and multiplicative sunspots," *American Economic Review*, 109, 1805–42.

Bansal, R., D. Kiku, and A. Yaron (2016): "Risks for the long run: Estimation with time aggregation," *Journal of Monetary Economics*, 82, 52 – 69.

Bansal, R. and I. Shaliastovich (2010): "Confidence risk and asset prices," *American Economic Review*, 100, 537–41.

——— (2011): "Learning and asset-price jumps," *The Review of Financial Studies*, 24, 2738–2780.

Bansal, R. and A. Yaron (2004): "Risks for the long run: A potential resolution of asset pricing puzzles," *The journal of Finance*, 59, 1481–1509.

Barberis, N., R. Greenwood, L. Jin, and A. Shleifer (2018): "Extrapolation and bubbles," *Journal of Financial Economics*, 129, 203–227.

Barro, R. J. and J. F. Ursúa (2012): "Rare macroeconomic disasters," *Annu. Rev. Econ.*, 4, 83–109.

Blanchard, O. (1979): "Backward and Forward Solutions for Economies with Rational Expectations," *American Economic Review*, 69, 114–118.

Blanchard, O. and C. Kahn (1980): "The solution of linear difference models under rational expectations," *Econometrica*, 48, 1305–11.

Boldrin, M., L. J. Christiano, and J. D. Fisher (2001): "Habit persistence, asset returns, and the business cycle," *American Economic Review*, 91, 149–166.

Boswijk, H. P., C. H. Hommes, and S. Manzan (2007): "Behavioral heterogeneity in stock prices," *Journal of Economic dynamics and control*, 31, 1938–1970.

Campbell, J. Y. and J. H. Cochrane (1999): "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107, 205–251.

Campbell, J. Y. and R. J. Shiller (1988): "The dividend-price ratio and expectations of future dividends and discount factors," *The Review of Financial Studies*, 1, 195–228.

Cassella, S. and H. Gulen (2018): "Extrapolation bias and the predictability of stock returns by price-scaled variables," *The Review of Financial Studies*, 31, 4345–4397.

Cochrane, J. H. (2009): *Asset pricing: Revised edition*, Princeton university press.

Cooley, T. F. and E. C. Prescott (1976): "Estimation in the presence of stochastic parameter variation," *Econometrica: journal of the Econometric Society*, 167–184.

Davidson, R., J. G. MacKinnon, et al. (2004): *Econometric theory and methods*, vol. 5, Oxford University Press New York.

Epstein, L. G., E. Farhi, and T. Strzalecki (2014): "How Much Would You Pay to Resolve Long-Run Risk?" *American Economic Review*, 104, 2680–97.

Epstein, L. G. and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–969.

Greenwood, R. and A. Shleifer (2014): "Expectations of returns and expected returns," *The Review of Financial Studies*, 27, 714–746.

Hansen, B. E. (1992): "Testing for parameter instability in linear models," *Journal of policy Modeling*, 14, 517–533.

Haruvy, E., Y. Lahav, and C. N. Noussair (2007): "Traders' expectations in asset markets: experimental evidence," *American Economic Review*, 97, 1901–1920.

Hirshleifer, D. (2015): "Behavioral finance," *Annual Review of Financial Economics*, 7, 133–159.

Jonides, J., R. L. Lewis, D. E. Nee, C. A. Lustig, M. G. Berman, and K. S. Moore (2008): "The mind and brain of short-term memory," *Annual Review of Psychology*, 59, 193–224.

Lucas, R. (1978): "Asset prices in an exchange economy," *Econometrica: Journal of the Econometric Society*, 1429–1445.

Malmendier, U. and S. Nagel (2016): "Learning from inflation experiences," *The Quarterly Journal of Economics*, 131, 53–87.

Mehra, R. and E. C. Prescott (1985): "The equity premium: A puzzle," *Journal of monetary Economics*, 15, 145–161.

Muth, J. F. (1960): "Optimal Properties of Exponentially Weighted Forecasts," *Journal of the American Statistical Association*, 55, 299–306.

——— (1961): "Rational expectations and the theory of price movements," *Econometrica: Journal of the Econometric Society*, 29, 315–335.

Nagel, S. and Z. Xu (2021): "Asset Pricing with Fading Memory," *The Review of Financial Studies*, 35, 2190–2245.

Nyblom, J. (1989): "Testing for the constancy of parameters over time," *Journal of the American Statistical Association*, 84, 223–230.

Sargent, T. J. (2008): "Evolution and intelligent design," *American Economic Review*, 98, 5–37.

Shiller, R. J. (1980): "Do stock prices move too much to be justified by subsequent changes in dividends?" Tech. rep., National Bureau of Economic Research.

Timmermann, A. (1996): "Excess volatility and predictability of stock prices in autoregressive dividend models with learning," *The Review of Economic Studies*, 63, 523–557.

Timmermann, A. G. (1993): "How learning in financial markets generates excess volatility and predictability in stock prices," *The Quarterly Journal of Economics*, 108, 1135–1145.

Woodford, M. (2018): *Monetary Policy Analysis When Planning Horizons Are Finite*, University of Chicago Press, 1–50.

Appendix

A The simple example: derivations

Muth's (1961) original formulation states that the RE solution should be a function of all the past, present and expected future structural shocks as described in (4). Plug this back to the expectation difference equation (1) we have

$$\sum_{j=1}^{\infty} u_j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \, \mathbb{E}_t \, \varepsilon_{t+j} = \theta \, \mathbb{E}_t \Big(\sum_{j=1}^{\infty} u_j \varepsilon_{t+1-j} + b\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_j \, \mathbb{E}_{t+1} \, \varepsilon_{t+1+j} \Big) + \varepsilon_t \tag{A1}$$

where u_j, c_j and b are coefficients to be determined. Equal coefficients of ε_{t-j} gives the expression for u's:

$$\begin{aligned} \varepsilon_t : &\quad b = \theta u_1 + 1 \Rightarrow u_1 = \frac{1}{\theta}(b-1); \\ \varepsilon_{t-1} : &\quad u_1 = \theta u_2 \Rightarrow u_2 = \frac{1}{\theta} u_1; \\ &\vdots \\ \varepsilon_{t-T} : &\quad u_T = \theta u_{T+1} \Rightarrow u_{T+1} = \frac{1}{\theta} u_T; \\ &\vdots \end{aligned}$$

and solve for c's:

$$\begin{aligned} \varepsilon_{t+1} : &\quad c_1 = \theta b \\ \varepsilon_{t+2} : &\quad c_2 = \theta c_1 \\ &\vdots \\ \varepsilon_{t+T} : &\quad c_T = \theta c_{T-1} \\ &\vdots \end{aligned}$$

The coefficients u's and c's can be represented by the parameter b, and thus the set of solution can be parameterized by b

$$y_t = \sum_{j=1}^{\infty} (\frac{1}{\theta})^j (b-1)\varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} b\theta^j \, \mathbb{E}_t \, \varepsilon_{t+j} \tag{A2}$$

For white noise shocks, the solution becomes

$$y_t = \sum_{j=1}^{\infty} (\frac{1}{\theta})^j (b-1)\varepsilon_{t-j} + b\varepsilon_t$$

Model (1) has an infinite number of solutions (each one corresponds to a particular value of b) due to the presence of the forward-looking component. In fact, the expectation term in that model, under the rational expectations hypothesis, is a conditional mean that can be written as a weighted average of the past observations (see Muth (1961))

$$\mathbb{E}_t\, y_{t+1} = \sum_{i=1}^{\infty} V_i y_{t+1-i} = \sum_{i=1}^{\infty} (b-1) \left(\frac{1}{\theta b}\right)^i y_{t+1-i} \tag{A3}$$

Proof. Without loss of generality we assume white noise shocks. From the solution

$$y_t = \sum_{j=1}^{\infty} \left(\frac{1}{\theta}\right)^j (b-1)\varepsilon_{t-j} + b\varepsilon_t \tag{A4}$$

