Testing monotonicity of treatment effects: a transformation model approach

Arek Szydłowski

University of Leicester

#StandWithUkraine

Introduction

Testing problem

- Does a randomly assigned treatment X have a diminishing/increasing effect on the outcome Y?
- Examples:
  - Is demand for loans convex in the interest rate?
  - Does a subsidy have a diminishing effect on production?

Transformation model approach

$$T(Y) = X'\beta + \varepsilon$$

where:

- Y - a scalar dependent variable
- X - a vector of non-degenerate explanatory variables with continuously distributed first component
- $T(\cdot)$ - an increasing function
- ε - an unobserved error term, independent of X

Location and scale normalizations:

- $T(y_0) = 0$
- $\beta_1 = 1$

Motivation for testing $T'' \lesseqgtr 0$

$Y = T^{-1}(X'\beta + \varepsilon)$ implies:

$$\frac{\partial^2 E(Y|X)}{\partial X_k^2} = -\beta_k^2 E\left[\frac{T''(T^{-1}(X'\beta+\varepsilon))}{T'(T^{-1}(X'\beta+\varepsilon))^3}\middle| X\right] \qquad \text{(mean regression)}$$

and

$$\frac{\partial^2 Q(Y|X)}{\partial X_k^2} = -\beta_k^2 \frac{T''(T^{-1}(X'\beta))}{T'(T^{-1}(X'\beta))^3} \qquad \text{(quantile regression)}$$

As $T'(\cdot) > 0$ the curvature of the mean/quantile effect depends on $T''(\cdot)$.

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- Shows that the critical values can be approximated by bootstrap.
- Test requires only one-dimensional kernel smoothing under weak conditions on the bandwidth rate.
- Application to loan demand shows that loan size is (mostly) convex in interest rate.
- Other applications of the test: specification search, testing monotonicity of hazard in duration models.

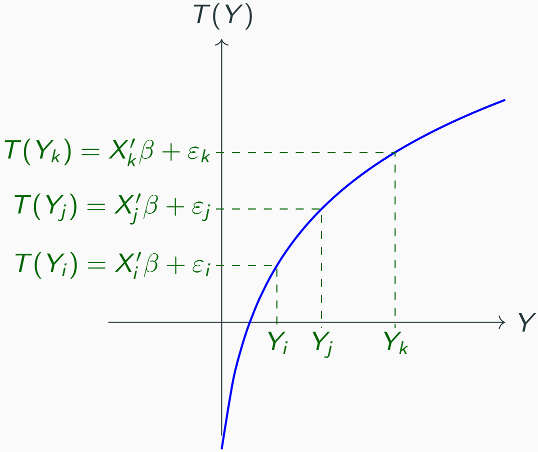
Literature review

- Testing monotonicity and curvature in the nonparametric regression model $Y = g(X) + \varepsilon$:
  Ghosal et al. (2000), Abrevaya, Jiang (2005), Gutknecht (2016), Chetvertikov (2019), Komarova, Hidalgo (2022)
- Testing for causal effects in a generalised regression model:
  Abrevaya et al. (2010)
- Specification testing in transformation models:
  Neumeyer et al. (2016), Szydłowski (2020)
- Specification testing in duration models: Hall, van Keilegom (2005)

Test statistic and critical value

Main idea

- Take $Y_i < Y_j < Y_k$ such that $T(Y_k) - T(Y_j) = T(Y_j) - T(Y_i)$:



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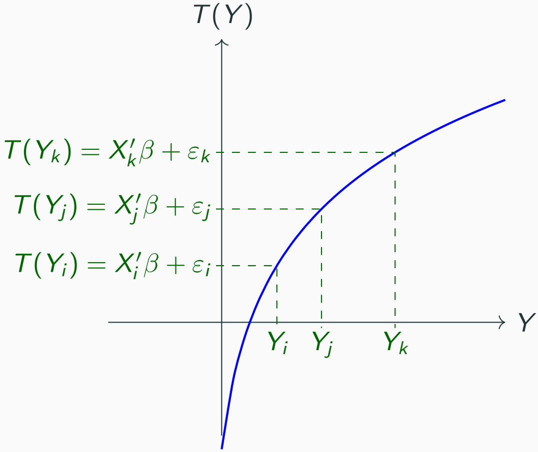

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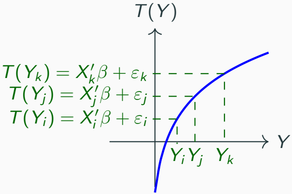

- Concavity of T implies: $Y_k - Y_j > Y_j - Y_i$.
- $T(Y_k) - T(Y_j) = T(Y_j) - T(Y_i)$ is equivalent to:

$$(X_k - X_j)'\beta + \varepsilon_k - \varepsilon_j = (X_j - X_i)'\beta + \varepsilon_j - \varepsilon_i$$

