Belief change, rationality, and strategic reasoning in sequential games

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Introduction

- **Sequential games** are situations of strategic interaction with sequential moves [although in some stages moves may be simultaneous]. **Information sets** represent what players observe about previous moves.
- Extant theories of rationality and strategic thinking in sequential games assume that players would hold subjective beliefs about co-players' behavior conditional on each information set, and carry out strategies that seem optimal given such subjective beliefs.
- Here we focus on the *implications about behavior and plans* of such extant theories, characterized by versions of a set-valued solution concept called "rationalizability." [These theories justify Nash or Subgame Perfect equilibrium only in special cases.]

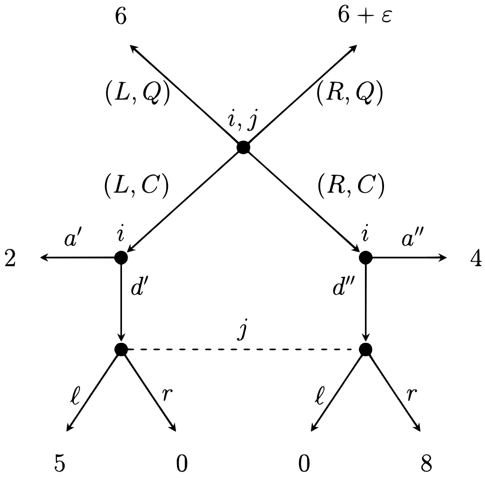
Introduction (cont.)

- *Key question*: How to model cognitively rational belief change and its impact on behavior.
- **Updating**: When new information is consistent with earlier beliefs, some external states (behaviors of others) are ruled out and *the relative probabilities of external states that are still deemed possible do not change.*
- **Revision:** When new information was previously deemed impossible, *new probabilistic beliefs are unconstrained by earlier beliefs*.

Introduction (cont.)

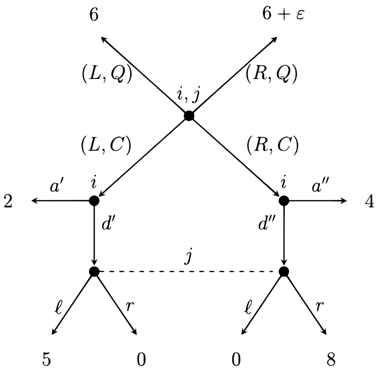
- Extant theories assume *fully introspective players* who know how they would change their beliefs if given new information [maybe "hypothetical information" that cannot be possibly observed as the play unfolds (i.e., does not correspond to information sets)].
  - Does it matter that players are fully introspective? Or is it enough to assume that they always plan and choose according to their current beliefs, applying belief updating when possible?
- **Belief System (BS)**: information $\mapsto$ conditional belief (both updating and revision). We consider different notions of BS complying with weak/strong requirements of cognitive rationality, i.e., weak/strong versions of the *chain rule* of conditional probability: $E \subseteq D \subseteq C \Rightarrow \mu(E|D)\,\mu(D|C) = \mu(E|C)$.
- Then we look at what behavior is *justifiable* as "sequentially optimal" under different types of BSs, and assuming that players hold on to some assumptions about co-players—e.g., that co-players are rational—as long as possible.

- **Common-interest game between Isa (i) and Joe (j)**



  - *Same payoffs for Isa and Joe*. Initial simultaneous move by Isa (L or R) and Joe (Q or C). If C, Isa (who observes it and recall own action) can go across (end) or *d*own. If *d*own, ...

- Suppose Isa is **initially certain** of *Q*uit, but Joe *C*ontinues, surprising her: same belief revision after (L, C) and (R, C)?



  - It only matters for "folding-back planning": $a'.a''$ unjustifiable if same conditional belief after (L, C) and (R, C).
  - If Isa goes *L*eft (e.g., because $\varepsilon < 0$), the planned action after (R, C) does not matter: if $\mu^i(\ell | (L, C)) < \frac{2}{5}$ she carries out the "forward plan" [reduced strategy] $L.a'$.

Sequential games played by agents with perfect recall

- Mathematical representation (we skip most details) with the following primitive and derived elements (everything is assumed finite):
  - **Players**: for simplicity *in slides*, focus on *two players*.
  - For each player i, collection H_i of **information sets** [including "no information" when the game starts at root]. Each $h_i \in H_i$ corresponds to a *personal history of signals received and actions played* by i ["no information"="nothing has happened yet"=root]. Signals may reveal, fully, partially, or not at all, actions previously chosen by $j \neq i$.
  - In example, $h_j = \{((L, C), d'), ((R, C), d'')\}$=You (Joe) Continued, then Isa went *d*own. The first choice of Isa (*L*eft or *R*ight) is not revealed.
  - **Strategies** $s_i \in S_i$ describe information-dependent behavior: *both* how a rational i *plans* to behave, and possible *"ways of behaving"* (descriptions of behavior) considered by j when he thinks about i.
  - **Information about strategies/behavior:** $S_i(h_j)$ (reduced) strategies of i compatible with (not ruled out by) h_j. [In example, $S_i(h_j) = \{L.d', R.d''\}$.]