The expectation is then

$$\begin{aligned}
\mathbb{E}_t\, y_{t+1} &= \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\frac{1}{\theta}\right)^j (b-1)\varepsilon_{t+1-j} + b\varepsilon_{t+1}\right] = \sum_{j=1}^{\infty} \left(\frac{1}{\theta}\right)^j (b-1)\varepsilon_{t+1-j} \\
&= \left(\frac{1}{\theta}\right)(b-1)\varepsilon_t + \left(\frac{1}{\theta}\right)^2 (b-1)\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^3 (b-1)\varepsilon_{t-2} + \dots \\
&= \left(\frac{1}{\theta b}\right)(b-1)\left[b\varepsilon_t + \left(\frac{1}{\theta}\right) b\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2 b\varepsilon_{t-2} + \left(\frac{1}{\theta}\right)^3 b\varepsilon_{t-3} + \dots\right] \\
&= \left(\frac{1}{\theta b}\right)(b-1)\left[\begin{array}{c} b\varepsilon_t + \left(\frac{1}{\theta}\right)(b-1)\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2(b-1)\varepsilon_{t-2} + \left(\frac{1}{\theta}\right)^3(b-1)\varepsilon_{t-3} + \dots \\ + \left(\frac{1}{\theta}\right)\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2\varepsilon_{t-2} + \left(\frac{1}{\theta}\right)^3\varepsilon_{t-3} + \dots \end{array}\right] \\
&= \left(\frac{1}{\theta b}\right)(b-1)y_t + \left(\frac{1}{\theta b}\right)(b-1)\left[\left(\frac{1}{\theta}\right)\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2\varepsilon_{t-2} + \left(\frac{1}{\theta}\right)^3\varepsilon_{t-3} + \dots\right] \\
&= \left(\frac{1}{\theta b}\right)(b-1)y_t + \left(\frac{1}{\theta b}\right)^2(b-1)\left[\begin{array}{c} b\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)(b-1)\varepsilon_{t-2} + \left(\frac{1}{\theta}\right)^2(b-1)\varepsilon_{t-3} + \dots \\ + \left(\frac{1}{\theta}\right)\varepsilon_{t-2} + \left(\frac{1}{\theta}\right)^2\varepsilon_{t-3} + \dots \end{array}\right] \\
&= \left(\frac{1}{\theta b}\right)(b-1)y_t + (b-1)\left(\frac{1}{\theta b}\right)^2 y_{t-1} + (b-1)\left(\frac{1}{\theta b}\right)^2\left[\left(\frac{1}{\theta}\right)\varepsilon_{t-2} + \left(\frac{1}{\theta}\right)^2\varepsilon_{t-3} + \dots\right] \\
&= (b-1)\sum_{i=1}^{\infty}\left(\frac{1}{\theta b}\right)^i y_{t+1-i}
\end{aligned}$$

Finite memory

Under finite memory and the time-varying expectation formation process, the original Muth (1961)'s formulation (4) becomes

$$y_t = \sum_{j=1}^{T} u_{j,t}\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} c_{j,t}\,\mathbb{E}_t\,\varepsilon_{t+j} \tag{A5}$$

At time t, the information set of the agent is given by $\mathcal{I}_t = \{\varepsilon_t, \varepsilon_{t-1}, \dots, \varepsilon_{t-T}\}$. Based on this information set, she forms her expectations

$$\mathbb{E}_t\, y_{t+1} = \mathbb{E}\left(y_{t+1}|\mathcal{I}_t\right) = \mathbb{E}\left(y_{t+1}|\varepsilon_t, \varepsilon_{t-1}, \dots, \varepsilon_{t-T}\right)$$

$$= \mathbb{E}_t \left(\sum_{j=1}^{T+1} u_{j,t+1}\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1} \mathbb{E}_{t+1} \varepsilon_{t+1+j} \right) \tag{A6}$$

This particular form of expectation implies that the agent does not internalize the limited memory.

Now use (A5) and (A6) to substitute for y_t and $\mathbb{E}_t y_{t+1}$ in the expectational difference equation (1),

$$\sum_{j=1}^{T} u_{j,t}\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} c_{j,t} \mathbb{E}_t \varepsilon_{t+j} =$$

$$\theta \mathbb{E}_t \left(\sum_{j=1}^{T+1} u_{j,t+1}\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1} \mathbb{E}_{t+1} \varepsilon_{t+1+j} \right) + \varepsilon_t$$

Again, equal coefficients to find an expression for the u's:

$$\begin{aligned}
\varepsilon_t: &\quad b_t = \theta \mathbb{E}_t u_{1,t+1} + 1 \Rightarrow \mathbb{E}_t u_{1,t+1} = \frac{1}{\theta}(b_t - 1); \\
\varepsilon_{t-1}: &\quad u_{1,t} = \theta \mathbb{E}_t u_{2,t+1} \Rightarrow \mathbb{E}_t u_{2,t+1} = \frac{1}{\theta} u_{1,t}; \\
&\quad \vdots \\
\varepsilon_{t-T+1}: &\quad u_{T-1,t} = \theta \mathbb{E}_t u_{T,t+1} \Rightarrow \mathbb{E}_t u_{T,t+1} = \frac{1}{\theta} u_{T-1,t}; \\
\varepsilon_{t-T}: &\quad u_{T,t} = \theta \mathbb{E}_t u_{T+1,t+1} \Rightarrow \mathbb{E}_t u_{T+1,t+1} = \frac{1}{\theta} u_{T,t};
\end{aligned}$$

and for the c's:

$$\begin{aligned}
\varepsilon_{t+1}: &\quad c_{1,t} = \theta \mathbb{E}_t b_{t+1} \\
\varepsilon_{t+2}: &\quad c_{2,t} = \theta \mathbb{E}_t c_{1,t+1} \\
&\quad \vdots \\
\varepsilon_{t+T}: &\quad c_{T,t} = \theta \mathbb{E}_t c_{T-1,t+1} \\
&\quad \vdots
\end{aligned}$$

Therefore, parameters $u_{j,t}$ and $c_{j,t}$ are defined by the path of b_t. If b is constant, we have $u_j = \frac{1}{\theta^j} b$ and $c_j = \theta^j b$ are constants as well. And the solution is

$$y_t = \sum_{j=1}^{T} (\frac{1}{\theta})^j (b-1)\varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} b\theta^j \mathbb{E}_t \varepsilon_{t+j} \tag{A7}$$

If b_t follows a random walk, i.e., $b_t = b_{t-1} + \sigma_b \xi_t$, with ξ_t i.i.d $N(0,1)$, and therefore $\mathbb{E}_t b_{t+1} = b_t$. The solution for the expectational difference equation can be parameterized by b_t. To derive the solution, assume that $u_{1,t+1} = F(b_{t+1})$, we need to find the formulation of F such that $b_t = \frac{1}{\theta} \mathbb{E}_t u_{1,t+1} + 1$ is satisfied, given the stochastic process for b_t. Guess that F is linear,

$$u_{1,t+1} = a_0 + a_1 b_{t+1}$$

then

$$b_t = \theta\,\mathbb{E}_t[a_0 + a_1 b_{t+1}] + 1$$
$$= \theta[a_0 + a_1 b_t] + 1$$

gives

$$a_0 = -\frac{1}{\theta} \qquad a_1 = \frac{1}{\theta}$$

which gives $u_{i,t+1} = -\frac{1}{\theta} + \frac{1}{\theta}b_{t+1}$, bring one period backward we get $u_{i,t} = -\frac{1}{\theta} + \frac{1}{\theta}b_t$. Analogously, assuming $u_{2,t+1} = F(b_{t+1})$ is linear, i.e. $u_{2,t+1} = d_0 + d_1 b_{t+1}$ then

$$u_{1,t} = \theta\,\mathbb{E}_t[d_0 + d_1 b_{t+1}]$$
$$= \theta[d_0 + d_1 b_t]$$

gives

$$d_0 = -(\frac{1}{\theta})^2 \qquad d_1 = (\frac{1}{\theta})^2$$

By the same reasoning, we can rewrite c_t's as functions of b_t. Then the set of solutions can be represented by b_t.

$$y_t = \sum_{j=1}^{T}(\frac{1}{\theta})^j(b_t - 1)\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} b_t\theta^j\,\mathbb{E}_t\,\varepsilon_{t+j}. \tag{A8}$$

Decay memory

Under decay memory, the original Muth (1961)'s formulation (4) becomes

$$y_t = \sum_{j=1}^{\infty} u_{j,t}\lambda^j\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} c_{j,t}\,\mathbb{E}_t\,\varepsilon_{t+j} \tag{A9}$$

At time t, the information set of the agent is given by $\mathcal{I}_t = \{\varepsilon_t, \lambda\varepsilon_{t-1}, \lambda^2\varepsilon_{t-2}, \ldots\}$. Based on this information set, she forms her expectations

$$\begin{aligned}\mathbb{E}_t\,y_{t+1} &= \mathbb{E}(y_{t+1}|\mathcal{I}_t) = \mathbb{E}(y_{t+1}|\varepsilon_t, \lambda\varepsilon_{t-1}, \lambda^2\varepsilon_{t-2}, \ldots)\\ &= \mathbb{E}_t\big(\sum_{j=1}^{\infty} u_{j,t+1}\lambda^{j-1}\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1}\,\mathbb{E}_{t+1}\,\varepsilon_{t+1+j}\big)\end{aligned} \tag{A10}$$