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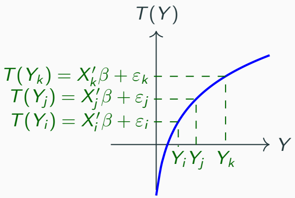

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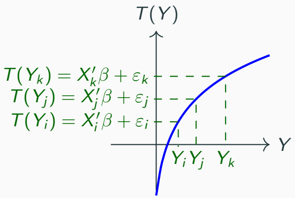

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- Hence, "on average" equally spaced $T(Y)$'s means equally spaced index values $X'\beta$.
- We can detect deviations from concavity by:

$$-\sum_{i \neq j \neq k} \mathbb{1}\{Y_i < Y_j < Y_k\}\mathbb{1}\{Y_k - Y_j < Y_j - Y_i\}\mathbb{1}\{(X_k - X_j)'\beta = (X_j - X_i)'\beta\}$$

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- With continuous $X'\beta$ and unknown β we need to use:

$$-\sum_{i\neq j\neq k} \mathbb{1}\{Y_i < Y_j < Y_k\}\mathbb{1}\{Y_k - Y_j < Y_j - Y_i\}K_h\left((X_k - X_j)'\hat{\beta} - (X_j - X_i)'\hat{\beta}\right)$$

  where $K_h(\cdot) = h^{-1}K(\cdot/h)$ is a smooth kernel and $\hat{\beta}$ is a consistent estimator of β.

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- Similarly, deviations from **convexity** can be detected by using:

$$\sum_{i\neq j\neq k} \mathbb{1}\{Y_i < Y_j < Y_k\}\mathbb{1}\{Y_k - Y_j > Y_j - Y_i\}K_h\left((X_k - X_j)'\hat{\beta} - (X_j - X_i)'\hat{\beta}\right)$$

Test statistic: "global" test

- Test statistic S_n:

$$S_n = \frac{\sqrt{n}}{n(n-1)(n-2)} \sum_{i \neq j \neq k} \mathbb{1}\{Y_i < Y_j < Y_k\} sgn(Y_k - 2Y_j + Y_i) K_h \left((X_{kj} - X_{ji})' \hat{\beta} \right)$$

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- Null and alternative hypotheses:

$$H_0 : \begin{cases} T(\cdot) \text{ is concave} \\ T(\cdot) \text{ is linear} \\ T(\cdot) \text{ is convex} \end{cases} \quad vs \quad H_A : \begin{cases} T(\cdot) \text{ is non-concave} \\ T(\cdot) \text{ is non-linear} \\ T(\cdot) \text{ is non-convex} \end{cases}$$

Rejection rule

Proposition 1

Assume that the distribution of ε is symmetric. Then, as $n \to \infty$, $n^{-1/2} S_n \to^p \theta$ where:

(i) $\theta \geq 0$ *if* $T(\cdot)$ *is globally concave,*

(ii) $\theta = 0$ *if* $T(\cdot)$ *is globally linear,*

(iii) $\theta \leq 0$ *if* $T(\cdot)$ *is globally convex.*

For example, reject concavity if S_n large negative.

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    and calculate S_n^* on this bootstrap sample.

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    and calculate S_n^* on this bootstrap sample.
  - Approximate the critical value c_α^* by the α quantile of S_n^* across bootstrap samples.

Critical values

- Assume $nh^4/\log^4 n \to \infty$ and other standard conditions.
- Under H_0: S_n approximately distributed $N(0, \sigma^2)$.
- At 5% level we conclude that:
  - T non-concave if $S_n < c^*_{0.05}$
  - T non-convex if $S_n > c^*_{0.95}$
  - T non-linear if $S_n < c^*_{0.025}$ or $S_n > c^*_{0.975}$

Global versus local test

Power of "global" test

$$S_n(y) = \frac{\sqrt{n}}{n(n-1)(n-2)} \sum_{i \neq j \neq k} \mathbb{1}\{Y_i < Y_j < Y_k\} sgn(Y_k - 2Y_j + Y_i) K_h \left((X_{kj} - X_{ji})' \hat{\beta} \right)$$

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- Need a test that will have power against local deviations.