Systems of beliefs and the forward consistency

- **Systems of Beliefs:** maps

$$\begin{array}{llll} \mu^i : & H_i & \rightarrow & \Delta(S_j) \\ & h_i & \mapsto & \mu^i(\cdot|h_i) \end{array}$$

such that $\mu^i(\cdot|h_i)$ assigns probability 1 to $S_j(h_i)$:

$$\mu^i(S_j(h_i)|h_i) \overset{(\text{def})}{=} \sum_{s_j \in S_j(h_i)} \mu^i(s_j|h_i) = 1.$$

- **Forward consistency**: IF $h_i \prec \bar{h}_i$ [hence, more behaviors of j are ruled out by $\bar{h}_i$, $S_j(\bar{h}_i) \subseteq S_j(h_i)$] then, for each $s_j \in S_j(\bar{h}_i) \subseteq S_j(h_i)$,

$$\mu^i(s_j|h_i) = \mu^i(s_j|\bar{h}_i)\,\mu^i(S_j(\bar{h}_i)|h_i),$$

that is, if $h_i \prec \bar{h}_i$ and updating is possible, update in the standard way:

$$h_i \prec \bar{h}_i \wedge \mu^i(S_j(\bar{h}_i)|h_i) > 0 \Rightarrow \mu^i(s_j|\bar{h}_i) = \frac{\mu^i(s_j|h_i)}{\mu^i(S_j(\bar{h}_i)|h_i)}$$

Systems of beliefs and the complete consistency

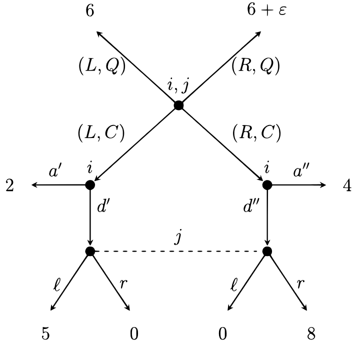
- **Complete consistency:** also "hypotetical updating/revision", what would I believe if I hypotetically knew that $s_j \in C \subseteq S_j$?
- *Key:* condition on subsets of S_j (including those that do not correspond to information sets, see Myerson 1986). For all $\emptyset \neq D \subseteq C \subseteq S_j$ and $s_j \in D$,

$$\bar{\mu}^i (s_j|C) = \bar{\mu}^i (s_j|D)\, \bar{\mu}^i (D|C)\,.$$

- Given this, from $\left(\bar{\mu}^i (\cdot|C)\right)_{\emptyset \neq C \subseteq S_j}$ derive **completely consistent** BS $\left(\mu^i (\cdot|h_i)\right)_{h_i \in H_i}$ letting $\mu^i (\cdot|h_i) = \bar{\mu}^i (\cdot|S_j (h_i))$.
- Thus, in particular, *beliefs about j depend only on information about j* (not on what i did):

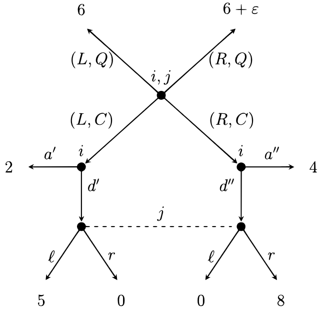
$$S_j (h_i') = S_j (h_i'') \Rightarrow \mu^i (\cdot|h_i') = \mu^i (\cdot|h_i'')\,.$$

- **How much consistency? Issue**



- Consider $h_i = \{\text{root}\}$, $h_i' = \{(L, C)\}$, $h_i'' = \{(R, C)\}$.
- h_i comes before h_i' and h_i'', h_i' and h_i'' are unrelated, but they reveal the *same information about j*: $S_j(h_i') = \{C.\ell, C.r\} = S_j(h_i'')$;
- if $\mu^i(S_j(h_i')|h_i) = 0$,forward consistency does not bite: $\mu^i(\cdot|h_i')$ and $\mu^i(\cdot|h_i'')$ are unrelated; instead, complete consistency implies $\mu^i(C.\ell|h_i') = \mu^i(C.\ell|\{((L, C))\}) = \mu^i(C.\ell|h_i'')$.

- **How much consistency? Discussion**

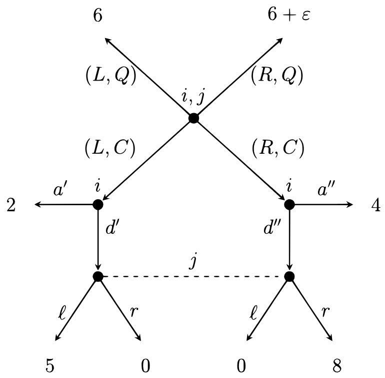


- Different interpretive assumptions about introspection: given that μ^i represents how *i* forms beliefs:
- **Partial introspection:** *Forward consistency* if *i* is only aware of her current beliefs (and understands that she would update) ⇒ not forced to compare *ex ante* $\mu^i(C.\ell|\{(L,C)\})$ with $\mu^i(C.\ell|\{(R,C)\})$.
- **Complete introspection:** *Complete consistency* if *i* knows what she would believe conditional on any (observable or virtual) information ⇒ forced to make all comparisons.