Now use (A9) and (A10) to substitute for y_t and $\mathbb{E}_t\,y_{t+1}$ in the expectational difference equation (1),

$$\sum_{j=1}^{\infty} u_{j,t}\lambda^j\varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} c_{j,t}\,\mathbb{E}_t\,\varepsilon_{t+j} =$$
$$\theta\,\mathbb{E}_t(\sum_{j=1}^{\infty} u_{j,t+1}\lambda^{j-1}\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1}\,\mathbb{E}_{t+1}\,\varepsilon_{t+1+j}) + \varepsilon_t$$

Again, equal coefficients to find an expression for the u's:

$$\varepsilon_t: \quad b_t = \theta\,\mathbb{E}_t\,u_{1,t+1} + 1 \Rightarrow \mathbb{E}_t\,u_{1,t+1} = \frac{1}{\theta}(b_t - 1);$$

$$\varepsilon_{t-1}: \quad \lambda u_{1,t} = \theta\lambda\, \mathbb{E}_t\, u_{2,t+1} \Rightarrow \mathbb{E}_t\, u_{2,t+1} = \frac{1}{\theta} u_{1,t};$$
$$\vdots$$

and for the c's:

$$\begin{aligned} \varepsilon_{t+1}: &\quad c_{1,t} = \theta\, \mathbb{E}_t\, b_{t+1} \\ \varepsilon_{t+2}: &\quad c_{2,t} = \theta\, \mathbb{E}_t\, c_{1,t+1} \\ &\vdots \end{aligned}$$

For constant $b_t = b$, the coefficient for ε_{t-j}, $\forall j$ is $u_j = (b-1)\left(\frac{\lambda}{\theta}\right)^j$, and the coefficient for $\mathbb{E}_t\, \varepsilon_{t+j}$, $\forall j$ is $c_j = b\theta^j$.

$$y_t = (b-1)\sum_{j=1}^{\infty}(\frac{\lambda}{\theta})^j \varepsilon_{t-j} + b\varepsilon_t + b\sum_{j=1}^{\infty}\theta^j \mathbb{E}_t \varepsilon_{t+j} \tag{A11}$$

For $b_t = b_{t-1} + \sigma_b \xi_t$ follows a random walk process, the solution is

$$y_t = (b_t-1)\sum_{j=1}^{\infty}(\frac{\lambda}{\theta})^j \varepsilon_{t-j} + b_t\varepsilon_t + b_t\sum_{j=1}^{\infty}\theta^j \mathbb{E}_t \varepsilon_{t+j} \tag{A12}$$

As $\lambda < 1$, the past shocks have dampened impacts on current equilibrium just because the memory loss. In cases where $\lambda > \theta$, transversality condition gives the unique solution, coinciding with $b = 1$. In the case where $\lambda < \theta$, transversality condition cannot help us pick a unique solution.

Under the time-varying expectation formation, the expectation is written as (with white noises),

$$\bar{\mathbb{E}}_t y_{t+1} = (b_t - 1)\sum_{i=1}^{\infty}\left(\frac{1}{\theta}\right)^i \left(\prod_{j=1}^{i} b_{t+1-j}\right)^{-1} \lambda^{i-1} y_{t+1-i} \tag{A13}$$

Derivation:
Start with the solution (A12) and assume $\mathbb{E}_t\, \varepsilon_{t+j} = 0$, $\forall j$

$$y_t = \sum_{j=1}^{\infty}(\frac{\lambda}{\theta})^j (b_t - 1)\varepsilon_{t-j} + b_t\varepsilon_t$$

Bring the solution one-period forward and take expectation

$$\begin{aligned} \bar{\mathbb{E}}_t y_{t+1} =& \bar{\mathbb{E}}_t\left(\sum_{j=1}^{\infty}\left(\frac{1}{\theta}\right)^j \lambda^{j-1}(b_{t+1}-1)\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1}\right) = \sum_{j=1}^{\infty}\left(\frac{1}{\theta}\right)^j \lambda^{j-1}(b_t-1)\varepsilon_{t+1-j} \\ =& \left(\frac{1}{\theta}\right)(b_t-1)\varepsilon_t + \left(\frac{1}{\theta}\right)^2 \lambda(b_t-1)\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^3 \lambda^2(b_t-1)\varepsilon_{t-2} + \dots \\ =& \left(\frac{1}{\theta b_t}\right)(b_t-1)\left[b_t\varepsilon_t + \left(\frac{1}{\theta}\right) b_t\lambda\varepsilon_{t-1} + \left(\frac{1}{\theta}\right)^2 b_t\lambda^2\varepsilon_{t-2} + \left(\frac{1}{\theta}\right)^3 b_t\lambda^3\varepsilon_{t-3} + \dots\right] \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{\theta b_t}\right)(b_t-1)\left[\begin{array}{c} b_t\varepsilon_t + \left(\frac{\lambda}{\theta}\right)(b_t-1)\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)^2(b_t-1)\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^3(b_t-1)\varepsilon_{t-3} + \ldots \\ + \left(\frac{\lambda}{\theta}\right)\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)^2\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^3\varepsilon_{t-3} + \ldots \end{array}\right] \\
&= \left(\frac{1}{\theta b_t}\right)(b_t-1)y_t + \left(\frac{1}{\theta b_t}\right)(b_t-1)\left[\left(\frac{\lambda}{\theta}\right)\varepsilon_{t-1} + \left(\frac{\lambda}{\theta}\right)^2\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^3\varepsilon_{t-3} + \ldots\right] \\
&= \left(\frac{1}{\theta b_t}\right)(b_t-1)y_t + \left(\frac{1}{\theta b_t}\right)(b_t-1)\left(\frac{\lambda}{\theta b_{t-1}}\right)\left[\begin{array}{c} b_{t-1}\varepsilon_{t-1} + \frac{\lambda}{\theta}(b_{t-1}-1)\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^2(b_{t-1}-1)\varepsilon_{t-3} \\ + \left(\frac{\lambda}{\theta}\right)\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^2\varepsilon_{t-3} + \ldots \end{array}\right] = \\
&= (b_t-1)\left[\left(\frac{1}{\theta b_t}\right)y_t + \left(\frac{1}{\theta}\right)^2\left(\frac{1}{b_t b_{t-1}}\right)\lambda y_{t-1} + \left(\frac{1}{\theta}\right)^2\left(\frac{1}{b_t b_{t-1}}\right)\lambda\left[\left(\frac{\lambda}{\theta}\right)\varepsilon_{t-2} + \left(\frac{\lambda}{\theta}\right)^2\varepsilon_{t-3} + \ldots\right]\right] \\
&= (b_t-1)\sum_{i=1}^{\infty}\left(\frac{1}{\theta}\right)^i\left(\prod_{j=1}^{i} b_{t+1-i}\right)^{-1}\lambda^{i-1}y_{t+1-i}
\end{aligned}
$$

For constant b, the expectation under decay memory becomes, this is equation (11)

$$
\bar{\mathbb{E}}_t y_{t+1} = (b-1)\sum_{i=1}^{\infty}\left(\frac{1}{\theta b}\right)^i\lambda^{i-1}y_{t+1-i} \tag{A14}
$$

B A Decay Memory Asset-pricing Model

B.1 Model derivations

Recall that the Euler condition in equation (18) implies that any asset i should satisfy the following pricing restriction, (this is just the log form of the Euler equation (18))

$$
\mathbb{E}_t\left[\exp\left(\theta\log(\delta) - \frac{\theta}{\psi}g_{c,t+1} + \theta r_{a,t+1}\right)\right] = 1 \tag{A15}
$$

where the lowercase letters refer logs.