Local test

- Define the local statistic at point y by:

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- Test for **concavity** with power against local deviations from the null:

$$S_n^{conc} = \inf_{y \in \mathcal{Y}} S_n(y)$$

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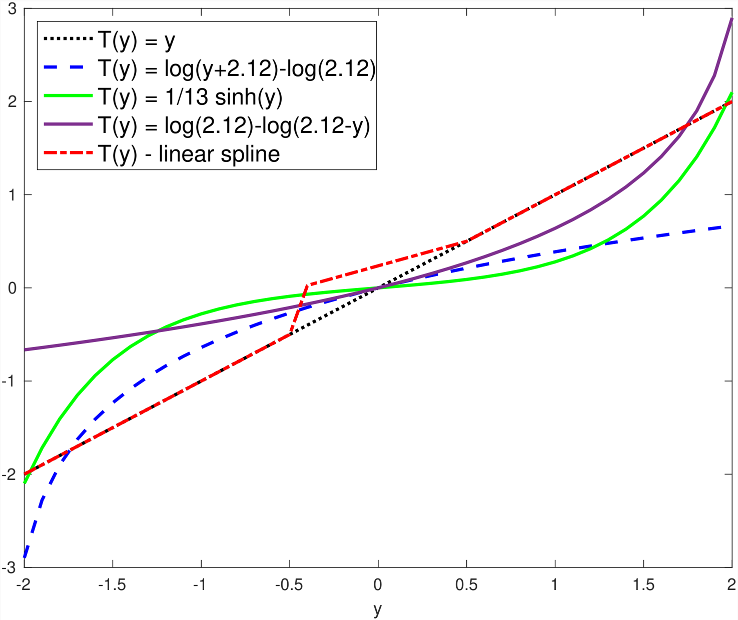
- Critical value: percentile bootstrap as the one for the "global" statistic.

Monte Carlo simulations

Monte Carlo design

- $X \sim N(0,1)$, $\varepsilon \sim N(0,1)$
- 500 bootstrap replications
- 1000 MC repetitions
- 5% level
- Rule-of-thumb bandwidths
- Testing concavity

Figure 1: Monte Carlo design



Monte Carlo simulations

Table 1: Rejection probabilities, 5% level

		Global test			Local test		
		$n = 100$	$n = 250$	$n = 500$	$n = 100$	$n = 250$	$n = 500$
H0 true	D0	0.072	0.057	0.050	0.038	0.048	0.051
H0 true	D1	0.000	0.000	0.000	0.000	0.000	0.000
H0 false	D2	0.093	0.090	0.093	0.896	0.994	1.000
H0 false	D3	1.000	1.000	1.000	1.000	1.000	1.000
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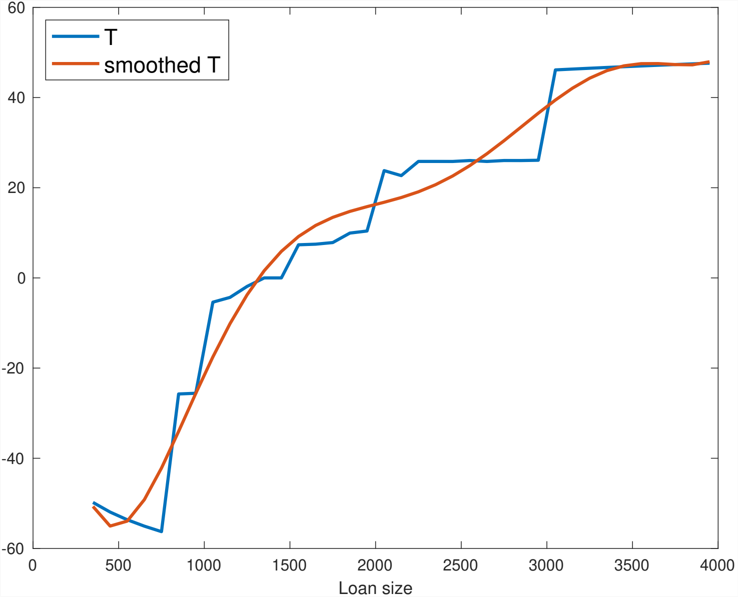
Application: Shape of loan demand in developing countries

Experimental data from South Africa

- Data from Karlan, Zinman, AER 2008.
- Randomised trials conducted by a consumer lender in South Africa.
- High-risk consumer loan market.
- Three mailer waves.
- Interest rate randomisation stratified by three risk categories.
- Controls: risk, wave.
- 2325 observations.

Figure 2: Estimated transformation



Curvature test

Table 2: Testing curvature of loan demand, 5% level

H_0	Global test		Local test		Local test, *loan* ≥ 1000	
	Statistic	Reject H_0?	Statistic	Reject H_0?	Statistic	Reject H_0?
convexity	14.11	No	-4.42	Yes	-2.64	No
linearity	14.11	Yes				
concavity	14.11	Yes				

Conclusion

- Easy-to-apply test for transformation curvature in linear models.
- Similar idea can be applied to a single-index model:

$$Y = T(X'\beta) + \varepsilon$$

e.g. test of non-linearity.

Thank you for your attention!