Rational planning

- A belief system (BS) $\mu^i = \left(\mu^i\left(\cdot|h_i\right)\right)_{h_i \in H_i}$ allows to determine "sequential best replies". Two approaches:
  - **Backward planning** starts from "last opportunities to choose", predicts an expected-utility (EU) maximizing choice, and works backward toward the root. This makes sense if player i is completely introspective and knows how she would revise her beliefs when surprised. $\Rightarrow$ Complete consistency is germane to this approach.
  - **Forward planning** starts at the beginning and picks what looks like the best plan; if later i is surprised by j given information set h_i, she picks the best continuation plan from h_i according to revised belief $\mu^i\left(\cdot|h_i\right)$. This makes sense even i is not fully introspective because she does not know how she would revise if surprised, so that BS μ^i is a representation of how i would update/revise, but is not fully known to i. $\Rightarrow$ Forward consistency is enough.
  - **Result:** *These two approaches are equivalent, i.e., they have the same behavioral implications. Behaviorally, the difference between forward and complete consistency doe not matter.*

- **Rational planning, example**



- $\mu^i(\{Q.\ell, Q.r\}) = 1$ (but j picks C), $\mu^i(C.\ell | \{((L,C))\}) = \frac{1}{3} < \frac{2}{5}$.
- If $\varepsilon < 0$, forward planning yields $L.a'$ (Isa need not anticipate $\mu^i(C.\ell | \{((L,C))\})$, she first picks L, when surprised she picks a').
- Backward planning depends on $\mu^i(C.\ell | \{((R,C))\})$, complete consistency yields $\mu^i(C.\ell | \{((R,C))\}) = \frac{1}{3}$ and $L.a'.d''$, but part [d'' if (R,C)] does not matter!

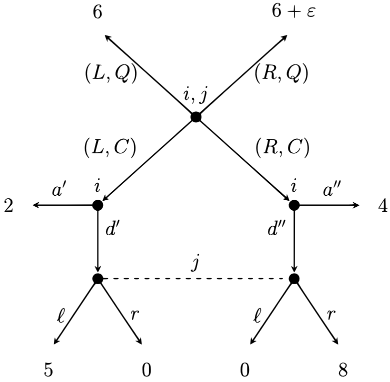
Strong rationalizability

- We consider solution concepts that can be *derived as characterizations of the behavioral implications of assumptions about rationality and strategic reasoning*.
  - Here, we focus on the **strong rationalizability,** an iterated elimination procedure, capturing the "best-rationalization principle" (e.g., Battigalli 1996, Battigalli & Siniscalchi 2002).
  - μ^i **strongly believes** $E_j \subseteq S_j$ [written $\mu^i \in \mathrm{SB}_i(E_j)$] if it assigns prob. 1 to E_j whenever possible: $S_j(h_i) \cap E_j \neq \emptyset \Rightarrow \mu^i(E_j|h_i) = 1$. Let Δ_i=set of forward consistent BSs or set of completely consistent BSs each i); initialize $S_i^{\Delta,0} = S_i$, then recursively define:
  - $S_i^{\Delta,m+1} = \{s_i \in S_i : \exists \mu^i \in \Delta_i \cap (\cap_{k=0}^m \mathrm{SB}_i(S_j^{\Delta,k})), s_i \text{ is } \mu^i\text{-forward-optimal}\}$
  - The set of **strongly rationalizable** strategies for i (given Δ) is $S_i^{\Delta,\infty} = \cap_{k=0}^\infty S_i^{\Delta,k}$.
  - *Interpretation*: each player always ascribes to co-player the highest degree of strategic sophistication consistent with evidence.

Theorem

(**Invariance**) *Whether forward of complete consistency holds does not matter, the strong rationalizability procedure is the same.*

- **Strongly rationalizable behavior: example, $\varepsilon > 0$**



- (1) For Joe, $C.\ell$ is dominated by Q.
- (2) By strong belief in rationality (behaviorally, in $\{Q, C.r\}$), Isa goes *R*ight; since C signals r; even if surprised, Isa after (R, C) would go *d*own. Her belief given (L, C) does not matter.
- (3) Since Joe is certain of Isa's rationality and strong belief in his own rationality, he *C*ontinues and then goes *r*ight.

Discussion

- Our interpretation of the results is that *partial introspection and forward consistency are sufficient to obtain the behavioral implications of rationality and common strong belief in rationality*.
- We also analyze an intermediate form of consistency (called "standard") whereby only conditioning events corresponding to information sets are considered (equivalent to complete consistency in the running example).
- *Similar results can be obtained for other solution concepts characterizing the behavioral implications of rationality and different assumptions about strategic reasoning*, like **weak/initial rationalizability** (Ben Porath 1997), and **backwards rationalizability** (Perea 2014, Battigalli & De Vito 2021).
- The invariance result may break down if extra (contextual) restrictions on conditional beliefs are added.
  - But invariance holds if extra restrictions only concern initial beliefs.

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