Solution for Wealth Return $r_{a,t+1}$:
Note that when substituting $r_{i,t+1} = r_{a,t+1}$ then (A15) becomes

$$
\mathbb{E}_t\left[\exp\left(\theta\log(\delta) - \frac{\theta}{\psi}g_{c,t+1} + \theta r_{a,t+1}\right)\right] = 1 \tag{A16}
$$

We start by conjecturing that the solutions for the endogenous variables z_t are a linear function of the *discounted* past, present and expected future values of x's but subject to decay memory constraint, i.e. the time t information set is $\mathbb{I}_t = \{x_t, \lambda x_{t-1}, \lambda^2 x_{t-2}, \ldots\}$. Guess the solution of log price consumption ratio $z_t = \log(P_t/C_t)$ has the following form:

$$
z_t = A_{0,t} + \left(1 - \frac{1}{\psi}\right)\left(\sum_{j=1}^{\infty} u_{j,t}\lambda^j x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t x_{t+j}\right)
$$

where the parameters $A_{0,t}$, $u_{j,t}$, b_t and $c_{j,t}$ are coefficients to be determined. Followed by that the expectation of z_{t+1} at time t given information set $\mathbb{I}_t$ has the form

$$\bar{\mathbb{E}}_t z_{t+1} = \mathbb{E}(z_{t+1}|x_t, \lambda x_{t-1}, \ldots)$$
$$= \mathbb{E}_t\left[A_{0,t+1} + \left(1-\frac{1}{\psi}\right)\left(\sum_{j=1}^{\infty} u_{j,t+1}\lambda^{j-1}x_{t+1-j} + b_{t+1}x_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1}\mathbb{E}_{t+1}x_{t+1+j}\right)\right]$$

Armed with the endogenous variable z_t and its expectation, we plug the approximation $r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1}$ into the Euler equation (A16). The solution coefficients can be derived by collecting the terms on the corresponding state variables.

Using the undetermined coefficient methods as in Appendix A, the log price-consumption ratio is given by

$$z_t = A_{0,t} + (1-\frac{1}{\psi})\left[\sum_{j=1}^{\infty}(\frac{1}{\kappa_1})^j\lambda^j(b_t-1)x_{t-j} + b_t x_t + b_t\sum_{j=1}^{\infty}(\kappa_1\rho)^j x_t\right]$$

with

$$A_{0,t} = \frac{1}{1-\kappa_1}\left(\log(\delta) - (\frac{1}{\psi}-1)\mu + \kappa_0 + \frac{\theta}{2}(\frac{1}{\psi}-1)^2\left[\sigma^2 - \sigma_b^2\kappa_1 X_t^2 + \kappa_1^2\left(\frac{\varphi_e\sigma}{1-\kappa_1\rho}\right)^2 b_t^2\right]\right)$$

where

$$X_t = \sum_{j=1}^{\infty}(\frac{1}{\kappa_1})^j\lambda^j x_{t-j} + \frac{1}{1-\kappa_1\rho}x_t$$

Solution for Market Return $r_{m,t+1}$:
When $r_{i,t+1} = r_{m,t+1}$ the log-linearized Euler equation becomes

$$\mathbb{E}_t\left[\exp\left(\theta\log\delta - \frac{\theta}{\psi}g_{c,t+1} + (\theta-1)\,r_{a,t+1} + r_{m,t+1}\right)\right] = 1 \tag{A17}$$

Plug in the linearized expression for $r_{i,t+1}$ and $r_{m,t+1}$

$$1 = \mathbb{E}_t\left[\exp\left(\theta\log(\delta) - \frac{\theta}{\psi}g_{c,t+1} + (\theta-1)(\kappa_0+\kappa_1 z_{t+1} - z_t + g_{c,t+1}) + \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + g_{d,t+1}\right)\right] \tag{A18}$$

Guess that the solution for the endogenous variables $z_{m,t}$ are a linear function of the discounted past, present and expected future values of x's, i.e. the time t information set is $\mathbb{I}_t = \{x_t, \lambda x_{t-1}, \lambda^2 x_{t-2}, ...\}$. Analogously, plugging the guess into the pricing equation (A18) and equating the coefficients of the state variables and constant. Replacing the consumption and dividend growth processes and of the price-consumption and price-dividend ratios, and solving for the expectations, we obtain the solution for $z_{m,t}$:

$$z_{m,t} = A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j(b_t-1)x_{t-j} + b_t x_t + b_t\sum_{j=1}^{\infty}(\kappa_{1,m}\rho)^j x_t\right]$$

Solution for the risk-free rate $r_{f,t+1}$:

To solve for the risk-free rate, we substitute $R_{i,t+1} = R_{f,t+1}$ then (A15) becomes

$$\mathbb{E}_t \left[\exp \left(\theta \log \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1) \, r_{a,t+1} + r_{f,t+1} \right) \right] = 1$$

In logarithms, the Euler equation is:

$$\begin{aligned} r_{f,t+1} &= - \log(\mathbb{E}_t(\exp(m_{t+1}))) \\ &= - \log \left(\mathbb{E}_t(\exp(\theta \log \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1) \, r_{a,t+1} \right) \end{aligned}$$

Further solve the above expression gives

$$\begin{aligned} r_{f,t+1} = & - \log(\delta) + \frac{1}{\psi}\mu + \frac{1}{\psi}x_t - (1-\theta)\frac{\theta}{2}(\frac{1}{\psi} - 1)^2\sigma^2 - \frac{1}{2}\left[(\frac{\theta}{\psi} + 1 - \theta)\sigma \right]^2 \\ & - \frac{1}{2}(1-\theta)(1-\frac{1}{\psi})^2 \left[\kappa_1^2 (\frac{1}{1-\kappa_1\rho})^2 \varphi_e^2 \sigma^2 b_t^2 - \sigma_b^2 \left[X_t\right]^2 - \kappa_1^2 \sigma_b^2 (\frac{1}{1-\kappa_1\rho})^2 \varphi_e^2 \sigma^2 \right] \end{aligned}$$

B.2 Analytical results: derivations

This section provides the derivation of the analytical results in the main paper.[27]

Price-dividend ratio:
To derive the persistence of price-dividend ratio (i.e., equation (34)), we define $\hat{z}$ as the deviation from the usual RE solution,

$$\begin{aligned} \hat{z}_{m,t} \equiv & z_{m,t} - z_{m,t}^{RE} \\ = & (b_t - 1) \left(\sum_{j=1}^{\infty} (\frac{\lambda}{\kappa_{1,m}})^j (\phi - \frac{1}{\psi}) x_{t-j} + \frac{1}{1-\kappa_{1,m}\rho}(\phi - \frac{1}{\psi}) x_t \right) \end{aligned} \tag{A19}$$

[27] The time-varying component in the A_0 and $A_{0,m}$ were abbreviated when deriving the analytical solutions as it does not affect the main results, but this impact was considered when doing the quantitative analysis.

Moving the above equation (A19) one-period forward we get,

$$
\begin{aligned}
\hat{z}_{m,t+1} &\equiv z_{m,t+1} - z^{RE}_{m,t+1} \\
&= (b_{t+1}-1)\left(\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j(\phi-\frac{1}{\psi})x_{t+1-j} + \frac{1}{1-\kappa_{1,m}\rho}(\phi-\frac{1}{\psi})x_{t+1}\right) \\
&= (b_{t+1}-1)\left(\frac{\lambda}{\kappa_{1,m}}\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j(\phi-\frac{1}{\psi})x_{t-j} + \frac{\lambda}{\kappa_{1,m}}(\phi-\frac{1}{\psi})x_t + \frac{1}{1-\kappa_{1,m}\rho}(\phi-\frac{1}{\psi})x_{t+1}\right) \\
&\overset{[A]}{=} (b_{t+1}-1)\left(\frac{\lambda}{\kappa_{1,m}}\left(\frac{\hat{z}_{m,t}}{(b_t-1)} - \frac{1}{1-\kappa_{1,m}\rho}(\phi-\frac{1}{\psi})x_t\right) + \frac{\lambda}{\kappa_{1,m}}(\phi-\frac{1}{\psi})x_t + \frac{1}{1-\kappa_{1,m}\rho}(\phi-\frac{1}{\psi})x_{t+1}\right) \\
&= \frac{\lambda}{\kappa_{1,m}}\frac{b_{t+1}-1}{b_t-1}\hat{z}_{m,t} + (b_{t+1}-1)\left(\phi-\frac{1}{\psi}\right)\left(\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}e_{t+1} + \frac{1}{1-\kappa_{1,m}\rho}(1-\lambda)\rho x_t\right) \\
&\overset{[B]}{\approx} \frac{\lambda}{\kappa_{1,m}}\frac{b_{t+1}-1}{b_t-1}\hat{z}_{m,t} + (b_{t+1}-1)\left(\phi-\frac{1}{\psi}\right)\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}e_{t+1} \qquad \text{as } \lambda \to 1
\end{aligned}
$$

