Identification of Nonlinear Time Series Models with Additive Noise

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Introduction

- Statistical identificaton in a SVAR model have mainly focused on exploiting:
  - Heteroskedasticity (see e.g. Rigobon 2003, Normadin and Phaneuf 2004, Lanne and Saikkonen 2007).
  - Non-Gaussianity (see e.g. Gourieroux et al. 2017, Lanne et al. 2017, Herwartz 2018).
- In this paper, we emphasize another statistical property which can be used for the sake of identification, namely nonlinearity.

Outline

- How nonlinearity may contribute to identification
- Specific identification scheme (RESIT)
- Non Linear impulse response functions
- Simulation results

Statistical identification

- Statistical identification by non-Gaussianity is an application of ICA (Independent Component Analysis), where the other **key assumption is independence**.

- Identification by nonlinearity relies on a similar idea.

Statistical identification

Independence-based identification

Starting point: Linear VAR identification by non-Gaussianity

$$\begin{cases} y_t = \sum_{\ell=1}^{p} A_\ell y_{t-\ell} + u_t, \\ u_t = A_0 \varepsilon_t \end{cases}$$

- If ε_t is a vector of non-Gaussian and independent shocks, A_0 can be identified up to a (generalized) permutation matrix (see Eriksson and Koivunen 2004; Gourieroux et al. 2017).
- Additionaly assume recursiveness, i.e. A_0 essentialy triangular. Then (under mild conditions) A_0 is uniquely identified (see LiNGAM algorithm in Shimizu et al. 2006; Moneta et al. 2013).

Statistical identification

Independence-based identification

For the sake of illustration, consider the following bivariate model:

$$\boldsymbol{u_{t,1}} = \alpha \boldsymbol{u_{t,2}} + \varepsilon_{t,1}, \quad \varepsilon_{t,1} \perp\!\!\!\perp \boldsymbol{u_{t,2}}$$

Then there exists $\beta \in \mathbb{R}$ and a random variable $\varepsilon_{t,2}$ such that

$$\boldsymbol{u_{t,2}} = \beta \boldsymbol{u_{t,1}} + \varepsilon_{2t}, \quad \varepsilon_{t,2} \perp\!\!\!\perp \boldsymbol{u_{t,1}}$$

if and only if $\varepsilon_{t,1}$ and $u_{t,2}$ are Gaussian.
(Proof follows from Darmois-Skitovic theorem, on which ICA is based; see Peters et al. 2017).

- Thus, under non-Gaussianity, in a bivariate setting, $\varepsilon_{t,1} \perp\!\!\!\perp u_{t,2}$ allows detecting the correct causal direction (NB independence test accounting for higher order statistics, not just correlation)
- This can be extended to a multivariate setting
- But also to a **nonlinear** one.

Statistical identification

Nonlinear additive noise model

(Going nonlinear)

- **Definition**: the joint distribution $P(u_t)$ is said to admit an ANM from $u_{t,j}$ to $u_{t,i}$ if we have

$$u_{t,i} = f_i(u_{t,j}) + \varepsilon_{t,i}, \quad \varepsilon_{t,i} \perp\!\!\!\perp u_{t,j}$$

- **Identifiability of ANM**: Under mild assumptions (f_i 3-times differentiabile, strictly positive densities, etc.) if there is ANM from $u_{t,j}$ to $u_{t,i}$, then there is no backward ANM from $u_{t,i}$ to $u_{t,j}$.
  - Except for few cases, most remarkably, the Gaussian linear case (see Hoyer et al. 2008, Theorem 1, Peters et al. 2014, Prop. 23).