[A] follows that by simply rearranging (A19)

$$
\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j(\phi-\frac{1}{\psi})x_{t-j} = \frac{\hat{z}_{m,t}}{(b_t-1)} - \frac{1}{1-\kappa_{1,m}\rho}(\phi-\frac{1}{\psi})x_t
$$

[B] follows that as $\lambda \to 1$, the memory loss term

$$
(b_{t+1}-1)\left(\phi-\frac{1}{\psi}\right)\left(\frac{1}{1-\kappa_{1,m}\rho}(1-\lambda)\rho x_t\right) \to 0
$$

The stochastic volatility (35) in the text is derived as followed: for $b_t \neq 1$, the variance of $z_{m,t+1}$

$$
\begin{aligned}
\mathbb{V}ar_t(z_{m,t+1}) &= \mathbb{V}ar_t\left((\phi-\frac{1}{\psi})\left[\sum_{j=1}^{\infty}(\frac{1}{\kappa_{1,m}})^j\lambda^{j-1}(b_{t+1}-1)x_{t+1-j} + b_{t+1}\frac{1}{1-\kappa_{1,m}\rho}x_{t+1}\right]\right) \\
&= \mathbb{V}ar_t\left((\phi-\frac{1}{\psi})\left[b_{t+1}\left(\sum_{j=1}^{\infty}(\frac{1}{\kappa_{1,m}})^j\lambda^{j-1}x_{t+1-j} + \frac{1}{1-\kappa_{1,m}\rho}\rho x_t\right) + b_{t+1}\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}e_{t+1}\right]\right) \\
&\overset{[C]}{=} \mathbb{V}ar_t\left(\left[b_{t+1}\frac{1}{\kappa_{1,m}}\left(z^{RE}_{m,t} - z^b_{m,t}\right) + b_{t+1}(\phi-\frac{1}{\psi})\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}e_{t+1}\right]\right) \\
&= \frac{\sigma_b^2}{\kappa_{1,m}^2}\left(z^{RE}_{m,t} - z^b_{m,t}\right)^2 + b_t^2(\phi-\frac{1}{\psi})^2\left(\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}\right)^2 + \sigma_b^2(\phi-\frac{1}{\psi})^2\left(\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}\right)^2 \\
&\overset{[D]}{=} \frac{\sigma_b^2}{\kappa_{1,m}}\left(\frac{z_{m,t} - z^{RE}_{m,t}}{1-b_t}\right)^2 + b_t^2(\phi-\frac{1}{\psi})^2\left(\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}\right)^2 + \sigma_b^2(\phi-\frac{1}{\psi})^2\left(\frac{\varphi_e\sigma}{1-\kappa_{1,m}\rho}\right)^2
\end{aligned}
$$

[C] follows that

$$z^b_{m,t} = -\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j x_{t-j} \qquad \text{and} \qquad z^{RE}_{m,t} = \frac{1}{1-\kappa_{1,m}\rho}x_t$$

[D] follows that

$$z_{m,t} = (1-b_t)z^b_{m,t} + b_t z^{RE}_{m,t} \qquad \Rightarrow z_{m,t} - z^{RE}_{m,t} = (b_t - 1)(z^{RE}_{m,t} - z^b_{m,t})$$

Equity premium: derivations

First, stochastic discount factor is a function of $r_{a,t+1}$ and $g_{c,t+1}$.

$$\begin{aligned} m_{t+1} &= \theta \log\delta - \frac{\theta}{\psi}g_{c,t+1} + (\theta-1)r_{a,t+1} \\ &= \theta\log\delta - \frac{\theta}{\psi}g_{c,t+1} + (\theta-1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1}) \end{aligned}$$

Substituting the equilibrium return for $r_{a,t+1}$ into the equation, it is straightforward to show that the innovation to the pricing kernel is

$$\begin{aligned} m_{t+1} - E_t m_{t+1} &= -\frac{\theta}{\psi}g_{c,t+1} + (\theta-1)(\kappa_1 z_{t+1} + g_{c,t+1}) - \mathbb{E}_t\left[-\frac{\theta}{\psi}g_{c,t+1} + (\theta-1)(\kappa_1 z_{t+1} + g_{c,t+1})\right] \\ &= \left(-\frac{\theta}{\psi} + \theta - 1\right)(g_{c,t+1} - \mathbb{E}_t g_{c,t+1}) + (\theta-1)\kappa_1(z_{t+1} - \mathbb{E}_t z_{t+1}) \\ &= -\left(1-\theta+\frac{\theta}{\psi}\right)\sigma\eta_{t+1} - (1-\theta)\kappa_1(1-\frac{1}{\psi})\left[\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_1})^j x_{t+1-j} + \frac{1}{1-\kappa_1\rho}\rho x_t\right]\sigma_b\xi_{t+1} \\ &\quad - (1-\theta)\kappa_1(1-\frac{1}{\psi})b_{t+1}\frac{\varphi_e}{1-\kappa_1\rho}\sigma e_{t+1} \\ &= -\lambda_{m,\eta}\sigma\eta_{t+1} - \lambda_{m,\xi,t+1}\sigma_b\xi_{t+1} - \lambda_{m,e,t+1}\sigma e_{t+1} \end{aligned} \tag{A20}$$

where $\lambda_{m,\eta}$, $\lambda_{m,e,t+1}$ and $\lambda_{m,\xi,t+1}$ captures the pricing kernel's exposure to the independent consumption shocks, η_{t+1}, to the expected growth rate shock, e_{t+1}, and to the time-varying expectation shock, ξ_{t+1}.

Equation (A20) already provides the innovation in m_{t+1}. We now proceed to derive the innovation in the market return. Recall that the return

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + g_{d,t+1}$$

Then

$$\begin{aligned} r_{m,t+1} - E_t r_{m,t+1} =& \kappa_{1,m}(z_{m,t+1} - E_t z_{m,t+1}) + (g_{d,t+1} - E_t g_{d,t+1}) \\ =& \kappa_{1,m}(\phi - \frac{1}{\psi})\left[\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j x_{t+1-j} + \frac{1}{1-\kappa_{m,1}\rho}\rho x_t\right]\sigma_b\xi_{t+1} \\ &+ \kappa_{1,m}(\phi - \frac{1}{\psi})\frac{1}{1-\kappa_{m,1}\rho}b_{t+1}\varphi_e\sigma e_{t+1} + \varphi_d\sigma u_{t+1} \\ =& \beta_{m,\xi,t+1}\sigma_b\xi_{t+1} + \beta_{m,e,t+1}\sigma e_{t+1} + \beta_{m,u}\sigma u_{t+1} \end{aligned} \tag{A21}$$

where

$$\beta_{m,u} = \varphi_d \tag{A22a}$$

$$\beta_{m,\xi,t+1} = \kappa_{1,m}(\phi - \frac{1}{\psi}) \left[\sum_{j=1}^{\infty} (\frac{\lambda}{\kappa_{1,m}})^j x_{t+1-j} + \frac{1}{1-\kappa_{m,1}\rho} \rho x_t \right] \tag{A22b}$$

$$\beta_{m,e,t+1} = \kappa_{1,m}(\phi - \frac{1}{\psi}) \frac{1}{1-\kappa_{m,1}\rho} b_{t+1} \tag{A22c}$$

Moreover, it follows that

$$\mathbb{V}ar_t(r_{m,t+1}) = (\beta_{m,u} + \beta_{m,e,t+1})^2 \sigma^2 + \beta^2_{m,\xi,t+1} \sigma^2_b \tag{A23}$$

The risk premium for any asset is determined by the conditional variance between the return and m_{t+1}. Thus the risk premium for the market portfolio $r_{m,t+1}$ is equal to $\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = -cov(m_{t+1} - \mathbb{E}_t\, m_{t+1}, r_{m,t+1} - \mathbb{E}_t\, r_{m,t+1}) - 0.5\mathbb{V}ar_t(r_{m,t+1})$. Using the innovations in the market return and the pricing kernel, the expression for the equity premium is

$$\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = \beta_{m,e,t}\lambda_{m,e,t}\sigma^2 + \beta_{m,\xi,t}\lambda_{m,\xi,t}\sigma^2_b - 0.5\mathbb{V}ar_t(r_{m,t+1}) \tag{A24}$$

C A Finite Memory Asset-pricing Model

C.1 Model derivations

In this section, we follow the same steps as in Section B.1. Recognize that, in both decay memory and finite memory, the agent has the same preferences and utility function, and thus both model have the same log Euler equation (A15).

Solution for Wealth Return $r_{a,t+1}$:
We start by conjecturing that the solutions for the endogenous variables z_i are a linear function of the *finite* past, present and expected future values of x's but subject to finite memory constraint, i.e. the time t information set is $\mathbb{I}_t = \{x_t, x_{t-1}, x_{t-2}, ...x_{t-T}\}$.

$$z_t = A_{0,t} + (1 - \frac{1}{\psi}) \left[\sum_{j=1}^{T} u_{j,t} x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\, \mathbb{E}_t\, x_{t+j} \right] \tag{A25}$$

where the parameters $A_{0,t}$, $u_{j,t}$, b_t and $c_{j,t}$. Under the finite memory assumption, the expectation at time t given information set $\mathbb{I}_t$ would have the following form

$$\begin{aligned} \bar{\mathbb{E}}_t z_{t+1} =& \mathbb{E}(z_{t+1} | x_t, x_{t-1}, x_{t-2}, ...x_{t-T}) \\ =& \mathbb{E}_t \left[A_{0,t+1} + (1 - \frac{1}{\psi}) \sum_{j=1}^{T+1} u_{t+1,j} x_{t+1-j} + b_{t+1} x_{t+1} + \sum_{j=1}^{\infty} \mathbb{E}_{t+1}\, c_{t+1,j} x_{t+1+j} \right] \end{aligned} \tag{A26}$$

Approximate that $r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1}$, and by the same procedure as described in Section B.1, we get

$$\begin{aligned} z_t &= A_{0,t} + (1-\frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_1})^j(b_t-1)x_{t-j} + b_t x_t + b_t\sum_{j=1}^{\infty}(\kappa\rho)^j x_t\right] \\ &= A_{0,t} + (1-\frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_1})^j(b_t-1)x_{t-j} + b_t\frac{1}{1-\kappa_1\rho}x_t\right] \end{aligned} \tag{A27}$$

with

$$A_{0,t} = \frac{1}{1-\kappa_1}\left(\log(\delta) + (1-\frac{1}{\psi})\mu + \kappa_0 + \frac{\theta}{2}(1-\frac{1}{\psi})^2\left(\sigma^2 + \kappa_1^2(\frac{\varphi_e\sigma}{1-\kappa_1\rho})^2 b_t^2 - \sigma_b^2\Sigma^2\right)\right)$$

where $\Sigma = \sum_{j=1}^{T+1}(\frac{1}{\kappa_1})^j x_{t+1-j} + \frac{\rho x_t}{1-\kappa_1\rho}$.

Solution for Market Return $r_{m,t+1}$:
When $r_{i,t+1} = r_{m,t+1}$ the log-linearized Euler equation has the same form as equation (A17). We conjecture that the solutions for the endogenous variables $z_{m,t}$ are a function of the *finite* past, present and expected future values of x's but subject to finite memory constraint, i.e. the time t information set is $\mathbb{I}_t = \{x_t, x_{t-1}, x_{t-2}, ...x_{t-T}\}$.

$$z_{m,t} = A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{T} u_{j,t}x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\,\mathbb{E}_t\, x_{t+j}\right] \tag{A28}$$

where the parameters $A_{0,m,t}$, $u_{j,t}$, b_t and $c_{j,t}$ are coefficients to be determined. To use the method of undetermined coefficients, we start by plugging the expressions for the two returns (i.e., $r_{t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1}$ and $r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + g_{d,t+1}$) into the above Euler equation

$$1 = \mathbb{E}_t\left[\exp\left(\theta\log(\delta) - \frac{\theta}{\psi}g_{t+1} + (\theta-1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}) + \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + g_{d,t+1}\right)\right]$$

Plugging the conjectured solution of logarithm of price-dividend ratio (A28) into the above equation and collect the terms on the corresponding state variables (i.e., the same procedure as described in Section B.1)

$$\begin{aligned} z_{m,t} &= A_{0,m,t} + (\phi-\frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_{1,m}})^j(b_t-1)x_{t-j} + b_t x_t + b_t\sum_{j=1}^{\infty}(\kappa_{1,m}\rho)^j x_t\right] \\ &= A_{0,m,t} + (\phi-\frac{1}{\psi})\left[\sum_{j=1}^{T}(\frac{1}{\kappa_{1,m}})^j(b_t-1)x_{t-j} + b_t\frac{1}{1-\kappa_{1,m}\rho}x_t\right] \end{aligned} \tag{A29}$$

C.2 Analytical results: derivations

This section provides the derivation of the analytical results for the finite memory model.[28] In general it has similar procedure as the decay memory case.

[28] The time-varying component in the A_0 and $A_{0,m}$ were abbreviated when deriving the analytical solutions as it does not affect the main results, but this impact was considered when doing the quantitative analysis.

Price-dividend ratio: Derivations

Notice that we can rewrite the solution for $z_{m,t}$ (i.e., equation (A29)) as the sum of the usual RE result and a backward-looking component

$$z_{m,t} = \underbrace{b_t \left[A_{0,m} + \frac{\phi - \frac{1}{\psi}}{1-\kappa_{1,m}\rho} x_t \right]}_{\text{usual RE model results, } z^{RE}_{m,t}} + \underbrace{(b_t - 1)\left[A_{0,m} + \sum_{j=1}^{T} (\frac{1}{\kappa_{1,m}})^j \left(\phi - \frac{1}{\psi}\right) x_{t-j} \right]}_{\text{backward-looking eq., } z^{b}_{m,t}} \tag{A30}$$

When $b_t = 1$, the result is back to the usual RE solution, while for $b_t \neq 1$, the asset price can deviate from their fundamental values. Moreover, the deviation from the stable solution is very persistent. To see this, define $\hat{z}$ as the deviation from the usual RE solution, and therefore we have

$$\begin{aligned}
\hat{z}_{m,t+1} &\equiv z_{m,t+1} - z^{RE}_{m,t+1} \\
&= \sum_{j=1}^{T} (\frac{1}{\kappa_{1,m}})^j \, (b_{t+1}-1) \left(\phi - \frac{1}{\psi}\right) x_{t+1-j} + (b_{t+1}-1) \frac{1}{1-\kappa_{1,m}\rho} \left(\phi - \frac{1}{\psi}\right) x_{t+1} \\
&= (b_{t+1}-1) \left(\phi - \frac{1}{\psi}\right) \left(\frac{1}{\kappa_{1,m}} \sum_{j=1}^{T} \left(\frac{1}{\kappa_{1,m}}\right)^j x_{t-j} + \frac{1}{1-\kappa_{1,m}\rho} x_{t+1} + \frac{1}{\kappa_{1,m}} x_t \right) - \mathit{\Gamma}_{t-T} \\
&= (b_{t+1}-1) \left(\frac{1}{\kappa_{1,m}} \left(\frac{1}{(b_t - 1)} \hat{z}_{m,t} - \frac{1}{1-\kappa_{1,m}\rho} (\phi - \frac{1}{\psi}) x_t \right) + \frac{\phi - \frac{1}{\psi}}{1-\kappa_{1,m}\rho} x_{t+1} + \frac{1}{\kappa_{1,m}} (\phi - \frac{1}{\psi}) x_t \right) - \mathit{\Gamma}_{t-T} \\
&= \frac{1}{\kappa_{1,m}} \frac{b_{t+1}-1}{b_t - 1} \hat{z}_{m,t} + (b_{t+1}-1) \left(\phi - \frac{1}{\psi}\right) \frac{1}{1-\kappa_{1,m}\rho} \varphi_e \sigma e_{t+1} - \mathit{\Gamma}_{t-T}
\end{aligned}$$

where $\mathit{\Gamma}_{t-T}$ denotes the memory loss, and it is

$$\mathit{\Gamma}_{t-T} = \frac{1}{\kappa^T_{1,m}} (b_{t+1}-1) \left(\phi - \frac{1}{\psi}\right) x_{t-T}$$

The last two steps follow that

$$\hat{z}_{m,t} \equiv z_{m,t} - z^{RE}_{m,t} = (b_t - 1) \sum_{j=1}^{T} (\frac{1}{\kappa_{1,m}})^j (\phi - \frac{1}{\psi}) x_{t-j} + (b_t - 1) \frac{1}{1-\kappa_{1,m}\rho} (\phi - \frac{1}{\psi}) x_t$$

And simple rearrange we get

$$\sum_{j=1}^{T} (\frac{1}{\kappa_{1,m}})^j (\phi - \frac{1}{\psi}) x_{t-j} = \frac{1}{(b_t - 1)} \hat{z}_{m,t} - \frac{1}{1-\kappa_{1,m}\rho} (\phi - \frac{1}{\psi}) x_t$$