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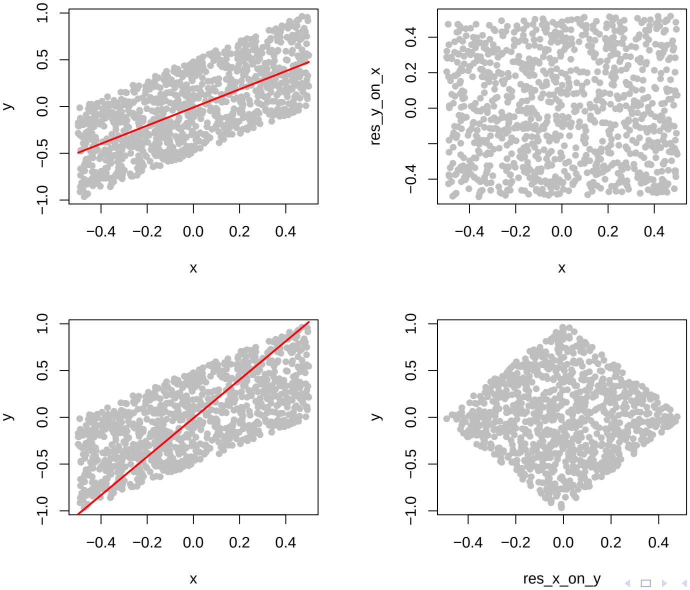
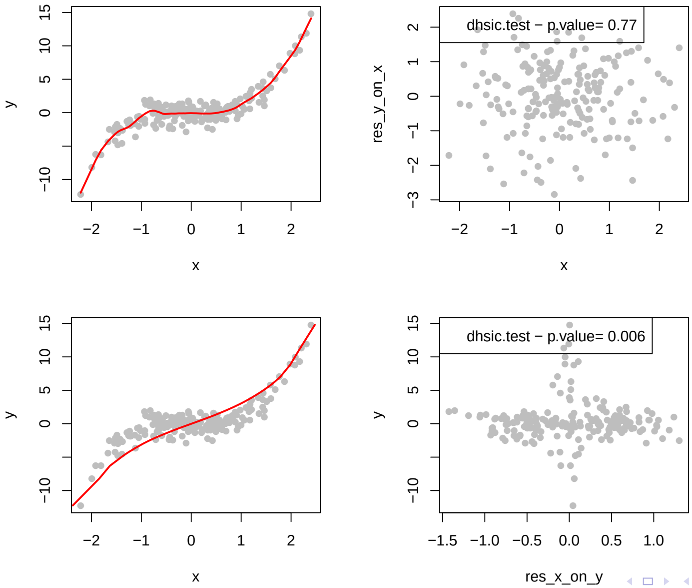
Linear non-Gaussian case

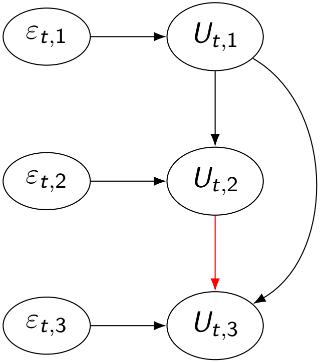


Nonlinear case



Statistical identification

DAG and SEM representations



$$
\begin{cases}
u_{t,1} &= \varepsilon_{t,1} \\
u_{t,2} &= \varphi_2(u_{t,1}) + \varepsilon_{t,2} \\
u_{t,3} &= \varphi_3(u_{t,1}, {\color{red}u_{t,2}}) + \varepsilon_{t,3}
\end{cases}
$$

- DAG and SEM representations of a generic ANM with recursive/topological ordering $u_{t,1}, u_{t,2}, u_{t,3}$.
- In red: elements to remove from each representation to eliminate the direct causation between $u_{t,2}$ and $u_{t,3}$.

Statistical identification

Nonlinear ANM and VAR

(Back to time series modelling)

- Let's consider this class of nonlinear VAR (see e.g. Kilian and Lütkepohl, 2017, ch. 18):

$$\begin{cases} y_t = F_t(y_{t-1}, \ldots, y_{t-p}) + u_t, \\ u_t = G_t(\varepsilon_t) \end{cases}$$

- Assume recursive causal structure among contemporaneous variables, i.e. it can be represented by a DAG (directed acyclic graph) over u_t, so that

$$u_{t,i} = \varphi_i(Pa(u_{t,i})) + \varepsilon_{t,i}$$

where $Pa(u_{t,i})$ is the set of graphical *parents* (i.e. direct causes) of $u_{t,i}$ (NB: only noise additivity is actually required)

Statistical identification

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Statistical identification

Nonlinear SVAR

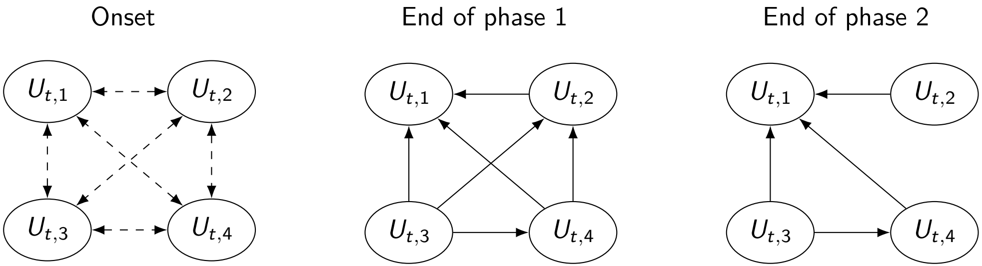
- Then the following nonlinear structural VAR is identifiable:

$$\begin{cases} y_t = F_t(y_{t-1}, \ldots, y_{t-p}) + u_t \\ u_{t,i} = \varphi_i(Pa(u_{t,i})) + \varepsilon_{t,i}, \quad \text{for } i \text{ in } 1, \ldots, k, \end{cases}$$

- Identifiability of the contemporaneous causal order follows from the requirement that in the true structural model, shocks are independent of covariates, i.e. $\varepsilon_{t,i} \perp\!\!\!\perp Pa(u_{t,i})$
- Any order different from the correct one would not satisfy this requirement

RESIT identification scheme

illustration: RESIT phases



RESIT identification scheme

Identification algorithm: Regression with subsequent independence test (RESIT); Peters et al. (2014).
Input: estimated reduced-form VAR residuals u_t.
Output: DAG

- **Phase 1**: determine a fully connected DAG (topological order within u_t).
  - Iterative procedure: in each step we identify a *sink* node. This is done, given a set of variables S, by regressing each variable in S on all the other variables in S and measuring dependence between residuals and covariates (p-value of HSIC independence test). The variable for which the corresponding residual display the weakest dependence on the covariates is denoted as sink and eliminated from S in the next step (until all the variables are ordered).
  - Main idea: in a DAG underlying an ANM for each node $u_{t,i}$ the noise $\varepsilon_{t,i}$ is independent of all non-effects of $u_{t,i}$

RESIT identification scheme

- **Phase 2**: remove superfluous edges.
- Main idea here: one edge from a putative cause x to an effect is superfluous if regressing the effect on the putative causes omitting x one still obtain residuals independent of covariates.
- Thus, if (i) in the DAG output of phase 1, $u_{t,i}$ is a child of $u_{t,j}$ and $u_{t,k}$, and if (ii) $\varepsilon_{t,i} \perp\!\!\!\perp u_{t,k}$, where $\varepsilon_{t,i}$ is obtained in a regression of $u_{t,i}$ on $u_{t,k}$ only (without $u_{t,j}$), then the edge, e.g., $u_{t,j} \rightarrow u_{t,i}$ is cut off.

Impulse Response Functions

Relaxing the linearity assumption implies a need for a more general IRF definition

$\Rightarrow$ Nonlinear impulse responses as differences of conditional expectations:

$$\mathrm{IRF}(h, \delta, \Omega_{t-1}) = E(y_{t+h}|\varepsilon_{t,i} = \delta, \Omega_{t-1}) - E(y_{t+h}|\Omega_{t-1}),$$

- where δ is the (positive or negative) magnitude of the shock $\varepsilon_{t,i}$ one wants to study, Ω_{t-1} is the history of the model data up to time $t-1$, and h is the horizon point up to which the impulse response functions are studied.

- Computed via Monte Carlo integration approach suggested in Kilian and Lutekopohl (2017, Ch. 18)

1. Statistical identification
2. RESIT identification scheme
3. IRF
4. Simulations
   - Causal learning
   - IRFs
5. Conclusions

Simulations

Simulations targeted at two distinct aspects

1. **Causal learning** (or causal discovery)
   - Assess the ability of RESIT to retrieve the true causal structure of the innovations
   - Comparisons with other causal discovery methods
2. Added value of **accounting for short-run non-linearities** for IRFs
   Contrast our IRF identification method with a standard benchmark (Choleski decomposition)

Simulations

Reduced form models for simulation

In all simulation exercises, we opted for simple **linear** VAR(1) models

$$y_t = A_1 y_{t-1} + u_t$$

Motivations:

- Emphasis on modelling contemporaneous structure
- Better use a simple/neutral time series model to highlight effects of contemporaneous linearities

Simulations

Causal learning

Goal: Assess the performance of the ANM principle (*near*-impossibility of backward model) in identifying the contemporaneous causal structure (sufficient for identification), formalized by a DAG.