The conditional variance of $z_{m,t+1}$ under finite memory is given by

$$\mathbb{V}ar_t(z_{m,t+1}) = \left[\frac{z_{m,t} - z^{RE}_{m,t}}{\kappa_{1,m}(b_t - 1)} \right]^2 \sigma_b^2 + \left[b_t (\phi - \frac{1}{\psi}) \left(\frac{\varphi_e \sigma}{1-\kappa_{1,m}\rho} \right) \right]^2 \sigma^2, \quad \text{for } b_t \neq 1$$

$$\mathbb{V}ar_t(z_{m,t+1}) = (\phi - \frac{1}{\psi})^2 \left(\frac{\varphi_e \sigma}{1-\kappa_{1,m}\rho} \right)^2 \sigma^2, \quad \text{for } b_t = 1$$

It is derived as followed: for $b_t \neq 1$, the variance of $z_{m,t+1}$

$$\begin{aligned}
\mathbb{V}ar_t(z_{m,t+1}) &= \mathbb{V}ar_t \left((\phi - \frac{1}{\psi}) \left[\sum_{j=1}^{T+1} (\frac{1}{\kappa_{1,m}})^j (b_{t+1}-1) x_{t+1-j} + b_{t+1} \frac{1}{1-\kappa_{1,m}\rho} x_{t+1} \right] \right) \\
&= \mathbb{V}ar_t \left((\phi - \frac{1}{\psi}) \left[b_{t+1} \left(\sum_{j=1}^{T+1} (\frac{1}{\kappa_{1,m}})^j x_{t+1-j} + \frac{1}{1-\kappa_{1,m}\rho} \rho x_t \right) + b_{t+1} \frac{\varphi_e \sigma}{1-\kappa_{1,m}\rho} e_{t+1} \right] \right) \\
&= \mathbb{V}ar_t \left(\left[b_{t+1} \frac{1}{\kappa_{1,m}} \left(z^{RE}_{m,t} - z^b_{m,t} \right) + b_{t+1} (\phi - \frac{1}{\psi}) \frac{\varphi_e \sigma}{1-\kappa_{1,m}\rho} e_{t+1} \right] \right) \\
&= \frac{\sigma_b^2}{\kappa_{1,m}^2} \left(z^{RE}_{m,t} - z^b_{m,t} \right)^2 + b_t^2 (\phi - \frac{1}{\psi})^2 \left(\frac{\varphi_e \sigma}{1-\kappa_{1,m}\rho} \right)^2 + \sigma_b^2 (\phi - \frac{1}{\psi})^2 \left(\frac{\varphi_e \sigma}{1-\kappa_{1,m}\rho} \right)^2 \\
&= \frac{\sigma_b^2}{\kappa_{1,m}} \left(\frac{z_{m,t} - z^{RE}_{m,t}}{1-b_t} \right)^2 + b_t^2 (\phi - \frac{1}{\psi})^2 \left(\frac{\varphi_e \sigma}{1-\kappa_{1,m}\rho} \right)^2 + \sigma_b^2 (\phi - \frac{1}{\psi})^2 \left(\frac{\varphi_e \sigma}{1-\kappa_{1,m}\rho} \right)^2 \\
&\approx \frac{\sigma_b^2}{\kappa_{1,m}} \left(\frac{z_{m,t} - z^{RE}_{m,t}}{1-b_t} \right)^2 + b_t^2 (\phi - \frac{1}{\psi})^2 \left(\frac{\varphi_e \sigma}{1-\kappa_{1,m}\rho} \right)^2
\end{aligned}$$

Equity Premium: derivations

The derivation of the equity premium under finite memory follows the same steps as the decay memory model. First, stochastic discount factor under finite memory can be derived as

$$\begin{aligned}
m_{t+1} &= \theta \log \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1) r_{a,t+1} \\
&= \theta \log \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + g_{c,t+1})
\end{aligned} \tag{A31}$$

Substituting the equilibrium return for $r_{a,t+1}$ into the equation, it is straightforward to show that the innovation to the pricing kernel is

$$\begin{aligned}
m_{t+1} - \bar{\mathbb{E}}_t m_{t+1} &= -\frac{\theta}{\psi} g_{c,t+1} + (\theta - 1)(\kappa_1 z_{t+1} + g_{c,t+1}) - \bar{\mathbb{E}}_t \left[-\frac{\theta}{\psi} g_{c,t+1} + (\theta - 1)(\kappa_1 z_{t+1} + g_{c,t+1}) \right] \\
&= \left(-\frac{\theta}{\psi} + \theta - 1 \right) (g_{c,t+1} - \mathbb{E}_t g_{c,t+1}) + (\theta - 1)\kappa_1 (z_{t+1} - \mathbb{E}_t z_{t+1}) \\
&= -\left(1 - \theta + \frac{\theta}{\psi} \right) \sigma \eta_{t+1} - (1-\theta)\kappa_1 (1 - \frac{1}{\psi}) \left[\sum_{j=1}^{T} (\frac{1}{\kappa_1})^j x_{t+1-j} + \frac{1}{1-\kappa_1 \rho} \rho x_t \right] \sigma_b \xi_{t+1} \\
&\quad - (1-\theta)\kappa_1 (1 - \frac{1}{\psi}) b_{t+1} \frac{\varphi_e}{1-\kappa_1\rho} \sigma e_{t+1} \\
&= -\lambda_{m,\eta} \sigma \eta_{t+1} - \lambda_{m,\xi,t+1} \sigma_b \xi_{t+1} - \lambda_{m,e,t+1} \sigma e_{t+1}
\end{aligned} \tag{A32}$$

The expressions $\lambda_{m,\eta}$, $\lambda_{m,e,t+1}$ and $\lambda_{m,\xi,t+1}$ captures the pricing kernel's exposure to the independent consumption shocks, η_{t+1}, to the expected growth rate shock, e_{t+1}, and to the time-varying expectation shock, ξ_{t+1}.

The risk premium for any asset is determined by the conditional variance between the innovations in return and the innovations in the stochastic discount factor. Thus the risk premium for the market portfolio $r_{m,t+1}$ is equal to $\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = -cov(m_{t+1} - \mathbb{E}_t\, m_{t+1}, r_{m,t+1} - \mathbb{E}_t\, r_{m,t+1}) - 0.5\mathbb{V}ar_t(r_{m,t+1})$.

Equation (A32) already provides the innovation in m_{t+1}. We now proceed to derive the innovation in the market return.

$$
\begin{aligned}
r_{m,t+1} - \bar{\mathbb{E}}_t(r_{m,t+1}) =& \kappa_{1,m}(z_{m,t+1} - \bar{\mathbb{E}}_t z_{m,t+1}) + \left(g_{d,t+1} - \bar{\mathbb{E}}_t g_{d,t+1}\right) \\
=& \varphi_d \sigma u_{t+1} + \kappa_{1,m}(\phi - \frac{1}{\psi}) \left[\sum_{j=1}^{T} (\frac{1}{\kappa_{1,m}})^j x_{t+1-j} + \frac{1}{1-\kappa_{1,m}\rho} x_{t+1}\right] \sigma_b \xi_{t+1} \\
&+ \kappa_{1,m} b_{t+1} \frac{1}{1-\kappa_{1,m}\rho}(\phi - \frac{1}{\psi})\varphi_e \sigma e_{t+1} \\
=& \beta_{m,u}\sigma u_{t+1} + \beta_{m,e,t+1}\sigma e_{t+1} + \beta_{m,\xi,t+1}\sigma_b \xi_{t+1}
\end{aligned} \tag{A33}
$$

Moreover, it follows that

$$
\mathbb{V}ar_t(r_{m,t+1}) = (\beta_{m,u} + \beta_{m,e,t+1})^2\sigma^2 + \beta^2_{m,\xi,t+1}\sigma_b^2 \tag{A34}
$$

Using the innovations in the market return and the pricing kernel, the expression for the equity premium is time-varying

$$
\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = \lambda_{m,e,t}\beta_{m,e,t}\sigma^2 + \lambda_{m,\xi,t}\beta_{m,\xi,t}\sigma_b^2 - 0.5\mathbb{V}ar_t(r_{m,t+1}) \tag{A35}
$$

The market compensation for expectation variation risks is determined by $\lambda_{m,\xi,t}\beta_{m,\xi,t}$.

D Derivation of the asset pricing model in a simple Lucas (1978) setting

This section introduces the time-varying expectation formation process with decay memory in a simple version of the Lucas (1978) model.

The usual asset pricing equation

$$
\bar{\mathbb{E}}_t\left[M_{t+1}R_{i,t+1}\right] = 1 \tag{A36}
$$

where M_{t+1} is the stochastic discount factor (SDF) that in case of CRRA preferences is given by $\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$.

We can write the asset pricing equation in terms of price

$$
\mathbb{E}_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right)\right] = 1
$$

Rearrange gives

$$\frac{P_t}{D_t} = \mathbb{E}_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{D_{t+1}}{D_t}\right)\left(\frac{P_{t+1}}{D_{t+1}}\right)\right] + \mathbb{E}_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{D_{t+1}}{D_t}\right)\right] \tag{A37}$$

Denote that

$$\eta_{t+1} \equiv \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{D_{t+1}}{D_t}\right) = \alpha^{1-\gamma}\left(\varepsilon^c_{t+1}\right)^{-\gamma}\varepsilon^d_{t+1}$$

with $\mathbb{E}_t\eta_{t+1} = \alpha^{1-\gamma}\rho_\varepsilon$. where

$$\rho_\varepsilon = \mathbb{E}_t[(\varepsilon^c_{t+1})^{-\gamma}\varepsilon^d_{t+1}] = e^{\gamma(1+\gamma)\frac{s_c^2}{2}}e^{-\gamma\rho_{c,d}s_c s_d} \tag{A38}$$

The equation (A37) becomes

$$\frac{P_t}{D_t} = \delta\mathbb{E}_t\left[\eta_{t+1}\left(\frac{P_{t+1}}{D_{t+1}}\right)\right] + \delta\mathbb{E}_t\left[\eta_{t+1}\right] \tag{A39}$$

Notice that the above equation has both the current and expected value of price-dividend ratio, the solution of the above equation given by

$$\frac{P_t}{D_t} = (b_t - 1)\left(\sum_{j=0}^{\infty}\left(\frac{\lambda}{\delta}\right)^j\left(\prod_{i=0}^{j}\eta_{t-i}\right)^{-1}\eta_{t-j}\right) + b_t\frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon}$$

Derivation:
Starting from the following equation

$$\frac{P_t}{D_t} = \delta\mathbb{E}_t\left[\eta_{t+1}\left(\frac{P_{t+1}}{D_{t+1}}\right)\right] + \delta\mathbb{E}_t\left[\eta_{t+1}\right]$$

Knowing that the solution can be written as a combination of backward-looking solution and a forward-looking one, the forward-looking solution is straight forward

$$PD^f = \frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon}$$

and the backward-looking solution under decay memory is derived by moving the equation (A39) one period backward, and under the decay memory, apply a decay factor to past observation

$$\lambda\frac{P_{t-1}}{D_{t-1}} = \delta\eta_t\left(\frac{P_t}{D_t}\right) + \delta\eta_t$$

$$\begin{aligned}
\Rightarrow \quad \frac{P_t}{D_t} =& \frac{\lambda}{\delta\eta_t}\frac{P_{t-1}}{D_{t-1}} - 1 \\
=& \frac{\lambda}{\delta\eta_t}\left(\frac{\lambda}{\delta\eta_{t-1}}\frac{P_{t-2}}{D_{t-2}} - 1\right) - 1 \\
=& \frac{\lambda}{\delta\eta_t}\frac{\lambda}{\delta\eta_{t-1}}\left(\frac{\lambda}{\delta\eta_{t-2}}\frac{P_{t-3}}{D_{t-3}} - 1\right) - 1 - \frac{\lambda}{\delta\eta_t}
\end{aligned}$$

$$
\begin{aligned}
&\dots \\
&= -1 - \frac{\lambda}{\delta}(\eta_t)^{-1} - \left(\frac{\lambda}{\delta}\right)^2 (\eta_t \eta_{t-1})^{-1} - \dots \\
&= -\sum_{j=0}^{\infty} \left(\frac{\lambda}{\delta}\right)^j \left(\prod_{i=0}^{j} \eta_{t-i}\right)^{-1} \eta_{t-j}
\end{aligned}
$$

Proof:
Plug the solution back to the differential equation

$$
\begin{aligned}
&(b_t - 1)\left(\sum_{j=0}^{\infty} \left(\frac{\lambda}{\delta}\right)^j \left(\prod_{i=0}^{j} \eta_{t-i}\right)^{-1} \eta_{t-j}\right) + b_t \frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon} \\
&= \delta\mathbb{E}_t \left[\eta_{t+1}\left((b_{t+1}-1)\left(\sum_{j=0}^{\infty} \left(\frac{1}{\delta}\right)^j \left(\prod_{i=0}^{j} \eta_{t+1-i}\right)^{-1} \lambda^{j-1}\eta_{t+1-j}\right) + b_{t+1}\frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon}\right)\right] + \delta\mathbb{E}_t[\eta_{t+1}]
\end{aligned}
$$

The R.H.S can be rewritten as

$$
\begin{aligned}
LHS =& \delta\mathbb{E}_t \left[\left((b_{t+1}-1)\left(\sum_{j=0}^{\infty} \left(\frac{1}{\delta}\right)^j \left(\prod_{i=1}^{j} \eta_{t+1-i}\right)^{-1} \lambda^{j-1}\eta_{t+1-j}\right) + b_{t+1}\eta_{t+1}\frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon}\right)\right] + \delta\mathbb{E}_t[\eta_{t+1}] \\
=& \delta\left[\left((b_t-1)\left(\frac{1}{\delta}\sum_{j=0}^{\infty} \left(\frac{1}{\delta}\right)^j \left(\prod_{i=0}^{j} \eta_{t-i}\right)^{-1} \lambda^{j}\eta_{t-j} + \alpha^{1-\gamma}\rho_\varepsilon\right) + b_t\alpha^{1-\gamma}\rho_\varepsilon\frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon}\right)\right] + \delta\alpha^{1-\gamma}\rho_\varepsilon
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(b_t - 1)\left(\sum_{j=0}^{\infty} \left(\frac{\lambda}{\delta}\right)^j \left(\prod_{i=0}^{j} \eta_{t-i}\right)^{-1} \eta_{t-j}\right) + b_t \frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon} \\
&= (b_t - 1)\left(\sum_{j=0}^{\infty} \left(\frac{\lambda}{\delta}\right)^j \left(\prod_{i=0}^{j} \eta_{t-i}\right)^{-1} \eta_{t-j}\right) + (b_t-1)\,\delta\alpha^{1-\gamma}\rho_\varepsilon + b_t\delta\alpha^{1-\gamma}\rho_\varepsilon\frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon} + \delta\alpha^{1-\gamma}\rho_\varepsilon
\end{aligned}
$$

A simple calculation would give

$$
b_t\frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon} = (b_t-1)\,\delta\alpha^{1-\gamma}\rho_\varepsilon + b_t\delta\alpha^{1-\gamma}\rho_\varepsilon\frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon} + \delta\alpha^{1-\gamma}\rho_\varepsilon
$$

$$
b_t\frac{1}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon} = b_t + b_t\frac{\delta\alpha^{1-\gamma}\rho_\varepsilon}{1-\delta\alpha^{1-\gamma}\rho_\varepsilon}
$$

$$
b_t = b_t
$$

Q.E.D.

Risk-free rate solves

$$1 = \delta\left(1+r_t\right)\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]$$

Therefore the risk-free rate is given by

$$\begin{aligned} 1+r_t &= \left(\delta\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]\right)^{-1} \\ &= \left(\delta\alpha^{-\gamma}\mathbb{E}_t\left[\exp\left(-\gamma\log\varepsilon^c_{t+1}\right)\right]\right)^{-1} \end{aligned}$$

As ε^c_{t+1}follows a log-normal distribution, i.e., $\log\varepsilon^c_{t+1} \sim N\left(-\frac{s_c^2}{2}, s_c^2\right)$, which implies that $-\gamma\log\varepsilon^c_{t+1} \sim N\left(-\frac{\gamma s_c^2}{2}, \gamma^2 s_c^2\right)$, and applying the log-normal propertiy,

$$\mathbb{E}_t\left[\exp\left(-\gamma\log\varepsilon^c_{t+1}\right)\right] = \exp\left(-\frac{\gamma s_c^2}{2} + \frac{1}{2}\gamma^2 s_c^2\right)$$

Therefore, the risk-free rate is given by

$$1+r_t = \left(\delta\alpha^{-\gamma}\exp\left(-\frac{\gamma s_c^2}{2} + \frac{1}{2}\gamma^2 s_c^2\right)\right)^{-1}$$