Simulation model:

- Innovation structure characterized by a sparse (randomly drawn) DAG
- Corresponding dependencies embodied as ANM of two possible classes:
  - Linear effect and non-Gaussian noise (**NG**)
  - Non-linear effect randomly generated from a GAM-GP model (**NL**)

Simulations

Causal learning

- 5 **identification algorithms** (RESIT, LiNGAM, PC, CPC, RAND)
- Performance assessed by calculating distance between true DAG and inferred DAG via '**structural Hamming distance**' (SHD) and '**structural intervention distance**' (SID) (see Peters et al. 2014).

Simulations

Causal learning

Table: Average **SHD** between the estimated and the true DAG, varying the number of observations T For a selected model (columns) the average and standard errors (in round brackets) over 500 simulations are reported for each employed method (rows).

	T = 250		T = 500		T = 1000	
$k = 8$	VAR+NL	VAR+NG	VAR+NL	VAR+NG	VAR+NL	VAR+NG
RESIT	**6.14**	6.15	**5.82**	4.91	**6.57**	3.3
	(0.149)	(0.189)	(0.167)	(0.18)	(0.203)	(0.155)
LINGAM	8.71	**2.53**	9.67	**2.14**	10.53	**1.5**
	(0.143)	(0.094)	(0.16)	(0.098)	(0.178)	(0.094)
PC	6.79	5.57	6.73	5.16	6.75	4.23
	(0.124)	(0.115)	(0.127)	(0.106)	(0.133)	(0.104)
CPC	7.66	6.07	7.55	5.48	7.47	4.53
	(0.134)	(0.125)	(0.131)	(0.117)	(0.137)	(0.112)
RANDOM	16.21	16.05	15.94	15.51	16.21	16.2
	(0.24)	(0.237)	(0.244)	(0.234)	(0.233)	(0.231)

Simulations

Causal learning

Table: Average **SID** between the estimated and the true DAG, varying the number of observations T For a selected model (columns) the average and standard errors (in round brackets) over 500 simulations are reported for each employed method (rows).

	T = 250		T = 500		T = 1000	
$k = 8$	VAR+NL	VAR+NG	VAR+NL	VAR+NG	VAR+NL	VAR+NG
RESIT	**10.65**	13.3	**7.63**	9.71	**6.18**	5.4
	(0.372)	(0.5)	(0.346)	(0.415)	(0.327)	(0.31)
LINGAM	24.29	**7.15**	24.39	**5.76**	23.77	**3.5**
	(0.478)	(0.307)	(0.474)	(0.302)	(0.46)	(0.253)
PC	21.61	23.69	20.54	22.22	19.38	18.36
	(0.512)	(0.576)	(0.512)	(0.541)	(0.502)	(0.518)
CPC	26.9	26.04	25.67	23.44	23.92	19.64
	(0.582)	(0.628)	(0.575)	(0.595)	(0.541)	(0.578)
RANDOM	20.38	21.12	20.04	21.16	19.71	20.16
	(0.499)	(0.494)	(0.469)	(0.479)	(0.48)	(0.496)

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Simulations

IRFs

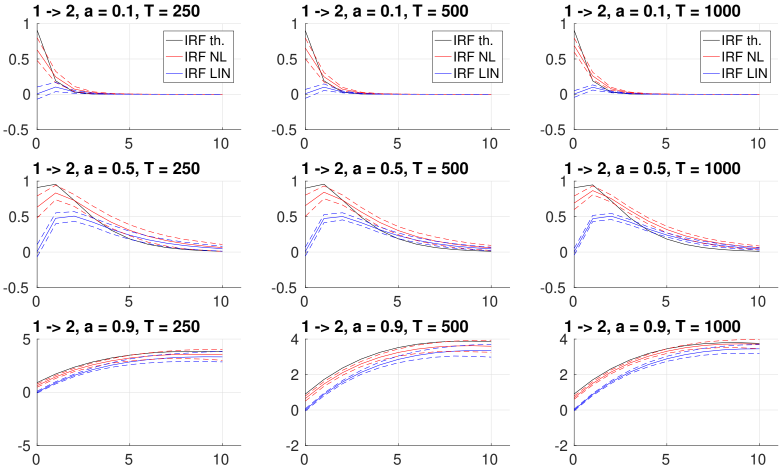
A **nonlinear** causal model for contemporaneous effects:

$$\begin{cases} u_{t,1} &= \varepsilon_{t,1} \\ u_{t,2} &= |u_{t,1}|^{\alpha} + \varepsilon_{t,2} \\ u_{t,3} &= sin(|u_{t,1}|^{\beta}) + \varepsilon_{t,3} \end{cases}$$

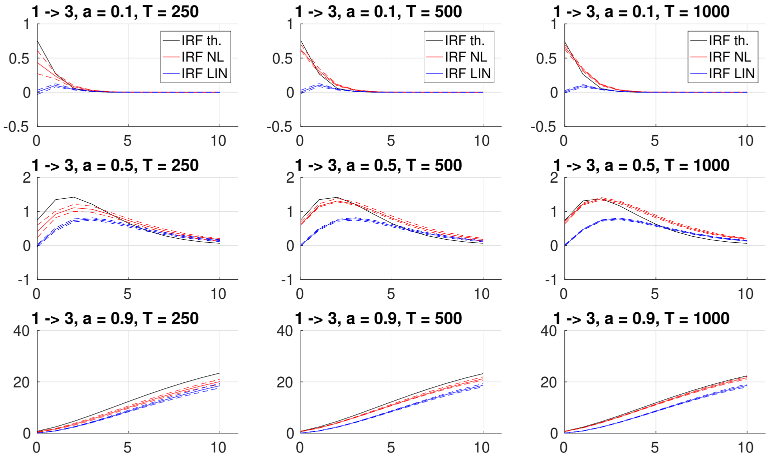
where:

- α, β are drawn from a uniform random variable with support $(1, 4)$, independently from each other
- $\varepsilon_{t,i}$ are the i.i.d. zero-mean uncorrelated Gaussian structural shocks

NB: $u_{t,3} \leftarrow u_{t,1} \rightarrow u_{t,2}$



Theoretical, linear and non-linear IRFs by varying the parameter a and number of observations T. Red (Blue) lines exhibits the average (among 200 simulations) for the non-linear (linear) IRFs response for the **response of variable 2** to a unitary shocks of $\varepsilon_{t,1}$, confidence interval at 68% are reported in dotted lines.



Theoretical, linear and non-linear IRFs by varying the parameter a and number of observations T. Red (Blue) lines exhibits the average (among 200 simulations) for the non-linear (linear) IRFs response for the **response of variable 3** to a unitary shocks of $\varepsilon_{t,1}$, confidence interval at 68% are reported in dotted lines.

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Conclusions

- Identification method:
  - Nonlinearity instrumental for identification in ANM, with similarities and differences with non-Gaussianity in the linear case.
  - RESIT scheme based on independence test between residuals and covariates.
- Simulations:
  - Showed the merits of RESIT approach applied to reduced form residuals
  - Highlighted the discrepancies Linear and non-linear IRF specifications
- Limitations and future research:
  - IRF estimation approach meant to cover the whole class of model

    → Probably better solutions in more specific settings

Thank you for listening

Conclusions

Bonus Slide

Table: Average (%) contemporaneous causal structural estimated by RESIT by varying T. Computed by averaging on the persistence parameters a over 200 simulations.

Var.	T 250	Pa(1)	Pa(2)	Pa(3)	T 500	Pa(1)	Pa(2)	Pa(3)
1		0	48.83	55.83		0	74.17	69.33
2		0.33	0	11.00		0.83	0	13.50
3		1.50	15.50	0		0.33	8.17	0

Var.	T 1000	Pa(1)	Pa(2)	Pa(3)
1		0	91.17	91.67
2		0.50	0	5.33
3		0.83	5.83	0